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Milnor-Thurston homology theory for some one-dimensional wild topological spaces

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1 Introduction

In various fields of mathematics man sometimes approaches topological spaces with bad local properties. Some examples may be: fractals, attractors in dynamical systems or tiling spaces. We call them "wild" because standard algebraic invariants does not work good here.

A good representative may be the space constructed by Milnor and Barett in 1962 [5], which has nonzero homology groups in dimensions exceeding dimension of the space. Recently people are looking for homology theories that are suitable for "wild" topological spaces [6, 7].

One of the problems we would like to overcome when constructing such theories is finiteness of cycles, which can restrict some constructions. There are some examples of the theories that admit infinite cycles; among them Milnor-Thurston homology theory.

This theory behaves well on CW-complexes and manifolds (it gives the same results as the singular homology, since it satisfies Eilenberg-Steenrod axioms), so it is natural, in order to investigate it, to consider one of the simplest examples of a space which is not a CW-complex: the Warsaw Circle.

In this paper we calculate nonzero Milnor-Thurston homology spaces for the Warsaw Circle, the Two Sided Warsaw Circle (see below) and one simple example of space from [2] (which we shall call the Convergent Arcs Space).

2 Milnor-Thurston homology theory

Milnor-Thurston homology theory is a version of the homology theory which admits infinite chains. It has first appeared in connection with hyperbolic geometry where it was used to prove Gromov theorem [3, Theorem 6.4], which states that two three-dimensional closed oriented hyperbolic manifolds with the same volume are isometric if they admit degree one maps. Few decades later it was formalised by Zastrow [2] and Hansen [1] independently. It has been defined for all topological spaces and it satisfies Eilenberg-Steenrod axioms with weakened Excision Axiom, which is equivalent to the classical one for well behaved spaces (including all normal spaces) [2].

Going more into details, in Milnor-Thurston homology theory k-dimensional chains are finite signed Borel measures on the space of singular simplices $C^0(\Delta^k, X)$ (with compact-open topology), having compact support (support of a measure is a set D such that any $A \subset C^0(\Delta^k, X) \setminus D$ is a zero set). We can see that these chains form a vector space over $\mathbb{R}$. The boundary operator is a natural extension of ∂ in the singular homology theory; that is:

$$\partial := \sum_{i=0}^{n} (-1)^i \partial_i,$$

where ∂_i is defined for measure μ as:

$$\partial_i \mu(A) := \mu(\delta_i^{-1}(A)),$$

with $\delta_i : C^0(\Delta^n, X) \to C^0(\Delta^{n-1}, X)$, which is a map induced by the inclusion of Δ^{n-1} as a face of Δ^n.

Consequently we get a chain-complex of real vector spaces and we refer to its homologies $\mathcal{H}_*$ as Milnor-Thurston homology spaces.

3 Geometric intuition

In this paper we calculate higher Milnor-Thurston homology spaces for three topological spaces: the Warsaw Circle, the Double Warsaw Circle and the Convergent Arcs Space.

The Warsaw Circle (see Figure 1) is a well known space that serves as a counterexample in many cases. It is a subset of $\mathbb{R}^2$ that consists of:

- a part of "Topologists Sine Curve" $\{(x, y) \in \mathbb{R}^2 \mid y = \sin 1/x\}$ from the line $x = 0$ up to the rightmost minimum,
- an accumulation line $\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$,
- an arc connecting point $(0, -1)$ with the rightmost minimum.

By the Double Warsaw Circle (see Figure 3) we mean the space that is a copy of two Warsaw Circles overlapping at the accumulation line.

The Convergent Arcs Space (see Figure 2) is a space built of a countable number of arcs connecting two given vertices. They converge to a line segment between which is also a part of the space. It was used by Zastrow [2]

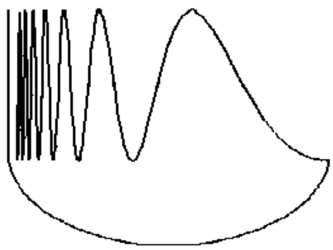

Figure 1: The Warsaw Circle

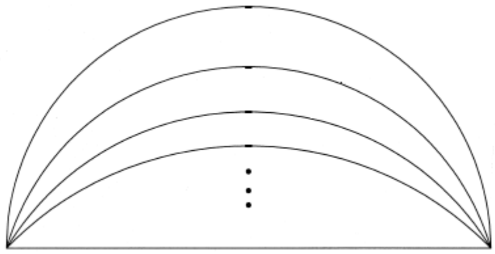

Figure 2: The Convergent Arcs Space

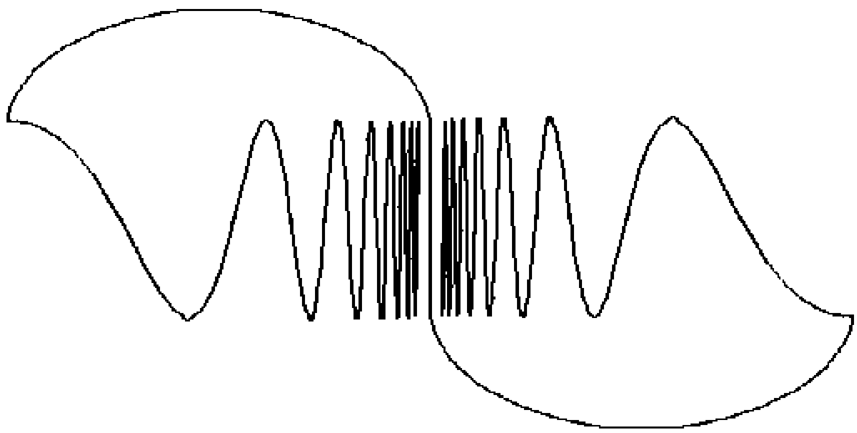

Figure 3: The Double Warsaw Circle

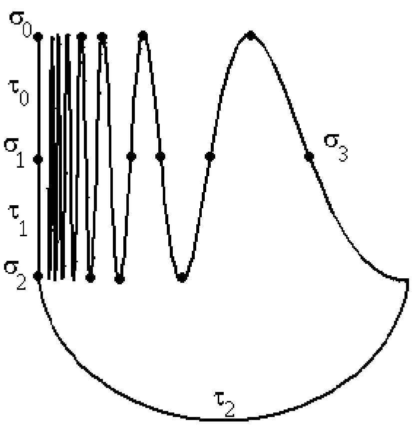

Figure 4: The Warsaw Circle subdivided into simplices

to show that the canonical inclusion of singular chains to Milnor-Thurston chains does not induce isomorphism on the level of homology.

The reason we took these spaces is that they are the simplest examples of non-triangulable spaces. Additionally, one can expect that the results should be different than in the singular homology case, since our theory admits infinite cycles. In particular, the reasonable question may be: "Does the Milnor-Thurston homology theory detects circular shape of the Warsaw Circle?". Being more precise, we ask whether $\mathcal{H}_1$ for this space equals $\mathbb{R}$. As we shall see it is not the case. In fact, the first Milnor-Thurston homology space for both the Warsaw Circle and the Double Warsaw Circle are equal zero (so they overlap with the singular homology case).

On the other hand, Convergent Arcs Space is a good example that $\mathcal{H}_1$ of a non-triangulable space may be different than in the singular homology theory. As we shall see it is an uncountable-dimensional space.

The geometric intuition behind this statements is the following. We can divide this spaces into a countable number of simplices in the natural way (the case for the Warsaw Circle is presented on Figure 4). Then, we can represent chains by the real functions supported on simplices τ_i (see Figure 4). The boundary is calculated in the usual way, so, for instance, coefficient of σ_1 is equal to the difference of the coefficients of τ_0 and τ_1. However, there is, one 0-simplex that does not belong to two 1-simplices (it is denoted by σ_0 on Figure 4), so the condition to be a cycle implies that coefficient of σ_0 and, consequently, coefficient of τ_0 is zero. By induction we see that all coefficients of τ_i should be zero, so there should not be any "fundamental class" for the

Warsaw Circle.

The above problem is caused by the non-paired 0-simplex, so it is reasonable to consider the case where no such simplex exists. This leads us to the idea of the Double Warsaw Circle (which can be divided into simplices in a similar manner as the Warsaw Circle). The cycle condition implies that coefficients for all 1-simplices should be the same. However, this contradicts finiteness of the corresponding measures. Hence, we have no “fundamental class” again.

The case of Converging Arcs is different. Of course, one should expect nontrivial cycles, yet the number of homology classes would be different than in singular case. The space is divided into simplices in the most natural way (every arc and the line segment is a 1-simplex). If we know coefficients of all the arcs, the cycle condition gives us coefficient for the line segment. So we can identify all the cycles with a vector of coefficients of the arcs.

In case of the singular homology the vector of coefficients should have finitely many nonzero coefficients, so we see that the first homology space should be isomorphic to $\mathbb{R}^\infty$ (the space of infinite real sequences with finite number of nonzero entries).

On the other hand, in Milnor-Thurston homology theory the vector of coefficients can have infinitely many nonzero entries. But the finiteness of measures gives us some restriction – they should form coefficients of the absolutely convergent sequence. Hence, the first homology space is isomorphic to ℓ^1.

The statements of this section can be formally proved using the Mayer-Vietoris theorem, as can be seen in the following paragraphs. The next section is devoted to this theorem and its purpose is to justify that it is valid in Milnor-Thurston homology theory. Then we are ready to calculate $\mathcal{H}_n$ for the three given spaces (we assume that $n > 0$). Because the calculations are similar, we will focus on the Warsaw Circle and then we will point out the differences.

4 Mayer-Vietoris theorem

The Mayer-Vietoris theorem is a way to relate the homology groups H_* (in some homology theory) of the space X with the homology groups of its two subspaces A and B

Theorem 1 *(Mayer-Vietoris) Let A and B be open subspaces such that $X =$*

$A \cup B$. Then the following sequence is exact:

$$\cdots \xrightarrow{(i_{*n}, j_{*n})} H_n(A) \oplus H_n(B) \xrightarrow{k_{*n} - l_{*n}} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \longrightarrow$$
$$\cdots \to H_0(A \cap B) \xrightarrow{(i_{*0}, j_{*0})} H_0(A) \oplus H_0(B) \xrightarrow{k_{*n} - l_{*n}} H_0(X) \longrightarrow 0$$

where $i : A \cap B \to A$, $j : A \cap B \to B$, $k : A \to X$, $l : B \to X$ are inclusion maps.

It is well known that it is implied by Eilenberg-Steenrod axioms [4]. Namely, the Mayer-Vietoris theorem is a consequence of the Excision Axiom and the Exactness Axiom, therefore the it is true in Milnor-Thurston homology theory for any space for which the Excition Axiom is true (that is, in particular, normal spaces). In the following paragraphs, we shall use this theorem to calculate Milnor-Thurston homology spaces for the Warsaw Circle.

Remark. The Mayer-Vietoris theorem can also be proved more directly. Let X be a topological space with subspaces A and B. We shall denote the space of Milnor-Thurston chains by $\mathcal{C}_\bullet$. According to [2, Lemma 4.10] the inclusion

$$\mathcal{C}_\bullet(A) + \mathcal{C}_\bullet(B) \to \mathcal{C}_\bullet(X)$$

induces isomorphism on the level of homology if there exist V such that $\overline{X \setminus A} \subset \overset{\circ}{V} \subset \overline{V} \subset B$ (when X is a normal space it suffices that A and B are open) and $A \cup B = X$.

Using this identity we can construct the short sequence of chain complexes

$$0 \longrightarrow \mathcal{C}_\bullet(A \cap B) \xrightarrow{(i_\bullet, j_\bullet)} \mathcal{C}_\bullet(A) \oplus \mathcal{C}_\bullet(B) \xrightarrow{k_\bullet - l_\bullet} \mathcal{C}_\bullet(A) + \mathcal{C}_\bullet(B) \longrightarrow 0,$$

and then its exactness yields Mayer-Vietoris theorem by homological algebra.

5 Higher dimensional homology spaces for the Warsaw Circle

In this section we consider X to be the Warsaw Circle. Since calculations of Milnor-Thurston homology spaces are similar in our three examples we shall focus on this one.

We cover X by two open subsets L and U. Let L be an intersection of X with the halfplane $\{(x, y) \mid y < \eta\}$ where $0 < \eta < 1$. Similarly, let U be an intersection of X with $\{(x, y) \mid y > -\eta\}$. We can see that L is not a path connected space. Let us denote its path components by L_i, for $i = 0, 1, ...$

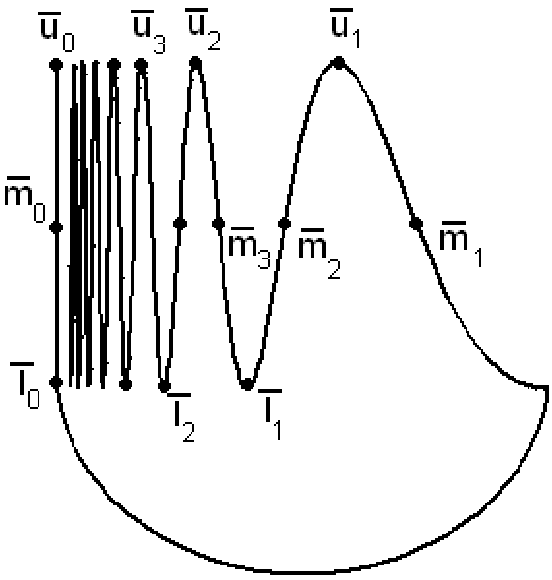

Figure 5: The Warsaw Circle with distinguished points

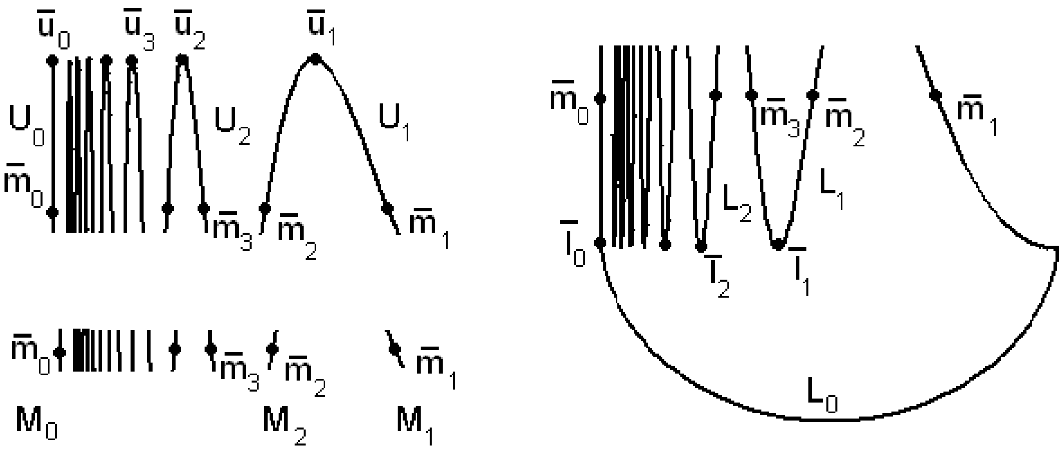

Figure 6: Three covering sets for the Warsaw Circle

(see Figure 6). In the same way U and $U \cap L$ can be decomposed into its path components U_i and M_i respectively.

We shall pick up one point from each of these components, which will be useful in the following proofs. Namely, let $\bar{m}_1$ by the first zero of $\sin 1/x$ after the rightmost minimum. The next zero we shall denote by $\bar{m}_2$, and so on (see Figure 5 and Figure 6). Additionally, let $\bar{m}_0 = (0, 0)$. We see that all $\bar{m}_i \in M_i \subset U \cap L$.

Next, let $\bar{u}_1$ denote first maximum after the rightmost minimum, $\bar{u}_2$ denote the next maximum, and so on. In addition, let $\bar{u}_0 = (0, 1)$. We see that all $\bar{u}_i \in U_i \subset U$.

Finally, we do the same for L: let $\bar{l}_1$ denote the first minimum on the left of rightmost minimum, let $\bar{l}_2$ denote the first minimum on the left of $\bar{l}_1$, and so on. Then, let $\bar{l}_0 = (0, -1)$. We get $\bar{l}_i \in L_i \subset L$.

We can see that the covering by sets U and L give us our desired subdivision of the Warsaw Circle into simplices. We will use the Mayer-Vietoris theorem to do the calculations, but first, in order to deal with the variety of possible singular simplices, we shall need an auxiliary result. We will prove that U, L and $U \cap L$ all have homotopy type of a convergent sequence with its limit. So let S denote a convergent sequence $\{x_n\}_{n=1}^{\infty}$ with its limit x_0 (with topology induced from the plane).

Lemma 2 *The spaces $U \cap L$, U and L have homotopy type of S.*

Proof. Let us start with proving this lemma for $U \cap L$. We can construct a map which sends each path component to the respective point of a sequence. Namely, we define a function $f_M : U \cap L \to S$ in the following way: let $x \in M_i$, than we put $f_M(x) = x_i$. Next, we can define $g_M : S \to U \cap L$ by $g(x_i) = \bar{m}_i$. We can see that $f_M \circ g_M = \mathrm{id}_S$ and $g_M \circ f_M$ is a map that sends points in M_i to $\bar{m}_i$. This function is homotopic to $\mathrm{id}_{U \cap L}$

Next, we prove the lemma for U. We define functions $f_U : U \to S$ and $g_U : S \to U$ in the similar way as in the previous case. That is: $f_U(x) = x_i$ for $x \in U_i$ and $g_U(x_i) = \bar{u}_i$. We can see, that $f_U \circ g_U = \mathrm{id}_S$ and $g_U \circ f_U \simeq \mathrm{id}_U$.

Finally, we prove the lemma for L. The functions $f_L : L \to S$ and $g_L : S \to L$ are defined in the similar manner as before. That is: $f_L(x) = x_i$ for $x \in L_i$ and $g_L(x_i) = \bar{l}_i$. We can see that $f \circ g = \mathrm{id}_S$ and $g_L \circ f_L \simeq \mathrm{id}_L$.

□

Since homology modules are homotopy invariant the next lemma allows us to calculate them for U, L and $U \cap L$.

Lemma 3 *If $n > 0$, then $\mathcal{H}_n(S) = 0$ and $\mathcal{H}_0(S) \cong \ell^1$. Where ℓ^1 denotes the space of sequences which form an absolutely convergent series.*

Proof. We can see that

$$C^0(\Delta^n, S) = \{x_k^n : \Delta^n \to S \mid x_k^n \text{ sends any point of } \Delta^n \text{ to } x_k, k \in \mathbb{N}_0\}.$$

The topology of $C^0(\Delta^n, S)$ is the same as for S, because x_k^n is a convergent sequence with a limit x_0^n. Hence, every set of this space is Borel and every two Borel measures equal on singletons $\{x_k^n\}$ are equal. Therefore, we can identify a sequence of real numbers $(a_k)_{k=0}^\infty$ with a measure $\mu(\{x_k^n\}) = a_k$. Additionally, we can see that

$$\|(a_k)_{k=0}^\infty\| = \sum_{k=0}^\infty |a_k|,$$

and every measure has a compact support (that is the whole space). Consequently,

$$\mathcal{C}_n(S) \cong \ell^1 := \{(a_k)_{k=0}^\infty \mid a_k \in \mathbb{R}, \sum_{k=0}^\infty |a_k| < \infty\}.$$

We have $\partial_i x_k^n = x_k^{n-1}$, which implies $\partial_i (a_k)_{k=0}^\infty = (a_k)_{k=0}^\infty$. Consequently,

$$\partial (a_k)_{k=0}^\infty = \sum_{i=0}^n (-1)^i \partial_i (a_k)_{k=0}^\infty = (a_k)_{k=0}^\infty \sum_{i=0}^n (-1)^i.$$

From here, $\partial = 0$ when n is odd and $\partial = \mathrm{id}$ when n is even. Thus, homology spaces are trivial for $n > 0$.

On the other hand, we have $\partial = 0$, for $n = 0$. Hence, every element in ℓ^1 is a cycle. Because $\partial = 0$, for $n = 1$, there are no boundaries and $\mathcal{H}_0(S) = \mathcal{C}_0(S) \cong \ell^1$.

□

Finally, using the Mayer-Vietoris sequence we can calculate homology spaces.

Theorem 4 *If $n > 0$, then $\mathcal{H}_n(X) = 0$.*

Proof. The Mayer-Vietoris sequence

$$\cdots \xrightarrow{(i_{*n}, j_{*n})} \mathcal{H}_n(U) \oplus \mathcal{H}_n(L) \xrightarrow{k_{*n} - l_{*n}} \mathcal{H}_n(X) \xrightarrow{\partial_*} \mathcal{H}_{n-1}(U \cap L) \longrightarrow$$
$$\cdots \to \mathcal{H}_0(U \cap L) \xrightarrow{(i_{*0}, j_{*0})} \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \xrightarrow{k_{*n} - l_{*n}} \mathcal{H}_0(X) \longrightarrow 0$$

is exact. Hence by Lemma 3, we have $\mathcal{H}_n(X) = 0$ for $n > 1$. So we only have to investigate the case $n = 1$.

By exactness of the Mayer-Vietoris sequence and the fact that $\mathcal{H}_1(U) = \mathcal{H}_1(L) = 0$ we see that $\partial_* : \mathcal{H}_1(X) \to \mathcal{H}_0(U \cap L)$ is a monomorphism. Consequently,

$$\mathcal{H}_1(X) \cong \ker(i_{*0}, j_{*0}).$$

Therefore we need to find the kernel of (i_{*0}, j_{*0}).

Hence, we can see that

$$\mathcal{H}_0(U) \cong \mathcal{H}_0(L) \cong \mathcal{H}_0(U \cap L) \cong \mathcal{C}_0(S) \cong \ell^1,$$

so we can identify elements of all these homology modules with real sequences that form an absolutely convergent series. We shall investigate the form of homomorphisms i_{*0} and j_{*0}.

Let $(m_i) \in \ell^1$ denote a homology class for $U \cap L$. It represents the measure supported on respective points $\bar{m}_i$ (m_i are values of the measure on singletons). Similarly, every homology class for U is represented by some $(u_i) \in \ell^1$ which we identify with the measure supported on $\bar{u}_i$.

In order to investigate i_{*0}, we have to interpret the measure supported on $\bar{m}_i$ as a measure supported on $\bar{u}_i$ with respect to homology class. We see that only $\bar{m}_0 \in U_0$. Hence, we can observe that

$$u_0 = m_0. \tag{1}$$

Furthermore, only $\bar{m}_{2k}, \bar{m}_{2k-1} \in U_k$, consequently we have

$$u_k = m_{2k} + m_{2k-1} \quad \text{for } k > 0. \tag{2}$$

In the similar way, we can see how j_{*0} works. Notice that only $\tilde{m}_{2k}, \tilde{m}_{2k+1} \in L_k$, hence

$$l_k = m_{2k} + m_{2k+1}. \tag{3}$$

It is a simple fact, that $\ell^1 \oplus \ell^1 \cong \ell^1$. To see that take any absolutely convergent sequence, we can divide it into the two sequences taking odd and even elements. And conversely, the sum of two absolutely convergent series is an absolutely convergent series. This observation can help us describe (i_{*0}, j_{*0}) in an elegant way. So let $x_{2k} = u_k$ and $x_{2k-1} = l_k$. We see that equations (2), (3) yields

$$x_k = \begin{cases} m_0 & \text{for } k = 0, \\ m_k + m_{k-1} & \text{for } k > 0. \end{cases} \tag{4}$$

Now, we have that the kernel of (i_{*0}, j_{*0}) and, consequently, $\mathcal{H}_1(X)$ are trivial.

□

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6 Corresponding calculations for two other examples

In the case of the Double Warsaw Circle the process of calculations is similar as in the case of a single one. We use the Mayer-Vietoris theorem in order to divide simplices (we divide sinusoid along the horizontal axis), and then we can identify every chain with sequences of real numbers. The exactness of Mayer-Vietoris sequence gives us relations which should be satisfied by a chain in order to be a cycle, and tells us which cycles represent the same homology class. Consequently, we get the equation of form (4). Nevertheless, the Double Warsaw Circle "expands" in both directions, so there is no initial condition. However, we can see that equation (4) yields that elements in the $\ker(i_{*0}, j_{*0})$ should have coefficients of the same absolute value but with alternating signs. So this time it is the finiteness of measure that gives us triviality of the kernel of (i_{*0}, j_{*0}) (in contrast to the previous case where we used the initial condition).

In the case of the Convergent Arcs space, we should vertically divide the space in half. Then, we can use intersections with right and left halfplanes as the subspaces in the Mayer-Vietoris theorem. This yields the subdivision of our space into simplices and the corresponding representation of our homology classes by the real sequences.

Each arc consists of only two vertices, so we can merge them. Then, as we noticed before, every homology class can be identified with a sequence of real coefficients (each coefficient for each arc). The finiteness of measures implies that the sequence should consist of elements of an absolutely convergent sequence, so $\mathcal{H}_1(X) = \ell^1$.

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