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Zeroth Milnor-Thurston homology for the Warsaw Circle

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1 Introduction

Algebraic topology is a theory that tries to capture subtle properties of topological spaces, creating simplified algebraic constructions from a more complicated geometric picture. It allowed to prove very important theorems in the case of spaces with a good local behaviour (e.g. Borsuk-Ulam Theorem [1, Theorem 7.2.3], Hairy Ball Theorem [4, Theorem 16.5] and many more [1]).

The results we mentioned above can be proved by means of specific range of techniques called homology theory. Its various modifications work well for spaces with good local behaviour. However, there are certain problems when one tries to apply them to the wild topological spaces.

One of the first examples of counterintuitive phenomena is by Milnor and Barratt [2], where the authors construct a two-dimensional space which has infinitely many nonzero homology groups (according to intuition all groups of dimensions greater than two should be zero for a two-dimensional space).

Because of such pathological examples algebraic topology aimed at spaces with a good local behaviour for a long time. There were only some separated results on wild topological spaces. However, lately there started systematic study of the field.

In this article we continue our considerations from semester paper [8] on Milnor-Thurston homology theory in the case of wild topological spaces (the Warsaw Circle and its modifications shall be the examples considered here). We will focus on investigating the zeroth Milnor-Thurston homology group.

In Section 2 we recall basic definitions and facts from Milnor-Thurston homology theory. Then, in Section 3 we prove that the zeroth Milnor-Thurston homology group is infinite dimensional. In Section 4 we introduce Berlanga topology and we describe Zastrow's construction of a space where zero order Milnor-Thurston homology group is non-Hausdorff. Finally, in Section 5 we prove that the zeroth Milnor-Thurston homology group of the Warsaw Circle is also non-Hausdorff.

2 Basics of Milnor-Thurston homology theory

Milnor-Thurston homology theory is a version of the homology theory which admits chains with infinite number of simplices. Its first appeareance was in connection with hyperbolic geometry in the proof of Gromov theorem [9, Theorem 6.4], which content is that two three-dimensional closed oriented hyperbolic manifolds with the same volume are isometric if they admit degree one maps.

Two decades later the theory was formalised by Zastrow [10] and Hansen [5] independently, and its definition was generalized to all topological spaces. In addition, it was proved that this theory satisfies Eilenberg-Steenrod axioms with weakened Excision Axiom, which is equivalent to the classical one for well behaved spaces (including all normal spaces) [10].

We shall start construction of Milnor-Thurston homology theory with defining its chain complex $\mathcal{C}_\bullet$. In this paper we shall use calligraph letters ($\mathcal{C}$, $\mathcal{H}$, etc.) for constructions in Milnor-Thurston homology theory and ordinary letters for the corresponding constructions in singular homology theory (C, H, etc.)

We will be concerned with singular simplices (continuous functions from the standard simplex Δ^k to our topological space X) as in the case of the singular homology theory. We endow the space of singular simplices $C^0(\Delta^k, X)$ with the *compact-open topology*. On this space we consider Borel sets $B(C^0(\Delta^k, X))$ – the smallest σ-algebra generated by open sets. A measure defined for all Borel sets on the given space is called a *Borel measure*. A *carrier* of Borel measure μ is a set $D \subset C^0(\Delta^k, X)$ such that all measurable subsets of $C^0(\Delta^k, X) \setminus D$ are zero sets.

Given a continuous function $f : C^0(\Delta^k, X) \to C^0(\Delta^l, Y)$ and an arbitrary Borel measure μ on $C^0(\Delta^k, X)$ we define the image measure $f\mu$ on $C^0(\Delta^l, Y)$ with the formula:

$$(f\mu)(A) = \mu(f^{-1}(A)).$$

Next, we define the sequence $\mathcal{C}_k(X)$ of real vector spaces consisting of signed Borel measures on $C^0(\Delta^k, X)$ with some *compact carrier*. Now, we construct the boundary operator ∂ in the usual way

$$\partial = \sum_{i=0}^{k} (-1)^k \partial_i,$$

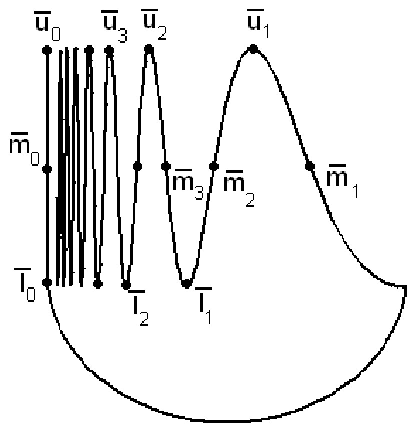

Figure 1: The Warsaw Circle with distinguished points

where ∂_i sends a measure to the image measure under the map $\sigma \mapsto \sigma \circ \delta_i$ with δ_i the usual inclusion of Δ_{k-1} as a face of Δ_k. We can prove that $\mathcal{C}_\bullet(X)$ with the boundary operator is a chain-complex.

The Milnor-Thurston homology groups $\mathcal{H}_*(X)$ are then defined as homology groups of this chain complex $\mathcal{C}_\bullet(X)$. Additionally, $\mathcal{C}_\bullet$ can be treated as a functor from the category of topological spaces to the category of chain-complexes. Thus, we can define relative homology groups $\mathcal{H}_*(X, A)$ in a natural way.

For more details on Milnor-Thurston homology theory see my semester theses [7, 8].

3 Zero-dimensional homology for the Warsaw Circle

The aim of semester thesis [8] was to investigate higher order (greater then zero) Milnor-Thurston homology groups of some spaces. The main example that was considered there was the Warsaw Circle. In this section we study zero order Milnor-Thurston homology group for the Warsaw Circle.

The Warsaw Circle (see Figure 1) is a well known space that serves as a counterexample in many cases. It is a subset of $\mathbb{R}^2$ that consists of:

- the part of Topologist Sine Curve $\{(x, y) \in \mathbb{R}^2 \mid y = \sin 1/x\}$ between the line $x = 0$ and the rightmost minimum,

- the "accumulation line" $\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$,
- an arc connecting the point $(0, -1)$ with the rightmost minimum.

Throughout this paper W will denote the Warsaw Circle. We distinguish three families of points $\bar{l}_k$, $\overline{m}_k$, $\overline{u}_k$ in W (see Figure 1). We shall refer to them as "minima of the sinusoid", "zeros of the sinusoid" and "maxima of the sinusoid" respectively.

Now we shall prove the following theorem

Theorem 1 *The vector space $\mathcal{H}_0(W)$ is uncountably-dimensional.*

Proof. The algebraic tool we shall use in this proof is Mayer-Vietoris theorem. It is true in Milnor-Thurston homology theory, because it is implied by Eilenberg-Steenrod axioms (see the proof of Mayer-Vietoris theorem in Greenberg's book [4, Theorem 17.6]). It is also possible to prove in a more direct manner, similarly as in [6, p. 149].

The Warsaw Circle can be divided into an upper part U and a lower part L. The upper part is an intersection of W with the halfplane $\{(x, y) \in \mathbb{R}^2 \mid y > -\varepsilon\}$ for some $0 < \varepsilon < 1$. The lower part is created in a similar manner (the intersection $U \cap L$ should be nonempty).

It has been proved that U, L and $U \cap L$ have homotopy type of a convergent sequence [8, Lemma 2], so their higher Milnor-Thurston homology groups are trivial and zero order homology groups are equal ℓ^1 [8, Lemma 3] (here ℓ^1 denotes *the vector space* of absolutely summable sequences, we do not, however, consider it as a Banach space with ℓ^1-norm).

Consequently the Mayer-Vietoris sequence is

$$0 \to \mathcal{H}_0(U \cap L) \xrightarrow{(i_{*0}, j_{*0})} \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \xrightarrow{s_{*0} - t_{*0}} \mathcal{H}_0(W) \longrightarrow 0,$$

where $i : U \cap L \to U$, $j : U \cap L \to L$, $s : U \to X$, $t : L \to X$ are inclusions.

In order to describe the arrows in the above diagram we should make some remarks on the notation. In each of the components of L we have distinguished a point (minimum of the sinusoid). By the proof of [8, Lemma 3] we see that each homology class of $\mathcal{H}_0(L)$ is represented by a measure supported on points $\bar{l}_k$ (the respective coefficients of the measure shall be denoted by l_k). We can proceed in an analogous manner with U and $U \cap L$.

In [8] we have described formulae for (i_{*0}, j_{*0}), in the notation introduced above they are

$$\begin{aligned} u_0 &= m_0, \\ u_k &= m_{2k} + m_{2k-1}, \quad \text{for } k > 0, \\ l_k &= m_{2k} + m_{2k+1}. \end{aligned}$$

The form of this equation is natural because (i_{*0}, j_{*0}) is simply reinterpretation of a measure in $U \cap L$ as a measure in U and in L.

If we introduce a unified notation for the distinguished points in U and L the above formulae will become simpler. So, let $x_{2k} = u_k$ and $x_{2k-1} = l_k$. Then we get

$$x_k = \begin{cases} m_0 & \text{for } k = 0, \\ m_k + m_{k-1} & \text{for } k > 0. \end{cases} \tag{1}$$

Now we can see that $\mathcal{H}_0(W)$ is a quotient $\ell^1/h(\ell^1)$ where $h : \ell^1 \to \ell^1$ is the map defined by equation (1).

The above equation (1) can be inverted so that, given an arbitrary sequence $x.$, we can find unique numbers m_k^x that satisfy it:

$$m_k^x = \sum_{i=0}^{k} (-1)^{i+k} x_i. \tag{2}$$

An element $x. \in \ell^1$ represents a nonzero homology class in $\mathcal{H}_0(W)$ if it is not in the image of (i_{*0}, j_{*0}) or, equivalently, when the corresponding $m_\cdot^x$ is not an absolutely convergent sequence.

For any space X there is the natural inclusion of singular chains into Milnor-Thurston chains: $C_k(X, \mathbb{R}) \to \mathcal{C}_k(X)$ [10]. It induces homomorphism on the level of homology. Thus, we can form the following definition

Definition 2 *A homology class in $\mathcal{H}_k(X)$ shall be called* singular homology class *if it lies in the image of $H_k(X, \mathbb{R}) \to \mathcal{H}_k(X)$. Otherwise it shall be called* non-singular homology class.

Now we can find a one dimensional subspace of $\mathcal{H}_0(W)$ which corresponds to singular homology classes. In singular homology theory we consider chains with only finite numbers of simplices, so for the sake of this argument assume that the sequence $x.$ has only finitely many nonzero elements. We will prove that such an element $x. \in \ell^1$ represents the same homology class as $y. \in \ell^1$ of the form $y. = (\alpha, 0, 0, 0, \dots)$ for some $\alpha \in \mathbb{R}$. Let N denote the biggest index of nonzero element in $x.$, then for $k > N$ we have

$$m_k^{x-y} = (-1)^k \left(\sum_{i=0}^{N} (-1)^i x_i - \alpha \right).$$

So putting $\alpha = \sum_{i=0}^{N} (-1)^i x_i$, yields $m_k = 0$. Thus it is absolutely summable and $x. - y.$ represents the zero homology class.

Now we shall prove that $\mathcal{H}_0(W)$ is much bigger than one-dimensional subspace of singular homology classes. In fact, as was stated in our theorem, its dimension is uncountable.

Let us start with some sequence of positive numbers n_k which is monotonically decreasing with $\lim n_k = 0$. From now on, up to the end of this proof, let $x.$ have a special form

$$x_k = (-1)^k(n_{k+1} - n_k).$$

We can see that:

$$\sum_{k=0}^{N} |x_k| = n_0 - n_{N+1},$$

hence the series is absolutely convergent, so $x. \in \ell^1$.

Let us calculate m_k^x:

$$m_k^x = \sum_{i=0}^{k}(-1)^{i+k}x_i = (-1)^k\sum_{i=0}^{k}(n_{i+1} - n_i) = (-1)^k(n_{k+1} - n_0). \quad (3)$$

Since $|m_k^x|$ does not fulfil the necessary condition it is not absolutely summable. Hence, $x.$ does not correspond to the zero homology class

More generally, we will check what conditions should x satisfy in order to be a non-singular homology class. So let $y. = (\alpha, 0, 0, 0, ...)$ (for $\alpha \in \mathbb{R}$) be a sequence corresponding to some singular homology class. In this case

$$m_k^{x-y} = (-1)^k(n_{k+1} - n_0 - \alpha),$$

if we take $\alpha = -n_0$ the sequence satisfies the necessary condition of series convergence. Then, we see that sufficient condition for x to be a non-singular homology class is

$$\sum_{k=0}^{\infty} n_k = \infty,$$

so we are interested in sequences converging to zero but not too fast.

As an example of such sequence we consider:

$$n_k^\beta = \frac{1}{(k+1)^\beta},$$

with $0 < \beta < 1$.

Now we shall prove that the homology classes as constructed before: $x_k^\beta = (-1)^k(n_{k+1}^\beta - n_k^\beta)$ form a set of linearly independent vectors. So take a finite sequence of numbers $0 < \beta_i < 1$ in an increasing order, and some finite sequence of real numbers b_i. We shall prove that the homology class of $z. = \sum_i b_i x_.^{\beta_i}$ is nontrivial.

In order to do this we need to prove that the sequence

$$m_k^z = (-1)^k \sum_i b_i \left(\frac{1}{(k+2)^{\beta_i}} - 1 \right)$$

is not absolutely summable. To obtain the above formula we use the fact that $m_\cdot^x$ is linear with respect to x, and the equation (3).

First, we notice that for the necessary condition of series $\sum_{k=0}^{\infty} |m_k^z|$ convergence to be satisfied, we should have $\sum_i b_i = 0$. Then, the study of the absolute summability of the above sequence can be reduced to the study of

$$\sum_{k=0}^{\infty} \left| \sum_i \frac{b_i}{(k+2)^{\beta_i}} \right|.$$

For sufficiently big k the expression in $|\cdot|$ has the sign of b_0 (since β_0 is the smallest of the numbers), so we can consider:

$$\sum_{k=0}^{\infty} \sum_i \frac{b_i}{(k+2)^{\beta_i}}.$$

This series is divergent. The easiest way to see this is to use integral criterion. First, we need to notice, that it is for monotonic sufficiently big k. Then, the application of the criterion is straightforward.

□

4 Berlanga topology on Milnor-Thurston homology groups

Berlanga equipped Milnor-Thurston homology groups with a topology [3]. He showed that this homology groups are a sequence functors from the category of second countable and separable topological spaces to the category of locally convex topological vector spaces (not necessarily Hausdorff!).

The topology is given in a natural way. Let X be a second countable separable topological space. Given any function $f : \mathcal{C}_k(X) \to \mathbb{R}$ we can define a linear functional

$$\Lambda_f(\mu) = \int_{\mathcal{C}_k(X)} f d\mu,$$

for $\mu \in \mathcal{C}_k(X)$. We shall work with the weakest topology on $\mathcal{C}_k(X)$ for which all such functionals are continuous.

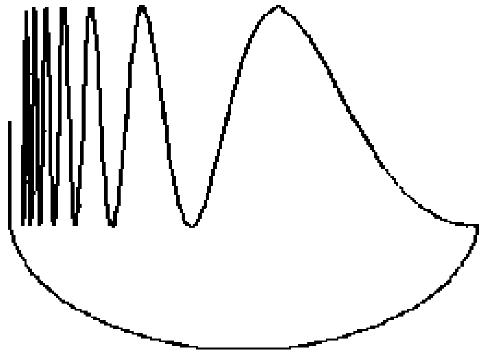

Figure 2: The Modified Warsaw Circle V

Berlanga proved that the boundary operator ∂ is continuous [3, Assertion 2.1]. Consequently the homology groups

$$\mathcal{H}_k(X) = \mathcal{Z}_k(X)/\mathcal{B}_k(X)$$

can be endowed with the structure of locally convex topological vector space. We shall call it *Berlanga topology*.

The question is whether Milnor-Thurston homology groups are Hausdorff in Berlanga topology. There are two results in this direction. Firstly, Berlanga's paper [3] ends with a proof that $\mathcal{H}_1$ is always Hausdorff for spaces that are homotopy equivalent to CW-complexes. Secondly, Zastrow constructed an example of a space V where $\mathcal{H}_0(V)$ is not Hausdorff [11]. This space V is the Warsaw Circle with a part of the accumulation line removed (see Figure 2). We shall present this construction here.

We shall follow Zastrow's argument to prove that $\mathcal{H}_0(V)$ is non-Hausdorff.

Theorem 3 *The topological vector space $\mathcal{H}_0(V)$ is non-Hausdorff.*

Proof. The idea of the proof is to show that boundaries $\mathcal{B}_0(V) := \partial\mathcal{C}_1(V)$ forms a set that is not closed in $\mathcal{C}_0(V)$. We will construct a sequence of measures $\mu_n \in \mathcal{C}_0(V)$ (which we identify with measures on V), such that there exist a sequence $\nu_n \in \mathcal{C}_1(V)$ with $\partial\nu_n = \mu_n$. However, we will show that $\mu = \lim \mu_n$ is not a boundary.

Just as in the previous sections $\{\bar{l}_k\}_{k=1}^{\infty}$ denotes the sequence of minima of the sinusoid (see Section 3). Let us define

$$\mu_n = (1 - 2^{-n})\delta_{\overline{l}_0} - \sum_{k=1}^{n} 2^{-k}\delta_{\overline{l}_k},$$

where δ denotes Dirac measure.

The natural candidate for the limit is

$$\mu = \delta_{\overline{l}_0} - \sum_{k=1}^{\infty} 2^{-k}\delta_{\overline{l}_k}$$

Indeed, it is sufficient to show that for every continuous function $f : V \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_V f d(\mu - \mu_n) = 0.$$

This is equivalent to

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} 2^{-k} f(\overline{l}_k) = 0,$$

which is true because tails of convergent series converge to zero.

Now we shall prove that μ is not a boundary. So suppose there is $\nu \in \mathcal{C}_1(V)$ such that $\partial\nu = \mu$. Then, we want to show that the carrier of ν cannot omit two consecutive maxima of the sinusoid. Being more specific, let D be a carrier of ν (which is compact), then we have continuous evaluation function

$$\begin{aligned} F \quad : \quad & D \times \Delta_k \to V \\ & \sigma \times q \mapsto \sigma(q). \end{aligned}$$

We want to show that $F(D \times \Delta^1)$ must contain infinitely many maxima of the sinusoid.

To the contrary, suppose that $\overline{u}_k$ and $\overline{u}_{k+1}$ are maxima of the sinusoid such that $\overline{u}_k, \overline{u}_{k+1} \notin F(D \times \Delta^1)$. Then let $Y = V \setminus \{\overline{u}_k, \overline{u}_{k+1}\}$. We can interpret μ and ν as elements of $\mathcal{C}_0(Y)$ and $\mathcal{C}_1(Y)$ respectively. Naturally, $\partial\nu = \mu$ still holds.

Then, we can embed Y into $Z = Y \cup S \cup J$, where S is an open rectangle with opposite vertices $\overline{u}_{k+1}$ and $\overline{l}_0$ and its faces paralell to coordinate axes, and J is a line segment between u_0 and u_{k+1}. This allows us to identify μ and ν with measures in $\mathcal{C}_0(Z)$ and $\mathcal{C}_1(Z)$ respectively. Still the condition $\partial\nu = \mu$ holds, hence μ represents the zero homology class in $\mathcal{H}_0(Z)$.

On the other hand we see that Z is triangulable (that was the reason behind including J in the definition of Z). By [10] we know that in this case Milnor-Thurston theory coincides with singular theory, so we can see that μ represents the same homology class in Z as $2^{-k}\delta_{\bar{l}_0} - 2^{-k}\delta_{\bar{l}_k}$. This homology class is not zero since points $\bar{l}_0$ and $\bar{l}_k$ lie in a different components of Z. Therefore, we got a contradiction, and we see that $F(D \times \Delta^1)$ contains infinitely many maxima of the sinusoid.

Since F is continuous, the set $F(D \times \Delta^1)$ must be compact, so it cannot contain infinitely many maxima of the sinusoid (because sequence containing infinitely many maxima does not have any convergent subsequence). Again, we have a contradiction. So there cannot exist a measure ν, such that $\partial\nu = \mu$, and consequently $\mathcal{H}_0(V)$ is not Hausdorff.

□

5 Non-Hausdorffness of the zeroth homology group for the Warsaw Circle

The result of the previous section was obtained with geometrical methods. These are strong enough to prove the fact that $\mathcal{H}_0(V)$ is non-Hausdorff, but we cannot obtain the analogous result for the Warsaw Circle W. The reason is that we used the non-compactness of V in our argument and, unfortunatelly, W is a compact space. In this section we will use metods of Section 3 to prove that $\mathcal{H}_0(V)$ is non-Hausdorff.

These methods can also be applied to the Modified Warsaw Circle V can assuring us that $\mathcal{H}_0(V)$ is also uncountably-dimensional. Notice that we have more homology classes here, since, as we can see from compactness condition, the measures supported on distinguished 1-simplices should have finitely many nonzero coefficients (almost all numbers m_k should be zero). Thus, homology class of every measure with support containing infinitely many minima of the sinusoid is a non-singular homology class.

The following theorem is the main result of this Section

Theorem 4 *The Milnor-Thurston homology group $\mathcal{H}_0(W)$ is non-Hausdorff in Berlanga topology*

Proof. We already know that $\mathcal{H}_0(W)$ is isomorphic to ℓ^1, so let a sequence of elements $x^n_\cdot \in \mathcal{H}_0(W)$ be described in the following way

$$x^n_k = \begin{cases} -\sum_{i=1}^{n}(-1)^i(n_{i+1} - n_i) & \text{for } k = 0 \\ (-1)^k(n_{k+1} - n_k) & \text{for } 0 < k \leq n, \\ 0 & \text{for } k > n. \end{cases}$$

where $n_\cdot$ is a decreasing sequence of positive numbers converging to zero (the same as in the proof of Theorem 1).

By the considerations of Section 3 we see that each of $x^n_.$ represents the zero homology class. However, the limit has the form (the proof is analogous as is in the previous section):

$$x_k = \begin{cases} -\sum_{i=1}^{\infty}(-1)^i(n_{i+1} - n_i) & \text{for } k = 0 \\ (-1)^k(n_{k+1} - n_k) & \text{for } k > 0. \end{cases}$$

Assume that the homology class described by the above sequence is a boundary. Define $y_k = (-1)^k(n_{k+1} - n_k)$. Then the difference

$$y_k - x_k = \begin{cases} \sum_{i=0}^{\infty}(-1)^i(n_{i+1} - n_i) & \text{for } k = 0 \\ 0 & \text{for } k > 0. \end{cases}$$

On the level of homology $x_.$ represents zero, so $y_.$ should represent singular homology class. However it is exactly the form of a sequence considered in the proof of Theorem 1, and we know that it is not the case. Hence, we got a contradiction. Consequently $x_.$ is not a boundary and $\mathcal{H}_0(W)$ is not Hausdorff.

□

Remark. A proof of the result of the previous section can be also achieved with these methods. Indeed, if we describe homology classes by the sequences (see Section 3) the homology class of measures μ_n of the previous Section is described by a sequence $z^n_.$, where nonzero elements are: $z^n_{2k+1} = -2^{-k}$ for $0 < k \leq n$, $z_0 = 1 - 2^{-n}$. We see that these measures are boundaries (see proof of Theorem 1 in Section 3). However, the limit measure μ is not a boundary since it is described by the sequence $z_.$ with infinitely many nonzero elements. Hence, we proved that $\mathcal{H}_0(V)$ is non-Hausdorff.

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