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Milnor-Thurston homology of some wild topological spaces

PHD DISSERTATION

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Streszczenie

Celem niniejszej pracy jest zbadanie zachowania niezmienników topologii algebraicznej w zastosowaniu do przestrzeni o skomplikowanej lokalnej strukturze. Przestrzenie takie nazywamy tu „dzikimi przestrzeniami topologicznymi" (nie jest to formalnie zdefiniowany termin, a stosujemy go głównie mając na myśli przestrzenie nie posiadające struktury CW-kompleksu).

Kluczowym problemem, który napotykamy, próbując stosować metody topologii algebraicznej do nietriangulowalnych przestrzeni, jest skończoność konstrukcji algebraicznych. Na przykład grupy homologii są opisywane przez skończone kombinacje liniowe sympleksów, natomiast klasyczne metody obliczania grupy podstawowej skupiają się na reprezentowaniu jej elementów poprzez skończonej długości słowa.

W naszym przypadku powyższe podejście jest nieefektywne, gdyż grupy podstawowe dzikich przestrzeni są często nieprzeliczalnie generowane, co jest spowodowane faktem, iż tego typu przestrzenie zawierają często dowolnie małe nietrywialne pętle. Zatem najbardziej naturalnym rozwiązaniem tej trudności wydaje się być opisywanie grup za pomocą przeliczalnych, a nie skończonych, słów. W ostatnich dwóch dekadach opublikowano kilka prac wykorzystujących tę ideę. Przykładowo, mamy prace, w których autorzy rozważają grupy podstawowe: Kolczyka Hawajskiego [9, 13], Przestrzeni Griffithsa [8] albo Trójkąta Sierpińskiego [1, 12, 16].

Kwestia skończoności jest również istotna jeśli chodzi o teorię homologii. Dlatego teoria homologii, dopuszczająca nieskończone łańcuchy sympleksów jest warta zbadania w kontekście dzikich przestrzeni topologicznych. Teo-

ria homologii Milnora-Thurstona jest takim przykładem. Łańcuchy są w tym przypadku miarami określonymi na przestrzeni sympleksów singularnych (formalną definicję znajdzie czytelnik w Sekcji 1.3).

Widzimy, że łańcuchy singularne, czyli skończone kombinacje liniowe sympleksów singularnych, mogą być również rozumiane jako łańcuchy w sensie teorii Milnora-Thurstona. Wystarczy skończone kombinacje sympleksów identyfikować z miarami skupioną na skończonej liczbie punktów (to utożsamienie prowadzi do definicji kanonicznego homomorfizmu pomiędzy homologiami singularnymi a homologiami Milnora-Thurstona, patrz Sekcja 1.3).

Jest jeszcze jeden powód by zająć się teorią Milnora-Thurstona w kontekście dzikich przestrzeni topologicznych. Mianowicie wiadomo, iż owa teoria spełnia aksjomaty Eilenberga-Steenroda, a zatem jest tożsama z teorią singularną dla przestrzeni traingulowalnych (patrz Sekcja 1.2). Jednakże jej zachowanie dla dzikich przestrzeni jest w dużej mierze niezbadane. Pierwsze rezultaty w tym kierunku zostały otrzymane przez Zastrowa [33, Section 6] [34], natomiast pierwszy opublikowany rezultat dotyczył obliczenia grup homologii Milnora-Thurstona dla Okręgu Warszawskiego [26] i został opisany w niniejszej pracy.

Rozdział 1 zawiera opis znanych wyników oraz krótką historię i formalną definicję teorii homologii Milnora-Thurstona. Rozdział 2 jest poświęcony wynikom opublikowanym przez autora w pracy [26] – dotyczy obliczenia grup homologii Milnora-Thurstona dla Okręgu Warszawskiego i wynikającego stąd rozwiązaniu problemu postawionego przez Berlangę [5]. Rozdział 3 zawiera dalsze wyniki dotyczące zerowej grupy homologii Milnora-Thurstona. Przedstawiono w nim dowód, iż zerowa grupa homologii dla kontinuów Peano jest jednowymiarowa, a kanoniczny homomorfizm jest iniektywny dla przestrzeni z borelowskimi składowymi łukowymi. Ponadto przedstawiony jest kontrprzykład, iż ostatni wynik nie zachodzi dowolnych przestrzeni topologicznych.

0.1 Podstawowe definicje

Ponieważ definicja grup homologii Milnora-Thurstona oparta jest o teorię miary, przedstawimy tutaj jej podstawowe pojęcia i koncepcje.

Definicja 0.1. Rodzinę podzbiorów zbioru Ω nazywamy *σ-algebrą* nad zbiorem Ω jeżeli zwiera ona zbiór pusty i jest zamknięta ze względu na dopełnienia i przeliczalne sumy.

Zauważmy, że przekrój dowolniej liczby σ-algebr jest również σ-algebrą. Stąd wynika, że dla każdej rodziny $\mathcal{S}$ podzbiorów zbioru Ω istnieje najmniejsza σ-algebra nad Ω zawierająca rodzinę $\mathcal{S}$. Nazywamy tę σ-algebrę *generowaną przez rodzinę $\mathcal{S}$* i oznaczamy ją $\sigma(\mathcal{S})$.

Definicja 0.2. Parę uporządkowaną $(\Omega, \mathcal{F})$ gdzie $\mathcal{F}$ jest σ-algebrą nad Ω nazywamy *przestrzenią mierzalną*.

Definicja 0.3. Niech $(\Omega, \mathcal{F})$ będzie przestrzenią mierzalną. Funkcję $\mu : \mathcal{F} \to \mathbb{R}$ nazywamy *skończoną miarą ze znakiem* jeżeli jest przeliczalnie addytywna i znika na zbiorze pustym.

W niniejszej pracy rozpatrujemy jedynie skończone miary ze znakiem, dlatego dalej będziemy je nazywać po prostu miarami.

Każda przestrzeń topologiczna w naturalny sposób jest przestrzenią mierzalną. Niech więc (X, τ) będzie przestrzenią topologiczną. Wówczas σ-algebra generowana przez τ jest nazywana σ-algebrą zbiorów borelowskich i oznaczamy ją $\mathcal{B}(X)$. Miary określone na $\mathcal{B}(X)$ nazywamy *miarami borelowskimi.*

Definicja 0.4. Niech $(\Omega_i, \mathcal{F}_i)$ dla $i = 1, 2$ będą przestrzeniami mierzalnymi. Funkcja $f : \Omega_1 \to \Omega_2$ nazywalna jest *funkcją mierzalną* jeżeli przeciwobraz każdego zbioru z $\mathcal{F}_2$ jest zawarty $\mathcal{F}_1$.

Definicja 0.5. Mając daną funkcję mierzalną $f : \Omega_1 \to \Omega_2$ i miarę μ na Ω_1 definiujemy *miarę przetransportowaną $f\mu$* następującym wzorem

$$(f\mu)(A) = \mu(f^{-1}(A)), \qquad \text{dla każdego mierzalnego zbioru A.}$$

Definicja 0.6. Niech μ będzie miarą na przestrzeni mierzalnej $(\Omega, \mathcal{F})$. *Nośnikiem miary* μ określamy zbiór $D \subset \Omega$ taki, że $\mu(A) = 0$ dla każdego $\mathcal{F} \ni A \subset \Omega \setminus D$.

Poniższy fakt będzie pomagał nam radzić sobie z miarami ze znakiem:

Twierdzenie 0.7. (Hahn [18, Theorem A, p. 121]) *Niech* μ *będzie miarą na* $(\Omega, \mathcal{F})$. *Wówczas istnieją dwa rozłączne zbiory* Ω^+, $\Omega^- \in \mathcal{F}$ *takie, że* $\Omega = \Omega^+ \cup \Omega^-$ *oraz dla każdego* $F \in \mathcal{F}$ *mamy* $\mu(F \cap \Omega^+) \geq 0$, $\mu(F \cap \Omega^-) \leq 0$.

Rozkład przestrzeni Ω na dwa podzbiory Ω^+, Ω^- nie jest jednoznaczny. Jednakże w przypadku dwóch różnych rozkładów Ω_i^+, Ω_i^-, $i = 1, 2$, można pokazać, że dla dowolnego $F \in \mathcal{F}$ mamy $\mu(F \cap \Omega_1^+) = \mu(F \cap \Omega_2^+)$, $\mu(F \cap \Omega_1^-) = \mu(F \cap \Omega_2^-)$ [18, p. 122]. Stąd też miara ze znakiem może być jednoznacznie rozłożona na następującą różnicę miar nieujemnych

$$\mu = \mu^+ - \mu^-,$$

gdzie $\mu^+(\cdot) := \mu(\cdot \cap \Omega_+)$, $\mu^-(\cdot) := -\mu(\cdot \cap \Omega_-)$.

Definicja 0.8. Niech μ będzie miarą na przestrzeni X, *wariację* $|\mu|$ miary μ określamy wzorem

$$|\mu| = \mu^+ + \mu^-.$$

Całkowitą wariację $\|\mu\|$ definiujemy jako

$$\|\mu\| = |\mu|(X).$$

0.2 Teoria homologii Milnora-Thurstona

Teraz pokrótce przedstawimy konstrukcję teorii homologii Milnora-Thurstona. Będziemy używać liter kaligraficznych ($\mathcal{C}$, $\mathcal{H}$, itp.) do oznaczenia odpowiednich konstrukcji w teorii Milnora-Thurstona, natomiast zwykłe litery (C, H, itp.) oznaczać będą odpowiednie grupy w teorii singularnej.

Na początek skonstruujemy kompleks łańcuchowy $\mathcal{C}_*(X)$ dla danej przestrzeni topologicznej X. Niech $C^0(\Delta^k, X)$ oznacza przestrzeń sympleksów singularnych (tj. ciągłych funkcji z sympleksu standardowego Δ^k w X, gdzie k jest całkowitą liczbą nieujemną). Będziemy rozpatrywać $C^0(\Delta^k, X)$ jako przestrzeń topologiczną wyposażoną w topologię zwarto-otwartą. Przestrzeń wektorową $\mathcal{C}_k(X)$ zawierającą k-wymiarowe łańcuchy definiujemy jako zbiór skończonych miar borelowskich ze znakiem posiadających zwarty nośnik.

W następnym kroku uczynimy z $\mathcal{C}_*(X)$ kompleks łańcuchowy. Niech $\delta_i : \Delta^{k-1} \hookrightarrow \Delta^k$, dla $i = 0, 1, ..., k$, oznaczają włożenia sympleksu Δ^{k-1} jako ściany sympleksu Δ^k. Odwzorowania δ_i indukują ciągłe odwzorowania $\partial_i : C^0(\Delta^k, X) \to C^0(\Delta^{k-1}, X)$ na poziomie sympleksów singularnych. Są one definiowane jako złożenia funkcji $\partial_i : \sigma \mapsto \sigma \circ \delta_i$. Nietrudno pokazać, że z definicji topologii zwarto-otwartej wynika ich ciągłość [33, Lemma 2.8].

Z kolei ciągłe funkcje ∂_i indukują odwzorowania $\partial_i : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$, gdzie ∂_i działa poprzez transport miary ze względu na ciągłą (a więc mierzalną) funkcję ∂_i (patrz Definicja 0.4). Ostatecznie operator brzegu jest definiowany typowym wzorem:

$$\partial = \sum_{i=0}^{k} (-1)^i \partial_i.$$

W pracy [33, Corollary 2.9] pokazano, że $\mathcal{C}_*(X)$ z tak zdefiniowanym operatorem brzegu jest istotnie kompleksem łańcuchowym.

Grupy homologii Milnora-Thurstona $\mathcal{H}_*(X)$ są definiowane w jako grupy homologii kompleksu łańcuchowego $\mathcal{C}_*(X)$. Ponadto widzimy, że $\mathcal{C}_*$ jest funktorem z kategorii przestrzeni topologicznych do kategorii kompleksów łańcuchowych. Rzeczywiście, odwzorowanie łańcuchowe $f_\bullet : \mathcal{C}_*(X) \to \mathcal{C}_*(Y)$ indukowane przez ciągłą funkcję $f : X \to Y$ jest definiowane podobnie jak operator brzegu. Możemy traktować $C^0(\Delta^k, -)$ jako funktor kowariantny, a wówczas $f_\bullet$ odwzorowuje każdą miarę na miarę przetransportowaną przez f (szczegółową analizę przedstawiono w Sekcji 1.3).

Niech X będzie przestrzenią topologiczną, natomiast A jej podprzestrzenią. Wówczas relatywny kompleks łańcuchowy $\mathcal{C}_*(X, A)$ jest definiowany jako iloraz $\mathcal{C}_*(X)$ przez $i_\bullet(\mathcal{C}_*(A))$, gdzie $i : A \hookrightarrow X$ jest włożeniem. Relatywne grupy homologii Milnora-Thurstona to grupy homologii komplesku $\mathcal{C}_*(X, A)$.

Istnieje kanoniczny homomorfizm łańcuchów singularnych w łańcuchy Milnora-Thurstona

$$\begin{aligned} C_k(X;\mathbb{R}) &\to \mathcal{C}_k(X), \\ \sum_i \alpha_i \sigma_i &\mapsto \sum_i \alpha_i \delta_{\sigma_i}, \end{aligned}$$

gdzie δ oznacza miarę Kroneckera. Ten homomorfizm jest monomorfizmem wtedy i tylko wtedy gdy X spełnia aksjomat oddzielania T_0. Ponadto, powyższy homomorfizm jest przemienny z operatorem brzegu, a zatem indukuje on odwzorowanie na poziomie homologii

$$H_k(X;\mathbb{R}) \to \mathcal{H}_k(X).$$

Odwzorowanie to jest izomorfizmem gdy X ma typ homotopijny CW-kompleksu [33, Section 5]. Ponadto okazuje się, że jest to monomorfizm dla wielu dzikich przestrzeni (np. w przypadku zerowych homologii dla Okręgu Warszawskiego lub w przypadku przykładowej przestrzeni zdefiniowanej w [33, Section 6]).

0.3 Topologia Berlangi

Berlanga wyposażył grupy homologii Milnora-Thurstona w topologię, która jest zgodna z jej strukturą liniową [5]. Co więcej można udowodnić, że jest ona lokalnie wypukła kiedy przestrzeń topologiczna spełnia drugi aksjomat przeliczalności i jest ośrodkowa. A zatem homologie stanowią funktory z kategorii ośrodkowych przestrzeni topologicznych spełniających drugi aksjomat przeliczalności do kategorii lokalnie wypukłych przestrzeni liniowo topologicznych (niekoniecznie spełniających aksjomat Hausdorffa!).

Owa topologia jest określona w naturalny sposób. Niech X będzie ośrodkową przestrzenią topologiczną spełniającą drugi aksjomat przeliczalności. Mając daną funkcję $f : \mathcal{C}_k(X) \to \mathbb{R}$ możemy określić następujący funkcjonał liniowy

$$\Lambda_f(\mu) = \int_{\mathcal{C}_k(X)} f d\mu,$$

gdzie $\mu \in \mathcal{C}_k(X)$. Będziemy pracować z najsłabszą topologią na $\mathcal{C}_k(X)$ taką, że wszystkie powyższe funkcjonały są ciągłe. Berlanga udowodnił, że operator brzegowy ∂ jest ciągły [5, Assertion 2.1]. A zatem grupy homologii

$$\mathcal{H}_k(X) = \mathcal{Z}_k(X)/\mathcal{B}_k(X)$$

mogą być wyposażone w strukturę lokalnie wypukłej przestrzeni liniowo topologicznej. Jej topologię będziemy nazywać *topologią Berlangi.*

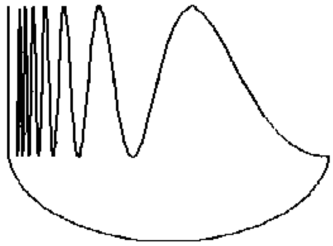
R. Berlanga postawił pytanie czy grupy homologii Milnora-Thurstona spełniają aksjomat Hausdorffa. W pracy [5] autor przedstawia dowód, iż $\mathcal{H}_1(X)$ jest przestrzenią Hausdorffa, jeżeli X jest homotopijnie równoważna z CW-kompleksem. Z kolei Zastrow pokazał przykład przestrzeni V gdzie $\mathcal{H}_0(V)$ nie jest Hausdorffa [34]. Ta przestrzeń V to Okrąg Warszawski z usuniętym fragmentem linii akumulacji (patrz Theorem 2.6).

0.4 Teoria homologii Milnora-Thurstona dla dzikich przestrzeni topologicznych

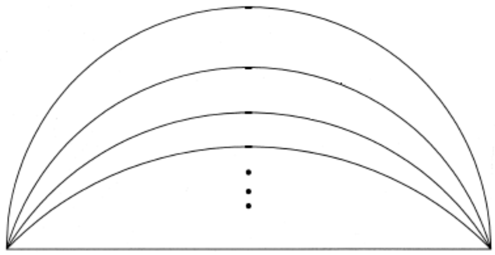
Mianem *dzikie przestrzenie topologiczne* określamy przestrzenie o skomplikowanej lokalnej strukturze. Nie przywołujemy żadnej formalnej definicji „dzikości", a podstawową cechą, która odróżnia przestrzenie dzikie od oswojonych jest ich nietriangulowalność.

Wiadomo, że kanoniczny homomorfizm pomiędzy homologiami singularnymi a homologiami Milnora-Thurstona jest izomorfizmem, gdy przestrzeń ma typ homotopijny CW-kompleksu. Dlatego też badanie homologii Milnora-Thurstona dla tego typu przestrzeni sprowadza się do badania homologii singularnych.

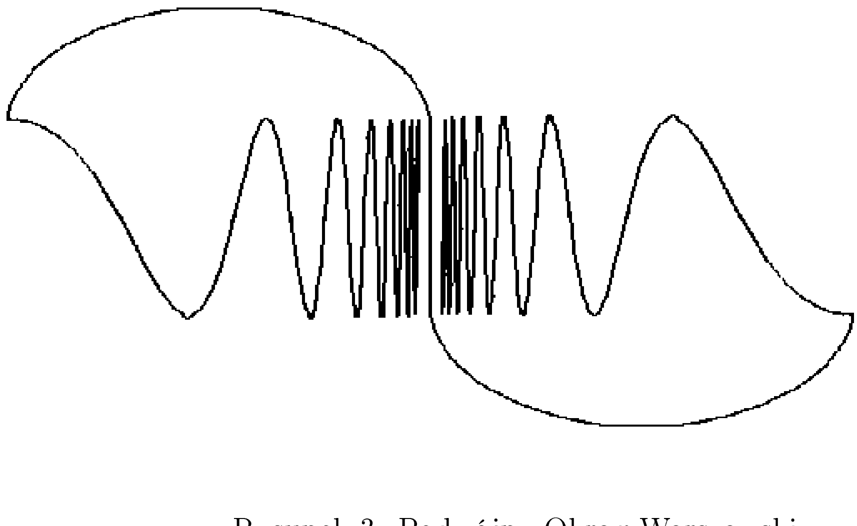
Sprawa wygląda inaczej w przypadku dzikich przestrzeni. Można podać przykłady przestrzeni (np. Okrąg Warszawski, patrz dalej), gdzie obie teorie homologii się różnią. Celem niniejszej pracy jest badanie tych różnic i, bardziej ogólnie, zbadanie własności grup homologii Milnora-Thurstona dla dzikich przestrzeni topologicznych.



Rysunek 1: Okrąg Warszawski



Rysunek 2: Przestrzeń Zbieżnych Łuków



Rysunek 3: Podwójny Okrąg Warszawski

Przykładowymi dzikimi przestrzeniami na których skupiliśmy się w tej pracy są: Okrąg Warszawski W, Przestrzeń Zbieżnych Łuków CA i Podwójny Okrąg Warszawski DW.

Okrąg Warszawski, przedstawiony na Rysunku 1 jest zdefiniowany jako podzbiór $\mathbb{R}^2$ składający się z:

- części Sinusoidy Warszawskiej $\{(x, y) \in \mathbb{R}^2 \mid y = \sin 1/x\}$, zawierającej się pomiędzy prostą $x = 0$ a najbardziej wysuniętym na prawo minimum,
- linii akumulacji $\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$,
- łuku łączącego punkt $(0, -1)$ z minimum wysuniętym najbardziej na prawo.

Podwójny Okrąg Warszawski przedstawiono na Rysunku 3, jest on sklejeniem dwóch kopii Okręgu Warszawskiego wzdłuż linii akumulacji. Przestrzeń Zbieżnych Łuków przedstawiona na Rysunku 2 jest zbudowana z przeliczalnej liczby łuków łączących dwa dane punkty i zbiegających do odcinka euklidesowego pomiędzy tymi punktami.

Przystąpimy teraz do prezentacji wyników pracy

Twierdzenie 0.9. (patrz Theorem 2.3) *Niech* $n > 0$*, wówczas* $\mathcal{H}_n(W) = 0$.

Szkic dowodu. Z geometrycznego punktu widzenia idea dowodu polega na podziale sympleksów singularnych w taki sposób, żeby każdy przechodził przez co najwyżej jedno maksimum Sinusoidy Warszawskiej. Jest to możliwe, gdyż nośnik łańcuchów Milnora-Thurstona jest zwarty. Taki podział pozwala pokazać, że grupy homologii dają się opisać za pomocą absolutnie sumowalnych ciągów. A stąd, wykonując odpowiednie obliczenia, pokazujemy wynik.

Technicznym narzędziem wykorzystywanym w tym dowodzie jest twierdzenie Mayera-Vietorisa. Jeżeli podzielimy Okrąg Warszawski na dwie połówki, górną i dolną, uzyskamy opisany powyżej efekt podziału sympleksów singularnych. Następnie dzięki homotopijnej niezmienniczości grup homologii, widzimy że obie połówki mają te same grupy homologii do ciąg punktów z granicą. Dla tego typu przestrzeni nietrudno policzyć grupy homologii Milnora-Thurstona wprost z definicji. Okazuje się, że 0-łańcuchy są izomorficzne z przestrzenią ciągów absolutnie sumowalnych. Stąd widać, że ciągi absolutnie sumowalne opisują homologie Okręgu Warszawskiego W.

Wykorzystując taki opis grup homologii możemy napisać wzór określający operator brzegu (patrz równanie (2.4)). Stąd odczytujemy, że nie istnieją nietrywialne 1-cykle, a zatem pierwsza grupa homologii jest trywialna.

□

Twierdzenie 0.10. (patrz Theorem 2.4) *Przestrzeń liniowa $\mathcal{H}_0(W)$ jest kontinuum-wymiarowa.*

Szkic dowodu. Powyższe twierdzenie dowodzi się wykorzystując techniki przedstawione w dowodzie Twierdzenia 0.9. Jak było wspomniane, grupy homologii są opisywane przez ciągi absolutnie sumowalne. Z tego opisu, możemy zauważyć, że niezerowe klasy homologii odpowiadają ciągom zbiegają do z dostatecznie powoli (są sumowalne, ale ciąg sum częściowych już nie jest). Możemy podać wiele takich ciągów, pisząc odpowiednie kombinacje liniowe ciągów postaci $1/k^{\alpha}$. Ponieważ parametr α może być zmieniany w przedziale $(0,1)$ w dowolny sposób, możemy tak wygenerować kontinuum wiele liniowo niezależnych klas homologii.

□

Analogicznymi metodami wyliczamy grupy homologii pozostałych rozpatrywanych przez nas przestrzeni (patrz Theorem 2.8 i Theorem 2.9). W szczególności grupy homologii Podwójnego Okręgu Warszawskiego DW są takie same jak W. Ponadto $\mathcal{H}_1(CA) \cong \bigoplus_{\mathfrak{c}} \mathbb{R}$. Natomiast dla homologii singularnych mamy $H_1(CA) \cong \bigoplus_{\aleph_0} \mathbb{R}$, stąd gołym okiem widać brak izomorfizmu pomiędzy teorią Milnora-Thurstona a teorią singularną. Natomiast w wymiarze zero obie teorie homologii przystają dla przestrzeni CA. Pokazujemy to podobnie jak w przypadku Okręgu Warszawskiego, ale można też na to patrzeć jako na wniosek z następującego twierdzenia

Twierdzenie 0.11. (patrz Theorem 3.2) *Niech X będzie kontinuum Peano, wówczas $\mathcal{H}_0(X) \cong \mathbb{R}$.*

Szkic dowodu. Kontinuum Peano jest to zwarta przestrzeń metryczna, która jest lokalnie spójna. W dowodzie wykorzystamy twierdzenie Hahna-Mazurkiewicza, które powiada iż istnieje ciągła suriekcja $f : [0,1] \to X$.

Należy wykazać, że dowolny 0-łańcuch Milnora-Thurstona μ na przestrzeni X jest homologiczny z miarą skupioną w jednym punkcie. Można pokazać, że istnieje miara $\tilde{\mu}$ na $[0,1]$ taka, że $f\tilde{\mu} = \mu$. Następnie każdemu punktowi $t \in [0,1]$ możemy przypisać 1-sympleks, który zaczyna się w $f(0)$ a kończy w $f(t)$. Stąd mamy odwzorowanie $[0,1] \to C^0(\Delta^1, X)$. Dalej transportując miarę $\tilde{\mu}$ poprzez to odwzorowanie, dostajemy miarę ν której brzegiem jest różnica μ i miary skupionej w punkcie $f(0)$.

□

Zauważyliśmy już, że pierwsza grupa homologii Milnora-Thurstona dla przestrzeni CA nie jest izomorficzna z odpowiednią grupą homologii singularnych. Możemy zauważyć jednak, że kanoniczny homomorfizm jest tutaj iniektywny (jest to naturalne włożenie $\bigoplus_{\aleph_0} \mathbb{R}$ w $\bigoplus_{\mathfrak{c}} \mathbb{R}$). Podobnie sprawa się ma w przypadku Okręgu Warszawskiego. Okazuje się, że mamy następujące twierdzenie:

Twierdzenie 0.12. (patrz Theorem 3.3) *Niech X będzie przestrzenią, której wszystkie łukowe składowe są borelowskie. Wówczas homomorfizm kanoniczny $H_0(X) \to \mathcal{H}_0(X)$ jest iniekcją.*

Szkic dowodu. Niech $\mu \in \mathcal{C}_0(X)$ będzie miarą skupioną na skończonej liczbie punktów reprezentującą nietrywialną klasę homologii singularnych. To znaczy, że nie istnieje miara $\nu \in \mathcal{C}_1(X)$ skupiona na skończonej liczbie punktów spełniająca

$$\partial\nu = \mu.$$

Musimy wykazać, że żadna miara $\nu \in \mathcal{C}_1(X)$ nie spełnia powyższego równania.

Dowód przeprowadzimy dla przypadku gdy μ jest skupiona na dwóch punktach. Czyli $\mu = \alpha\delta_x + \beta\delta_y$, gdzie punkty $x, y \in X$ leżą w różnych składowych spójności, a współczynniki $\alpha, \beta \neq 0$.

Załóżmy, że istnieje ν taka, że $\mu = \partial\nu$. Niech Y będzie składową spójności zawierającą x. Wówczas prosty rachunek pokazuje, iż $\mu(Y) = (\partial\nu)(Y) = 0$. Jednakże $\mu(Y) = (\alpha\delta_x + \beta\delta_y)(Y) = \alpha$, co daje sprzeczność, gdyż $\alpha \neq 0$.

Dowód dla miar skupionych na większej liczbie punktów przeprowadza się analogicznie.

□

Założenie o borelowskich składowych łukowych jest istotne w powyższym twierdzeniu. Skonstruujemy teraz przestrzeń, dla której kanoniczny homomorfizm nie jest iniektywny.

Twierdzenie 0.13. (patrz Theorem 3.13) *Istnieje przestrzeń X, dla której kanoniczny homomorfizm $H_0(X) \to \mathcal{H}_0(X)$ nie jest iniektywny.*

Szkic dowodu. Istnieje rozbicie $[-1, 1] = X_0 \cup X_1$ takie, że każdy borelowski podzbiór zbioru X_0 lub zbioru X_1 jest miary Lebesgue'a zero (patrz Lemma 3.5). Rzecz jasna zbiory X_0 i X_1 nie są mierzalne w sensie Lebesgue'a.

Rozważmy teraz dwa stożki CX_0 i CX_1, nad zbiorem X_0 i X_1 odpowiednio. Odcinek $[-1, 1]$ wraz ze stożkami traktujemy jako podzbiór płaszczyzny z topologią indukowaną.

Do tak skonstruowanej przestrzeni doklejamy dwa odcinki I_0, I_1. Łączymy je z wierzchołkami odpowiednich stożków. Topologię na dodatkowych odcinkach zadajemy przez podanie bazy otoczeń punktów. Mianowicie otoczenia punktów odcinków I_i, dla $i = 0, 1$, składają się z pododcinków odcinka I_i i z odpowiednich pododcinków prawie wszystkich odcinków stożka CX_i. Tak otrzymujemy przestrzeń X.

Przestrzeń X ma dwie składowe spójności, z których każda zawiera jeden ze stożków. Rozważmy teraz miarę $\mu = \delta_{x_0} - \delta_{x_1}$ gdzie x_i jest wierzchołkiem stożka CX_i. Istnieje miara ν taka, że $\partial\nu = \mu$. Miara ta jest jednorodnie skupiona na odcinkach (traktowanych jako sympleksy singularne) stożka CX_0 i stożka CX_1. Powodem dla którego żaden punkt odcinka $[0, 1]$ nie znajduje się w nośniku miary $\partial\nu$ jest fakt, że każdy podzbiór borelowski zbiorów X_0 i X_1 ma miarę zero. Dlatego w brzegu miary ν znajdują się tylko punkty x_0 i x_1, leżące w różnych łukowych składowych spójności.

$\square$

Introduction

The aim of this thesis is to investigate the behaviour of invariants from algebraic topology when applied to topological spaces with a complicated local structure. For such spaces the term “wild topological spaces” is used (this is not a formally defined notion, here it refers mostly to topological spaces with no CW-complex structure).

The crucial problem when we try to apply methods of algebraic topology to non-triangulable spaces is finiteness of basic algebraic constructions. For example, the homology groups are described by finite linear combinations of simplices, and the classical methods for computing the fundamental groups focus on decomposing each element of the group into finite words of generators.

This approach seems ineffective, since fundamental groups of non-tame spaces are often uncountably generated caused by the fact that such spaces contain infinitely many small non-nullhomotopic loops. Consequently, the structure of the fundamental group of such a space can only be adequately reflected by infinite multiplication. We see that the most natural solution to this problem is to describe the group by countable infinite words instead of finite ones. In the last two decades some papers in this direction were published. For example, there were articles published where the authors consider such a description of the fundamental groups of the Hawaiian Earring [9, 13], the Griffiths space [8] or the Sierpiński Gasket [1, 12, 16].

The issue of finiteness is also important when it comes to homology theory. Therefore, a homology theory with infinite chains of simplices is worth being investigated in perspective for wild topological spaces.

Milnor-Thurston homology theory is a particular example of a homology theory that admits infinite chains. They are by definition Borel measures on the space of singular simplices (a formal definition can be found in Section 1.3). We see that singular chains, which are finite linear combinations of singular simplices, can also be interpreted as Milnor-Thurston chains. We just have to identify finite linear combinations with measures concentrated on a finite number of points (this identification leads to the definition of a canonical homomorphism between singular homology and Milnor-Thurston homology, see Section 1.3).

This homology theory was invented to in order to have a more convenient representation of cycles. It was supposed to coincide with singular homology for hyperbolic manifolds. And in fact, as it was proved, it satisfies the Eilenberg-Steenrod axioms. However, its calculation for spaces more complicated than CW-complexes is by no means automatic. The first results in this direction was provided by Zastrow [34] [33, Section 6.], and the first concrete computation of Milnor-Thurston homology groups was done for the Warsaw Circle by the author of this thesis [26].

Chapter 1 contains a presentation of known results, a brief history and a formal definition of Milnor-Thurston homology. Chapter 2 presents calculation of Milnor-Thurston homology groups of the Warsaw Circle and some other similar spaces. Moreover, it also contains an answer to Berlanga's question whether Milnor-Thurston homology groups are Hausdorff [5]. Chapter 3 contains further results on the zeroth Milnor-Thurston homology group – a proof that for Peano continua it is one-dimensional, a proof that the canonical homomorphism is injective for spaces satisfying some technical conditions (see Theorem 3.3) and finally a counterexample that the canonical homomorphism need not be injective in general. Results of Chapter 2 have already been published by the author of this dissertation [26] and results of Chapter 3 are contained in the preprint [27] which is currently under review.

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Chapter 1

Preliminaries

This chapter is devoted to recalling results that exist in literature. In the first section we define some notions and recall several results that will be used in this thesis. The purpose of the second section is to present Milnor-Thurston homology theory along with some of its applications.

1.1 Results from analysis and measure theory

A *σ-algebra* over a set Ω is a family of subsets of Ω that contains the empty set and is closed with respect to complements and countable unions. An intersection of any number of σ-algebras is also a σ-algebra. Thus, for every family $\mathcal{S}$ of subsets of Ω there exists the smallest σ-algebra containing $\mathcal{S}$. We call it *the σ-algebra generated by* $\mathcal{S}$, and it is denoted by $\sigma(\mathcal{S})$.

Definition 1.1. A pair $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra over Ω is called *measurable space.*

Definition 1.2. Let $(\Omega, \mathcal{F})$ be a measurable space. A function $\mu : \mathcal{F} \to \mathbb{R}$ is called a *finite signed measure* if it is countably additive and vanishes on the empty set.

Remark. In this thesis we consider only finite signed measures, thus for simplicity we shall call them measures.

Every topological space is a measurable space in the following natural way: Let (X, τ) be a topological space. The σ-algebra generated by τ is called *the Borel σ-algebra* and it is denoted by $\mathcal{B}(X)$.

Let $(\Omega_i, \mathcal{F}_i)$ for $i = 1, 2$ be measurable spaces. A function $f : \Omega_1 \to \Omega_2$ is called *measurable* if the preimage of every set in $\mathcal{F}_2$ is contained in $\mathcal{F}_1$.

Definition 1.3. Given a measurable function $f : \Omega_1 \to \Omega_2$ and a measure μ on Ω_1 we define *the image measure $f\mu$* by the formula

$$(f\mu)(A) = \mu(f^{-1}(A)), \qquad \text{for any measurable set A}$$

Definition 1.4. Let μ be a measure on a measurable space $(\Omega, \mathcal{F})$. *A carrier of measure μ* is a set $D \subset \Omega$ such that $\mu(A) = 0$ for any $\mathcal{F} \ni A \subset \Omega \setminus D$.

The following result helps us to deal with signed measures.

Theorem 1.5. (Hahn [18, Theorem A, p. 121])) *Let μ be a signed measure on $(\Omega, \mathcal{F})$. Then there exist two disjoint sets Ω^+, $\Omega^- \in \mathcal{F}$ such that $\Omega = \Omega^+ \cup \Omega^-$ and such that for every $F \in \mathcal{F}$ we have $\mu(F \cap \Omega^+) \geq 0$, $\mu(F \cap \Omega^-) \leq 0$.*

The decomposition of our space Ω into sets Ω^+, Ω^- is not unique. Nevertheless, for two distinct decompositions: Ω_i^+, Ω_i^-, $i = 1, 2$, one can prove that, given any $F \in \mathcal{F}$ it is $\mu(F \cap \Omega_1^+) = \mu(F \cap \Omega_2^+)$, $\mu(F \cap \Omega_1^-) = \mu(F \cap \Omega_2^-)$ [18, p. 122]. Therefore the signed measure μ can be uniquely decomposed into the following difference of unsigned measures

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+(\cdot) = \mu(\cdot \cap \Omega_+)$, $\mu^-(\cdot) = -\mu(\cdot \cap \Omega_-)$.

Definition 1.6. Let μ be a measure on a space X, *the variation* $|\mu|$ of the measure μ shall be defined as

$$|\mu| = \mu^+ + \mu^-.$$

The total variation $\|\mu\|$ shall be defined as

$$\|\mu\| = |\mu|(X).$$

Definition 1.7. Let μ be a signed finite Borel measure. We say that μ is *regular* if for every Borel set B

- $|\mu|(B)$ is the supremum of $|\mu|(K)$ where $K \subset B$ is compact,
- $|\mu|(B)$ is the infimum of $|\mu|(U)$ where $U \supset B$ is open.

The space of regular finite Borel measures on a topological space X shall be denoted by $M(X)$. It is a normed space equipped with the total variation norm. Let $C(X)$ denote the space of real continuous functions on a topological space X. We have

Theorem 1.8. (Compact version of Riesz Representation Theorem [10, Chapter III, Theorem 5.7]) *Let X be a compact Hausdorff space and let $\mu \in M(X)$. Define $F_\mu : C(X) \to \mathbb{R}$ by:*

$$F_\mu(f) = \int_{C(X)} f d\mu.$$

Then $F_\mu \in C(X)^*$ and the map $\mu \mapsto F_\mu$ is an isometric isomorphism of $M(X)$ onto $C(X)^*$.

Here "$()^*$" denotes the continuous dual.

We define the following notions as in [7, p. 41]:

Definition 1.9. A non-empty family of sets is called a *π-system* if it is closed under finite intersections.

Obviously any topology is a π-system.

Definition 1.10. A non-empty family of subsets of space X is called *λ-system* if: it contains X, it is closed under complements and it is closed under countable disjoint unions.

Notice, that any σ-algebra is a λ-system.

Theorem 1.11. (Dynkin's lemma [7, Theorem 3.2]) *Let $\mathcal{D}$ be a λ-system and let $\mathcal{P} \subset \mathcal{D}$ be a π-system. Then $\sigma(P) \subset \mathcal{D}$.*

Corollary 1.12. *Let μ and ν be Borel measures on a topological space X. Suppose μ and ν are equal on open sets, then $\mu = \nu$.*

Proof. Let $\mathcal{D}$ be the subset of Borel σ-algebra such that for every $A \in \mathcal{D}$ we have $\mu(A) = \nu(A)$. We see that $\mathcal{D}$ is a λ-system. The topology τ of X is a π-system such that $\tau \subset \mathcal{D}$. So by Dynkin's lemma we see that $\mathcal{D}$ is in fact the Borel σ-algebra and hence $\mu = \nu$.

□

The definition of an *algebra* of subsets is analogous to the definition of a σ-algebra but with finite unions instead of countable unions. In construction of measures we shall use the following result of Constatin Carathéodory [2, Theorem 1.3.10]:

Theorem 1.13. (Carathéodory Extension Theorem) *Let μ be an unsigned measure on an algebra of sets $\mathcal{F}_0$. Then, μ has a unique extension to a measure on $\sigma(\mathcal{F}_0)$.*

In fact, if we want to construct a measure it is convenient to define it on some "smaller" family of sets:

Definition 1.14. We say that a family $\mathcal{S}$ of subsets of X is a *semi-algebra* if it contains the empty set, it is closed under finite intersections and for any set $E \in \mathcal{S}$ there exists a finite disjoint collection of sets $C_i \in \mathcal{S}$, such that $X \setminus E = \bigcup_i C_i$.

Remark. An example of a semi-algebra over $[-1, 1]$ may be the family of *semi-closed intervals* of the form when $[a, b)$ intersected with $[-1, 1]$.

Corollary 1.15. *If μ is a non-negative countably additive set function on a semi-algebra $\mathcal{S}$ such that $\mu(\varnothing) = 0$, then there exists an extension of μ to $\sigma(\mathcal{S})$.*

Proof. The algebra of sets $\mathcal{F}_0$ that is generated by $\mathcal{S}$ has a simple description:

$$\mathcal{F}_0 = \left\{ \bigcup_i E_i \mid E_i \text{ is a finite collection of subsets of } \mathcal{S} \right\}$$

It is easy to see that every element of $\mathcal{F}_0$ is in fact a disjoint union of elements in $\mathcal{S}$. Hence, μ has a natural (and well defined!) extension to an additive set function on $\mathcal{F}_0$.

We will prove that it is in fact countably additive. Take a countable collection of subsets $F_j \in \mathcal{F}_0$ such that $F = \bigcup_j F_j \in \mathcal{F}_0$. Each of these sets is a finite disjoint union of elements in $\mathcal{S}$. Namely, $F = \bigcup_i E_i$, $F_j = \bigcup_i E_i^j$. By the intersection property of a semi-algebra we can assume that each E_i^j is a subset of some E_k. Thus, we have

$$E_i = \bigcup_{E_i^j \subset E_i} E_i^j.$$

Hence, countable additivity of μ on $\mathcal{S}$ implies countable additivity of μ on $\mathcal{F}_0$. Finally, by the Carathéodory Extension Theorem we know that there exists an extension of μ on $\sigma(\mathcal{F}_0) = \sigma(\mathcal{S})$.

□

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of X and let $Y \subset X$, then $Y \cap \mathcal{A}$ denotes $\{Y \cap A \mid A \in \mathcal{A}\}$ and $\mathcal{A} \cup \mathcal{B}$ denotes $\{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Moreover notice that if $\mathcal{F}$ is a σ-algebra over X then $A \cap \mathcal{F}$ is a σ-algebra over A.

Lemma 1.16. *Let $A \subset X$ be a subset of a measurable space $(X, \mathcal{F})$. Let $\mathcal{F}$ be generated by a semi-algebra $\mathcal{S}$. Then $A \cap \mathcal{F} = \sigma(A \cap \mathcal{S})$ as a σ-algebra over A.*

Proof. The idea of this proof is a slight generalisation of the proof of [33, Proposition 1.10] (proofs by this method can also be found in some standard texts on measure theory [4, I.1 (1.4)], [20, 1.5(Satz 8)]). So let $\mathcal{G}$ be the

σ-algebra over A generated by $A \cap \mathcal{S}$. Obviously, we have $\mathcal{G} \subset A \cap \mathcal{F}$. In order to prove the other inclusion notice that $\mathcal{G} \cup ((X \setminus A) \cap \mathcal{F})$ is a σ-algebra over X containing $\mathcal{S}$. Thus, $\mathcal{F} \subset \mathcal{G} \cup ((X \setminus A) \cap \mathcal{F})$. Now, applying to both sides of this inclusion $A\cap$ we obtain $A \cap \mathcal{F} \subset \mathcal{G}$.

□

Lemma 1.17. *Let $f : X \to Y$ be a map between a set X and a measurable space $(Y, \mathcal{G})$. Let $\mathcal{G}$ be generated by a semi-algebra $\mathcal{S}$. Then $f^{-1}(\mathcal{G}) = \sigma(f^{-1}(\mathcal{S}))$ as a σ-algebra over X.*

Proof. Without loss of generality we can assume that f is a surjection. This follows from Lemma 1.16 and the fact that $f^{-1}(f(X) \cap \mathcal{A}) = f^{-1}(\mathcal{A})$, for every family $\mathcal{A}$ of subsets of Y.

Let $\mathcal{F} \subset f^{-1}(\mathcal{G})$ be the σ-algebra generated by $f^{-1}(\mathcal{S})$. First, we will prove that $f(\mathcal{F}) := \{f(B) \mid B \in \mathcal{F}\}$ is a σ-algebra. Countable additivity is proved using good behaviour of images with respect to unions. Finally, let $A = f(B)$ for some $B \in \mathcal{F}$, then $Y \setminus A = f(X \setminus B)$ because f is a surjection and every set in $\mathcal{F}$ is a preimage of a set in $\mathcal{G}$.

We can see that $\mathcal{S} \subset f(\mathcal{F})$, thus $\mathcal{G} \subset f(\mathcal{F})$. Applying the operation f^{-1} to this equation we obtain $f^{-1}(\mathcal{G}) \subset \mathcal{F}$, which proves our lemma.

□

Lemma 1.18. *Let G be an open set of a metric space (X, d). Then there exists a sequence of continuous functions converging pointwise from below to the characteristic function of G.*

Proof. Let χ_G denote the characteristic function of G and let f be a continuous function on $[0, \infty)$ such that $f(0) = 0$, $f(t) = 1$ for $t \geq 1$ and $0 \leq f \leq 1$. Then $f_n(x) = f(n \cdot d(x, X \setminus G))$ converge pointwise to χ_G and $f_n \leq \chi_G$ for all n.

□

Theorem 1.19. (Lebesgue Dominated Convergence Theorem [28, p.229]) *Let $(X, \mathcal{F}, \mu)$ be a measure space, let $E \in \mathcal{F}$ and let f_n be a sequence of measurable functions on E such that*

$$|f_n(x)| \leq g(x), \qquad \text{for } x \in E$$

and for an integrable function g on E. Suppose

$$f_n(x) \to f(x)$$

almost everywhere on E. Then f is integrable, and

$$\int_E f d\mu = \lim \int_E f_n d\mu.$$

Theorem 1.20. (Hahn-Banach Theorem [28, p.187]) *Let p be a real valued function defined on a vector space W satisfying $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$. Suppose that λ is a linear functional defined on a subspace $V \subset W$ and that $\lambda(v) \leq p(v)$ for all $v \in V$. Then there is a linear functional Λ defined on W such that $\Lambda(w) \leq p(w)$ for all $w \in W$ and $\Lambda(v) = \lambda(v)$ for all $v \in V$.*

Corollary 1.21. *Let W be a normed real vector space and let $V \subset W$ be its subspace. Then any bounded linear functional $V \to \mathbb{R}$ has a bounded extension to W of the same norm.*

The last well known result we mention here is purely topological. We use it in Chapter 3.

Definition 1.22. *A Peano continuum* is a compact and locally connected metric space.

Theorem 1.23. (Hahn-Mazurkiewicz [21, Theorem 3-30]) *Let X be a Peano continuum, then there exists a continuous surjection $f : [0,1] \to X$.*

1.2 A brief history of Milnor-Thurston homology theory

The idea of Milnor-Thurston homology emerged from Gromov's proof of the Mostow Rigidity Theorem. The first mention of this theory can be found in circulated lecture notes [31, Chapter 6]. Thurston remarks that the proof presented in the notes is different from Gromov's original proof and that it is to be published in his paper with Milnor *Characteristic numbers for three-manifolds*. The paper, however, never appeared.

Simplicial volume, introduced by Gromov in the proof of the Mostow Rigidity Theorem, is a topological invariant deeply connected with the geometric structure of a hyperbolic manifold. Let M be a closed orientable smooth manifold. There is a natural ℓ^1-norm on the space $C_k(M;\mathbb{R})$ generated by singular k-simplices (the norm of a linear combination of simplices is defined to be sum of absolute values of the coefficients). This norm induces *the Gromov seminorm* on the level of homology – it is the infimum to the norm of cycles in the particular homology class (or equivalently the distance of the given homology class to the subspace of boundaries). Now, *the simplicial volume* is defined to be the Gromov seminorm of the fundamental class of M.

Since simplicial volume is defined via homology groups, it is a homotopy invariant. Moreover, Thurston, following Gromov's ideas, proved that for orientable closed hyperbolic manifolds it is proportional to the hyperbolic volume [31, Theorem 6.2]. Thus, any hyperbolic manifold homotopically equivalent to M must have the same volume.

In the proof of Theorem 6.2 in [31] Thurston represents the fundamental class by a measure supported on geodesic simplices of arbitrarily large volume. Thus, there was a need of homology theory, where chains are measures supported on simplices. Thurston creates such a theory and extends the Gromov seminorm to chains of that type (it is simply defined to be the total variation of a measure). The fact that the fundamental class is represented by a measure supported on isometric simplices allowed to calculate the integral of the volume form in an automatic way (it just yields an integral of a constant function!), and thus finding the relation between the simplicial volume and the hyperbolic volume.

In this proof Thurston used the obvious fact that his measure homology and singular homology coincide for hyperbolic manifolds in an isometric way (with respect to Thurston's seminorm on measure homology, and the Gromov seminorm on singular homology). Recently it has been proved even more. First, coincidence result of measure homology (called here Milnor-Thurston homology) was shown by Hansen and Zastrow independently [19, 33]. The authors prove that the homology theory in principal satisfies Eilenberg-Steenrod axioms and thus it coincides with singular homology for CW-complexes. The next essential step, was to prove that Thurston's seminorm and the Gromov seminorm coincide for spaces more general then hyperbolic manifolds. This was done by Clara Löh [24].

Another application of Milnor-Thurston homology groups was found by Ricardo Berlanga. *The mass flow* is a homomorphism from the universal covering of the group of measure preserving homomorphisms to first homology group with real coefficients. Fathi [15] attributes it to Schwartzman [29]. Application of Milnor-Thurston homology instead of singular homology allowed Berlanga to extend Fathi's results on the mass flow and simplify his arguments. In particular, Berlanga introduced a structure of topological vector space on Milnor-Thurston homology groups [5] and proved that the mass flow is continuous with respect to this topology and the Whitney topology on the space of homeomorphisms of a given manifold [6].

1.3 Milnor-Thurston homology theory

Now, we shall present the construction of Milnor-Thurston homology theory. Here we use calligraphic letters ($\mathcal{C}$, $\mathcal{H}$, etc.) for constructions in Milnor-Thurston homology theory and ordinary letters for the corresponding constructions in singular homology theory (C, H, etc.).

First, we will construct the chain complex $\mathcal{C}_*(X)$ for a given topological space X. Let $C^0(\Delta^k, X)$ denote the set of singular simplices (continuous functions from the standard simplex Δ^k to X, where k is a non-negative integer). We shall consider $C^0(\Delta^k, X)$ as a topological space equipped with a compact-open topology. The vector space $\mathcal{C}_k(X)$ of k-dimensional chains shall consist of finite measures with a compact carrier (cf. Definition 1.4; in this thesis the notion of compactness does not require Hausdorffness, this is a different terminology than the one used by Zastrow in [33, Section 1.8]).

Next, in order to make the sequence of vector spaces $\mathcal{C}_k(X)$ a chain complex, we shall define a boundary operator. We can see that the natural inclusions of faces $\delta_i : \Delta^{k-1} \hookrightarrow \Delta^k$, for $i = 0, 1, ..., k$, induce continuous maps $\partial_i : C^0(\Delta^k, X) \to C^0(\Delta^{k-1}, X)$ on the level of singular simplices. These functions are constructed just by using the composition: $\partial_i : \sigma \mapsto \sigma \circ \delta_i$. It can be easily proved that ∂_i are continuous, since the spaces of singular simplices are endowed with the compact-open topology [33, Lemma 2.8].

Now, the continuous functions ∂_i induce maps $\partial_i : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ (denoted by the same symbol!). The operator $\partial_i : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ by definition sends a measure to its image measure (cf. Definition 1.3) with respect to continuous (and hence, measurable) function $\partial_i : C^0(\Delta^k, X) \to C^0(\Delta^{k-1}, X)$. Finally, the boundary operator $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ is given with the usual formula:

$$\partial = \sum_{i=0}^{k} (-1)^i \partial_i.$$

We have the following theorem:

Theorem 1.24. (see [33, Corollary 2.9]) *The sequence $\mathcal{C}_k(X)$ of real vector spaces together with the boundary operators defined above forms a chain complex $\mathcal{C}_*(X)$.*

We can see that $\mathcal{C}_*$ is a functor from the category of topological spaces to the category of chain complexes. Indeed, the chain map $f_\bullet : \mathcal{C}_*(X) \to \mathcal{C}_*(Y)$ induced by a continuous function $f : X \to Y$ is defined in a similar way as the boundary operator. Namely, on the level of singular simplices we have a function $f : C^0(\Delta^k, X) \to C^0(\Delta^k, Y)$ denoted by the same symbol f and defined by the composition

$$f : \sigma \mapsto f \circ \sigma.$$

This function is continuous (again see [33, Lemma 2.8]). Finally, $f_{\bullet k} : \mathcal{C}_k(X) \to \mathcal{C}_k(Y)$ is defined as an operator sending a measure to its image measure with respect to f.

Now, in order to see that $\mathcal{C}_*$ is a functor, we have to prove that it behaves well with respect to a composition of morphisms. It is an immediate consequence of distributivity of composition operation that $C^0(\Delta^k, -)$ is a covariant functor. Thus, it is sufficient to prove that the image measure construction behaves well. But it is implied by the following lemma:

Lemma 1.25. *Let $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ be measurable maps and let μ be a measure on Ω_1. Then we have*

$$(g \circ f)(\mu) = g(f\mu)$$

Proof. Take a measurable set $A \subset \Omega_1$. Then we have $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$. Thus, we have

$$(g \circ f)(\mu)(A) = \mu(f^{-1}(g^{-1}(A))) = (f\mu)(g^{-1}(A)) = g(f\mu)(A).$$

From that, the assertion of our lemma follows.

□

From the same lemma we see that $f_\bullet$ is in fact a chain mapping. We need to prove that $f_{\bullet k-1} \circ \partial_i = \partial_i \circ f_{\bullet k}$, for $i = 0, 1, ..., k$. But from the lemma we see that the operators on the both sides of this equation are induced by the mapping $\sigma \mapsto f \circ \sigma \circ \delta_i$ on singular simplices, and thus they are equal.

Definition 1.26. *The Milnor-Thurston homology groups $\mathcal{H}_*(X)$* are defined as homology groups of this chain complex $\mathcal{C}_*(X)$:

$$\mathcal{H}_k(X) := \frac{\mathcal{Z}_k(X)}{\mathcal{B}_k(X)} = \frac{\ker\{\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)\}}{\mathrm{im}\{\partial : \mathcal{C}_{k+1}(X) \to \mathcal{C}_k(X)\}}.$$

Moreover, we see that $\mathcal{H}_*$ is a functor.

We can also define relative Milnor-Thurston homology groups. Let X be a topological space and let A be its subspace. The relative chain complex $\mathcal{C}_*(X, A)$ is defined as a quotient of $\mathcal{C}_*(X)$ by $i_\bullet(\mathcal{C}_*(A))$ where $i : A \hookrightarrow X$ is the inclusion map. The relative Milnor-Thurston homology groups $\mathcal{H}_*(X, A)$ are by definition homology groups of $\mathcal{C}_*(X, A)$.

Definition 1.27. Let X be a topological space, and let $x \in X$. *The Kronecker measure* concentrated on x is a measure δ_x such that $\delta_x(B) = 1$ if $x \in B$, and $\delta_x(B) = 0$ otherwise.

There is a canonical homomorphism from singular chains to Milnor-Thurston chains

$$\begin{aligned} C_k(X; \mathbb{R}) &\to \mathcal{C}_k(X), \\ \sum_i \alpha_i \sigma_i &\mapsto \sum_i \alpha_i \delta_{\sigma_i}, \end{aligned}$$

where δ denotes the Kronecker measure.

This homomorphism is a monomorphism if and only if X satisfies the separation axiom T_0. Indeed, suppose there are two points $x_1, x_2 \in X$ that have the same neighbourhoods. Let σ_1 and σ_2 be the singular k-simplices that map the whole standard simplex into x_1 or x_2, respectively. Both of these simplices have the same neighbourhoods in $C^0(\Delta^k, X)$. Now, notice that δ_{σ_1} and δ_{σ_2} are the same Borel measures, even though the chains σ_1 and σ_2 are different.

On the other hand assume that X is T_0. Take a linear combination $\sum_i \alpha_i \sigma_i$ that is mapped to zero by the canonical homomorphism. There exists a neighbourhood of σ_1 that does not contain any of σ_i for $i \neq 1$. The value of $\sum_i \alpha_i \delta_{\sigma_i}$ on this neighbourhood is α_1. But, it is zero by the assumption, thus $\alpha_1 = 0$. In the same way we prove that all $\alpha_i = 0$, and thus the kernel of the canonical homomorphism is trivial.

Let σ be a singular k-simplex. It is easy to see, that $\partial_i \delta_\sigma = \delta_{\partial_i \sigma}$. From that, we have

$$\partial \delta_\sigma = \sum_{i=0}^{k} (-1)^i \delta_{\partial_i \sigma}.$$

The right hand side of this formula is the value of the canonical homomorphism on $\sum_{i=0}^{k}(-1)^i \partial_i \sigma = \partial\sigma$. Thus, we see that the canonical homomorphism of chains commutes with the boundary operator, and therefore it induces a canonical homomorphism on the level of homology

$$H_k(X;\mathbb{R}) \to \mathcal{H}_k(X).$$

As was mentioned before, this homomorphism is an isomorphism when X is a CW-complex (it is a consequence of the Eilenberg-Steenrod axioms, see [33]).

1.4 The Mayer-Vietoris theorem

The Mayer-Vietoris theorem is a way to relate the homology groups of a space X with the homology groups of two of its subspaces A and B.

Theorem 1.28. (Mayer-Vietoris) *Let H_* be a homology theory that satisfies the Eilenberg-Steenrod axioms and let A and B be open subspaces such that $X = A \cup B$. Then the following sequence is exact:*

$$\cdots \xrightarrow{(i_{*n}, j_{*n})} H_n(A) \oplus H_n(B) \xrightarrow{k_{*n} - l_{*n}} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \longrightarrow$$
$$\cdots \to H_0(A \cap B) \xrightarrow{(i_{*0}, j_{*0})} H_0(A) \oplus H_0(B) \xrightarrow{k_{*n} - l_{*n}} H_0(X) \longrightarrow 0$$

where $i : A \cap B \to A$, $j : A \cap B \to B$, $k : A \to X$, $l : B \to X$ are inclusion maps.

The proof of this theorem can be found in [14, Theorem 14.6 of Chapter I]. In fact, it is the modern proof. The original result by Walther Mayer [25, IV. Abschnitt] concerned only Betti numbers. One year later it was generalised to homology groups by Leopolod Vietoris [32], but still it was far before formulation of the notion of exact sequence [11, p. 345].

Eilenberg's and Steenrod's proof of the Mayer-Vietoris theorem used the Excision Axiom and the Exactness Axiom. Therefore, the result is true in Milnor-Thurston homology theory for any space for which the Excision Axiom is fulfilled (at least for normal spaces; see [33, Section 4]). In the next chapter we shall use this theorem to calculate Milnor-Thurston homology groups for the Warsaw Circle and some other wild topological spaces.

Remark. The Mayer-Vietoris theorem can also be proved more directly. Let X be a topological space with subspaces A and B. According to [33, Lemma 4.10] the inclusion

$$\mathcal{C}_*(A) + \mathcal{C}_*(B) \to \mathcal{C}_*(X)$$

induces an isomorphism on the level of homology if there exist V such that $\overline{X \setminus A} \subset \overset{\circ}{V} \subset \overline{V} \subset B$ (when X is a normal space it suffices that A and B are open) and $A \cup B = X$.

Using this identity we can construct the short sequence of chain complexes

$$0 \longrightarrow \mathcal{C}_*(A \cap B) \xrightarrow{(i_\bullet, j_\bullet)} \mathcal{C}_*(A) \oplus \mathcal{C}_*(B) \xrightarrow{k_\bullet - l_\bullet} \mathcal{C}_*(A) + \mathcal{C}_*(B) \longrightarrow 0,$$

and then its exactness yields Mayer-Vietoris theorem by homological algebra [23, Theorem 2.1 of Chapter XX].

1.5 Berlanga topology on Milnor-Thurston homology groups

Berlanga equipped Milnor-Thurston homology groups with a topology consistent with their linear space structure [5]. Moreover, it is proved that this topology is locally convex when the underlying topological space is second countable and separable (it is discussed below in more details). Consequently, we obtain a functor from the category of second countable and separable topological spaces to the category of locally convex topological vector spaces (not necessarily Hausdorff!):

Let X be a second countable separable topological space. Given any continuous function $f : C^0(\Delta^k, X) \to \mathbb{R}$ we define a linear functional $\Lambda_f : \mathcal{C}_k(X) \to \mathbb{R}$:

$$\Lambda_f(\mu) = \int_{C^0(\Delta^k,X)} f d\mu,$$

for every $\mu \in \mathcal{C}_k(X)$. The above functional is well defined, since f is continuous and every measure in $\mathcal{C}_k(X)$ has a compact carrier. We shall work with the weakest topology on $\mathcal{C}_k(X)$ for which all such functionals are continuous. It has been proved, that this weak topology is locally convex and Hausdorff because X is second countable and separable [5, Assertion 2.2].

Berlanga proved that the boundary operator ∂ is continuous [5, Assertion 2.1]. Consequently the homology groups

$$\mathcal{H}_k(X) = \mathcal{Z}_k(X)/\mathcal{B}_k(X)$$

can be endowed with the structure of locally convex topological vector space. We call this topology *Berlanga topology.*

Remark. Notice, that the notion of local convexity does not include Hausdorfness here. There is no reason to think that $\mathcal{B}_k(X)$ are closed subspaces, and thus $\mathcal{H}_k(X)$ need not to be Hausdorff. In fact, Berlanga asked a question whether Milnor-Thurston homology groups are Hausdorff in this topology [5].

Berlanga himself was able to show that $\mathcal{H}_1$ is always Hausdorff for spaces that are homotopy equivalent to CW-complexes. Moreover, Frigerio extended this result to every dimension [17]. On the other hand, Zastrow constructed an example of the space V where $\mathcal{H}_0(V)$ is not Hausdorff [34]. This space V is the Warsaw Circle with a part of accumulation line removed (see Figure 2.7). We present a proof of this fact in Chapter 2 (see Theorem 2.6).

Chapter 2

Milnor-Thurston homology for wild topological spaces

We know that Milnor-Thurston homology theory coincides with singular homology for CW-complexes (see Section 1.3). Additionally, Zastrow constructed a space where the canonical homomorphism is not an isomorphism [33, p. 393]. This space, that we call here the Convergent Arcs Space, is not a CW-complex, and therefore its study naturally fits our topic, since the general question of this thesis is: "What is the behaviour of Milnor-Thurston homology for spaces that are not CW-complexes?".

Another interesting research problem within this topic, is comparing Milnor-Thurston homology with Čech homology. There is the well known example of the Warsaw Circle W (it is formally defined below in Section 2.1) that has the same Čech homology groups as a circle [22, Remark 2.7]. Moreover, first singular homology group of W is trivial. The natural question is: "Does Milnor-Thurston homology detect the circular shape of the Warsaw Circle like Čech homology does?".

The techniques we present in this chapter are powerful enough to understand the structure of Milnor-Thurston homology groups of the Warsaw Circle and the Convergent Arcs Space. Additionally, we can also answer the question of Berlanga: "Are Milnor-Thurston homology groups Hausdorff in

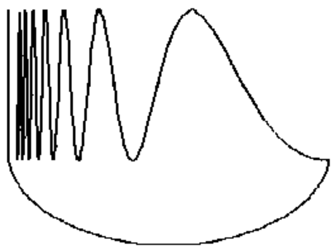

Figure 2.1: The Warsaw Circle

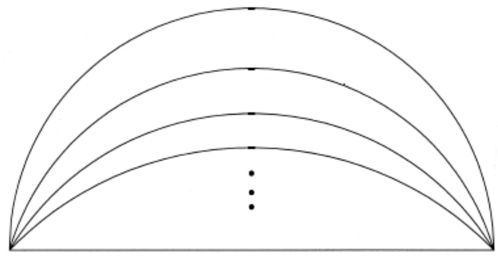

Figure 2.2: The Convergent Arcs Space

Berlanga topology" [5, p. 367].

2.1 Spaces we are interested in

In this chapter we focus on three different examples of spaces: the Warsaw Circle W, the Convergent Arcs Space CA and the Double Warsaw Circle DW. We define them formally in this section.

The Warsaw Circle (see Figure 2.1) is defined as the subset of $\mathbb{R}^2$ that consists of:

- the part of "Topologists Sine Curve" $\{(x, y) \in \mathbb{R}^2 \mid y = \sin 1/x\}$ between the line $x = 0$ and the rightmost minimum,
- the "accumulation line" $\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$,
- an arc connecting the point $(0, -1)$ with the rightmost minimum.

By *the Double Warsaw Circle* (see Figure 2.3) we mean the space that is a copy of two Warsaw Circles overlapping at the accumulation line.

The Convergent Arcs Space (see Figure 2.2) is a space built of a countable number of arcs connecting two given vertices. They converge, in the topology induced from the plane, to a line segment that is also a part of this space.

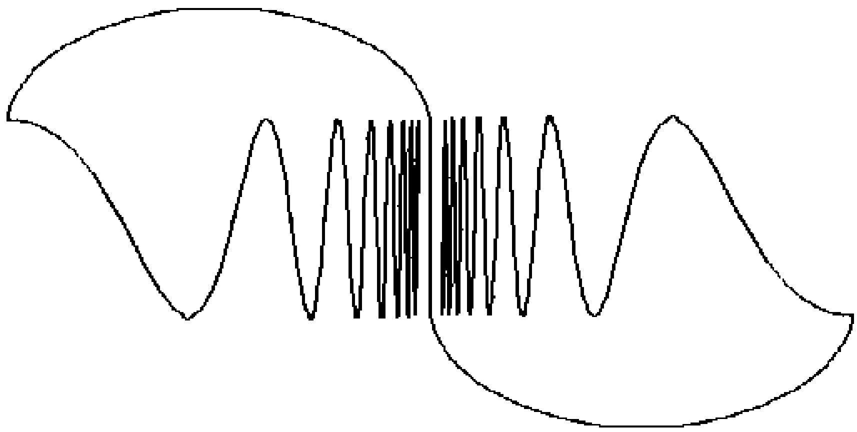

Figure 2.3: The Double Warsaw Circle

2.2 Geometric intuition

This section explains the geometric intuition behind the results of this chapter. They will be proved by formal arguments in the next sections.

First, we try to understand why the canonical homomorphism (cf. Section 1.3) from singular homology to Milnor-Thurston homology may not be an isomorphism. More generally, we will see that there is no isomorphism between first homology groups for *the Convergent Arcs Space CA*.

Let us denote the building arcs of CA by l_i for $i = 1, 2, ...$ The limit arc is denoted by l_0, and we denote endpoints of those arcs by P and Q. For every arc l_i we choose some singular simplex σ_i that parametrises it. Let δ_i denote the Kronecker measure on σ_i.

Now, pick some $\mu \in \mathcal{Z}_1(CA)$. Every singular 1-simplex in CA can be homotoped relative to its vertices to a 1-simplex that passes through P and Q only finite number of times. Thus, μ is homological to some cycle μ_1 supported on such simplices.

Next, every 1-simplex in a carrier of μ_1 can be divided into paths such that at least one of its vertices is P or Q. Therefore, there exists $\mu_2 \in \mathcal{Z}_1(CA)$ homological to μ_1, and such that each 1-simplex in its carrier is attached

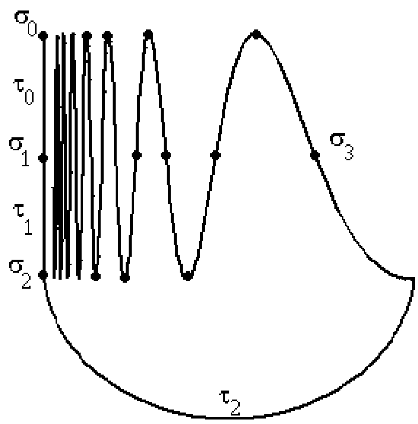

Figure 2.4: The Warsaw Circle subdivided into simplices

to P or Q. Notice, that μ_2 is a finite measure. This is a consequence of compactness of a carrier D of μ_1 that implies existence of a uniform bound to the number of occurrences of P and Q in 1-simplices in D.

In the carrier of μ_2 there are simplices with only one vertex in $\{P, Q\}$. However, since μ_2 is a cycle, we can merge such simplices together. Thus, we get a measure μ_3 that is supported only on simplices connecting P and Q.

Finally, by homotopy relative to the endpoints P and Q (and change of orientation if necessary) we construct measure μ_4 that is supported on $\{\delta_i\}_{i=0}^{\infty}$. Hence, we see that every 1-cycle is homological to a measure of the form

$$\sum_{i=0}^{\infty} a_i \delta_i,$$

where $(a_i)_{i=0}^{\infty}$ is an absolutely summable sequence.

An analogous reasoning shows that every singular 1-cycle is homological to a finite linear combination of σ_{l_i}. Additionally, we see that the canonical homomorphism is

$$\sum_{i=0}^{n} a_i \sigma_{l_i} \mapsto \sum_{i=0}^{n} a_i \delta_i.$$

This clearly shows, that the canonical homomorphism is an injection, but it is not isomorphism. Moreover,

$$H_1(CA) \cong \mathbb{R}^\infty \cong \bigoplus_{\aleph_0} \mathbb{R}, \quad \mathcal{H}_1(CA) \cong \ell^1 \cong \bigoplus_{\mathfrak{c}} \mathbb{R},$$

where ℓ^1 denotes the vector space of absolutely summable sequences and $\mathbb{R}^\infty$ denote the vector space of sequences with almost all elements zero. Thus, we see that these groups cannot be isomorphic.

The next problem posed by us was, whether the first Milnor-Thurston homology group of *the Warsaw Circle* is a one-dimensional vector space. Again, we address this question in this section in an intuitive manner and we postpone a formal argument to the next section.

Let us divide the Warsaw Circle into arcs as presented on Figure 2.4. Choose a family $\{\tau_i\}_{i=0}^\infty$ of singular 1-simplices that parametrise the corresponding arcs and let $\{\sigma_i\}_{i=0}^\infty$ denote the vertices of the corresponding 1-simplices.

The argument analogous as in the case of the Convergent Arcs Space shows us, that we can represent chains by absolutely summable real functions supported on simplices τ_i. The boundary is calculated in the usual way, so, for instance, the coefficient of σ_1 is equal to the difference of coefficients of τ_0 and τ_1.

However, there is one 0-simplex that is a face of only one 1-simplex (it is denoted by σ_0 on Figure 2.4), so the condition to be a cycle implies that coefficient of σ_0 and, consequently, coefficient of τ_0 is zero. By induction we see that coefficients of τ_i should be zero, for every $i \in \mathbb{N}$. As a consequence, there is no "fundamental class" for the Warsaw Circle.

The above problem is caused by the 0-simplex that does not belong to two 1-simplices. Therefore, it is reasonable to consider the case where no such simplex exists. This leads us to the idea of *the Double Warsaw Circle* (this space can be divided into simplices in a similar manner as the Warsaw Circle). Here, the cycle condition implies that coefficients for all 1-simplices should be the same. However, this contradicts finiteness of the corresponding measures. Hence, we have no "fundamental class" again.

Now, let us focus on $\mathcal{H}_0(W)$ for a moment. By the above argument, we see that every 0-chain can be represented by a measure concentrated on $\{\sigma_i\}_{i=0}^{\infty}$. Every such a measure is a cycle, by definition. In order to find $\mathcal{H}_0(W)$ we need to find cycles that are in the image of the boundary operator, and then mod out by these cycles.

At first glance it is hard to see what we get, but there are some things we can say right away. The zeroth singular homology group of W is one-dimensional, since W is a path-connected space. From that, we see that every 0-cycle concentrated on finite number of σ_i will be homological to a chain $\alpha\sigma_0$, for some $\alpha \in \mathbb{R}$.

The natural question is whether there exist cycles concentrated on infinite number of σ_i which are not homological to $\alpha\sigma_0$. We will show that the answer is positive and the condition such cycles need to satisfy is a convergence of coefficients of σ_i to zero that is slow enough. Consequently, the group $\mathcal{H}_0(W)$ is not one-dimensional. To see the above facts, one has to write down the formulae for the boundary operator. However, we postpone it to the next section.

The arguments given in this section can be formalised. Although, there already exists an algebraic technique, that can be used to prove the above results in a formal way. It is the Mayer-Vietoris theorem. However, the intuition presented here can help us to understand how this abstract method really works, as we can see in the next section.

2.3 Higher dimensional homology groups for the Warsaw Circle

The goal of this section is to prove that Milnor-Thurston homology groups of the Warsaw Circle W are trivial in positive dimensions. The algebraic technique we use is the Mayer-Vietoris theorem applied in a proper way.

We cover W by two open subsets L and U. Both of them are constructed using the embedding of W in the plane. Let L be an intersection of W with

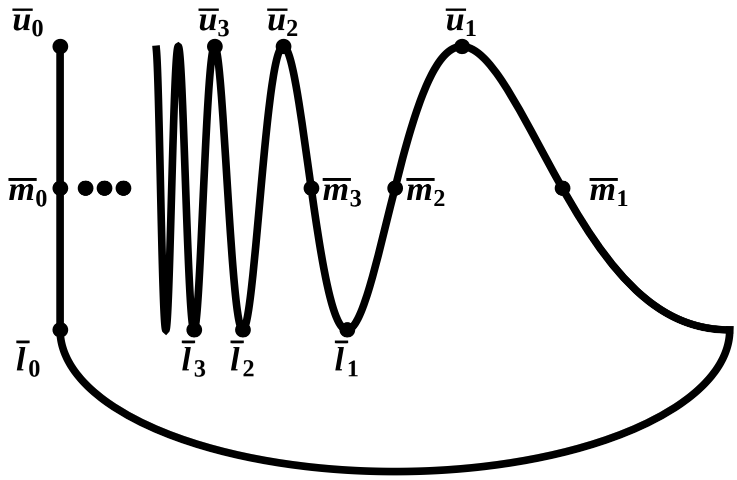

Figure 2.5: The Warsaw Circle with distinguished points

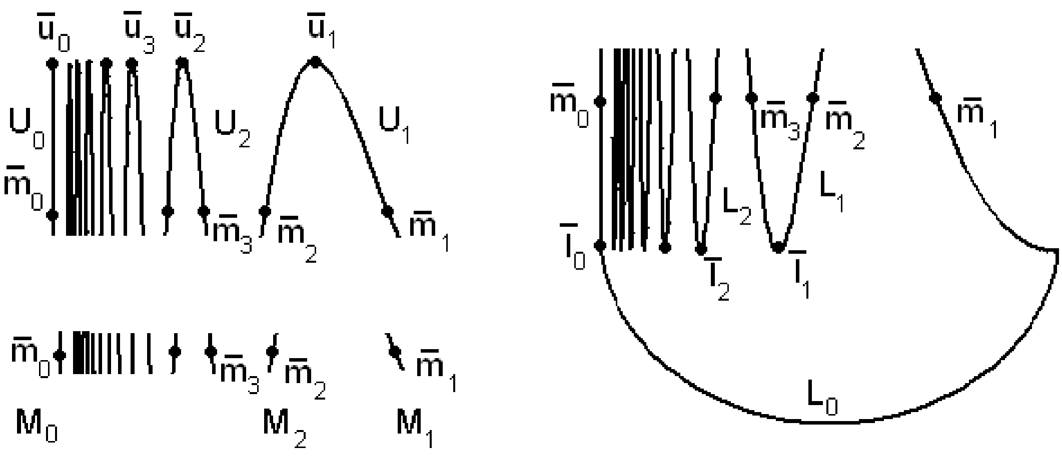

Figure 2.6: Three covering sets for the Warsaw Circle

the halfplane $\{(x, y) \mid y < \eta\}$, where $0 < \eta < 1$. Similarly, the subset U is an intersection of W with $\{(x, y) \mid y > -\eta\}$. Let us denote the path components of L by L_k, for $k = 0, 1, ...$ (see Figure 2.6). In the same way U and $U \cap L$ is decomposed into its path components denoted by U_k and M_k, respectively.

We pick up one point from each of these components; this will be useful in the following proofs. Namely, let $\overline{m}_1$ be the first zero of $\sin 1/x$ after the rightmost minimum. The next zero is denoted by $\overline{m}_2$, and so on (see Figure 2.5 and Figure 2.6). Additionally, let $\overline{m}_0 = (0, 0)$. We see that all $\overline{m}_k \in M_k \subset U \cap L$.

Similarly, let $\overline{u}_1$ denote the first maximum after the rightmost minimum, let $\overline{u}_2$ denote the next maximum, and so on. Moreover, let $\overline{u}_0 = (0, 1)$. Again, we see that all $\overline{u}_k \in U_k \subset U$.

Finally, we do the same for L: let $\overline{l}_1$ denote the first minimum on the left of the rightmost minimum, let $\overline{l}_2$ denote the first minimum on the left of $\overline{l}_1$, and so on. Then, let $\overline{l}_0 = (0, -1)$. We get $\overline{l}_k \in L_k \subset L$.

According to our intuition as presented in the previous section, it is necessary to divide singular simplices into shorter ones. This process can be technically realised via the Mayer-Vietoris theorem. The key idea of this theorem is to divide all singular 1-simplices into their parts contained in U or L (this is done by the barycentric subdivision of simplices, which is used to prove the Excision Axiom [33, Section 4] or the Mayer-Vietoris theorem itself, cf. Remark on p. 14). After this process of division, every simplex is contained in one of L_k or U_k.

Moreover, we would like to reduce our attention to 1-simplices that have their endpoints in $\{\overline{m}_k\}_{k=0}^{\infty}$. In this case, however, the Mayer-Vietoris theorem is not much of a help – the only thing we know is that their endpoints lie in $\bigcup_{k=0}^{\infty} M_k$.

There is however another approach to this problem – we can prove that U, L and $U \cap L$ all have the homotopy type of a convergent sequence with its limit. For that kind of space the calculation of Milnor-Thurston homology groups is straightforward.

So, let S denote a convergent sequence $(x_k)_{k=0}^{\infty}$ with its limit x_0 (its topology is induced from the plane). This space is so simple that we can put our hands on the space of singular simplices, and also on the space of measures (cf. Lemma 2.2). Consequently, this will allow us to do our calculations.

Lemma 2.1. *The spaces $U \cap L$, U and L have the homotopy type of S.*

Proof. Let us start with proving this lemma for $U \cap L$. We define a function $f_M : U \cap L \to S$ in the following way: let $x \in M_k$, than we put $f_M(x) = x_k$. Next, we define $g_M : S \to U \cap L$ by $g(x_k) = \overline{m}_k$. We can see that $f_M \circ g_M = \mathrm{id}_S$ and $g_M \circ f_M$ is a map that sends each point in M_k to $\overline{m}_k$, for $k = 0, 1, \ldots$. This composition is homotopic to $\mathrm{id}_{U \cap L}$.

Next, we prove the lemma for U. We define functions $f_U : U \to S$ and $g_U : S \to U$ in the similar way as in the previous case. That is: $f_U(x) = x_k$ for $x \in U_k$ and $g_U(x_k) = \overline{u}_k$. We can see, that $f_U \circ g_U = \mathrm{id}_S$ and $g_U \circ f_U \simeq \mathrm{id}_U$.

Finally, we prove the lemma for L. The functions $f_L : L \to S$ and $g_L : S \to L$ are defined in a similar manner as before. That is: $f_L(x) = x_k$ for $x \in L_k$ and $g_L(x_k) = \bar{l}_k$. We can see that $f \circ g = \mathrm{id}_S$ and $g_L \circ f_L \simeq \mathrm{id}_L$.

$\square$

Since Milnor-Thurston homology groups are homotopy invariant (because they satisfy the Eilenberg-Steenrod axioms, cf. Section 1.3), the next lemma allows us to calculate them for U, L and $U \cap L$.

Lemma 2.2. *If $n > 0$, then $\mathcal{H}_n(S) = 0$ and $\mathcal{H}_0(S) \cong \ell^1$, where ℓ^1 denotes the space of sequences which form an absolutely convergent series.*

Proof. We can see that

$$C^0(\Delta^n, S) = \{x_k^n : \Delta^n \to S \mid x_k^n \text{ sends any point of } \Delta^n \text{ to } x_k, k \in \mathbb{N}_0\}.$$

The space $C^0(\Delta^n, S)$ is homeomorphic to S, because $(x_k^n)_{k=0}^{\infty}$ is a convergent sequence with limit x_0^n. From that, every subset of this space is Borel, and every two Borel measures which are equal on singletons $\{x_k^n\}$ are equal.

Therefore, we can identify a sequence of real numbers $(a_k)_{k=0}^{\infty}$ with a measure μ such that $\mu(\{x_k^n\}) = a_k$. Additionally, we can see that

$$\|(a_k)_{k=0}^{\infty}\| := \|\mu\| = \sum_{k=0}^{\infty} |a_k|,$$

and every measure has a compact carrier (that is the whole space). Consequently,

$$\mathcal{C}_n(S) \cong \ell^1 := \{(a_k)_{k=0}^{\infty} \mid a_k \in \mathbb{R}, \sum_{k=0}^{\infty} |a_k| < \infty\}.$$

We have $\partial_i x_k^n = x_k^{n-1}$, which implies that $\partial_i (a_k)_{k=0}^{\infty} = (a_k)_{k=0}^{\infty}$. From that,

$$\partial (a_k)_{k=0}^{\infty} = \sum_{i=0}^{n} (-1)^i \partial_i (a_k)_{k=0}^{\infty} = (a_k)_{k=0}^{\infty} \cdot \sum_{i=0}^{n} (-1)^i.$$

From here, $\partial = 0$ when n is odd, and $\partial = \mathrm{id}$ when $n > 0$ is even. Thus, homology groups are trivial for $n > 0$. Indeed, this implies that if n is odd $\mathcal{Z}_n(S) = \mathcal{C}_n(S)$, but on the other hand $\mathcal{B}_n(S) = \mathcal{C}_n(S)$. Hence, $\mathcal{H}_n(S) = 0$. If n is even, the subspace $\mathcal{Z}_n(S)$ of cycles is trivial, and so is $\mathcal{H}_n(S)$.

On the other hand, we have $\partial = 0$, for $n = 0$. Hence, every element in ℓ^1 is a cycle. Because $\partial = 0$, for $n = 1$, there are no boundaries and $\mathcal{H}_0(S) = \mathcal{C}_0(S) \cong \ell^1$.

□

Finally, using the Mayer-Vietoris sequence, we can calculate homology groups.

Theorem 2.3. *If* $n > 0$*, then* $\mathcal{H}_n(W) = 0$.

Proof. The Mayer-Vietoris sequence

$$\cdots \xrightarrow{(i_{*n}, j_{*n})} \mathcal{H}_n(U) \oplus \mathcal{H}_n(L) \xrightarrow{k_{*n} - l_{*n}} \mathcal{H}_n(W) \xrightarrow{\partial_*} \mathcal{H}_{n-1}(U \cap L) \longrightarrow$$
$$\cdots \to \mathcal{H}_0(U \cap L) \xrightarrow{(i_{*0}, j_{*0})} \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \xrightarrow{k_{*0} - l_{*0}} \mathcal{H}_0(W) \longrightarrow 0$$

is exact. Hence by Lemma 2.2, we have $\mathcal{H}_n(W) = 0$ for $n > 1$. So, we have to investigate the case $n = 1$ only.

By exactness of the Mayer-Vietoris sequence and the fact that $\mathcal{H}_1(U) \cong \mathcal{H}_1(L) \cong 0$ we see that $\partial_* : \mathcal{H}_1(W) \to \mathcal{H}_0(U \cap L)$ is a monomorphism. Consequently,

$$\mathcal{H}_1(W) \cong \ker(i_{*0}, j_{*0}).$$

Therefore, we need to find the kernel of (i_{*0}, j_{*0}).

By Lemma 2.2:

$$\mathcal{H}_0(U) \cong \mathcal{H}_0(L) \cong \mathcal{H}_0(U \cap L) \cong \mathcal{C}_0(S) \cong \ell^1,$$

so we can identify elements of all these homology groups with absolutely summable real sequences. This identification allows us to write down formulae for i_{*0} and j_{*0}.

Let $(m_k)_{k=0}^{\infty} \in \ell^1$ denote a homology class in $\mathcal{H}_0(U \cap L)$. This class is represented by a measure supported on the set $\{\overline{m}_k\}_{k=0}^{\infty}$, where m_k's are the values of the measure on the singletons $\{\overline{m}_k\}$. Similarly, every homology class in $\mathcal{H}_0(U)$ is described by some $(u_k)_{k=0}^{\infty} \in \ell^1$, and it is represented by a measure supported on $\{\overline{u}_k\}_{k=0}^{\infty}$.

In order to investigate i_{*0}, we have to associate a measure supported on $\{\overline{u}_k\}_{k=0}^{\infty}$ with a measure supported on $\{\overline{m}_k\}_{k=0}^{\infty}$ that represents the same homology class in U. So, let μ be a measure supported on $\{\overline{m}_k\}_{k=0}^{\infty}$ (cf. Figure 2.5) represented by the sequence $(m_k)_{k=0}^{\infty}$. We will construct a measure supported on $\{\overline{u}_k\}_{k=0}^{\infty}$ which belongs to the same $\mathcal{H}_0(U)$-homology class as μ.

Let σ_0 be a singular 1-simplex that connects $\overline{m}_0$ with $\overline{u}_0$. And, let σ_{2k} denote a singular 1-simplex connecting $\overline{m}_{2k}$ with $\overline{u}_k$ and let σ_{2k+1} be a singular 1-simplex connecting $\overline{m}_{2k+1}$ with $\overline{u}_k$. Now, let $\nu = \sum_{k=0}^{\infty} m_k \delta_{\sigma_k}$, where δ_{σ_k} is the Kronecker measure supported on σ_k. We can see, that $\nu \in \mathcal{C}_1(U)$, since ν is finite and has a compact carrier (because $(\sigma_k)_{k=0}^{\infty}$ is a convergent sequence). The measure $\mu + \partial\nu$ is supported on $\{\overline{u}_k\}_{k=0}^{\infty}$, its coefficients depend on $(m_k)_{k=0}^{\infty}$ as described below. From the definition of σ_0 we have that

$$u_0 = m_0. \tag{2.1}$$

Furthermore, from the definitions of σ_{2k} and σ_{2k+1} we have

$$u_k = m_{2k} + m_{2k-1} \quad \text{for } k > 0. \tag{2.2}$$

These are the equations that describe i_{*0}.

In the similar way, we can write down formulae for j_{*0}:

$$l_k = m_{2k} + m_{2k+1}. \tag{2.3}$$

We can describe (i_{*0}, j_{*0}) in a compact way. So let $x_{2k} = u_k$ and $x_{2k-1} = l_k$. From now on an absolutely summable sequence $(x_k)_{k=0}^{\infty}$ is identified with elements of $\mathcal{H}_0(U) \oplus \mathcal{H}_0(L)$. In this notation, equations (2.1), (2.2), (2.3) yield

$$x_k = \begin{cases} m_0 & \text{for } k = 0, \\ m_k + m_{k-1} & \text{for } k > 0. \end{cases} \tag{2.4}$$

Now, we have that the kernel of (i_{*0}, j_{*0}) and, consequently, $\mathcal{H}_1(W)$ is trivial.

□

2.4 Zeroth Milnor-Thurston homology group for the Warsaw Circle

The Mayer-Vietoris theorem allowed us to prove triviality of the first Milnor-Thurston homology group of the Warsaw Circle. Now, we shall focus on the zeroth homology group; it can also be calculated using this technique. Here, we use the notation defined in the Section 2.3. The following theorem unveils the structure of the zeroth homology group

Theorem 2.4. *The vector space $\mathcal{H}_0(W)$ is continuum-dimensional.*

Proof. We shall use results from the proof of Theorem 2.3; mostly equation (2.4). Once again we will use the Mayer-Vietoris theorem to do the calculations. The Mayer-Vietoris sequence is (cf. Theorem 1.28)

$$0 \to \mathcal{H}_0(U \cap L) \xrightarrow{(i_{*0}, j_{*0})} \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \xrightarrow{s_{*0} - t_{*0}} \mathcal{H}_0(W) \longrightarrow 0.$$

From that, we can see $\mathcal{H}_0(W)$ is the quotient $\ell^1/h(\ell^1)$, where $h : \ell^1 \to \ell^1$ is the map defined by equation (2.4). This equation can be inverted so that, given an arbitrary sequence $(x_k)_{k=0}^{\infty}$, we can find a unique sequence $(m_k^x)_{k=0}^{\infty}$ that satisfies it; a simple calculation yields

$$m_k^x = \sum_{i=0}^{k} (-1)^{i+k} x_i. \tag{2.5}$$

An element $(x_k)_{k=0}^{\infty} \in \ell^1$ represents a nonzero homology class in $\mathcal{H}_0(W)$ iff it is not in the image of (i_{*0}, j_{*0}) or, equivalently, if the corresponding $(m_k^x)_{k=0}^{\infty}$ is not an absolutely summable sequence.

Definition 2.5. A homology class in $\mathcal{H}_k(X)$ shall be called *singular homology class* if it lies in the image of the canonical homomorphism $H_k(X;\mathbb{R}) \to \mathcal{H}_k(X)$. Otherwise it shall be called *non-singular homology class.*

Now, we shall find a one dimensional subspace of $\mathcal{H}_0(W)$ corresponding to singular homology classes. In singular homology theory we consider chains with only finite numbers of simplices, so now restrict ourselves to considering a sequence $(x_k)_{k=0}^{\infty}$ with finitely many nonzero elements. We will prove that such an element $(x_k)_{k=0}^{\infty} \in \ell^1$ represents the same homology class as $(y_k)_{k=0}^{\infty} \in \ell^1$ of the form $(y_k)_{k=0}^{\infty} = (\alpha, 0, 0, 0, \dots)$, for some $\alpha \in \mathbb{R}$. Let N denote the biggest index of nonzero elements in $(x_k)_{k=0}^{\infty}$, then for $k > N$ we have

$$m_k^{x-y} = (-1)^k \left(\sum_{i=0}^{N} (-1)^i x_i - \alpha \right).$$

So putting $\alpha = \sum_{k=0}^{N} (-1)^i x_k$, yields $m_k^{x-y} = 0$. Thus, it is absolutely summable and $(x_k)_{k=0}^{\infty} - (y_k)_{k=0}^{\infty}$ represents the zero homology class.

This result is very intuitive. The Warsaw Circle is a path-connected space, thus its zeroth singular homology group is one-dimensional. Moreover, one can easily deduce this result using our intuitive model (cf. Section 2.2). A simple calculation shows that every measure concentrated on a finite number of points σ_i is homological to a measure concentrated on σ_0 (see Figure 2.4).

Now, we shall prove that $\mathcal{H}_0(W)$ is much bigger than the one-dimensional subspace of singular homology classes. In fact, as was stated in our theorem, its dimension is continuum.

We will start with some sequence of positive numbers n_k which is monotonically decreasing with $\lim n_k = 0$. From now on, up to the end of this proof, let $(x_k)_{k=0}^{\infty}$ have a special form:

$$x_k = (-1)^k(n_{k+1} - n_k).$$

We can see that

$$\sum_{k=0}^{N} |x_k| = n_0 - n_{N+1},$$

hence $(x_k)_{k=0}^{\infty} \in \ell^1$.

Let us calculate m_k^x using (2.5):

$$m_k^x = \sum_{i=0}^{k}(-1)^{i+k}x_i = (-1)^k\sum_{i=0}^{k}(n_{i+1} - n_i) = (-1)^k(n_{k+1} - n_0). \tag{2.6}$$

The sequence $(m_k^x)_{k=0}^{\infty}$ is not absolutely summable, since it does not fulfil the necessary condition $\lim_{k\to\infty} m_k^x = 0$. Hence, $(x_k)_{k=0}^{\infty}$ does not correspond to the zero homology class

More generally, we will check what conditions should be imposed on $(x_k)_{k=0}^{\infty}$ in order to make it a non-singular homology class. So let $(y_k)_{k=0}^{\infty} = (\alpha, 0, 0, ...)$, for $\alpha \in \mathbb{R}$, be a sequence corresponding to some singular homology class. In this case obviously:

$$m_k^{x-y} = (-1)^k(n_{k+1} - n_0 - \alpha);$$

we can easily see this when we notice that $(m_k^x)_{k=0}^{\infty}$ is linear with respect to x according to equation (2.5). So, if we take $\alpha = -n_0$ the sequence satisfies the necessary condition of series convergence. Then, we see that a sufficient condition for x to be a non-singular homology class is

$$\sum_{k=0}^{\infty} n_k = \infty,$$

so we are interested in sequences $(n_k)_{k=0}^{\infty}$ converging to zero but not too fast.

As an example of such a sequence we consider:

$$n_k^{\beta} = \frac{1}{(k+1)^{\beta}},$$

with $0 < \beta < 1$.

Now, we shall prove that the homology classes in $\mathcal{H}_0(W)$ corresponding to the family of sequences $(x_i^{\beta})_{i=0}^{\infty}$ defined by $x_k^{\beta} = (-1)^k(n_{k+1}^{\beta} - n_k^{\beta})$ form a set of linearly independent vectors. So, take a finite sequence of numbers $0 < \beta_i < 1$ in an increasing order, and some finite sequence of real numbers b_i. We shall prove that the homology class of $(z_k)_{k=0}^{\infty} = \sum_i b_i \cdot (x_k^{\beta_i})_{k=0}^{\infty}$ is nontrivial.

In order to do this we need to prove that the sequence

$$m_k^z = (-1)^k \sum_i b_i \left(\frac{1}{(k+2)^{\beta_i}} - 1 \right)$$

is not absolutely summable. To obtain the above formula we used the fact that $(m_i^x)_{i=0}^{\infty}$ is linear with respect to x, and the equation (2.6).

First, we notice that for the necessary condition of convergence for series $\sum_{k=0}^{\infty} |m_k^z|$ to be satisfied, we should have $\sum_i b_i = 0$. Then, the study of the absolute summability of the above sequence can be reduced to the study of

$$\sum_{k=0}^{\infty} \left| \sum_i \frac{b_i}{(k+2)^{\beta_i}} \right|.$$

For sufficiently big k the expression in $|\cdot|$ has the sign of b_0 (since β_0 is the smallest of the numbers), so we can consider:

$$\sum_{k=0}^{\infty} \sum_i \frac{b_i}{(k+2)^{\beta_i}}.$$

This series is divergent. The easiest way to see this is to use the integral criterion. First, we need to notice, that it is for monotonic sufficiently big k. Then, application of the criterion is straightforward.

□

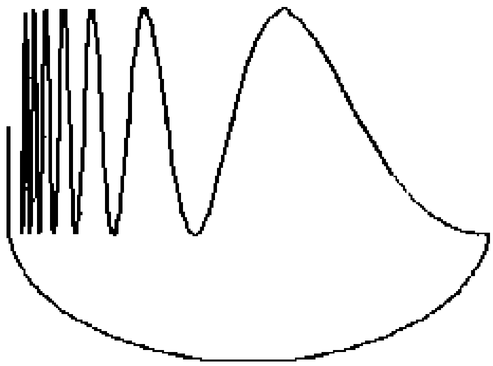

Figure 2.7: The Modified Warsaw Circle

2.5 On Hausdorffness of Berlanga topology

The question that was posed by Berlanga in [5] is whether Milnor-Thurston homology groups are Hausdorff with respect to a topology defined in this paper. There are three results in this direction. Firstly, Berlanga's paper that was mentioned above ends with a proof that $\mathcal{H}_1$ is always Hausdorff for spaces that are homotopy equivalent to countable CW-complexes. Secondly, Frigerio proved that Berlanga topology on all Milnor-Thurston homology groups of CW-complexes is the strongest weak topology, and thus Hausdorff [17].

Finally, Zastrow constructed an example of a space V where $\mathcal{H}_0(V)$ is not Hausdorff [34]. This space V is the Warsaw Circle with a part of the accumulation line removed (see Figure 2.7).

So let V denote the Warsaw Circle $W \subset \mathbb{R}^2$ with an interval $\{(0, y) \in \mathbb{R}^2 \mid 0 < y \leq 1\}$ removed. Since Zastrow's construction and the proof was not made public apart from the conference talk [34], we shall present Zastrow's proof that $\mathcal{H}_0(V)$ is non-Hausdorff. Here we use the notation introduced in Section 2.3.

Theorem 2.6. *The topological vector space $\mathcal{H}_0(V)$ is non-Hausdorff.*

Proof. The idea of the proof is to show that the boundary group $\mathcal{B}_0(V)$ is not a closed subset of $\mathcal{C}_0(V)$. We will construct a sequence of measures $\mu_n \in \mathcal{C}_0(V)$, such that there exists $\nu_n \in \mathcal{C}_1$ with $\partial\nu_n = \mu_n$. However, we will show that there is some $\mu \in \lim \mu_n$ which is not a boundary.

Just as in Section 2.3 let $\{\bar{l}_k\}_{k=1}^{\infty}$ denote the sequence of minima of the sinusoid. Moreover,

$$\mu_n := (1 - 2^{-n})\delta_{\bar{l}_0} - \sum_{k=1}^{n} 2^{-k}\delta_{\bar{l}_k},$$

where δ denotes the Kronecker measure.

The natural candidate for a limit is

$$\mu = \delta_{\bar{l}_0} - \sum_{k=1}^{\infty} 2^{-k}\delta_{\bar{l}_k}$$

Indeed, it is sufficient to show that for every continuous function $f : V \to \mathbb{R}$ (here we identify $\mathcal{C}_0(V)$ with appropriate measures on V) we have

$$\lim_{n\to\infty} \int_X f d(\mu - \mu_n) = 0.$$

This is equivalent to

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} 2^{-k} f(\bar{l}_k) = 0,$$

which is true because tails of convergent series converge to zero.

Now we shall prove that μ is not a boundary. So suppose there is $\nu \in \mathcal{C}_1(V)$ such that $\partial\nu = \mu$. Then, a compact carrier of ν cannot omit two consecutive maxima of the sinusoid. Being more specific, let D be a compact carrier of ν. Then we have a continuous evaluation function

$$\begin{aligned} F \;\; : \;\; & D \times \Delta^k \to X \\ & \sigma \times q \mapsto \sigma(q) \end{aligned}$$

we want to show that $F(D \times \Delta^1)$ must contain infinitely many maxima of the sinusoid.

To the contrary, suppose that $\overline{u}_k$ and $\overline{u}_{k+1}$ are maxima of the sinusoid such that $\overline{u}_k, \overline{u}_{k+1} \notin F(D \times \Delta^1)$. Then let $Y = V \setminus \{\overline{u}_k, \overline{u}_{k+1}\}$. We can interpret μ and ν as elements of $\mathcal{C}_0(Y)$ and $\mathcal{C}_1(Y)$ respectively. Naturally, $\partial\nu = \mu$ still holds.

Then, we can embed Y into $Z = Y \cup S$, where S is an open rectangle with opposite vertices $\overline{u}_{k+1}$ and $\bar{l}_0$. This allows us to identify μ and ν with measures in $\mathcal{C}_0(Z)$ and $\mathcal{C}_1(Z)$ respectively. Still, we have the condition $\partial\nu = \mu$, hence μ represents zero homology class in $\mathcal{H}_0(Z)$.

On the other hand, we can see that μ represents the same homology class in Z as $2^{-k}\delta_{\bar{l}_0} - 2^{-k}\delta_{\bar{l}_k}$ which is not zero since points $\bar{l}_0$ and $\bar{l}_k$ lie in a different components of Z. Therefore, we got a contradiction and we see that $F(D \times \Delta^1)$ contains infinitely many maxima of the sinusoid.

Since F is continuous, the set $F(D \times \Delta^1)$ must be compact, so it cannot contain infinitely many maxima of the sinusoid. Again, we have a contradiction. So, there cannot exist measure ν such that $\partial\nu = \mu$, and consequently $\mathcal{H}_0(V)$ is not Hausdorff.

□

Based on different arguments than in [34] we obtain the following result for the Warsaw Circle itself:

Theorem 2.7. *The Milnor-Thurston homology group $\mathcal{H}_0(W)$ is not Hausdorff in Berlanga topology.*

Proof. The groups of cycles $\mathcal{Z}_k(W)$ and the groups of boundaries $\mathcal{B}_k(W)$ are all Hausdorff as subspaces of the Hausdorff space $\mathcal{C}_k(W)$. Hence, their quotient will be Hausdorff if and only if $\mathcal{B}_k(W)$ is closed in $\mathcal{Z}_k(W)$. Consequently, we need find a sequence of boundaries such that the limit of this sequence is not a boundary. From the proof of Theorem 2.4 we know that chains in $\mathcal{C}_0(W)$ can be described by elements of ℓ^1. So, let our sequence of chains be described by elements $(x_k^n)_{k=0}^\infty \in \ell^1$ in the following way:

$$x_k^n = \begin{cases} -\sum_{i=1}^n (-1)^i (n_{i+1} - n_i), & \text{for } k = 0, \\ (-1)^k (n_{k+1} - n_k), & \text{for } 0 < k \le n, \\ 0, & \text{for } k > n. \end{cases}$$

where $(n_i)_{i=0}^\infty \notin \ell^1$ is a decreasing sequence of positive numbers converging to zero (compare with proof of Theorem 2.4).

Chains described by $(x_k^n)_{k=0}^\infty$ are boundaries (or, equivalently, they represent zero homology classes). To justify it, recall the proof of Theorem 2.4. From that, we know that an arbitrary sequence $(z_k)_{k=0}^\infty \in \ell^1$ with at most N nonzero elements represents the same homology class as the sequence $(\alpha, 0, 0, \dots)$, where $\alpha = \sum_{k=0}^N (-1)^k z_k$. Therefore, we see that for each n the sequence $(x_k^n)_{k=0}^\infty$ represents the zero homology class.

The natural candidate for the limit of $(x_k^n)_{k=0}^\infty$ is a sequence $(x_k)_{k=0}^\infty$ with

$$x_k = \begin{cases} -\sum_{i=1}^\infty (-1)^i (n_{i+1} - n_i), & \text{for } k = 0, \\ (-1)^k (n_{k+1} - n_k), & \text{for } k > 0. \end{cases}$$

In order to show that the above sequence is the limit of $(x_k^n)_{k=0}^\infty$ we need to prove that

$$\lim_{n\to\infty} \int_W f d(\mu - \mu_n) = 0,$$

for any continuous $f : W \to \mathbb{R}$. Here μ and μ_n are measures on W corresponding to $(x_k)_{k=0}^\infty$ and $(x_k^n)_{k=0}^\infty$, respectively (remember that we identify $C^0(\Delta^0, W)$ with W).

The measures μ and μ_n are concentrated on a countable set of points (namely the maxima $\overline{u}_k$ and minima $\overline{l}_k$ of the sinusoid), therefore the above integral can be calculated as an infinite series. The values of the continuous function f on that countable set of points form a bounded sequence $(a_k)_{k=0}^\infty$, so we need to prove that

$$\lim_{n\to\infty} \left(-a_0 \sum_{i=n+1}^\infty (-1)^i (n_{i+1} - n_i) + \sum_{i=n+1}^\infty (-1)^i a_i (n_{i+1} - n_i) \right) = 0.$$

We can easily see that it is true since tails of absolutely convergent series converge to zero.

Assume that the homology class described by $(x_k)_{k=0}^{\infty}$ is a boundary. Let $y_k = (-1)^k(n_{k+1} - n_k)$. Then, consider the difference

$$y_k - x_k = \begin{cases} \sum_{i=0}^{\infty}(-1)^i(n_{i+1} - n_i), & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases}$$

We assumed that on the level of homology $(x_k)_{k=0}^{\infty}$ represents zero, and thus it represents a singular homology class. On the other hand, from the above equation we see that $(y_k)_{k=0}^{\infty} - (x_k)_{k=0}^{\infty}$ also represents a singular homology class. Therefore, $(y_k)_{k=0}^{\infty}$ should also represent a singular homology class. However, $(y_k)_{k=0}^{\infty}$ is exactly the form of a sequence considered in the proof of Theorem 2.4, and we know that it represents a non-singular homology class (note that the sequence denoted here by $(y_k)_{k=0}^{\infty}$ was denoted by $(x_k)_{k=0}^{\infty}$ in the proof of that theorem). Hence, we got a contradiction. Consequently, we see that $(x_k)_{k=0}^{\infty}$ is not a boundary and $\mathcal{H}_0(W)$ is not Hausdorff.

□

2.6 Corresponding calculations for two other examples

The proof strategy in the case of two other examples: the Double Warsaw Circle DW and the Convergent Arcs Space CA is analogous as in the case of the Warsaw Circle.

The Warsaw Circle can be viewed as a halfline equipped with a topology that is weaker than the usual Euclidean topology. Roughly speaking, the fact that there are no Milnor-Thurston 1-cycles in the Warsaw Circle is a consequence of the fact that halfline has a starting point, so the measure cycle that is zero on this starting point is zero everywhere (cf. equation (2.4)). On the other hand, the Double Warsaw Circle can be interpreted as a line equipped with some special toplology. A line does not have a starting point, so one may suspect that there should exist some Milnor-Thurston cycles. However, this is not the case, as one can see in the proof of the following theorem:

Theorem 2.8. *Milnor-Thurston homology groups of the Double Warsaw Circle DW are trivial except for $\mathcal{H}_0(DW)$ which is a continuum-dimensional real vector space.*

Proof. The key idea is again to apply the Mayer-Vietoris theorem. Let us divide DW into the upper part U and the lower part L like we did for the Warsaw Circle in Section 2.3.

Again we can see that U, L and $U \cap L$ are homotopy equivalent to a convergent sequence with limit. Thus, by Lemma 2.2 the Mayer-Vietoris sequence reduces to (cf. proof of Theorem 2.3)

$$0 \longrightarrow \mathcal{H}_1(DW) \xrightarrow{\partial_*} \mathcal{H}_0(U \cap L) \xrightarrow{(i_{*0}, j_{*0})} \\ \to \mathcal{H}_0(U) \oplus \mathcal{H}_0(L) \xrightarrow{k_{*0} - l_{*0}} \mathcal{H}_0(DW) \longrightarrow 0,$$

and we see that higher Milnor-Thurston homology groups of DW vanish.

Next, we derive formulae for (i_{*0}, j_{*0}) in the above Mayer-Vietoris sequence. Again, we get the same answer (cf. equation 2.4)

$$x_k = m_k + m_{k-1}. \tag{2.7}$$

The notation is analogous to the one in the proof of Theorem 2.3. Here, however, k runs through all integers and there is no initial condition. Nevertheless, if look for a kernel of this mapping, we get the equation $m_k = -m_{k-1}$ and we know that nonzero sequences of this type cannot be absolutely summable. Thus, the kernel is trivial again, and the first Milnor-Thurston homology group vanishes.

Now, the dimension of $\mathcal{H}_0(DW)$ shall be found in an analogous way as in the proof of Theorem 2.4. From the Mayer-Vietoris sequence we see that $\mathcal{H}_0(DW)$ is again a quotient of ℓ^1 and the image of ℓ^1 by the map defined by equation (2.7).

We shall find continuum many sequences $(x^\beta)_{i=-\infty}^\infty$ in ℓ^1 such that any linear combination of these sequences is nontrivial in the quotient of ℓ^1 by ℓ^1. Let $0 < \beta < 1$, again we put $x_k^\beta = (-1)^k(n_{k+1}^\beta - n_k^\beta)$ where (cf. proof of Theorem 2.4)

$$n_k^\beta = \begin{cases} \frac{1}{k^\beta}, & \text{for } k > 0, \\ \frac{1}{(1-k)^\beta}, & \text{for } k \leq 0. \end{cases}$$

Next, for each β we derive formulae for the solution $(m_k^\beta)_{i=-\infty}^\infty$ of equation (2.7). After simple calculations we get

$$m_k^\beta = \begin{cases} (-1)^k \left(\frac{1}{(k+1)^\beta} - 1\right) + (-1)^k m_0^\beta, & \text{for } k > 0, \\ (-1)^k \left(1 - \frac{1}{(-k)^\beta}\right) + (-1)^k m_0^\beta, & \text{for } k \leq 0. \end{cases} \tag{2.8}$$

Now let us choose finite collection of numbers $0 < \beta_i < 1$ and for each i pick a real number b_i. Now, let us consider a linear combination of sequences $(x_k)_{k=-\infty}^\infty = \sum_i b_i(x_k^{\beta_i})_{k=-\infty}^\infty$. A possible solution $(m_k)_{k=-\infty}^\infty$ to equation (2.7) is a linear combination of sequences of the form (2.8). The most general solution depends on parameters $m_0^{\beta_i}$, however if we want $(m_k)_{k=-\infty}^\infty$ to be in ℓ^1 it has to satisfy the necessary condition of sequence convergence. Hence we get $\sum_i b_i = \sum_i m_0^{\beta_i}$, and from that

$$m_k = \sum_i (-1)^k \frac{1}{(k+1)^{\beta_i}},$$

for $k > 0$. Hence, we see that it is not absolutely summable (cf. proof of Theorem 2.4) and we see that $(x_k)_{k=-\infty}^\infty$ represents a nontrivial homology class. Thus, we constructed a family with continuum-many linearly independent vectors.

□

Finally, the case of the Convergent Arcs Space CA is done in a similar way.

Theorem 2.9. *The Milnor-Thurston homology groups of the Convergent Arcs Space CA are trivial except for $\mathcal{H}_1(CA) \cong \ell^1$ and $\mathcal{H}_0(CA) \cong \mathbb{R}$.*

Proof. This time the space shall be divided into left and right part, denoted L and R respectively. Both L and R are contractible, and hence their Milnor-Thurston homology groups are trivial, except for the zeroth group which is one-dimensional. Thus, the Mayer-Vietoris sequence is

$$0 \longrightarrow \mathcal{H}_1(CA) \xrightarrow{\partial_*} \mathcal{H}_0(L \cap R) \xrightarrow{(i_{*0}, j_{*0})} \\ \to \mathcal{H}_0(L) \oplus \mathcal{H}_0(R) \xrightarrow{k_{*0} - l_{*0}} \mathcal{H}_0(DW) \longrightarrow 0.$$

The intersection $L \cap R$ is homotopy equivalent to a convergent sequence with its limit, thus $\mathcal{H}_0(L \cap R) \cong \ell^1$ (see Lemma 2.2). The argument similar to the one in the proof of Theorem 2.3 allows us to write down equations for the homomorphism (i_{*0}, j_{*0}):

$$x = \sum_{k=0}^{\infty} m_k, \quad y = -\sum_{k=0}^{\infty} m_k,$$

where $m_k \in \mathcal{H}_0(L \cap R) \cong \ell^1$, $x \in \mathcal{H}_0(L) \cong \mathbb{R}$ and $y \in \mathcal{H}_0(R) \cong \mathbb{R}$. From that, we see that the kernel of (i_{*0}, j_{*0}) consists of sequences whose sum is equal zero. Yet, such a space is isomorphic to ℓ^1 itself. Moreover, we see that the quotient of $\mathcal{H}_0(L) \oplus \mathcal{H}_0(R)$ by the image of $\mathcal{H}_0(L \cap R)$ is one-dimensional.

□

Chapter 3

More on the zeroth Milnor-Thurston homology group

In the previous chapter the Milnor-Thurston homology groups of the Warsaw Circle were computed, with the surprising result that the zeroth Milnor-Thurston homology group is infinite-dimensional. Milnor-Thurston homology theory satisfies in principle the Eilenberg-Steenrod axioms, so that the coincidence of Milnor-Thurston homology with singular homology is guaranteed for spaces with homotopy type of CW-complexes. Since the example of the Warsaw Circle (i.e. of a metric compact space), implies that, although zeroth homology is usually related to the number of path-components, for non-triangulable spaces the canonical homomorphism from singular to Milnor-Thurston homology can even in this dimension fail to be an isomorphism (in particular: fail to be surjective). Moreover, for the Convergent Arcs space the canonical homomorphism is injective in every dimension. Hence, there are the following natural two questions:

- Is this homomorphism in general injective?
- Are there beyond triangulability sufficient criteria, when it will be an isomorphism?

In this chapter we provide the following answers to these questions:

- For Peano continua (cf. Definition 1.22) we have coincidence in dimension zero, i.e here the canonical homomorphism will be an isomorphism for any such space (cf. Section 3.1).
- For spaces with Borel path-components this homomorphism will be at least injective in dimension zero (cf. Section 3.2).
- However, we will also provide an example, where it will not even be injective (cf. Section 3.3).

Peano continua are in general not triangulable. Thus, the fact that the zeroth Milnor-Thurston homology group of a Peano continuum will in any case be one-dimensional does neither follow from the Eilenberg-Steenrod Axioms, nor, as the above mentioned example shows, from the fact that these spaces are path-connected. Nevertheless it holds, as we will show in this chapter (see Theorem 3.2).

3.1 Zeroth Milnor-Thurston homology for Peano continua

In the previous chapter it has been proved that the Warsaw Circle has uncountable-dimensional zeroth Milnor-Thurston homology group. We may suspect that the fact that this space is not locally connected is the reason behind this phenomenon. However, we may notice that there exist path-connected spaces that are not locally path connected and have one-dimensional zeroth homology group. The example may be the Broom Space (it is the cone over the space consisting of the sequence $1/n$ and its limit point).

Nevertheless, we may ask the opposite question: Does a connected and locally connected metric space have one-dimensional zeroth Milnor-Thurston homology group? In this section we prove that the answer is affirmative at least when the space is compact (see Theorem 3.2).

The Hahn-Mazurkiewicz theorem together with the following lemma, will allow us to prove one of the main results of this chapter (cf. Definition 1.22 and Theorem 1.23).

Lemma 3.1. *Let $f : [0,1] \to X$ be a continuous surjection on a metric space X. Suppose μ is a finite Borel measure on X, then there exists a measure $\tilde{\mu}$ on $[0,1]$ such that $f\tilde{\mu} = \mu$.*

Proof. Let $V = \{g \in C([0,1]) \mid \text{there exists } h \in C(X) \text{ such that } g = h \circ f\}$. We see that V is a nonempty linear space. Let $g \in V$, thanks to surjectivity of f the function $h \in C(X)$ such that $g = h \circ f$ is unique. We shall denote it by h_g. Notice, that h_g is linear with respect to g.

One can show that the linear functional defined below is bounded (it follows from the fact that the norm on V is supremum norm and that μ is finite):

$$V \to \mathbb{R}, \quad g \mapsto \int_X h_g d\mu.$$

By Corollary 1.21 there exists a bounded extension ξ to $C([0,1])$ of this linear functional. Then, by the Riesz Representation Theorem we know that there exists a Borel measure $\tilde{\mu}$ such that

$$\xi(g) = \int_{[0,1]} g d\tilde{\mu}.$$

Now, we shall prove that $f\tilde{\mu} = \mu$. By Corollary 1.12 it is sufficient to check this only for open sets. So, let $G \subset X$ be an arbitrary open set. By Lemma 1.18 there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of positive functions that is pointwise convergent to χ_G and such that $h_n \leq \chi_G$. Let $g_n = h_n \circ f$. Then for each n the function $g_n \in V$, and the sequence $(g_n)_{n\in\mathbb{N}}$ is pointwise convergent from below to $\chi_{f^{-1}(G)}$.

We know that

$$\int_{[0,1]} g_n d\tilde{\mu} = \xi(g_n) = \int_X h_n d\mu.$$

Using Theorem 1.19 on the both sides of the above equation we get

$$\int_{[0,1]} \chi_{f^{-1}(G)} d\tilde{\mu} = \int_X \chi_G d\mu,$$

which means that $\tilde{\mu}(f^{-1}(G)) = \mu(G)$, hence $f\tilde{\mu}(G) = \mu(G)$.

□

Theorem 3.2. *If X is a Peano continuum, then $\mathcal{H}_0(X) \cong \mathbb{R}$.*

Proof. Let $\mu \in \mathcal{C}_0(X)$ represent some homology class. From Lemma 3.1 we know that there exists a measure $\tilde{\mu}$ on $[0,1]$ such that $f\tilde{\mu} = \mu$.

Next, let us define $g : [0,1] \to C^0(\Delta^1, X)$ with the following formula: $g(x)(t) = f(tx)$. Let $\nu = g\tilde{\mu}$, we shall prove that $\partial\nu = \mu - \mu(X)\delta_{f(0)}$. Take any Borel subset $A \subset X$, then

$$\partial\nu(A) = \tilde{\mu}(g^{-1}(\partial_0^{-1}A)) - \tilde{\mu}(g^{-1}(\partial_1^{-1}A)). \tag{3.1}$$

Suppose $f(0) \notin A$. Then, $g^{-1}(\partial_0^{-1}A) = f^{-1}(A)$ and $g^{-1}(\partial_1^{-1}A)$ is empty, so equation (3.1) reduces to:

$$\partial\nu(A) = \tilde{\mu}(f^{-1}(A)) = \mu(A).$$

And when $f(0) \in A$, we have $g^{-1}(\partial_0^{-1}A) = f^{-1}(A)$ and $g^{-1}(\partial_1^{-1}A) = [0,1]$, then equation (3.1) reduces to:

$$\partial\nu(A) = \tilde{\mu}(f^{-1}(A)) - \tilde{\mu}(f^{-1}(X)) = \mu(A) - \mu(X).$$

From that, we see that every cycle $\mu \in \mathcal{C}_0(X)$ is homological to the measure $\mu(X)\delta_{f(0)}$.

The Kronecker measure $\delta_{f(0)}$ is non-trivial on the level of homology. Indeed, to the contrary suppose $\partial\alpha = \delta_{f(0)}$ for some measure α. By the obvious fact that every singular 1-simplex in X has both its endpoints in X we have the following equality between sets: $\partial_0^{-1}X = \partial_1^{-1}X$. Hence, $(\partial\alpha)(X) = \alpha(\partial_0^{-1}X) - \alpha(\partial_1^{-1}X) = \alpha(\partial_1^{-1}X) - \alpha(\partial_1^{-1}X) = 0$. That contradicts the fact that $\delta_{f(0)}(X) = 1$. Thus, our zeroth homology group is a one-dimensional vector space.

□

3.2 Is the canonical map from singular homology to Milnor-Thurston homology a monomorphism?

In Chapter 1 we have seen that there exists a canonical homomorphism from singular homology groups to Milnor-Thurston homology groups

$$H_k(X;\mathbb{R}) \to \mathcal{H}_k(X),$$

where X is a topological space and k is a non-negative integer.

When X is a CW-complex this canonical homomorphism is an isomorphism (see Section 1.3), thus it is also an injection. Additionally, for all the examples considered in Chapter 2 (the Warsaw Circle, the Double Warsaw Circle and the Convergent Arcs Space) it is also the case. In this section we will give a partial answer to the question, whether we always get an injection.

We shall prove the following theorem:

Theorem 3.3. *Let X be a topological space with Borel path-components. Then, the canonical map $H_0(X;\mathbb{R}) \to \mathcal{H}_0(X)$ is an injection.*

Lemma 3.4. *Let X be a topological space with Borel path-components. Let μ be a measure on $C^0(\Delta^1, X)$, such that $\partial\mu = \nu_{X_1} - \delta_{x_0}$, where ν_{X_1} is concentrated on a set $X_1 \subset X$ and $x_0 \notin X_1$. Then there exists a path starting at x_0 with its endpoint in X_1.*

Proof. Let Y be the path-component containing x_0. Notice that $\partial_0^{-1}(Y) = \partial_1^{-1}(Y)$. Thus, we have

$$(\partial\mu)(Y) = \mu(\partial_0^{-1}(Y)) - \mu(\partial_1^{-1}(Y)) = 0.$$

Now, assume that there is no path from x_0 to any point of X_1. That is, X_1 intersects Y in the empty set. As a consequence, $(\partial\mu)(Y) = -1$ which contradicts the above calculations.

□

Proof of Theorem 3.3. Our theorem states that the kernel of the canonical homomorphism is trivial. In other words, we have to show that every boundary in the sense of Milnor-Thurston homology is in fact a boundary in the sense of singular homology. Let

$$i : C_0(X;\mathbb{R}) \to \mathcal{C}_0(X)$$

denote the canonical homomorphism on the level of chains.

So, suppose we have a singular cycle $z = \sum_{i=1}^{k} \alpha_i x_i$ such that

$$i(z) = \partial\mu \tag{3.2}$$

for some $\mu \in \mathcal{C}_1(X)$. We will inductively show that z is a boundary of a singular chain.

Let us start with $z = \alpha_1 x_1$. Notice that $\partial_0^{-1}(X) = \partial_1^{-1}(X)$ implies that

$$\partial\mu(X) = \mu(\partial_0^{-1}(X)) - \mu(\partial_1^{-1}(X)) = 0. \tag{3.3}$$

From that, $\alpha_1 = 0$. Hence, no singular chain with one simplex can be a Milnor-Thurston boundary.

Suppose $z = \alpha_1 x_1 + \alpha_2 x_2$. Application of equation (3.3) implies that $\alpha_2 = -\alpha_1$. Moreover, by Lemma 3.4 there exists a path σ connecting x_1 and x_2. Hence, $\partial(\alpha_1\sigma) = z$.

Now, assume that every z satisfying (3.2) and having a number of 0-simplices less than k is a singular boundary. The measure μ/α_k satisfies the assumptions of the above Lemma 3.4, so there exists a path σ_k connecting x_k to, say, x_j. Let $z' = z - \alpha_k x_k + \alpha_k x_j$. We see, that $z' = z + \partial(\alpha_k\sigma_k)$. Moreover, z' has at most $k-1$ simplices, and its image with respect to homomorphism i is a boundary of a measure $\tilde{\mu} = \mu + \alpha_k\delta_{\sigma_k}$. Thus, there exists a singular 1-chain c' such that $\partial c' = z'$. From that, $c = c' - \alpha_k\sigma_k$ has the desired property $\partial c = z$, which ends our proof.

□

3.3 A space with a non-injective canonical homomorphism

The assumption that X has Borel path components was crucial in the proof of Theorem 3.3. Now, we will construct a counterexample showing that this assumption cannot be omitted. Namely, we will construct a topological space X, where there exists a measure $\nu \in \mathcal{C}_1(X)$ such that $\partial\nu = \delta_{x_1} - \delta_{x_0}$ where the points $x_1, x_0 \in X$ lie in separate path components. The concept of this construction was provided by my supervisor.

The following lemma will allow us to perform our construction

Lemma 3.5. *There exists a partition $[-1, 1] = A \cup B$, where A and B are not Lebesgue measurable and every Borel subset of A or B is of measure zero.*

Proof. First, we will find such a partition for $S^1 = \mathbb{R}/\mathbb{Z}$. It is enough to show that there exists a set $A \subset S^1$ with Lebesgue inner measure zero and full Lebesgue outer measure (here we normalise the Lebesgue measure λ in a way that $\lambda(S^1) = 2$). Indeed, if we have $\lambda_*(A) = 0$ and $\lambda^*(A) = 2$, then the set B can be defined as a complement of A. We see that

$$\lambda_*(B) = \sup_{B \supset O \in \mathcal{B}(S^1)} \lambda(O) = \sup_{A \subset O' \in \mathcal{B}(S^1)} (2 - \lambda(O')) = 2 - \lambda^*(A) = 0,$$

thus every Borel subset of B has indeed Lebesgue measure zero.

In order to construct the subset A, we will use the natural action of $G = \mathbb{Q}/\mathbb{Z}$ on S^1. It is known that $\mathcal{B}(S^1)$ has the cardinality of continuum [30, Theorem 3.3.18]. Let $(B_\alpha)_{\alpha<\mathfrak{c}}$ denote the family $\mathcal{B}(S^1)$ with a well-ordering. This well-ordering exists by the well-ordering theorem, which is equivalent to the Axiom of Choice. Using transfinite induction, we shall construct a sequence of elements $(x_\alpha)_{\alpha<\mathfrak{c}}$.

Suppose, we have chosen x_β for all $\beta < \alpha$. Then, we chose x_α that satisfy the following conditions:

- for every $\beta < \alpha$, the element x_α lies in a different orbit of G-action than x_β,

- if complement of B_α is uncountable, then $x_\alpha \in S^1 \setminus B_\alpha$.

Elements satisfying both of these conditions always exist. That is because, the number of G-orbits is continuum. Moreover, if κ denote the number of G-orbits that intersect $S^1 \setminus B_\alpha$ in a nonempty set, then the cardinality of $S^1 \setminus B_\alpha$ is less then $\aleph_0 \cdot \kappa = \max(\aleph_0, \kappa)$. Thus, if cardinality of $S^1 \setminus B_\alpha$ is uncountable then it is continuum, which is true for every uncountable Borel set [30, Theorem 3.2.7]. Consequently, we see that $\kappa = \mathfrak{c}$, so there are continuum-many orbits we can choose the element x_α from.

Now, we shall prove that the set $A := \{x_\alpha\}_{\alpha<\mathfrak{c}}$ has the desired properties. Suppose, we have a Borel set $O \subset A$, then both A and O intersect each orbit of G in a set with at most one element. From that, the family $G + O := \{g+O \mid g \in G\}$ consist of pairwise disjoint sets. Now, suppose $\lambda(O) > 0$, then $\lambda\left(\bigcup(G + O)\right) = \sum_{g\in G} \lambda(g+O) = \infty$, which is impossible. Hence, $\lambda(O) = 0$. On the other hand, consider $O \supset A$. If O has a countable complement, then it has full Lebesgue measure. Otherwise, from the fact that $O = B_\alpha$ for some $\alpha < \mathfrak{c}$, we know that $x_\alpha \notin O$, which contradicts $O \supset A$.

Finally, we can construct our decomposition of the interval $[-1, 1]$. There exists a continuous, measure preserving, map $f : [-1, 1] \to S^1$ which identifies both ends of the interval. In order to get a partition of $[-1, 1]$ we take preimages of $S^1 = A \cup B$. The properties of the partition are conserved, since f preserves measure.

□

Now, we will start our construction. Take $N = [-1, 1] \setminus \mathbb{Q}$ with the topology induced from the real line. By the above Lemma 3.5, there exist disjoint non-Lebesgue measurable sets such that $N = N_0 \cup N_1$ and for any Borel set $A \in N_i$ we have $\lambda(A) = 0$.

In order to get two connected components, the next stage of our construction will be taking cones over N_0 and N_1. So, identify N with the subset of $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$. We define the cone CN_0 as the union of affine intervals connecting points of N_0 with $x_0 := (0,1)$. Analogously, let CN_1 be the union of intervals connecting N_1 with $x_1 := (0,-1)$.

Notice, that the above construction of a cone is different than usual. Taking the Cartesian product with the interval, and then collapsing one face to a point yields a different neighbourhood system of the cone-point than the one induced from the plane.

Let $Y := CN_0 \cup CN_1$ and let I_0, I_1 be disjoint copies of $[0,1]$ and $[-1,0]$ respectively. Let us identify the point 1 of I_0 with vertex $x_0 \in Y$ and the point -1 of I_1 with $x_1 \in Y$. This is the underlying set of our topological space X.

The topology on Y is induced from $\mathbb{R}^2$. Thus, by choosing a neighbourhood basis of each point of I_i, for $i = 0,1$, we will complete the definition of the topology of X by the following lemma:

Lemma 3.6. *Let X be a set having a topological space Y as a subset. For each $t \in X \setminus Y$ choose a family $\mathcal{B}_t$ of subsets containing t. Assume that each $\mathcal{B}_t$ is closed with respect to finite intersections. Then, there exists a natural topology on X such that the topology on Y is the subspace topology induced from X. Moreover, if the intersection with Y of every element in $\mathcal{B}_t$ is open in Y, then $\mathcal{B}_t$ is a basis of open neighbourhoods for every $t \in X \setminus Y$.*

Proof. Let the topology τ on X consist of the empty set and subsets $U \subset X$ such that for every $t \in X \setminus Y$ there exists $\mathcal{B}_t \ni V \subset U$ and $U \cap Y$ is open in Y. We see that $X \in \tau$. Indeed, $X \cap Y = Y$ is open by the definition of topology and every element of $\mathcal{B}_t$ is a subset of X.

An arbitrary union of sets from τ is again a set in τ, it is a consequence of distributivity property of "$\cap$" and the obvious fact that $V \subset U_\alpha$ for some α, implies $V \subset \bigcup_\alpha U_\alpha$. Finally, let $U_1, U_2 \in \tau$. Then $U_1 \cap U_2 \cap Y$ is open in Y, which is a consequence of commutativity and associativity of "$\cap$". Moreover, if $\mathcal{B}_t \ni V_1 \subset U_1$ and $\mathcal{B}_t \ni V_2 \subset U_2$ then $V_1 \cap V_2 \subset U_1 \cap U_1$ and $V_1 \cap V_2 \in \mathcal{B}_t$ by the intersection property. Thus, τ is indeed a topology.

Moreover, if the intersection with Y of each element of $\mathcal{B}_t$ is open in Y, then $\mathcal{B}_t \subset \tau$. Hence, $\mathcal{B}_t$ is a basis of open neighbourhoods of $t \in X \setminus Y$.

□

Let $\mathcal{J}_i$ denote the family of finite subsets of N_i. Then, for each $J \in \mathcal{J}_i$ let CN_i^J denote the sub-cone $C(N_i \setminus J) \subset CN_i$. Now, let $t \in I_i$ (remember that t is identified with a real number). The basis of neighbourhoods of t shall be

$$\mathcal{B}_t = \{U^\varepsilon \cup U_J^\varepsilon \mid \varepsilon > 0, J \in \mathcal{J}_i\}$$

where $U^\varepsilon = (t - \varepsilon, t + \varepsilon) \cap I_i$ and $U_J^\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid t - \varepsilon < y < t + \varepsilon\} \cap (CN_i^J \cup CN_j)$, where $j = 1 - i$. We see that each family $\mathcal{B}_t$ is closed with respect to finite intersections, thus the assumptions of Lemma 3.6 are satisfied.

Let y_i denote the endpoint of I_i different form x_i and let $T = N \cup \{y_0, y_1\}$ with the topology induced from X.

Lemma 3.7. *Every continuous map $f : [0, 1] \to T$ is constant.*

Proof. The lemma is true if $f([0, 1]) \subset N$. So, suppose that $f^{-1}(\{y_0, y_1\})$ is not empty. First consider the case when $f^{-1}(N)$ is empty. Then $[0, 1]$ can be decomposed into the disjoint union of closed sets: $f^{-1}(\{y_0\}) \cup f^{-1}(\{y_1\})$, this contradicts connectivity of $[0, 1]$. Next, let $f^{-1}(N)$ be nonempty. Notice that it is an open set because N is open in T. Therefore, it must be a countable disjoint union of open nonempty intervals. Now, take (a, b) to be one of these intervals. By assumption, $f(a) = y_i$ for some i. Because (a, b) is connected, f should be constant on it with a value, say, $x \in N$. There exists a neighbourhood of y_i without x, therefore f is discontinuous at a.

□

Lemma 3.8. *The points x_0 and x_1 lie in different path-components.*

Proof. Suppose that there is a path $\alpha : [0, 1] \to X$ connecting x_0 and x_1. Notice that there is a supremum t_0 of points t such that $\alpha(t) = x_0$. From the continuity of α we see $\alpha(t_0) = x_0$. Similarly, there exists an infimum t_1 of points $t > t_0$ such that $\alpha(t) = x_1$. Now, we have that the points between t_0 and t_1 are mapped into $X \setminus \{x_0, x_1\}$.

Take a point $a \in [t_0, t_1]$ close enough to t_0 so that $\alpha(a) \in CN_0$ and take a point $b \in [t_0, t_1]$ close enough to t_1 so that $\alpha(b) \in CN_1$. We see that the interval $[a, b]$ is mapped into $X \setminus \{x_0, x_1\}$, so we can construct a path $\beta : [0, 1] \to X \setminus \{x_0, x_1\}$ connecting a point of CN_0 with a point of CN_1.

There is the obvious retraction $r : X \setminus \{x_0, x_1\} \to T$ that maps each point to the end-point of its ray in the respective cone. By the above lemma the function $r \circ \beta$ is constant, hence β maps the interval $[0, 1]$ into a single ray of one of the cones. Consequently, it cannot connect points in separate cones.

□

Now, we shall construct our measure ν on $C^0(\Delta^1, X)$. It will consist of two parts, one concentrated on simplices in CN_0 and the other concentrated on simplices in CN_1. Their carriers shall consist of simplices connecting the points of N with the respective vertices.

To get a convenient description of the carriers of our measures we shall still treat Y as a subset of $\mathbb{R}^2$ (in the way described above). Let $\sigma^x_{x_0}$ be the singular simplex such that $\sigma^x_{x_0}(t) = (f_x(t), 1 - t)$, where f_x is the unique affine function such that $\sigma^x_{x_0}(t) \in Y$ and $\sigma^x_{x_0}(1) = x$. In the analogous way we define the simplex $\sigma^{x_1}_x$ for $x \in N_1$ (the direction is such that $\sigma^{x_1}_x(0) = x$).

Now, our carriers shall be $S_0 = \{\sigma^x_{x_0} \in C^0(\Delta^1, X) \mid x \in N_0\}$ and $S_1 = \{\sigma^{x_1}_x \in C^0(\Delta^1, X) \mid x \in N_1\}$.

Notice that each of S_i is not compact, however if we add to S_i the respective paths connecting x_i with y_i (parametrised in the proper way) we shall get compact sets of simplices. Thus, our measures shall have compact carriers.

Lemma 3.9. *S_0 and S_1 are Borel sets in $C^0(\Delta^1, X)$.*

Proof. First, we will show that it is sufficient to prove that S_i are Borel in $C^0(\Delta^1, Y)$. To do this we show that $C^0(\Delta^1, Y)$ is Borel. It suffices, since every Borel subset of a Borel subspace is Borel with respect to the bigger space.

Take $i = 0, 1$, and let U_n^i denote a sequence of neighbourhoods of x_i such that $\bigcap_n U_n^i = \{x_i\}$. Now, let $Y_n = Y \cup U_n^0 \cup U_n^1$. We see that each Y_n is an open set in X and $\bigcap_n Y_n = Y$. By this fact and the definition of the compact-open topology, $C^0(\Delta^1, Y_n)$ is open in $C^0(\Delta^1, X)$. The intersection of $C^0(\Delta^1, Y_n)$ is $C^0(\Delta^1, Y)$, so it is a Borel subset of $C^0(\Delta^1, X)$.

Now, we shall prove that the each of S_i is closed in $C^0(\Delta^1, Y)$. The space $C^0(\Delta^1, Y)$ is metrizable, thus it is enough to show that the both S_i contain limit points of all convergent sequences. Let σ_n be a sequence of singular 1-simplices in $\mathbb{R}^2$ with affine parametrisation, say, $\sigma_n(t) = (a_n + b_n t, c_n + d_n t)$. Such a sequence is convergent iff the sequences of coefficients a_n, b_n, etc. are convergent.

Now, take a sequence of 1-simplices $(\sigma_n) \subset S_0 \subset C^0(\Delta^1, Y) \subset C^0(\Delta^1, \mathbb{R}^2)$ convergent in $C^0(\Delta^1, Y)$. By the above observation a limit of such a sequence is a 1-simplex with affine parametrisation that connects x_0 with a point of N. However, any such simplex is an element of S_0, hence S_0 is closed. Analogously, we prove that S_1 closed.

□

We preferred to state the following lemma in an abstract way. Its assumptions are satisfied in our case. Namely, take $Z = C^0(\Delta^1, Y)$, $f_i = \partial_i$, $M_i = S_i$ (this yields $R_i = N_i$, see below), and S_i is homeomorphic to N_i (cf. Lemma 3.11 and Lemma 3.12).

Lemma 3.10. *Let $f_i : Z \to [-1, 1]$ for $i = 0, 1$ be continuous functions on a topological space Z such that there exist two disjoint Borel subsets M_i of Z with the following properties:*

- *Every $R_i := f_i(M_i)$ is dense in $[-1, 1]$,*
- *R_i are disjoint,*
- *$R_0 \cup R_1$ is a full-measure Borel subset of $[-1, 1]$,*
- *Every subset of R_i being Borel in $[-1, 1]$ has Lebesgue measure zero,*

- *$f_i|M_i$ is a homeomorphism between M_i and R_i.*

Then

1. *Every Borel set in M_i has the form $M_i \cap f_i^{-1}(B)$ for some Borel subset of $[-1, 1]$,*

2. *The semi-algebra $\mathcal{I}_i = \{M_i \cap f_i^{-1}(I) \mid I \subset [-1, 1]$ is a semi-closed interval$\}$ generates the Borel subsets of M_i,*

3. *The set functions $\mu_i : M_i \cap f_i^{-1}(I) \mapsto \lambda(I)$, where λ denotes the Lebesgue measure and I is a semi-closed subinterval of $[-1, 1]$, can be extended to a Borel measure μ_i on M_i.*

Proof. To prove the first statement, take a Borel subset A of M_i. Then $f_i(A)$ is a Borel subset of R_i (because each of f_i is a homeomorphism). Notice, that $\mathcal{B}(R_i) = R_i \cap \mathcal{B}([-1, 1])$ which implies the first statement.

To prove the second statement we need to notice that $\mathcal{I}_i = \{M_i \cap f_i^{-1}(I) \mid I \subset [-1, 1]$ is a semi-closed interval$\}$ is a semi-algebra (cf. Definition 1.14 and a remark below). Then Lemma 1.17 and Lemma 1.16 combined together give us this result.

In order to prove the third statement it is enough to show that μ_i's are countably additive (see Corollary 1.15). So, let us take a pairwise disjoint countable family $\{A_j = M_i \cap f_i^{-1}(I_j)\}_{j \in \mathcal{J}} \subset M_i \cap f_i^{-1}(\mathcal{I})$, such that the union of this family is $A \in M_i \cap f_i^{-1}(\mathcal{I})$. Thus, the set A is of the form $M_i \cap f_i^{-1}(I)$ for some semi-closed interval I.

We claim that $\{I_j\}_{j \in \mathcal{J}}$ is a pairwise disjoint family. To the contrary, assume that two of these sets, say, I_1 and I_2, have the non-empty intersection $[a, b)$, for some real numbers $a < b$. Consequently, $A_1 \cap A_2 = M_i \cap f_i^{-1}([a, b))$ that is non-empty, since R_i is dense in $[-1, 1]$. However, our family of sets is disjoint, hence we got a contradiction.

Moreover, we claim that $I \setminus \bigcup_{j \in \mathcal{J}} I_j \in \mathcal{B}([-1, 1])$ is a null subset of $[-1, 1] \setminus R_i$. Indeed, the fact that A is the union of A_j means that $M_i \cap f_i^{-1}(\bigcup_j I_j) = M_i \cap f_i^{-1}(I)$. That implies $I \setminus \bigcup_{j \in \mathcal{J}} I_j \subset [-1, 1] \setminus R_i$ Moreover, we see that

$[-1,1] \setminus R_i = R_k \cup ([-1,1] \setminus R_0 \cup R_1)$ for $k = 1-i$. Thus, $I \setminus \bigcup_j I_j$ can be decomposed into two parts. The first one is a subset of $[-1,1] \setminus R_0 \cup R_1 \in \mathcal{B}([-1,1])$ and hence it is a null-set as a subset of a null-set. Moreover, it is Borel in $[-1,1]$ as an intersection with a Borel set. The second part is a subset of R_k that is Borel in $[-1,1]$ because it is a difference of two Borel sets. By the assumption, it has to be a null-set. As a consequence, $I \setminus \bigcup_j I_j$ is a null-set, which yields $\lambda(I) = \sum_j \lambda(I_j)$. This fact proves that the μ_i's are countably additive.

□

Lemma 3.11. *$\partial_0|_{S_0} : S_0 \to N_0$ is a homeomorphism.*

Proof. It is obviously a continuous bijection; in order to prove that its inverse is continuous, take a convergent sequence of $x_n \in N_0$ with a limit $x \in N_0$. Observe that for each n, the simplex $(\partial_0|_{S_0})^{-1}(x_n)$ connects x_0 with x_n. Because all simplices have affine parametrisation, the limit of $(\partial_0|_{S_0})^{-1}(x_n)$ is the simplex connecting x_0 with x that has affine parametrisation. Hence $(\partial_0|_{S_0})^{-1}$ is continuous.

□

Lemma 3.12. *$\partial_1|_{S_1} : S_1 \to N_1$ is a homeomorphism.*

The proof is analogous as in the previous case.

□

Now, let ν_i's be the measures on the Borel subsets of S_i that exists by Assertion 3 of Lemma 3.10. We can extend the measures ν_i for $i = 0, 1$ to the Borel σ-algebra of $C^0(\Delta^1, X)$ with the formula

$$\nu_i(A) = \nu_i(A \cap S_i), \quad \text{for any Borel subset } A \text{ of } C^0(\Delta^1, X),$$

which is well-defined thanks to Lemma 3.9.

Now, let us put $\nu = \nu_1 + \nu_0$. Finally, we can prove our result.

Theorem 3.13. *The canonical homomorhpism $h : H_0(X;\mathbb{R}) \to \mathcal{H}_0(X)$ is not a monomorhpism.*

Proof. The singular homology class of the cycle $z = x_1 - x_0$ is nontrivial in $H_0(X;\mathbb{R})$, since x_0 and x_1 lie in separate path components (see Lemma 3.8). The canonical homomorphism maps $[z]$ to the Milnor-Thurston class of the cycle $\delta_{x_1} - \delta_{x_0}$ in $\mathcal{H}_0(X)$. We shall prove that it is trivial. In fact, we will show that for the measure ν defined above we have

$$\partial\nu = 2(\delta_{x_1} - \delta_{x_0}). \tag{3.4}$$

The crucial step of our proof is to show that every Borel subset of N is of $\partial\nu$-measure zero. So, let $B \subset N \subset [-1, 1]$ be a Borel set. Notice, that $\nu_1(\partial_0^{-1}(B)) = 0$ because $S_1 \cap \partial_0^{-1}(B)$ is empty. Similarly, $\nu_0(\partial_1^{-1}(B)) = 0$. As a consequence we see

$$(\partial\nu)(B) = \nu_0(\partial_0^{-1}(B)) - \nu_1(\partial_1^{-1}(B)).$$

Now, notice that if $B = I \cap N$ where I is an interval, then we have $(\partial\nu)(B) = \nu_0(\partial_0^{-1}(I)) - \nu_1(\partial_1^{-1}(I)) = \lambda(I) - \lambda(I) = 0$. So the λ-system of Borel sets that satisfy $(\partial\nu)(B) = 0$ contains a semi-algebra generating Borel subsets of N. Every semi-algebra is a π-system, so by Theorem 1.11 we see that $(\partial\nu)(B) = 0$ for every Borel set $B \subset N$.

Next, let $B \subset X \setminus (N \cup \{x_1\})$ be a Borel set containing the point x_0. Then, we see that $\partial_0^{-1}(B) \cap (S_0 \cup S_1)$ and $\partial_1^{-1}(B) \cap S_1$ are empty, so

$$(\partial\nu)(B) = -\nu_0(\partial_1^{-1}(B))$$

follows. Moreover, we have that $\partial_1^{-1}(B) \cap S_0 = S_0 = S_0 \cap \partial_0([-1, 1])$. From that, we get $\nu_0(\partial_1^{-1}(B)) = \lambda([-1, 1]) = 2$. An analogous assertion is true for sets B containing x_1. Finally, there is no simplex in S_0 or S_1 that has its endpoint in $B \subset X \setminus (N \cup \{x_0\} \cup \{x_1\})$. These facts together imply equation (3.4).

□

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