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Joanna Ochremiak

Uniwersytet Warszawski

Nominal sets for symmetries with function symbols

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Opiekun pracy: Bartosz Klin

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

NOMINAL SETS FOR SYMMETRIES WITH FUNCTION SYMBOLS

JOANNA OCHREMIAK

Abstract. We study nominal sets in symmetries obtained from homogeneous structures and their groups of automorphisms. We prove the representation theorem for orbit-finite nominal sets, due to Bojańczyk, Klin and Lasota, in the general case when the function symbols are allowed.

1. Introduction

In [1] Bojańczyk, Klin and Lasota study languages and automata over infinite alphabets. Each alphabet $\mathbb{D}$, whose elements are called *data values*, comes with some structure which can be captured by the group of its automorphisms G. Given such a pair $(\mathbb{D}, G)$, called a *data symmetry*, one can consider an arbitrary G-set X, i.e. a set equipped with an action of the group G, and study the relations between the canonical action of G on $\mathbb{D}$ and the action of G on X.

We say that a G-set X is *nominal* if for any element $x \in X$ the result of an action of each π from G on x is determined by a set of finitely many data values. Formally, there exists a finite set $C \subseteq \mathbb{D}$, called a *support* of x, such that $\pi|_C = \sigma|_C$ implies $x \cdot \pi = x \cdot \sigma$. An example of a nominal set is $\mathbb{D}^*$, where a word is supported by the set of its letters. The set of all cofinite subsets of $\mathbb{D}$ is also nominal. One of the supports of a cofinite set is simply its complement.

In the special case of $\mathbb{D}$ being countably infinite and G being the group of all bijections of $\mathbb{D}$ the theory of nominal sets was introduced by Gabbay and Pitts [4]. In [1] Bojańczyk et al. study the concept of nominal sets in different symmetries and, among other results, provide a concrete representation of orbit-finite nominal sets. The representation theorem is obtained using the theory of Fraïssé limits (see e.g. [5]) and relates to the symmetries $(\mathbb{D}, G)$, where G is the group of automorphisms of a countable relational structure. The examples of such structures include $\langle \mathbb{Q}, < \rangle$ and the random graph.

In [2] Bojańczyk and Lasota use the theory of nominal sets to obtain a machine-independent characterization of the languages recognized by deterministic timed automata. To do so they introduce a data symmetry with a function symbol $+1$. It is therefore natural to ask if the representation theorem can be generalized to cover also structures with function symbols.

The main purpose of this paper is to answer the above question in the positive. The proof of the representation theorem for symmetries with function symbols turns out to be essentially the same as the proof given in [1]. There are, though, some subtleties, while instead of finite supports one has to consider finitely generated supports (which can be infinite). As a result it is not always obvious how to define the counterparts of some key notions like *fungibility*.

2. Data symmetries

A (right) *group action* of a group G on a set X is a binary operator $\cdot\colon X \times G \to X$ that satisfies following conditions:

$$\text{for all } x \in X \quad x \cdot e = x, \text{ where } e \text{ is the neutral element of } G,$$

$$\text{for all } x \in X \text{ and } \pi, \sigma \in G \quad x \cdot (\pi\sigma) = (x \cdot \pi) \cdot \sigma.$$

The set X equipped with such an action is called a *G-set*.

Example 2.1. For a set X let $\mathrm{Sym}(X)$ denote the symmetric group on X, i.e. the group of all bijections of X. Take any subgroup G of the symmetric group $\mathrm{Sym}(X)$. There is a natural action of the group G on the set X defined by $x \cdot \pi = \pi(x)$.

Definition 2.2. A *data symmetry* $(\mathbb{D}, G)$ is a set $\mathbb{D}$ of *data values*, together with a subgroup $G \leq \mathrm{Sym}(\mathbb{D})$ of the symmetric group on $\mathbb{D}$.

Example 2.3. Examples of data symmetries include:

- the *equality symmetry*, where $\mathbb{D}$ is a countably infinite set, say the natural numbers, and $G = \mathrm{Sym}(\mathbb{D})$ contains all bijections of $\mathbb{D}$,
- the *total order symmetry*, where $\mathbb{D} = \mathbb{Q}$ is the set of rational numbers, and G is the group of all monotone permutations,
- the *timed symmetry*, where $\mathbb{D} = \mathbb{Q}$ is the set of rational numbers, and G is the group of all permutations of rational numbers that preserve the order relation $<$ and the successor function $x \mapsto x + 1$[1].

For any element x of a G-set X the set

$$x \cdot G = \{x \cdot \pi \mid \pi \in G\} \subseteq X$$

is called the *orbit* of x. The set of orbits form a partition of X. The set X is called *orbit-finite* if the partition has finitely many parts. Each of the orbits can be perceived as a separate G-set. Therefore we can treat any G-set X as a disjoint union of its orbits.

Example 2.4. For any data symmetry $(\mathbb{D}, G)$ the action of G on $\mathbb{D}$ extends pointwise to an action of G on the set of tuples $\mathbb{D}^n$. In the equality symmetry, the set $\mathbb{D}^2$ has two orbits:

$$\{(d, d) \mid d \in \mathbb{D}\} \qquad\qquad \{(d, e) \mid d \neq e \in \mathbb{D}\}.$$

[1]The timed symmetry was originally defined in [2] for $\mathbb{D} = \mathbb{R}$. Considering the rational numbers instead of the reals makes little difference but is essential for our purposes, since we need the set of data values to be countable.

In the timed symmetry, the set $\mathbb{D}^2$ is not orbit-finite. Notice that for any $d \in \mathbb{Q}$ each of the elements $(d, d+1)$, $(d, d+2)$, ... is in a different orbit.

Let X be a G-set. A subset $Y \subseteq X$ is *equivariant* if $Y \cdot \pi = Y$ for every $\pi \in G$ i.e., it is preserved under group action. Considering a point-wise action of a group G on the Cartesian product $X \times Y$ of two G-sets X, Y we can define an *equivariant relation* $R \subseteq X \times Y$. In the special case when the relation is a function $f\colon X \to Y$ we obtain a following definition of an *equivariant function*

$$f(x \cdot \pi) = f(x) \cdot \pi \text{ for any } x \in X, \pi \in G.$$

For any x in a G-set X, the group

$$G_x = \{\pi \in G \mid x \cdot \pi = x\} \leq G$$

is called the *stabilizer* of x.

Lemma 2.5. *If $H \leq G$ is the stabilizer of an element x of a G-set X then $\pi^{-1}H\pi$ is the stabilizer of $x \cdot \pi$ for each $\pi \in G$.*

Proof. Let $K \leq G$ be the stabilizer of $x \cdot \pi$. Obviously $\pi^{-1}H\pi \subseteq K$. On the other hand, $x \cdot (\pi\sigma\pi^{-1}) = x$ for any $\sigma \in K$. Hence $\pi K \pi^{-1} \subseteq H$, which means that $K \subseteq \pi^{-1}H\pi$. As a result $K = \pi^{-1}H\pi$, as required. ☐

Proposition 2.6. *Let x be an element of a single-orbit G-set X. For any G-set Y equivariant functions from X to Y are in bijective correspondence with elements $y \in Y$ for which $G_x \leq G_y$.*

Proof. Given an equivariant function $f\colon X \to Y$, let $y = f(x)$. If $\pi \in G_x$ then

$$y \cdot \pi = f(x) \cdot \pi = f(x \cdot \pi) = f(x) = y,$$

thus $G_x \leq G_y$. On the other hand, given $y \in Y$ such that $G_x \leq G_y$, define a function $f\colon X \to Y$ by $f(x \cdot \pi) = y \cdot \pi$. Function f is well-defined. Indeed, if $x \cdot \pi = x \cdot \sigma$ then $\pi\sigma^{-1} \in G_x \subseteq G_y$, hence $y \cdot \pi = y \cdot \sigma$.

It is easy to check that the two above constructions are mutually inverse. ☐

3. Fraïssé symmetries

3.1. **Fraïssé limits.** A *signature* is a set of relation and function names together with (finite) arities. We will consider structures over a fixed finite signature. For two structures $\mathfrak{A}$ and $\mathfrak{B}$, an *embedding* $f\colon \mathfrak{A} \to \mathfrak{B}$ is an injective function from the carrier of $\mathfrak{A}$ to the carrier of $\mathfrak{B}$ that preserves and reflects all relations and functions in the signature.

Definition 3.1. A class $\mathcal{K}$ of finitely generated structures over some fixed signature is called a *Fraïssé class* if it:

- is closed under isomorphisms as well as finitely generated substructures and has countably many members up to isomorphism,

- has *joint embedding property*: if $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ then there is a structure $\mathfrak{C}$ in $\mathcal{K}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathfrak{C}$,
- has *amalgamation*: if $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{K}$ and $f_{\mathfrak{B}}\colon \mathfrak{A} \to \mathfrak{B}$, $f_{\mathfrak{C}}\colon \mathfrak{A} \to \mathfrak{C}$ are embeddings then there is a structure $\mathfrak{D}$ in $\mathcal{K}$ together with two embeddings $g_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{D}$ and $g_{\mathfrak{C}}\colon \mathfrak{C} \to \mathfrak{D}$ such that $g_{\mathfrak{B}} \circ f_{\mathfrak{B}} = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}$.

Examples of Fraïssé classes include:

- all finite structures over an empty signature, i.e. finite sets,
- finite total orders,
- all finite structures over a signature with a single binary relation symbol, i.e. directed graphs,
- finite Boolean algebras,
- finite groups,
- finite fields of characteristic p.

Classes that are not Fraïssé include:

- total orders of size at most 7 – due to lack of amalgamation,
- all finite fields – due to lack of joint embedding property.

Some Fraïssé classes admit a stronger version of amalgamation property. We say that a class $\mathcal{K}$ has *strong amalgamation* if it has amalgamation and moreover, $g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = g_{\mathfrak{B}}(\mathfrak{B}) \cap g_{\mathfrak{C}}(\mathfrak{C})$. It means that we can make amalgamation without identifying any more points than absolutely necessary.

Example 3.2. All the Fraïssé classes listed above, except for the class of finite fields of characteristic p, have the strong amalgamation property.

The *age* of structure $\mathfrak{U}$ is the class $\mathcal{K}$ of all structures isomorphic to finitely generated substructures of $\mathfrak{U}$. A structure $\mathfrak{U}$ is *homogeneous* if any isomorphism between finitely generated substructures of $\mathfrak{U}$ extends to an automorphism of $\mathfrak{U}$. The following theorem says that for a Fraïssé class $\mathcal{K}$ there exists a homogeneous structure of age $\mathcal{K}$, called its *Fraïssé limit* (see e.g. [5]).

Theorem 3.3. *For any Fraïssé class $\mathcal{K}$ there exist a unique, up to isomorphism, countable (finite or infinite) universal structure $\mathfrak{U}_{\mathcal{K}}$ such that $\mathcal{K}$ is the age of $\mathfrak{U}_{\mathcal{K}}$ and $\mathfrak{U}_{\mathcal{K}}$ is homogeneous.*

Example 3.4. The Fraïssé limit of the class of finite total orders is $\langle \mathbb{Q}, < \rangle$. For finite Boolean algebras it is the countable atomless Boolean algebra.

A structure $\mathfrak{U}$ is called *weakly homogeneous* if for any two finitely generated substructures $\mathfrak{A}, \mathfrak{B}$ of $\mathfrak{U}$, such that $\mathfrak{A} \subseteq \mathfrak{B}$, any embedding $f_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{U}$ extends to an embedding $f_{\mathfrak{B}}\colon \mathfrak{B} \to \mathfrak{U}$. It turns out that a countable structure $\mathfrak{U}$ is homogeneous if and only if it is weakly homogeneous (see [5]). Hence one way to obtain a Fraïssé class $\mathcal{K}$ is to take a weakly homogeneous, countable structure $\mathfrak{U}$ and simply consider its age.

Fact 3.5. *Every countable, weakly homogeneous structure* $\mathfrak{U}$ *is a Fraïssé limit of its age.*

Example 3.6. Consider a signature with a single unary function symbol $+1$ and a single binary relation symbol $<$. The structure $\langle \mathbb{Q}, <, +1 \rangle$ is countable and weakly homogeneous. Therefore it is the Fraïssé limit of its age.

From a Fraïssé class $\mathcal{K}$ we obtain a data symmetry $(\mathbb{D}_\mathcal{K}, G_\mathcal{K})$, where $\mathbb{D}_\mathcal{K}$ is the carrier of $\mathfrak{U}_\mathcal{K}$ and $G_\mathcal{K} = \mathrm{Aut}(\mathfrak{U}_\mathcal{K})$ is its group of automorphisms. Such a data symmetry is called a *Fraïssé symmetry.*

Example 3.7. All symmetries in Example 2.3 are Fraïssé symmetries. The equality symmetry arises from the class of all finite sets, the total order symmetry from the class of finite total orders and the timed symmetry from the class of all finitely generated substructures of $\langle \mathbb{Q}, <, +1 \rangle$ (see Example 3.6).

For simplicity we frequently identify the elements of age $\mathcal{K}$ with finitely generated substructures of $\mathfrak{U}_\mathcal{K}$.

3.2. **Least supports.** From now on, we focus on G-sets for groups arising from Fraïssé symmetries. Consider such a symmetry $(\mathbb{D}_\mathcal{K}, G_\mathcal{K})$ and a $G_\mathcal{K}$-set X.

Definition 3.8. A set $C \subseteq \mathbb{D}_\mathcal{K}$ *supports* an element $x \in X$ if $x \cdot \pi = x$ for all $\pi \in G_\mathcal{K}$ such that $\pi|_C = \mathrm{id}|_C$. A $G_\mathcal{K}$-set is *nominal* in the symmetry $(\mathbb{D}_\mathcal{K}, G_\mathcal{K})$ if its every element is supported by the carrier of a finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_\mathcal{K}$. We call $\mathfrak{A}$ a *finitely generated support* of x.

Example 3.9. For any Fraïssé symmetry $(\mathbb{D}_\mathcal{K}, G_\mathcal{K})$ the sets $\mathbb{D}_\mathcal{K}$ and $\mathbb{D}_\mathcal{K}^n$ are nominal. A tuple $(d_1, ..., d_n)$ is supported by the structure generated by its elements.

Lemma 3.10. *The following conditions are equivalent:*
(1) *C supports an element $x \in X$;*
(2) *for any $\pi, \sigma \in G_\mathcal{K}$ if $\pi|_C = \sigma|_C$ then $x \cdot \pi = x \cdot \sigma$.*

Proof. For the implication (1) $\implies$ (2), notice that if $\pi|_C = \sigma|_C$, then $\pi\sigma^{-1}$ act as identity on C, hence $x \cdot \pi\sigma^{-1} = x$ and $x \cdot \pi = x \cdot \sigma$, as required. The opposite implication follows immediately from the definition if we take $\sigma = \mathrm{id}$. ☐

It is easy to see that if an element $x \in X$ has a finitely generated support $\mathfrak{A}$ then it is also supported by the finite set C of its generators. Thus we can equivalently require x to be finitely supported.

Fact 3.11. *A $G_\mathcal{K}$-set is nominal if and only if its every element has a finite support.*

Example 3.12. Consider the timed symmetry. If the automorphism π preserves a data value $d \in \mathbb{Q}$, then it necessarily preserves also $d + i$ for any integer i. Therefore, if an element x of a nominal set is supported by a substructure generated by $\{1, 30\frac{1}{2}, 100\frac{5}{7}\}$ it is also supported by a substructure generated by $\{1000, 300\frac{1}{2}, 105\frac{5}{7}\}$. Hence in this case for any finitely generated support $\mathfrak{A}$ of an element x we can always find a finitely generated substructure $\mathfrak{B}$, which is properly contained in $\mathfrak{A}$ and still supports x.

An element of a nominal set has many supports. In particular, supports are closed under adding data values. If every element of a nominal set X has a unique least finitely generated support, we say that X *admits least supports.* As shown in Example 3.12 it is not always the case. It turns out that to check if a single-orbit nominal set admits least supports, one just needs to find out if any element of the set has the least finitely generated support.

Lemma 3.13. *If $\mathfrak{A} \subseteq \mathfrak{U}_{\mathcal{K}}$ is the least finitely generated support of an element $x \in X$, then $\mathfrak{A} \cdot \pi$ is the least finitely generated support of $x \cdot \pi$ for any $\pi \in G_{\mathcal{K}}$.*

Proof. First we prove that $\mathfrak{A} \cdot \pi$ supports $x \cdot \pi$. Indeed, if an arbitrary $\rho \in G_{\mathcal{K}}$ is an identity on $\mathfrak{A} \cdot \pi$, then $\pi\rho\pi^{-1}$ is an identity on $\mathfrak{A}$, hence $x \cdot (\pi\rho\pi^{-1}) = x$. As a result $(x \cdot \pi) \cdot \rho = x \cdot \pi$, as required.

Now let $\mathfrak{B} \subseteq \mathfrak{U}_{\mathcal{K}}$ be any finitely generated support of $x \cdot \pi$. W need to show that $\mathfrak{A} \cdot \pi \subseteq \mathfrak{B}$. A reasoning similar to the one above shows that $\mathfrak{B} \cdot \pi^{-1}$ supports x, from which we obtain $\mathfrak{A} \subseteq \mathfrak{B} \cdot \pi^{-1}$. Therefore, since π is a bijection, $\mathfrak{A} \cdot \pi \subseteq \mathfrak{B}$. □

Definition 3.14. A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ *admits least supports* if every nominal $G_{\mathcal{K}}$-set admits least supports.

We call a structure $\mathfrak{U}$ *locally finite* if all its finitely generated substructures are finite. Notice that if the universal structure $\mathfrak{U}_K$ is locally finite then admitting least supports is equivalent to finitely generated supports being closed under finite intersections. The same holds under more general assumption that any finitely generated structure has only finitely many finitely generated substructures.

Example 3.15. If we have only relation symbols in the signature it is obvious that any finitely generated structure is finite. One can prove that in the equality symmetry the intersection of two supports is a support itself. Hence the equality symmetry admits least supports. The same holds for the total order symmetry. Both facts are proved e.g. in [1].

Example 3.16. From the Example 3.12 we learned that the timed symmetry does not admit least supports (even though the finitely generated supports are closed under finite intersections). The situation changes if we add to the signature another unary function symbol -1 and consider a universal structure $\langle \mathbb{Q}, <, +1, -1 \rangle$. The group of automorphisms remains the same, but for any data value $d \in \mathbb{Q}$ we bind together all the elements $d+i$. As a result we obtain a Fraïssé symmetry that admits least supports.

Proposition 3.17. *The Fraïssé symmetry obtained from the universal structure $\langle \mathbb{Q}, <, +1, -1 \rangle$ admits least supports.*

Proof. Notice that any finitely generated substructure of $\langle \mathbb{Q}, <, +1, -1\rangle$ has only finitely many substructures. Hence it is enough to show that finitely generated supports are closed under finite intersections.

Take any two finitely generated substructures $\mathfrak{A}$, $\mathfrak{B}$ of $\langle \mathbb{Q}, <, +1, -1\rangle$. Let A and B be the sets of elements of $\mathfrak{A}$ and $\mathfrak{B}$ that are contained in the interval $[0, 1)$. As A and B are (finite) sets of generators, it is enough to show that if an automorphism π acts as identity on $A \cap B$, then

$$\pi = \sigma_1\tau_1\sigma_2\tau_2...\sigma_n\tau_n,$$

where σ_i act as identity on A and τ_i act as identity on B.

Let l be the smallest and h the biggest element of the set $A \cup B$. Notice that $h - l < 1$. Take two different open intervals (l_A, h_A), (l_B, h_B) of length 1 such that

$$[l, h] \subseteq (l_A, h_A) \text{ and } [l, h] \subseteq (l_B, h_B).$$

Now, consider sets $A' = A \cup \{l_A, h_A\}$, $B' = B \cup \{l_B, h_B\}$. Take an automorphism π that acts as identity on $A \cap B = A' \cap B'$. Obviously π is a monotone bijection of the set of rational numbers. Therefore, since the total order symmetry admits least supports,

$$\pi = \sigma'_1\tau'_1\sigma'_2\tau'_2...\sigma'_n\tau'_n,$$

where σ'_i, τ'_i are monotone bijections of $\mathbb{Q}$ and σ'_i act as identity on A', τ'_i act as identity on B'. For each of the permutations σ'_i, τ'_i take an automorphism σ_i, τ_i of the universal structure $\langle \mathbb{Q}, <, +1, -1\rangle$, such that

$$\sigma'_i|_{(l_A,h_A)} = \sigma_i|_{(l_A,h_A)}, \quad \tau'_i|_{(l_B,h_B)} = \tau_i|_{(l_B,h_B)}.$$

Then σ_i act as identity on A and τ_i act as identity on B. Moreover $\pi = \sigma_1\tau_1\sigma_2\tau_2...\sigma_n\tau_n$, as required. □

From now on, we assume a Fraïssé symmetry $(\mathbb{D}_\mathcal{K}, G_\mathcal{K})$ that admits least supports.

4. Structure representation

For any $C \subseteq \mathbb{D}$ and $G \leq \mathrm{Sym}(\mathbb{D})$, the restriction of G to C is defined by

$$G|_C = \{\pi|_C \mid \pi \in G,\ C \cdot \pi = C\} \leq \mathrm{Sym}(C).$$

Lemma 4.1. *Let $\mathfrak{A} \in \mathcal{K}$ be a finitely generated structure. The set of embeddings $u\colon \mathfrak{A} \to \mathfrak{U}_K$ with the $G_\mathcal{K}$-action defined by composition:*

$$u \cdot \pi = u\pi$$

is a single-orbit nominal set.

Proof. First notice that any embedding $u\colon \mathfrak{A} \to \mathfrak{U}_K$ is supported by its image $u(\mathfrak{A})$. Indeed, if an automorphism $\pi \in G_\mathcal{K}$ is an identity on $u(\mathfrak{A})$ then obviously $u \cdot \pi = u$. Hence the set of embeddings is a nominal set. Now take any two embeddings u and v. The images $u(\mathfrak{A})$, $v(\mathfrak{A})$ are finitely generated isomorphic substructures of $\mathfrak{U}_\mathcal{K}$. By extending any isomorphism between $u(\mathfrak{A})$ and $v(\mathfrak{A})$, we obtain an automorphism $\pi \in G_{\mathcal{K}}$ such that $u \cdot \pi = v$. □

Notice that the quotient of a G-set by an equivariant equivalence relation R has a natural structure of a G-set, with the action defined as follows:

$$[x]_R \cdot \pi = [x \cdot \pi]_R.$$

It is easy to see that if X has one orbit, then so does the quotient X/R. Moreover, any support C of an element $x \in X$ supports the equivalence class $[x]_R$, hence if X is nominal then X/R is also nominal.

Definition 4.2. A *structure representation* is a finitely generated structure $\mathfrak{A} \in \mathcal{K}$ together with a group of automorphisms $S \leq \mathrm{Aut}(\mathfrak{A})$ (the *local symmetry*). Its *semantics* $[\mathfrak{A}, S]$ is the set of embeddings of $u \colon \mathfrak{A} \to \mathfrak{U}_K$, quotiented by the equivalence relation:

$$u \equiv_S v \Leftrightarrow \exists \tau \in S \; \tau u = v.$$

A $G_{\mathcal{K}}$-action on $[\mathfrak{A}, S]$ is defined by composition:

$$[u]_S \cdot \pi = [u\pi]_S.$$

Proposition 4.3. *(1) $[\mathfrak{A}, S]$ is a single-orbit nominal $G_{\mathcal{K}}$-set. (2) Every single-orbit nominal $G_{\mathcal{K}}$-set X is isomorphic to some $[\mathfrak{A}, S]$.*

Proof. For (1), use Lemma 4.1. The set of embeddings $u \colon \mathfrak{A} \to \mathfrak{U}_K$ is a single-orbit nominal $G_{\mathcal{K}}$-set, and so is the quotient $[\mathfrak{A}, S]$.

For (2), take a single-orbit nominal set X and let $H \leq G_{\mathcal{K}}$ be the stabilizer of some element $x \in X$. Put $S = H|_{\mathfrak{A}}$ where $\mathfrak{A} \in \mathcal{K}$ is the least finitely generated support of x. Define $f \colon X \to [\mathfrak{A}, S]$ by $f(x \cdot \pi) = [\pi|_{\mathfrak{A}}]_S$. The function f is well defined: if $x \cdot \pi = x \cdot \sigma$ then $\pi\sigma^{-1} \in H$. As $\mathfrak{A} \cdot \pi\sigma^{-1}$ is the least finitely generated support of $x \cdot \pi\sigma^{-1} = x$, we obtain $\mathfrak{A} \cdot \pi\sigma^{-1} = \mathfrak{A}$. Therefore for $\tau = (\pi\sigma^{-1})|_{\mathfrak{A}} \in S$ we have $\tau\sigma|_{\mathfrak{A}} = \pi|_{\mathfrak{A}}$, hence $[\pi|_{\mathfrak{A}}]_S = [\sigma|_{\mathfrak{A}}]_S$. It is easy to check that f is equivariant.

It remains to show that f is bijective. For injectivity, assume $f(x \cdot \pi) = f(x \cdot \sigma)$. This means that there exists $\tau \in S$ such that $\tau\sigma|_{\mathfrak{A}} = \pi|_{\mathfrak{A}}$, then $(\pi\sigma^{-1})|_{\mathfrak{A}} \in S$, hence $(\pi\sigma^{-1})|_{\mathfrak{A}} = \rho|_{\mathfrak{A}}$ for some $\rho \in H$. Therefore $x \cdot \pi\sigma^{-1} = x \cdot \rho = x$, from which we obtain $x \cdot \pi = x \cdot \sigma$. For surjectivity of f, note that by universality of the structure $\mathfrak{U}_K$ any embedding $u \colon \mathfrak{A} \to \mathfrak{U}_K$ can be extended to an automorphism π of $\mathfrak{U}_K$, for which we have $f(x \cdot \pi) = [u]_S$. □

Example 4.4. Consider the Fraïssé symmetry obtained from the universal structure $\langle \mathbb{Q}, <, +1, -1 \rangle$ and a structure $\mathfrak{A}$ generated by $\{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\}$. Notice that mapping one of the generators, say $\frac{1}{2}$, to any element of $\mathfrak{A}$, say $\frac{1}{2} \mapsto 3\frac{3}{4}$, uniquely determines an automorphism π of $\mathfrak{A}$. The automorphism can be seen as a shift. It maps $\frac{1}{3}$ to $3\frac{1}{2}$ and $\frac{3}{4}$ to $4\frac{1}{3}$. This observation leads to a conclusion that $\mathrm{Aut}(\mathfrak{A}) = \mathbb{Z}$. Any subgroup S of $\mathrm{Aut}(\mathfrak{A})$ is therefore isomorphic to $\mathbb{Z}$ and generated by a single automorphism π of the form described above. The same holds for any finitely generated substructure

$\mathfrak{A}$. In this particular case Proposition 4.3 provides a very nice finite representation of single-orbit nominal sets.

4.1. **Fungibility.** Even if the symmetry admits least supports it may happen that some finitely generated structure is not the least finitely generated support of anything. Now we will introduce a condition which ensures that any finitely generated structure is the least finitely generated support of some x.

Definition 4.5. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is *fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exist $\pi \in G_{\mathcal{K}}$ such that:

- $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$,
- $\pi(\mathfrak{A}) \neq \mathfrak{A}$.

A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is fungible.

Example 4.6. The equality and total order symmetries are both fungible. The timed symmetry is not fungible. Take a structure $\mathfrak{A}$ generated by $\{0\}$ and its substructure $\mathfrak{B}$ generated by $\{1\}$. Obviously if an automorphism π acts as identity on $\mathfrak{B}$ then it acts as identity also on $\mathfrak{A}$. It is easy to see that by adding to the timed symmetry the function symbol -1 (see Example 3.16) we obtain a fungible symmetry.

In general, admitting least supports and being fungible are independent properties. The examples are given in [1].

Lemma 4.7. *(1) If $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ admits least supports then every finitely generated fungible $\mathfrak{A} \subseteq \mathfrak{U}_{\mathcal{K}}$ is the least finitely generated support of $[id|_{\mathfrak{A}}]_S$, for any $S \leq \mathrm{Aut}(\mathfrak{A})$.*

(2) If $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is fungible then every finitely generated $\mathfrak{A} \subseteq \mathfrak{U}_{\mathcal{K}}$ is the least finitely generated support of $[id|_{\mathfrak{A}}]_S$, for any $S \leq \mathrm{Aut}(\mathfrak{A})$.

Proof. For (1), recall from Lemma 4.1 that an embedding $u\colon \mathfrak{A} \to \mathfrak{U}_{\mathcal{K}}$ is supported by its image. Therefore $\mathfrak{A}$ supports $\mathrm{id}|_{\mathfrak{A}}$ and hence also $[\mathrm{id}|_{\mathfrak{A}}]_S$. Now consider any finitely generated structure $\mathfrak{B}$ properly contained in $\mathfrak{A}$. Since $\mathfrak{A}$ is fungible there exists an automorphism π from the Definition 4.5. The automorphism π acts as identity on $\mathfrak{B}$, but $[\mathrm{id}|_{\mathfrak{A}}]_S \cdot \pi = [\pi|_{\mathfrak{A}}]_S \neq [\mathrm{id}|_{\mathfrak{A}}]_S$ as the image of π is not $\mathfrak{A}$.

For (2), we first show that $\mathfrak{A}$ supports $[\mathrm{id}|_{\mathfrak{A}}]_S$ as in (1) above. Then let $\mathfrak{B}$ be another support of $[\mathrm{id}|_{\mathfrak{A}}]_S$ and assume $\mathfrak{A}$ is not contained in $\mathfrak{B}$ i.e., there exists some $a \in \mathfrak{A} \setminus \mathfrak{B}$. Since the structure $\mathfrak{C}$ generated by $\mathfrak{A} \cup \mathfrak{B}$ is fungible, there exists an automorphism π such that $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$ and $\pi(\mathfrak{C}) \neq \mathfrak{C}$, which means that also $\pi(\mathfrak{A}) \neq \mathfrak{A}$. Hence $[\mathrm{id}|_{\mathfrak{A}}]_S \cdot \pi = [\pi|_{\mathfrak{A}}]_S \neq [\mathrm{id}|_{\mathfrak{A}}]_S$ and we obtain a contradiction as it turns out that $\mathfrak{B}$ does not support $[\mathrm{id}|_{\mathfrak{A}}]_S$. □

Let us focus for a moment on relational structures. In this case to obtain a fungible symmetry it is enough to require an existence of π that is not an identity on $\mathfrak{A}$.

Definition 4.8. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is *weakly fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exist $\pi \in G_{\mathcal{K}}$ such that:

- $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$,
- $\pi|_{\mathfrak{A}} \neq \mathrm{id}|_{\mathfrak{A}}$.

A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is weakly fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is weakly fungible.

On the other hand, if we restrict ourselves to relational structures, we can also equivalently require an existence of automorphisms π that satisfy a stronger condition.

Definition 4.9. A finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is *strongly fungible* if for every finitely generated substructure $\mathfrak{B} \subsetneq \mathfrak{A}$, there exist $\pi \in G_{\mathcal{K}}$ such that:

- $\pi|_{\mathfrak{B}} = \mathrm{id}|_{\mathfrak{B}}$,
- $\pi(\mathfrak{A}) \cap \mathfrak{A} = \mathfrak{B}$.

A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is strongly fungible if every finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ is weakly fungible.

Fact 4.10. *Let $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ be a Fraïssé symmetry over a signature containing only relation symbols. The following conditions are equivalent:*

(1) $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is weakly fungible,
(2) $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is fungible,
(3) $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is strongly fungible.

The general picture is more complicated. When we introduce function symbols, the notions of weak fungibility, fungibility and strong fungibility differ from each other. Before showing this let us notice that the condition of strong fungibility is in fact equivalent to the strong amalgamation property.

Proposition 4.11. *A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is strongly fungible if and only if the age $\mathcal{K}$ of the universal structure $\mathfrak{U}_{\mathcal{K}}$ has strong amalgamation property.*

Proof. The *if* part is easily proved using homogenity. For the *only if* part take any finitely generated substructures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{U}_{\mathcal{K}}$ and embeddings $f_{\mathfrak{B}} \colon \mathfrak{A} \to \mathfrak{B}$, $f_{\mathfrak{C}} \colon \mathfrak{A} \to \mathfrak{C}$. Thanks to amalgamation there exists a finitely generated substructure $\mathfrak{D}$ of $\mathfrak{U}_{\mathcal{K}}$ together with two embeddings $g_{\mathfrak{B}} \colon \mathfrak{B} \to \mathfrak{D}$ and $g_{\mathfrak{C}} \colon \mathfrak{C} \to \mathfrak{D}$ such that $g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = \mathfrak{A}'$. Take $\pi \in G_{\mathcal{K}}$ for which $\pi|_{\mathfrak{A}'} = \mathrm{id}|_{\mathfrak{A}'}$ and $\pi(\mathfrak{D}) \cap \mathfrak{D} = \mathfrak{A}'$. Let $\mathfrak{D}'$ be a substructure generated by $\mathfrak{D} \cup \pi(\mathfrak{D})$. The embeddings $g_{\mathfrak{B}}$ and $g'_{\mathfrak{C}} = \pi \circ g_{\mathfrak{C}}$ into $\mathfrak{D}'$ are as needed:

$$g_{\mathfrak{B}} \circ f_{\mathfrak{B}}(\mathfrak{A}) = g'_{\mathfrak{C}} \circ f_{\mathfrak{C}}(\mathfrak{A}) = g_{\mathfrak{B}}(\mathfrak{B}) \cap g'_{\mathfrak{C}}(\mathfrak{C}).$$

□

Corollary 4.12. *A Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ over a signature containing only relation symbols is fungible if and only if the age $\mathcal{K}$ of the universal structure $\mathfrak{U}_{\mathcal{K}}$ has strong amalgamation property.*

Example 4.13. Consider a signature with unary function symbols F and G. For any integer i let $\mathbb{D}_i$ be the set of all infinite, binary sequences $\langle a_n \rangle$ defined for $n \geq i$ and equal 0 almost everywhere. Take $\mathbb{D} = \bigcup \mathbb{D}_i$ and define a structure $\mathfrak{U}$ with a carrier $\mathbb{D}$, where

$$F(\langle a_i, a_{i+1}, a_{i+2}, ...\rangle) = \langle a_{i+1}, a_{i+2}, ...\rangle, \quad G(w0) = w1, \quad G(w1) = w0.$$

Since the structure is weakly homogeneous, we obtain a Fraïssé symmetry. The symmetry is weakly fungible, but it is not fungible, as the structure generated by $\{w0, w1\}$ is not fungible for any $w \in \{0,1\}^*$.

Example 4.14. Consider a signature with a single unary function symbol F. For any integer i let $\mathbb{D}_i$ be the set of all infinite sequences $\langle a_n \rangle$ of natural numbers defined for $n \geq i$ and equal 0 almost everywhere. Take $\mathbb{D} = \bigcup \mathbb{D}_i$ and define a structure $\mathfrak{U}$ with a carrier $\mathbb{D}$, where

$$F(\langle a_i, a_{i+1}, a_{i+2}, ...\rangle) = \langle a_{i+1}, a_{i+2}, ...\rangle.$$

Notice that the age of $\mathfrak{U}$ is the class $\mathcal{K}$ of all finitely generated structures that satisfy the following axioms

- for any d, e there exist $m, n \in \mathbb{N}$ such that $F^m(d) = F^n(e)$,
- there are no loops i.e. $F^n(d) \neq d$ for all $n \in \mathbb{N}$.

Since the structure is weakly homogeneous, we obtain a Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$. It is easy to check that the symmetry is fungible.

Now, take any nonempty finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}$ and the empty substructure $\emptyset \subseteq \mathfrak{A}$. For any automorphism π of $\mathfrak{U}$ and $a \in \mathfrak{A}$ there exist $m, n \in \mathbb{N}$ for which $F^m(a) = F^n(a \cdot \pi)$. Hence there is no π for which $\pi(\mathfrak{A}) \cap \mathfrak{A} = \emptyset$ and the structure $\mathfrak{A}$ is not strongly fungible. Therefore the symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ is not strongly fungible.

From now on, we assume a Fraïssé symmetry $(\mathbb{D}_{\mathcal{K}}, G_{\mathcal{K}})$ that admits least supports and is fungible. We shall call such a Fraïssé symmetry *well behaved.*

4.2. **Representation of functions.** For any finitely generated substructure $\mathfrak{A}$ of $\mathfrak{U}_{\mathcal{K}}$ and any $S \leq \mathrm{Aut}(\mathfrak{A})$, the $G_{\mathcal{K}}$-extension of S is

$$ext_{G_{\mathcal{K}}}(S) = \{\pi \in G_{\mathcal{K}} \mid \pi|_{\mathfrak{A}} \in S\} \leq G_{\mathcal{K}}.$$

Notice that $ext_{G_{\mathcal{K}}}(S)$ is exactly the stabilizer of $[\mathrm{id}\,|_{\mathfrak{A}}]_S$ in $G_{\mathcal{K}}$.

Lemma 4.15. *For each embedding* $u\colon \mathfrak{A} \to \mathfrak{U}_{\mathcal{K}}$ *the group* $ext_{G_{\mathcal{K}}}(u^{-1}Su)$*, where* $u^{-1}Su \leq \mathrm{Aut}(u(\mathfrak{A}))$*, is the stabilizer of an element* $[u]_S \in [\mathfrak{A}, S]$*.*

Proof. For any $\pi \in G_{\mathcal{K}}$ that extends u we have $[u]_S = [\mathrm{id}\,|_{\mathfrak{A}}]_S \cdot \pi$. Hence, by Lemma 2.5, the stabilizer of $[u]_S$ is $\pi^{-1} ext_{G_{\mathcal{K}}}(S)\pi$. It is easy to check that

$$\pi^{-1} ext_{G_{\mathcal{K}}}(S)\pi = ext_{G_{\mathcal{K}}}(u^{-1}Su).$$

□

Lemma 4.16. *Let $\mathfrak{A}, \mathfrak{B}$ be finitely generated substructures of $\mathfrak{U}_{\mathcal{K}}$ and let $S \leq \mathrm{Aut}(\mathfrak{A})$, $T \leq \mathrm{Aut}(\mathfrak{B})$ then $ext_{G_{\mathcal{K}}}(S) \leq ext_{G_{\mathcal{K}}}(T)$ if and only if $\mathfrak{B} \subseteq \mathfrak{A}$ and $S|_{\mathfrak{B}} \leq T$.*

Proof. The *if* part is obvious. For the *only if* part, we first prove that $\mathfrak{B} \subseteq \mathfrak{A}$. Notice that if $\pi|_{\mathfrak{A}} = \mathrm{id}|_{\mathfrak{A}}$ then $\pi \in ext_{G_{\mathcal{K}}}(S)$ and hence $\pi \in ext_{G_{\mathcal{K}}}(T)$, which is the stabilizer of $[\mathrm{id}|_{\mathfrak{B}}]_T$. Therefore $\mathfrak{A}$ supports $[\mathrm{id}|_{\mathfrak{B}}]_T$. By Lemma 4.7 (2) the least support of $[\mathrm{id}|_{\mathfrak{B}}]_T$ is $\mathfrak{B}$. Hence $\mathfrak{B} \subseteq \mathfrak{A}$. Then we have

$$ext_{G_{\mathcal{K}}}(S) \leq ext_{G_{\mathcal{K}}}(T)$$

$$\Updownarrow$$

$$\forall \pi \in G_{\mathcal{K}}\ \pi|_{\mathfrak{A}} \in S \Longrightarrow \pi|_{\mathfrak{B}} \in T$$

$$\Updownarrow$$

$$\forall \pi \in G_{\mathcal{K}}\ \pi|_{\mathfrak{A}} \in S \Longrightarrow (\pi|_{\mathfrak{A}})|_{\mathfrak{B}} \in T$$

$$\Updownarrow$$

$$\forall \tau \in S\ \tau|_{\mathfrak{B}} \in T.$$

□

Proposition 4.17. *Let $X = [\mathfrak{A}, S]$ and $Y = [\mathfrak{B}, T]$ be single-orbit nominal sets. The set of equivariant functions from X to Y is in one to one correspondence with the set of embeddings $u \colon \mathfrak{B} \to \mathfrak{A}$, for which $uS \subseteq Tu$, quotiented by $\equiv_T$.*

Proof. By Proposition 2.6 and Lemma 4.15 equivariant functions from $[\mathfrak{A}, S]$ to $[\mathfrak{B}, T]$ are in bijective correspondence with those elements $[u]_T \in [\mathfrak{B}, T]$ for which

$$ext_{G_{\mathcal{K}}}(S) \leq ext_{G_{\mathcal{K}}}(u^{-1}Tu).$$

Hence, by Lemma 4.16, equivariant functions from $[\mathfrak{A}, S]$ to $[\mathfrak{B}, T]$ correspond to those elements $[u]_T \in [\mathfrak{B}, T]$ for which

$$u(\mathfrak{B}) \subseteq \mathfrak{A} \quad \text{and} \quad S|_{u(\mathfrak{B})} \leq u^{-1}Tu,$$

which means that u is an embedding from $\mathfrak{B}$ to $\mathfrak{A}$ and $uS \subseteq Tu$, as required. □

Let $G{-}\mathbf{Nom^1}$ denote the category of single-orbit nominal sets and equivariant functions. Propositions 4.3 and 4.17 can be phrased in the language of category theory:

Theorem 4.18. *In a well-behaved Fraïssé symmetry, the category $G{-}\mathbf{Nom^1}$ is equivalent to the category with:*

- *as objects, pairs $(\mathfrak{A}, S)$ where $\mathfrak{A} \in \mathcal{K}$ and $S \leq \mathrm{Aut}(\mathfrak{A})$,*
- *as morphisms from $(\mathfrak{A}, S)$ to $(\mathfrak{B}, T)$, those embeddings $u \colon \mathfrak{B} \to \mathfrak{A}$ for which $uS \subseteq Tu$, quotiented by $\equiv_T$.*

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Institute of Mathematics of the Polish Academy of Sciences, Warsaw, Poland
E-mail address: joanna.ochremiak@gmail.com