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Generalized small cancellation groups and asymptotic cones

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Abstract. We prove that for any separable metric space there exists a finitely generated group such that one of its asymptotic cones is an $\mathbb{R}$-tree while another asymptotic cone contains a subset isometric to the given metric space.

1. Introduction

The notion of an *asymptotic cone* was introduced by Gromov [3] and the usual definition involving ultrafilters was given in [12]. Roughly speaking, an asymptotic cone is what one sees when looking at a metric space from infinitely far away. Gromov [4] noted that asymptotic cones of groups, though rarely locally compact, exhibit low-dimensional properties, such as finiteness of covering dimension.

Gromov [4] also asked if a finitely generated group can have non-homeomorphic asymptotic cones. A positive answer [11] relies on certain small cancellation groups given by 'sparse' (*lacunary*) presentations. For such groups some asymptotic cones are $\mathbb{R}$-trees and some are not simply-connected, all of them however being *locally* homeomorphic to $\mathbb{R}$-trees (Chapter 4 of [8]). On the other hand, the fundamental group of an asymptotic cone of a finitely generated group can contain any countable subgroup [2]. The latter result is based on embedding certain metric spaces into asymptotic cones of groups, it is however not evident if a group arising from this construction has different asymptotic cones (the present paper proves this is the case). Similar considerations appear in [1], where spaces with non-isomorphic fundamental groups are embedded into fundamental groups of asymptotic cones.

The aim of this paper is to show a link between those two techniques — lacunary presentations giving rise to $\mathbb{R}$-trees as asymptotic cones and isometric embeddings of metric spaces into asymptotic cones based on generalized small cancellation theory. More precisely, we prove

Theorem 1.1. *For any metric space X approximable by finite sets (see Definition 3.1) there exists a group generated by 2 elements whose one asymptotic cone is an $\mathbb{R}$-tree and another asymptotic cone contains an isometric copy of X.*

Since the approximability condition is satisfied by any separable metric space (Proposition 3.2), we get the following.

Corollary 1.2. *There exists a group generated by 2 elements whose one asymptotic cone is an $\mathbb{R}$-tree and another asymptotic cone has infinite covering dimension.*

It seems plausible that the authors of [2] and [1] have been aware of this result (cf. Theorems 7.26 in 7.30 in [1]), but it does not appear in the literature as stated.

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2. Preliminaries

The purpose of this section is to review the background necessary for the proof of Theorem 1.1. The proof itself is postponed to the next section.

2.1. **Small cancellation theory.** Small cancellation groups are groups described by generators and relations such that the relations do not share long common parts. The consequence of this 'independence' is that in any word in the generators equal to the identity in the group, a long trace of some of the relations can be found.

There are many small cancellation conditions available (see Chapter V.2 of [6]). We choose to work with the following condition $C^*(\lambda)$.

Definition 2.1. *Let R be a set of cyclically reduced words in the alphabet S and let $\mathcal{R}$ be the symmetrization of R, i.e. the set of all cyclic shifts of words from R and their inverses.*

The set R and the presentation $\langle S|R\rangle$ is said to satisfy $C^*(\lambda)$ if

(1) *no word from R is a proper power of another word,*
(2) *a common initial segment of any two distinct words from $\mathcal{R}$ is shorter than λ times any of these words.*

If $u \equiv v$ denotes the letter-for-letter equality of the words u, v and $|u|$ denotes the length of u, then the statement (2) in Definition 2.1 can be written as follows: for distinct $r_1, r_2 \in \mathcal{R}$

$$\text{if } r_1 \equiv uv_1 \text{ and } r_2 \equiv uv_2, \quad \text{then } |u| < \lambda \cdot \min\{|r_1|, |r_2|\}.$$

To give a geometric description of $C^*(\lambda)$ we need the following notion.

Definition 2.2. *A van Kampen diagram over a presentation $\langle S|R\rangle$ is a connected planar 2-dimensional complex Δ with every 1-cell treated as a pair of opposite edges, subject to the following conditions:*

(1) *Every edge of Δ is labelled by an element of $S \cup S^{-1}$, and opposite edges have inverse labels.*
(2) *The boundary label of any 2-cell of Δ is an element of R (up to a cyclic shift and orientation).*

Note that a word in the alphabet S represents the identity of the group $\langle R|S\rangle$ if and only if it is the boundary label of some simply-connected van Kampen diagram ([6], Theorem V.1.1).

Suppose now that Δ is a van Kampen diagram over a $C^*(\lambda)$-presentation. Then for two distinct 2-cells $D_1, D_2 \subset \Delta$ and for any path $p \subset \partial D_1 \cap \partial D_2$ we have

$$|p| < \lambda \cdot \min\{|\partial D_1|, |\partial D_2|\}.$$

(Note that the intersection $\partial D_1 \cap \partial D_2$ might consist of several disjoint paths.) Similarly, if $D \subset \Delta$ is a 2-cell whose boundary cycle is 'glued to itself' and runs through a path p twice, once in every direction — so that ∂D is topologically not a circle — then $|p| < \lambda \cdot |\partial D|$.

Observe that, with the geometric meaning of $C^*(\lambda)$ explained in the previous paragraph, we can actually talk about $C^*(\lambda)$-*diagrams*, without any reference to van Kampen diagrams over presentations. The crucial property of such diagrams is that they always contain a face (i.e. a 2-cell) with a long segment of the boudary contained in the boundary of the whole diagram:

Theorem 2.3 ([7], Lemma 4)**.** *Suppose Δ is a $C^*(\lambda)$-diagram with at least one face, where $\lambda \leq \frac{1}{6}$. Then there exists a face D and a path $p \subset \partial D \cap \partial \Delta$ of length*

$$|p| > (1 - 3\lambda) \cdot |\partial D|.$$

Proof. Clearly we can remove from Δ all vertices and edges not belonging to the boundary of any face. Then we introduce a new structure of a 2-complex on Δ, which we denote Δ', by contracting to a single edge each maximal path contained in the boundary of two neighboring (not necessarily distinct) faces or in the outer boundary of a face. In other words, vertices of Δ of degree at most 2 are removed and vertices of Δ of degree at least 3 remain vertices of Δ' of the same degree.

Let V, E, E_{int}, E_{bd}, F, F_{int}, F_{bd} denote the number of vertices, edges, internal edges, boundary edges, faces, internal faces, boundary faces of Δ'. Moreover, let $F^1_{\leq 3}$, $F^1_{\leq 4}$, $F^{\geq 2}$ denote the number of boundary faces of Δ' respectively with: one boundary and at most 3 internal edges, one boundary and at least 4 edges, at least 2 boundary edges.

Each vertex of Δ' is of degree at least 3, thus

$$V - \tfrac{2}{3}E \leq 0. \tag{1}$$

Moreover, we clearly have

$$E_{bd} \geq F^1_{\leq 3} + F^1_{\leq 4} + 2F^{\geq 2}. \tag{2}$$

Now, the condition $C^*(\lambda)$ with $\lambda \leq \frac{1}{6}$ implies that every internal face of Δ' has more than 6 edges. Since every face with at least 2 boundary edges has at least 2 internal edges, by counting all internal edges we obtain

$$2E_{int} \geq F^1_{\leq 3} + 4F^1_{\geq 4} + 2F^{\geq 2} + 6F_{int}. \tag{3}$$

Finally, we apply Euler's formula and (1), (2), (3) to get

$$\begin{aligned}
1 &= V - E + F = (V - \tfrac{2}{3}E) - \tfrac{1}{3}E + F \leq -\tfrac{1}{3}E + F \\
&= -\tfrac{1}{3}E_{int} - \tfrac{1}{3}E_{bd} + F^1_{\leq 3} + F^1_{\geq 4} + F^{\geq 2} + F_{int} \\
&\leq -\tfrac{1}{6}(F^1_{\leq 3} + 4F^1_{\geq 4} + 2F^{\geq 2} + 6F_{int}) - \tfrac{1}{3}(F^1_{\leq 3} + F^1_{\geq 4} + 2F^{\geq 2}) + \\
&\quad F^1_{\leq 3} + F^1_{\geq 4} + F^{\geq 2} + F_{int} \\
&= \tfrac{1}{2}F^1_{\leq 3}.
\end{aligned}$$

Hence $F^1_{\geq 3} \geq 2$ and there exists a face D with boundary ∂D consisting of a path $p \subset \partial \Delta$ and at most 3 internal edges, each shorter than $\lambda \cdot |\partial D|$ in view of $C^*(\lambda)$. Consequently, $|p| > (1 - 3\lambda) \cdot |\partial D|$. □

Theorem 2.3 provides an efficient algorithm for solving the word problem in small cancellation groups: any word representing the identity contains more than half of some relation, so by replacing this occurence with the remaining part of the relation we are left with a shorter word. We will see that this reasoning proves a linear isoperimetric inequality and hyperbolicity of small cancellation groups.

However, ordinary small cancellation groups have a disadvantage: they do not admit quasi-isometric embeddings of arbitrary finite metric spaces. To remedy this, in the next section we will introduce *generalized small cancellation groups* which will retain all necessary properties of ordinary small cancellation groups.

To conclude this subsection we show that there are sufficiently large sets of words with sufficiently small cancellation:

Theorem 2.4 ([2], Proposition 3.3)**.** *There exists a set W of words over a 2-letter alphabet $\{a, b\}$ such that*

(1) *W satisfies $C^*(\frac{1}{100})$,*
(2) *the cardinality of $\{w \in W : |w| = n\}$ is a non-decreasing function of n diverging to ∞,*
(3) *for all n the set $\{w \in W : |w| \geq n\}$ satisfies $C^*(\lambda_n)$, where $\lambda_n \to 0$.*

Proof. The idea is to consider words of the form

$$(a^6bx_1b)(a^6bx_2b)(a^6bx_3b)\dots(a^6bx_mb), \tag{4}$$

where x_1, x_2, ..., x_m are distinct words of the same length not containing a subword a^6. Then every sufficiently long subword w of (4) contains $(a^6bx_jb)(a^6bx_{j+1}b)$, which due to the absence of a^6-subwords in x_j determines uniquely $|x_j|$ and the fragment of (4) where w occurs.

We turn to the details. A word with no a^6-subwords will be called a *good* word.

Note first that the number of good words of length $3k$ is at least 7^k. To see this, just divide $3k$ places for letters into k blocks of length 3; one may then choose any of the 7 words distinct from aaa for each block. It follows that the number of good words of length ℓ is at least $7^{\frac{1}{3}(\ell-2)}$.

Let U_ℓ be the set of all words (4), where $\{x_1, x_2, \dots, x_\ell\}$ runs through an arbitrary maximal family of pairwise disjoint ℓ-element sets of good words of length ℓ. Then all words in U_ℓ are of length $\ell(\ell + 8)$ and

$$|U_\ell| \geq \left[\frac{7^{\frac{1}{3}(\ell-2)}}{\ell}\right]. \tag{5}$$

Partition each U_ℓ into $2\ell+9$ pairwise disjoint sets $U_{\ell,i}$ $(i = 0, 1, \dots, 2\ell+8)$ with at least $[\frac{1}{2\ell+9}|U_\ell|]$ elements. Each number $n \in \mathbb{N}$ can be written as $n = \ell(\ell + 8) + i$, where ℓ, i are non-negative integers with ℓ minimal, so that $i < 2\ell + 9$. Set

$$V_n = \{wb^i : w \in U_{\ell,i}\};$$

this is a set of words of length n.

We will show that $V = \bigcup_{n=4700^2}^{\infty} V_n$ satisfies assertions (1) and (3) with $\lambda_n = \frac{47}{\sqrt{n}}$.

To this end, consider a word $v \in V$ of length n; it is of the form

$$v = (a^6bx_1b)(a^6bx_2b)(a^6bx_3b)\dots(a^6bx_\ell b)b^i \tag{6}$$

with x_1, x_2, ..., x_ℓ distinct good words of length ℓ and $n = \ell(\ell+8)+i$, $i < 2\ell+9$. Consider a cyclic subword w of v with $|w| \geq 6\ell + 41$. Then $|w| > 4(\ell + 8) + 1$, so w necessarily contains a subword $(a^6bx_jb)(a^6bx_{j+1}b)$. The only occurences of a^6 in (6) are immediately after opening parentheses, so by measuring the distance between the consecutive a^6's we recover $|x_j| = \ell$ and the word x_j itself, which in turn determines the values of i and n, since good words used for constructing V_n's were different.

Moreover, we have

$$\frac{6\ell + 41}{|v|} = \frac{6\ell + 41}{\ell(\ell + 8) + i} \leq \frac{47\ell}{\ell(\ell + 8) + i} \leq \frac{47}{\ell + 8} < \frac{47}{\sqrt{(\ell + 1)(\ell + 9)}} < \frac{47}{\sqrt{n}} = \lambda_n.$$

In other words, $|w| \geq \lambda_n \cdot |v|$ leads to $|w| \leq 6\ell + 41$ and the reasoning from the previous paragraph applies.

We conclude that a subword of any word from $V_{\geq n} = \{v \in V : |v| \geq n\}$ of length at least λ_n of this word occurs only in this word and only in one place. Words from

$V_{\geq n}$ do not contain inverses a^{-1}, b^{-1}, hence we conclude that $V_{\geq n}$ satisfies $C^*(\lambda_n)$. Since $\lambda_n \leq \frac{1}{100}$ for $n \geq 4700^2$, assertions (1) and (3) are proved.

It remains to refine the V_n's so that (2) is also satisfied. But (5) gives

$$|V_{\ell(\ell+8)+i}| \geq \left\lfloor \frac{1}{2\ell+9} \left\lfloor \frac{7^{\frac{1}{3}(\ell-2)}}{\ell} \right\rfloor \right\rfloor \to \infty \qquad \text{as } \ell \to \infty,$$

hence also $|V_n| \to \infty$ as $n \to \infty$. Now it suffices to replace each V_n with a subset of V_n so that $|V_n|$ is a *non-decreasing* function of n diverging to ∞. □

2.2. **Asymptotic cones, $\mathbb{R}$-trees, and hyperbolicity.** As we have mentioned in the introduction, an asymptotic cone of a metric space is the picture of the space from infinite distance. We are now going to formalize this loose description.

Recall that an *ultrafilter* on $\mathbb{N}$ is a proper family $\omega \subset \mathcal{P}(\mathbb{N})$ closed under taking finite intersections and oversets and maximal with respect to containment among such families. We also assume that ω is different from the family of all subsets of $\mathbb{N}$ containing a fixed element. Any bounded sequence (a_n) of real numbers has a unique limit $a = \lim_\omega a_n$ described by the property that $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\} \in \omega$ for all $\varepsilon > 0$; any sequence of real numbers has a limit, possibly an infinite one.

Definition 2.5. *Let (X, d) be a metric space, let $p = (p_n)$ be a sequence of points of X and $c = (c_n)$ a sequence increasing to ∞. Then for any ultrafilter ω the asymptotic cone $Cone_\omega(X, c, p)$ is defined as follows:*

(1) *Set $\Pi X = X^{\mathbb{N}}$, define an equivalence relation on ΠX:*

$$(x_1, x_2, \ldots) \sim (y_1, y_2, \ldots) \quad \Leftrightarrow \quad \lim_\omega \frac{d(x_n, y_n)}{c_n} = 0,$$

and denote by $\Pi X/_\omega$ the set of equivalence classes of $\sim$.

(2) *Define*

$$Cone_\omega(X, c, p) = \left\{ (x_1, x_2, \ldots)_\sim \in \Pi X/_\omega : \lim_\omega \frac{d(x_n, p_n)}{c_n} < \infty \right\}$$

with metric

$$d((x_1, x_2, \ldots)_\sim, (y_1, y_2, \ldots)_\sim) = \lim_\omega \frac{d(x_n, y_n)}{c_n}.$$

A finitely generated group can be considered as a metric space by means of the *word metric*: if G is generated by a finite set S, then the distance between $x, y \in G$ is the smallest number k for which there is an equality $x^{-1}y = s_1 s_2 \ldots s_k$ with each $s_i \in S \cup S^{-1}$. Hence we can talk about asymptotic cones of finitely generated groups. Note that they do not depend on the choice of S, since different finite generating sets yield bilipschitz equivalent metric spaces and hence bilipschitz equivalent (in particular homeomorphic) asymptotic cones. Moreover, the sequence (p_n) can be suppressed as a finitely generated group is a homogeneous metric space. One may therefore write $\mathrm{Cone}_\omega(G, c)$, where both the ultrafilter ω and the scaling sequence $c = (c_n)$ might be important.

Now we will briefly discuss $\mathbb{R}$-trees. Recall that a *geodesic* space is a metric space in which any two points can be connected by a *geodesic segment* isometric to a segment on the real line.

Definition 2.6. *An $\mathbb{R}$-tree is a geodesic space in which any two distinct points can be connected by a unique topological arc.*

It turns out that there are more convenient characterizations:

Proposition 2.7. *If a geodesic space contains no simple geodesic triangles, i.e. subsets homeomorphic to the circle and composed of three geodesic segments, then it is an $\mathbb{R}$-tree.*

Proof. Observe that the non-existence of simple geodesic triangles in a metric space (X, d) has the following direct consequences:

(1) Any two points of X are joined by a *unique* geodesic segment.
(2) A point p outside a geodesic segment Λ has a unique *geodesic projection* onto Λ — the point of Λ closest to p, and also the only point of Λ such that the geodesic joining it to p has no common internal points with Λ.
(3) The projection onto a fixed geodesic segment is a continuous function.
(4) Three geodesic segments joining every pair of three distinct points form a *tripod* and have exactly one common point, the *midpoint* of the tripod.

Suppose the proposition is false. Then there exist points $x, y \in X$, a geodesic parametrization $\lambda : [0, \ell] \to X$ of a geodesic segment Γ and a parametrization $c : [0, 1] \to X$ of an arc, such that $\gamma(0) = c(0) = x$, $\gamma(\ell) = c(1) = y$, but the image of c does not coincide with Γ. The following set is then non-empty:

$$U = \{t \in [0, 1] : c(t) \notin \Gamma\}.$$

For $s \in [0, 1]$ let $\pi(s)$ be the midpoint of the tripod determined by x, y, $c(s)$. The function $\pi : [0, 1] \to \Gamma$ is a geodesic projection from $c(s)$ onto Γ; therefore π is continuous and $\pi(s) = c(s)$ for $s \notin U$, so that π is injective on $[0, 1] \setminus U$.

But if $s \in U$ and s' is sufficiently close to s, then $d(c(s'), c(s)) < d(c(s'), \Gamma)$ and the projection of $c(s')$ onto the geodesic segment between $c(s)$ and $\pi(s)$ is not equal to $\pi(s)$, hence an inspection of the tripod with vertices $c(s')$, $c(s)$, $\pi(s)$ shows that $\pi(s') = \pi(s)$. In other words, for $s \in U$ the projection π is locally constant in the neighborhood of s, which contradicts the conclusion of the previous paragraph. □

Proposition 2.8. *Let (X, d) be a geodesic space such that for all $x, y, z, t \in X$ we have*

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\}.$$

Then X is an $\mathbb{R}$-tree.

Proof. Proposition 2.7 reduces the task to showing that X contains no simple geodesic triangles; suppose towards a contradiction that p, q, r are vertices of such a triangle. Consider 'the point of tangency of the incircle to the side $[q, r]$', i.e. the point s on the side $[q, r]$ such that

$$(7)\quad d(q, s) = \tfrac{1}{2}(d(p, q) + d(q, r) - d(p, r)), \quad d(r, s) = \tfrac{1}{2}(d(p, r) + d(q, r) - d(p, q)).$$

Set $(x, y, z, t) = (p, s, q, r)$ to obtain

$$d(p, s)+d(q, r) \leq \max\{d(p, q)+d(r, s), d(p, r)+d(q, s)\} = \tfrac{1}{2}(d(p, q)+d(q, r)+d(p, r)),$$

which after taking (7) into account gives

$$d(p, s) + d(s, q) = d(p, q).$$

Hence there exists a point $v \in [p, q]$ with $d(p, s) = d(p, v)$, $d(q, s) = d(q, v)$. Now set $(x, y, z, t) = (p, q, s, v)$; the conclusion is that

$$d(p, q) + d(s, v) \leq \max\{d(p, s) + d(q, v), d(p, v) + d(q, s)\} = d(p, q).$$

Therefore the points $s \in [q, r]$ and $v \in [p, q]$ coincide, so in fact $s = v = q$. But then the first equality in (7) shows that $d(p, q) + d(q, r) = d(p, r)$. By symmetry, any of the lengths between the points p, q, r is equal to the sum of the other two lengths. However, this implies that $p = q = r$, a clear contradiction. □

We conclude this subsection by giving a sufficient condition for a group to have an $\mathbb{R}$-tree as an asymptotic cone. For this purpose we need the notion of hyperbolicity in the sense of Gromov:

Definition 2.9. *A geodesic space (X, d) is δ-hyperbolic if for all $x, y, z, t \in X$ we have*

$$d(x, y) + d(z, t) \le \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\} + 2\delta.$$

This is not the usual characterization of δ-hyperbolicity that every side of a geodesic triangle is contained in the δ-neighborhood of the sum of two others. However, these two properties are equivalent ([10], Corollary 2.4). Note that Proposition 2.8 says that a 0-hyperbolic space is just an $\mathbb{R}$-tree.

Definition 2.10. *A (δ-)hyperbolic group is a group admitting a finite generating set with respect to which it is a δ-hyperbolic metric space. (We consider the Cayley graph here rather than the discrete metric space (G, d_G) itself.)*

Now we turn to the criterion in which *lacunarity* of relations plays a key role:

Proposition 2.11 (Part of Theorem 1.1 in [8])**.** *Consider a sequence of groups given by presentations*

$$G_n = \langle S | R_1 \cup R_2 \cup \ldots \cup R_n \rangle$$

and their direct limit $G = \langle S | R_1 \cup R_2 \cup \ldots \rangle$. Suppose that, with respect to the finite generating set S,

(1) *G_n is δ_n-hyperbolic,*
(2) *the natural epimorphism $G_n \to G$ is an isometry on every ball of radius r_n,*
(3) $\frac{\delta_n}{r_n} \to 0$.

Then some asymptotic cone of the group G is an $\mathbb{R}$-tree.

Proof. Take an arbitrary ultrafilter ω; by (3) there exists a scaling sequence $c = (c_n)$ with

$$\frac{\delta_n}{c_n} \to 0, \qquad \frac{c_n}{r_n} \to 0. \tag{8}$$

Consider arbitrary points $x = (x_1, x_2, \ldots)_\sim$, $y = (y_1, y_2, \ldots)_\sim$, $z = (z_1, z_2, \ldots)_\sim$, $t = (t_1, t_2, \ldots)_\sim$ of the asymptotic cone $X = \mathrm{Cone}_\omega(G, c)$. Assumption (1) and Definitions 2.9, 2.10 give

$$\begin{aligned} &d_n(x_n, y_n) + d_n(z_n, t_n) \\ &\qquad \le \max\{d_n(x_n, z_n) + d_n(y_n, t_n), d_n(x_n, t_n) + d_n(y_n, z_n)\} + 2\delta_n \end{aligned} \tag{9}$$

for each n, where d_n is the word metric of the group G_n.

On the other hand, Definition 2.5(2) shows that the sequence $\frac{1}{c_n} d_n(x_n, y_n)$ is ω-convergent to $d_X(x, y)$. Hence it is bounded for ω-almost all n and the second limit in (8) implies that $d_n(x_n, y_n) < r_n$ and, in view of assumption (2), $d_n(x_n, y_n) = d_G(x_n, y_n)$ for ω-almost all n. By an analogous reasoning with other symbols replacing x, y we see that (9) leads to

$$d_G(x_n, y_n) + d_G(z_n, t_n) \le \max\{d_G(x_n, z_n) + d_G(y_n, t_n), d_G(x_n, t_n) + d_G(y_n, z_n)\} + 2\delta_n$$

for ω-almost all n, whence dividing by c_n, using the first convergence in (8) and passing to an ω-limit, we get

$$d_X(x,y) + d_X(z,t) \le \max\{d_X(x,z) + d_X(y,t), d_X(x,t) + d_X(y,z)\}.$$

It remains to apply Proposition 2.8. □

3. The main construction

The purpose of this section is to prove Theorem 1.1. Before proceeding to the main construction we have to explain the expression 'approximable by finite sets' occuring in Theorem 1.1. For this purpose note that Definition 2.5 can be easily generalized to a sequence of spaces: if we are given a sequence of metric spaces X_n with chosen points $p_n \in X_n$, then all we have to change in Definition 2.5 is to replace $\Pi X = X^{\mathbb{N}}$ with $\Pi X = X_1 \times X_2 \times \ldots$, and suppress the scaling sequence (c_n) by setting $c_n = 1$. In that way we define the ω-limit $\lim_\omega(X_n, p_n)$; note that the asymptotic cone of a group $\mathrm{Cone}_\omega(G, c_n)$ is just the limit $\lim_\omega(G, \frac{1}{c_n} d_G)$, where d_G is the word metric of G.

Definition 3.1. *We will call a metric space A* approximable by finite sets *if there exists an ultrafilter ω and a sequence of finite metric spaces A_1, A_2, … with chosen points a_1, a_2, … such that A is isometric to a subspace of the ω-limit* $\lim_\omega(A_n, a_n)$.

This class of spaces includes all 'small' metric spaces:

Proposition 3.2. *Every separable metric space is approximable by finite sets.*

Proof. Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of a metric space X, and set $A_n = \{d_1, d_2, \ldots, d_n\}$ with the metric induced from X. Then we can easily define an isometric embedding $D \to \lim_\omega(A_n, d_1)$: just send $d_n \in D$ to the equivalence class of the sequence $(d_1, d_2, \ldots, d_{n-2}, d_{n-1}, d_n, d_n, d_n, \ldots)$.

Now, since any ω-limit is a complete metric space [12] and $\lim_\omega(A_n, d_1)$ contains an isometric copy of D, it also contains an isometric copy of the completion D, which is the same as the completion of X by uniqueness of completions. Hence X is isometric to a subspace of $\lim_\omega(A_n, d_1)$. □

However, there are also non-separable metric spaces approximable by finite sets: if A_n is the ball of radius n around the identity e in the free group $\mathbb{F}_2$ with the word metric rescaled by a factor $\frac{1}{n}$, then $\lim_\omega(A_n, e)$ is isometric to a ball of radius 1 in the asymptotic cone $\mathrm{Cone}_\omega(\mathbb{F}_2, \frac{1}{n})$. This cone is an $\mathbb{R}$-tree by Proposition 2.11; one easily shows that its valency at every point is equal to continuum. Thus the space $\lim_\omega(A_n, e)$ approximated by finite spaces A_n is not separable.

3.1. **Outline of the main construction.** In this subsection we sketch the construction of the group satisfying Theorem 1.1. The exposition will be complete up to proofs of a few statements, which will be given in the next subsection.

Let A_1, A_2, … be finite metric spaces approximating the space that we wish to embed into an asymptotic cone. For each $n \in \mathbb{N}$ let Γ_n be the full directed graph on $|A_n|$ vertices with a fixed bijection $A_n \leftrightarrow V(\Gamma_n)$. Let $c_1 < c_2 < \ldots$ be a rapidly increasing sequence of positive integers; it will be defined inductively in the course of the construction.

Let $E = \coprod_{n=1}^{\infty} E(\Gamma_n)$ be the disjoint sum of all edges and let W be the set of words given by Theorem 2.4. Then we can define a function $\varphi : E \to W \cup W^{-1}$ labelling the edges of all Γ_n's such that

(1) For every edge $e \in E$ we have $\varphi(e^{-1}) = \varphi(e)^{-1}$, where e^{-1} is opposite to e.
(2) For each edge $e \in E(\Gamma_n)$ with endpoints $x, y \in A_n$ we have

$$|\varphi(e)| = [c_n \cdot d_{A_n}(x, y)].$$

Note that (2) requires that φ be inductively defined on $E(\Gamma_1)$, $E(\Gamma_2)$, ... together with choosing $c_1 < c_2 < \ldots$. More precisely, if φ has already been defined on $E(\Gamma_1) \cup E(\Gamma_2) \cup \ldots \cup E(\Gamma_{n-1})$, then c_n has to be so large that the number

$$m_n = \min_{\substack{x,\, y \,\in\, A_n \\ x \neq y}} [c_n \cdot d_{A_n}(x, y)]$$

is greater than all the lengths $|\varphi(e)|$ for $e \in E(\Gamma_1) \cup E(\Gamma_2) \cup \ldots \cup E(\Gamma_{n-1})$ and the number of words in $\{w \in W : |w| = m_n\}$ is greater than the number of pairs of points of A_n. Later we will impose one additional restriction on c_n. For such a choice of c_n Theorem 2.4(2) certainly enables to extend the labelling φ to $E(\Gamma_n)$.

Finally, define R_n as the labels of all triangles of Γ_n, that is,

$$R_n = \{\varphi(e_1)\varphi(e_2)\varphi(e_3) : \triangle(e_1e_2e_3) \subset \Gamma_n\}$$

and let

$$G = \langle a, b | R_1 \cup R_2 \cup \ldots \rangle. \tag{10}$$

We will now prove that the group (10) satisfies the assertion of Theorem 1.1.

Lemma 3.3. *For each $n \in \mathbb{N}$ there exists a function $\psi_n : A_n \to G$, where d_G is the word metric of G with respect to $\{a, b\}$, such that*

$$\frac{1}{\kappa_n} \cdot d_{A_n}(x, y) \leq \frac{d_G(\psi_n(x), \psi_n(y))}{c_n} \leq \kappa_n \cdot d_{A_n}(x, y)$$

for all $x, y \in A_n$, where moreover $\kappa_n \to 1$.

Passing to ω-limits in the inequality from Lemma 3.3 we see that the map $\lim_\omega A_n \to \mathrm{Cone}_\omega(G, \frac{1}{c_n})$ defined as $(x_1, x_2, \ldots) \mapsto (\psi_1(x_1), \psi_2(x_2), \ldots)$ is an isometric embedding. It follows that G satisfies the second part of Theorem 1.1.

Lemma 3.4. *For each $n \in \mathbb{N}$ the group $G_n = \langle a, b | R_1 \cup R_2 \cup \ldots \cup R_n \rangle$ is hyperbolic.*

Lemma 3.5. *For each $n \in \mathbb{N}$ the natural epimorphism $G_n \to G$ is an isometry on every ball of radius $\frac{1}{8}m_{n+1}$.*

Note that, since we wish to deduce the first part of Theorem 1.1 from Proposition 2.11, we need only to ensure that the ratio of the injectivity radius $\frac{1}{8}m_{n+1}$ in Lemma 3.5 to the hyperbolicity constant of G_n tends to ∞ as $n \to \infty$, and that will be clearly satisfied if c_{n+1} is large enough.

Therefore Theorem 1.1 will follow once we have proved Lemmas 3.3, 3.4, 3.5.

3.2. **Proofs of the required properties.** The main idea is simple: on van Kampen diagrams over (10) we introduce a new structure of a 2-complex by merging initial 2-cells along their 'common sides' into larger blocks. Diagrams with the new structure will satisfy $C^*(\frac{1}{10})$, giving the way to apply Theorem 2.3.

In order to describe this merging of cells into blocks, we introduce some terminology. The notation from the previous subsection remains in force here.

Definition 3.6. *A* word of type k *is a word $\varphi(p) = \varphi(e_1)\varphi(e_2)\ldots\varphi(e_m)$, where $p = e_1e_2\ldots e_m$ is a reduced path in the directed graph Γ_k.*

Since W satisfies $C^*(\frac{1}{100})$, the cancellation between $\varphi(e_j)$ and $\varphi(e_{j+1})$ concerns at most $\frac{1}{100}$ of each of these two words. Therefore after performing all possible cancellations in $\varphi(p)$ at least $\frac{98}{100}$ of every word $\varphi(e_j)$ will remain untouched. It follows that there is a bijection between words of type k and paths in Γ_k.

Definition 3.7. *A* cell of type k *in a van Kampen diagram over the presentation (10) is a* 2*-cell whose boundary label is a relation from the set* R_k.

Obviously a boundary label of a cell of type k is a word of type k of the form $w = \varphi(e_1)\varphi(e_2)\varphi(e_3)$, where $e_1e_2e_3$ is a triangle in Γ_k. As above we see that after all cyclic reductions in w at least $\frac{98}{100}$ of each word $\varphi(e_j)$ is preserved. Hence we can treat such a cell as a triangle whose sides have labels equal to maximal parts of the words $\varphi(e_j)$ contained in the cyclic reduction of w.

Now, $C^*(\frac{1}{100})$ tells that two sides of two cells of type k either do not have a common segment of length greater than $\frac{1}{100}$ of the words assigned to these sides or they have a common segment of length greater than $\frac{98}{100}$ of the same word assigned to them (but not necessarily the whole word, due to the possibility of short cancellations near the endpoints). In the latter case we will say that the cells have a *common side*. Clearly this cannot happen for two cells of different types.

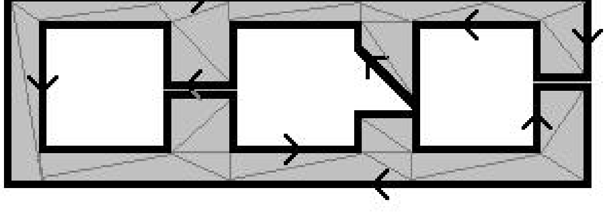
The relation of having a common side leads to the idea of *blocks*:

Definition 3.8. *A* block of type k *is a maximal subdiagram* Δ_b *of a van Kampen diagram over the presentation (10) such that any two cells of* Δ_b *are of type* k *and can be joined by a chain of cells in which two neighboring cells have a common side.*

In other words, we take the transitive closure of the relation of having a common side and a block is the union of all cells belonging to an equivalence class of this transitive closure.

Definition 3.9. *For a van Kampen diagram* Δ *over the presentation (10) we say that* $\mathcal{B}(\Delta)$ *is the* block-diagram *of* Δ *if it is the same subset of the plane and with the same vertices as* Δ *but with blocks as new* 2*-cells.*

Note that a priori blocks do not have to be simply connected and one needs to be careful about what the boundary label of a non-simply-connected block is. There is exactly one bounded region (the leftmost square on the figure below) in the complement of such a block such that the boundary cycle of that region is *one* subsegment of the boundary cycle of the block. This region will be called *distinguished region.*



In fact, we will see in the proof of Proposition 3.10 that such a picture is impossible and blocks are necessarily simply-connected, but that will require the use of their small cancellation properties.

Proposition 3.10. *Block-diagrams satisfy* $C^*(\frac{1}{10})$.

Proof. Suppose first that we have two distinct blocks $B_1, B_2 \subset \mathcal{B}(\Delta)$ and a path $p \subset \partial B_1 \cap \partial B_2$ of length

$$|p| \geq \tfrac{1}{10} \cdot \max\{|\partial_{red} B_1|, |\partial_{red} B_2|\}, \tag{11}$$

where ∂_{red} denotes the cyclic reduction of the boundary label of the respective block (recall that the reduction occurs only near the endpoints of the sides). The blocks B_1, B_2 have no common sides, so p is contained in the sum of two consecutive sides of $\partial_{red} B_1$ and at least half of p is contained in one side of $\partial_{red} B_1$. Similarly, at least half of that half of p is contained in one side of $\partial_{red} B_2$. Hence there is a subpath $q \subset p$ with $|q| \geq \frac{1}{4}|p|$ such that q is contained in one side of both $\partial_{red} B_1$ and $\partial_{red} B_2$. But (11) implies that $|q| \geq \frac{1}{40} \cdot \max\{|\partial_{red} B_1|, |\partial_{red} B_2|\}$, which in view of $C^*(\frac{1}{100})$ for the words from W clearly contradicts the fact that the blocks B_1, B_2 have no common sides.

It remains to verify that one block $B \subset \mathcal{B}(\Delta)$ cannot be 'adjacent to itself' along a segment of length at least $\frac{1}{10}|\partial_{red} B|$. This will immediately follow if we prove that blocks are simply-connected, which is what we are going to do.

To this end, suppose that $\mathcal{B}(\Delta)$ contains a non-simply-connected block. This block has a distinguished region. If there is a non-simply-connected block inside this region, it has its distinguished region and we can go deeper. We continue this process until we eventually find a distinguished region Δ_0 inside which all blocks are simply-connected. Moreover, $\partial \Delta_0$ is a segment in the boundary cycle of the surrounding non-simply connected block B_{sur}.

Note that Δ_0 is a block-diagram in its own right and the first paragraph of the proof establishes $C^*(\frac{1}{10})$ for this block-diagram. Hence by Theorem 2.3 there is a block $B \subset \Delta_0$ and a path $p \subset \partial_{red}\Delta_0 \cap \partial_{red} B$ such that $|p| > \frac{7}{10}|\partial_{red} B|$. However, $\partial \Delta_0$ after a suitable choice of the starting point is a subpath of ∂B_{sur}, so there is a path of length at least $\frac{1}{2}|p| > \frac{7}{20}|\partial_{red} B|$ contained in $\partial_{red} B \cap \partial_{red} B_{sur}$. This is excluded by the first paragraph of the proof. Thus blocks are simply-connected. □

We will need to know that the class of words of type k is closed under taking minimal representatives in the group G. Specifically, we have the following.

Proposition 3.11. *Let w be a word of type k. Let v be the shortest word in the alphabet $\{a, b\}$ representing the same element of the group G as w. Then v is cyclically equal to a word of type k.*

Proof. Consider all van Kampen diagrams over the presentation (10) whose boundary label is uv^{-1} with u a word of type k. (Such diagrams exist, for instance a diagram corresponding to the equality $w = v$ in G.) Among all such diagrams we choose one Δ with the smallest number of blocks.

Let uv^{-1} be the boundary label of Δ. It clearly suffices to prove that Δ does not contain 2-cells, since then v and u (which is of type k) are cyclically equal.

Suppose then that Δ contains 2-cells; we can assume that there are no edges not incident to any 2-cell. By Proposition 3.10 and Theorem 2.3 there is a block $B \subset \mathcal{B}(\Delta)$ and a path $p \subset \partial_{red} B \cap \partial_{red}\Delta$ with $|p| > \frac{7}{10}|\partial_{red} B|$.

As the boundary $\partial_{red}\Delta$ is the cyclic reduction of uv^{-1}, it can be decomposed into two segments q_u, q_v contained in u, v respectively.

Assume now that p and q_u have a common subpath of length at least $\frac{1}{10}|\partial_{red} B|$ or that q_u is contained in p. Then the fragment of the proof of Proposition 3.10

following (11) shows that p contains a side s of B. Hence B is of type k. Now remove the block B and the side s from Δ but keep the remaining part of ∂B. The result is a van Kampen diagram with fewer blocks and corresponding to the equality in G of v to a word of type k, which contradicts the choice of Δ.

The conclusion is that at every end of p there is a path of length at most $\frac{1}{10}|\partial_{red}B|$ common with q_u and, consequently, $p \cap q_v$ is a path of length greater than $\frac{1}{2}|\partial_{red}B|$. By replacing this path with the remaining shorter part of $\partial_{red}B$ and removing B from Δ we get a van Kampen diagram expressing the equality in G of u with a word shorter than v. This contradicts the minimality of v and the proof is complete. □

We are ready to prove Lemmas 3.3, 3.4, 3.5.

Proof of Lemma 3.3. Let $a_n \in A_n$ be the points used in Definition 3.1 of the limit $\lim_\omega(A_n, a_n)$ containing the metric space we wish to embed into $\mathrm{Cone}_\omega(G, c)$. Define the required maps $\psi_n : A_n \to G$ in the following way: for each $x \in A_n$ let $\psi_n(x) = \varphi(e_1)\varphi(e_2)\dots\varphi(e_m)$, where $e_1e_2\dots e_m$ is any path in Γ_n from a_n to x. This is clearly independent of the choice of the path since the boundary labels of all triangles are trivial in G. Now, the inequality

$$d_G(\psi_n(x), \psi_n(y)) \leq c_n \cdot d_{A_n}(x, y) \tag{12}$$

follows directly from the definition of φ. On the other hand, the cyclic reduction of a word of type n preserves at least $1 - 2\lambda_{m_n}$ of every subword of the form $\varphi(e)$ for $e \in E(\Gamma_n)$, where λ_{m_n} is as in Theorem 2.4(3). Hence Proposition 3.11 gives

$$d_G(\psi_n(x), \psi_n(y)) \geq c_n(1 - 2\lambda_{m_n}) \cdot d_{A_n}(x, y). \tag{13}$$

The assertion follows from (12), (13), and Theorem 2.4(3). □

In order to verify hyperbolicity of the groups G_n we recall the following isoperimetric notion:

Definition 3.12. *A group given by a finite presentation is said to satisfy a* linear isoperimetric inequality *if there exists a constant C with the following property: for every word w equal to the identity of the group there exists a van Kampen diagram over the presentation with boundary label w and at most $C \cdot |w|$ two-cells.*

It is known ([10], Theorem 2.5) that a finitely presented group is hyperbolic if and only if it satisfies a linear isoperimetric inequality.

Proof of Lemma 3.4. Note first that if $f_1f_2\dots f_r$ is a cycle in Γ_k, then one may construct a simply-connected block of type k with $r-2$ cells and boundary label $\varphi(f_1)\varphi(f_2)\dots\varphi(f_r)$. The existence of such a block is a straightforward induction based on the fact that $r = 3$ gives rise to a single cell.

Suppose that $w = \varphi(e_1)\varphi(e_2)\dots\varphi(e_m)$ is the boundary label of a block B of type $k \leq n$. If $m > k$, then some *proper* subpath $\gamma = e_se_{s+1}\dots e_{t-1}e_t$ is a cycle in Γ_k and the element $\varphi(e_s)\varphi(e_{s+1})\dots\varphi(e_{t-1})\varphi(e_t)$ is trivial in G_n. But the label of ∂B is also trivial in G_n, so the same holds for the complementary cycle $\gamma' = e_{t+1}e_{t+2}\dots e_{n-1}e_ne_1e_2\dots e_{s-2}e_{s-1}$. Now we can form two blocks B_1, B_2 of type k whose boundary labels are the labels of γ, γ', respectively, and join B_1, B_2 in a single vertex to obtain a diagram with boundary label corresponding to the original cycle $e_1e_2\dots e_m$.

Continuing this process we see that a block of type k can be trasformed, without affecting the boundary label, into some sort of 'wedge' of blocks of type k, each containing at most $|A_k|$ cells. The resulting diagram (which is no longer a single block) will be called the *refinement* of the original block.

There are only finitely many blocks of type k with at most $|A_k|$ cells, so there is an isoperimetric constant C such that for all these blocks B we have $\frac{|B|}{|\partial_{red}B|} \leq C$, where $|B|$ is the number of cells in B. The construction of the refinement of a block shows that this inequality is valid whenever B is a refinement.

We are now in a position to prove a linear isoperimetric inequality for G_n. Consider a van Kampen diagram Δ with boundary label w. Replace every block of $\mathcal{B}(\Delta)$ with its refinement to obtain a new block-diagram $\mathcal{B}_{ref}(\Delta)$. Now Proposition 3.10 and Theorem 2.3 guarantee the existence of a refinement $P \subset \mathcal{B}_{ref}(\Delta)$ with a subpath of ∂P of length greater than $\frac{7}{10}|\partial_{red}P|$ which is contained in $\partial\Delta$. The remaining part of ∂P is of length smaller than $\frac{3}{10}|\partial_{red}P|$. Hence the result of the removal of P from $\mathcal{B}_{ref}(\Delta)$ is the decrease in $|\partial_{red}\mathcal{B}_{ref}(\Delta)|$ by more than $\frac{4}{10}|\partial_{red}B|$. In other words, $|\partial_{red}P|$ is smaller than $\frac{5}{2}$ of the decrease in $|\partial_{red}\mathcal{B}_{ref}(\Delta)|$. But $\frac{|P|}{|\partial_{red}P|} \leq C$, so the ratio of the decrease in the number of blocks in $\mathcal{B}_{ref}(\Delta)$ to the decrease in the length $|\partial_{red}\mathcal{B}_{ref}(\Delta)|$ is at most $\frac{5}{2}C$. Continuing to remove consecutive refinements we infer that $\mathcal{B}_{ref}(\Delta)$ contains at most $\frac{5}{2}C \cdot |w|$ cells, which proves a linear isoperimetric inequality for G_n. □

Proof of Lemma 3.5. The content here is that a word w with $|w| \leq \frac{1}{8}m_{n+1}$ which is the shortest representative in G_n with respect to $\{a, b\}$ is also shortest in G.

Suppose towards a contradiction that the element represented by w in G has a shorter representative v. Hence wv^{-1} is trivial in G and non-trivial in G_n. Consider a van Kampen diagram Δ over the presentation (10) corresponding to the triviality of wv^{-1} in G; obviously Δ contains a block of type greater than n.

By Proposition 3.10 Δ contains a block B together with a path $p \subset \partial_{red}B \cap \partial_{red}\Delta$ such that $|p| > \frac{1}{2}|\partial_{red}B|$. In particular, we have $|\partial_{red}\Delta| > \frac{1}{2}|\partial_{red}B|$. If we remove B from Δ, we obtain a smaller diagram with shorter boundary. By removing consecutive blocks we conclude that $|\partial_{red}\Delta| > \frac{1}{2}|\partial B|$ for *all* blocks $B \subset \mathcal{B}(\Delta)$. But $|\partial_{red}\Delta| \leq |wv^{-1}| < 2 \cdot |w|$ and thus

$$|\partial_{red}B| < 4 \cdot |w| \leq \tfrac{1}{2}m_{n+1} \qquad \text{for all blocks } B \subset \mathcal{B}(\Delta). \tag{14}$$

Observe however that the reduced boundary of a block of type at least $n+1$ is of length at least $\frac{98}{100}m_{n+1}$. Hence (14) prevents Δ from containing blocks of type greater than n, a contradiction. □

4. Concluding remarks

The main result of this paper leads to a series of natural questions. It is clear that our methods cannot produce *finitely presented* groups and it would be of interest to investigate the behavior of different asymptotic cones in the class of finitely presented groups. We cannot demand one of the cones to be an $\mathbb{R}$-tree since a finitely presented group whose one asymptotic cone is an $\mathbb{R}$-tree must be hyperbolic [8] so that all asymptotic cones are $\mathbb{R}$-trees. Note also that there are only two known examples of finitely presented groups with non-homeomorphic asymptotic cones [9], [5], and [5] actually depends on set theory. Thus we ask the following.

Question 4.1. *Does there exist a finitely presented group that has two asymptotic cones which are not locally homeomorphic?*

In particular, can one obtain asymptotic cones of different covering dimensions?

On the other hand, asymptotic cones of lacunary groups are their only asymptotic invariant which is reasonably well-understood: they can exhibit arbitrarily complicated local topological properties but this is entirely due to the nature of the sets R_n used in the construction. If each of these sets consists of a single relation, then all asymptotic cones are locally homeomorphic to $\mathbb{R}$-trees (Chapter 4 of [8]). Other asymptotic invariants of groups of this type are unknown. In particular:

Question 4.2. *Is the asymptotic dimension* [4] *of generalized small cancellation lacunary groups finite? Do they admit a uniform embedding into Hilbert space?*

The question is open even for ordinary infinitely presented small cancellation groups. Note that our groups are of cohomological dimension 2 and in fact aspherical, for exactly the same reason as ordinary small cancellation groups (see [6], Chapter III.11), so this is a special case of the general problem of the equality of cohomological and asymptotic dimension. Note also that random groups of Gromov [7] admitting no uniform embeddings into Hilbert space arise in a related process of small cancellation on graphs. This makes our question even more interesting.

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