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Hyperbolic quotients of Coxeter groups

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HYPERBOLIC QUOTIENTS OF COXETER GROUPS

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Abstract. A conjecture of Januszkiewicz states that every non-affine Coxeter group admits a non-elementary word-hyperbolic quotient. In this paper we supply affirmative evidence for this conjecture in a few special cases.

1. Introduction

Coxeter groups are finitely generated groups defined by a presentation of the following form:

$$\langle \{s_i\}_{i\in I} | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle,$$

with *Coxeter exponents* $m_{ij} = m_{ji} \in \{2, 3, 4, \ldots, \infty\}$ for all distinct indices $i, j \in I$, where an infinite exponent m_{ij} means that there is no corresponding relation between the generators s_i and s_j. Note that $m_{ij} = 2$ if and only if s_i and s_j commute. The interest in these groups has a long history (see [4]) and the classification of affine, i.e. finite and Euclidean, Coxeter groups is well-known. However, the world of the non-affine ones happens to be more mysterious.

From a geometric point of view, Coxeter groups form an important class of *CAT(0) groups*: they act properly and cocompactly on non-positively curved spaces. This fundamental result is due to Moussong [9], who also gave a necessary and sufficient condition for a Coxeter group to act properly and cocompactly on a negatively curved space, as well as to be word-hyperbolic. Word-hyperbolicity is a geometric property of a group resembling negative curvature, but — conjecturally — not implying the existence of an action on a negatively curved space.

In this context hyperbolic behavior of those non-affine Coxeter groups that are not word-hyperbolic themselves comes up naturally. Hence, Januszkiewicz [5] asked the following question: Can every non-affine Coxeter group be mapped onto a non-elementary hyperbolic group?

By a result of Margulis and Vinberg [8] every non-affine Coxeter group is *large*, that is, a finite index subgroup admits a non-abelian free quotient. Since virtually free groups are word-hyperbolic, a natural point to start is to determine which Coxeter groups admit virtually free quotients. In this direction we show the following.

Theorem 1.1. *A non-affine Coxeter group admits a virtually free non-abelian quotient if and only if there exists an exponent $m_{ij} = \infty$ and moreover not all of the numbers m_{ik}, m_{jk} for $k \in I \setminus \{i, j\}$ are equal to 2.*

Note that if $m_{ij} = \infty$, then the elements s_i and s_j generate an infinite dihedral group D_∞. Then the additional condition from Theorem 1.1 simply states that the Coxeter group is not a direct product of this D_∞ subgroup and the subgroup generated by $\{s_k\}_{k\in I\setminus\{i,j\}}$.

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We are also able to work out the following example.

Theorem 1.2. *Every Coxeter group with at least* 4 *generators and all exponents* $m_{ij} = 3$ *for* $i \neq j$ *admits a non-elementary hyperbolic quotient.*

The construction is actually quite natural. The proof of hyperbolicity unexpectedly relies on simplicial non-positive curvature theory developed by Januszkiewicz and Świątkowski [6]. However, the potential of this construction is certainly not fully exploited here. We will speculate on a wider class of groups where the method might work and discuss possible generalizations of the method.

2. The virtually free case

The purpose of this section is to prove Theorem 1.1. In one direction the proof easily follows from Bass-Serre theory of group actions on trees [11]. In the other the essence is contained in a separability result from [3].

Proposition 2.1. *Suppose that there exists a surjection from a Coxeter group* W *onto a virtually free non-abelian group* V*. Then there exist three distinct indices* $i, j, k \in I$ *such that* $m_{ij} = \infty$ *and* $m_{ik} > 2$*.*

Proof. The virtually free group V admits an action on a simplicial tree T with finite vertex stabilizers [11]. Since V is infinite, this action has no fixed points. Thus a surjection $W \to V$ produces an action of the Coxeter group W on the tree T without fixed points.

Let $T_i = \{x \in T : s_i x = x\}$ for each $i \in I$. This is the set of fixed points of s_i acting as an automorphism of the tree T. It follows that T_i is a subtree of T.

Suppose that $m_{ij} < \infty$ for all pairs of indices $i, j \in I$. Then the generators s_i, s_j are subject to the relations $s_i^2 = s_j^2 = (s_i s_j)^{m_{ij}} = 1$ and generate a finite dihedral group. A finite group acting on a tree always has a fixed point [11], for example the circumcenter of any orbit of the action. Hence there exists $x \in T$ such that $s_i x = s_j x = x$. In other words, $T_i \cap T_j \neq \emptyset$. Therefore $\{T_i\}_{\in I}$ is a family of subtrees of T with pairwise non-empty intersection. Such a family has non-empty intersection ([11], Exercise 3, p. 66). But a point $x \in \bigcap_{i \in I} T_i$ satisfies $s_i x = x$ for each $i \in I$, and since the elements $\{s_i\}_{i \in I}$ generate the group W, we obtain $wx = x$ for all $w \in W$. This contradicts the fact that the action of W on the tree T has no fixed points.

Hence $m_{ij} = \infty$ for some $i, j \in I$. If there exists $k \in I \setminus \{i, j\}$ such that $m_{ik} > 2$ or $m_{jk} > 2$, then the proof is finished. On the other hand, if no such k exists, then the infinite dihedral subgroup D_∞ generated by s_i, s_j and the subgroup W' generated by $\{s_k\}_{k \in I \setminus \{i,j\}}$ commute. (Note that W' is a Coxeter group in its own right, with Coxeter exponents inherited from W, see Theorem 4.1.6(i) in [4].) Thus we get a surjection $\varphi : D_\infty \times W' \to V$. Then $\varphi(W') \subset V$ is virtually free since V is. Also $\varphi(W')$ cannot be elementary (finite or virtually cyclic), otherwise $\varphi(D_\infty)$ and $\varphi(W')$ would be two commuting elementary subgroups generating a virtually free non-abelian group, which is impossible. It follows that φ induces a surjection of W' onto a virtually free non-abelian subgroup of V. By the first part of the proof, there exists a pair of generators of W' with an infinite Coxeter exponent. This gives either a triple of indices as required in the statement of the Proposition or another D_∞ factor. But the process of separating D_∞ factors terminates since at each step we reduce the number of Coxeter generators by two. Thus we will finally find a D_∞ subgroup not commuting with all other generators, and the proof is complete. □

Now we proceed to the construction of a virtually free quotient. A *special subgroup* of a Coxeter group W with Coxeter generators $\{s_i\}_{i\in I}$ is any subgroup W_J generated by $\{s_i\}_{i\in J}$ for some subset $J \subset I$. As we have already mentioned in the proof of Proposition 2.1, W_J is itself a Coxeter group. It is our goal now to show that W_J is a well-behaved subgroup of W.

Recall that a subgroup $H \subset G$ is *separable* in G if for every element $g \in G \setminus H$ there exists a homomorphism $\varphi : G \to F$ where F is a finite group, such that $\varphi(g) \notin \varphi(H)$.

Lemma 2.2 ([3], Theorem 1.4)**.** *A special subgroup W_J of a Coxeter group W is separable in W.*

Proof. The idea is to prove that W_J is the fixed point set of an automorphism of a residually finite group.

Let W' be an isomorphic copy of W, with Coxeter generators $\{s'_i\}_{i\in I}$ in bijective correspondence with the generators $\{s_i\}_{i\in I}$ of W. The special subgroup $W'_J \subset W'$ generated by $\{s'_i\}_{i\in J}$ is then isomorphic to W_J. Now we can form the following free product with amalgamation: $K = W *_\psi W'$, where $\psi : W_J \to W'_J$ is the isomorphism defined by the condition $\psi(s_i) = s'_i$ for all $i \in J$. The group K is a Coxeter group, with the Coxeter generating set $\{s_i\}_{i\in I} \cup \{s'_i\}_{i\in I\setminus J}$. We will however still use the symbol s'_i for all $i \in I$, even though as an element of K it is equal to s_i for $i \in J$. The fact that K is really a Coxeter group follows from the presentation of a free product with amalgamation, indeed, the Coxeter exponents of K are: m_{ij} for any pair s_i, s_j or s'_i, s'_j with $i, j \in I$, and ∞ for any pair s_i, s'_j with $i, j \in I \setminus J$.

Note that there exists a homomorphism $\varphi : K \to K$ such that $\varphi(s_i) = s'_i$ and $\varphi(s'_i) = s_i$ for $i \in I$. The existence of φ follows from the evident fact that the Coxeter exponent of any pair of generators is equal to the Coxeter exponent of their images. Moreover, φ is actually an automorphism since $\varphi \circ \varphi = id_K$. Observe finally that the fixed point set $\{x \in K : \varphi(x) = x\}$ is equal to the amalgamated subgroup $W_J = W'_J$ of K. Indeed, all elements of this subgroup are obviously fixed by φ. On the other hand, any other element of K is of the form $x = z_1 z_2 \ldots z_n$ where $z_1, z_2, \ldots, z_n$ alternately belong to $W \setminus W_J, W' \setminus W'_J$. And since it is clear from the definition of φ that $\varphi(W \setminus W_J) = W' \setminus W'_J$ and $\varphi(W' \setminus W'_J) = W \setminus W_J$, we conclude that

$$x\varphi(x)^{-1} = z_1 z_2 \ldots z_{n-1} z_n \varphi(z_n)^{-1} \varphi(z_{n-1})^{-1} \ldots \varphi(z_2)^{-1} \varphi(z_1),$$

where the $2n$ factors on the right hand side belong alternately to $W \setminus W_J, W' \setminus W'_J$. Therefore, applying the normal form theorem for free products with amalgamation ([7], Theorem IV.2.6) to the element $x\varphi(x)^{-1}$ we see that it is nontrivial in K and hence $\varphi(x) \neq x$.

To sum up, we see that W_J is the fixed point set of the endomorphism φ of K.

We are now going to demonstrate that W_J is separable in K (and therefore in W since W is a subgroup of K containing W_J). To this end, recall that, by a theorem of Tits ([4], Corollary 6.12.11), Coxeter groups are linear and consequently residually finite. Take any element $g \in K \setminus W_J$. Then $g\varphi(g)^{-1} \neq 1$ in K. By residual finiteness of K there exists a homomorphism $\Phi : K \to F$ where F is a finite group, such that

$$\Phi(g\varphi(g)^{-1}) = \Phi(g)\Phi(\varphi(g))^{-1}$$

is nontrivial in F, which means that $\Phi(g) \neq \Phi(\varphi(g))$ in F. Now define a homomorphism $\beta : K \to F \times F$ by the formula

$$\beta(x) = (\Phi(x), \Phi(\varphi(x)))$$

for all $x \in K$. Then the element $\beta(g) = (\Phi(g), \Phi(\varphi(g)))$ does not belong to the diagonal subgroup

$$\Delta = \{(x, x) : x \in F\} \subset F \times F.$$

On the other hand, for every $x \in W_J$ we have $x = \varphi(x)$ and hence, by the definition of β, the element $\beta(x)$ belongs to Δ. This shows that

$$\beta(W_J) \subset \Delta \qquad \text{and} \qquad \beta(g) \notin \Delta.$$

As $g \in K \setminus W_J$ was arbitrary, the proof of separability is complete. □

Proposition 2.3. *Suppose that, for a Coxeter group W, there exist three distinct indices $i, j, k \in I$ such that $m_{ij} = \infty$ and $m_{ik} > 2$. Then W admits a surjection onto a virtually free non-abelian group.*

Proof. Define W_1, W_2, W' as the special subgroups of W generated by the sets $\{s_\ell\}_{\ell \in I \setminus \{j\}}, \{s_\ell\}_{\ell \in I \setminus \{i\}}, \{s_\ell\}_{\ell \in I \setminus \{i,j\}}$, respectively. Then W' is a common subgroup of W_1 and W_2. Moreover, we have a decomposition into a free product with amalgamation: $W = W_1 *_{W'} W_2$. Indeed, the presentation for such a product comes from a presentation of W_1 by adding one generator (corresponding to s_j) and those relations of W_2 in which at least one generator does not belong to W'. But the only such relations are the ones involving s_j and omitting s_i. It follows that the product $W_1 *_{W'} W_2$ recovers all Coxeter relations of W, except for the relation between s_i and s_j. However, we have $m_{ij} = \infty$, so there is no such relation.

Note now that the assumption $m_{ik} > 2$ guarantees that the index $[W_1 : W']$ is greater than 2. To see this, observe that for any group G we have

$$[W_1 \cap G : W' \cap G] \leq [W_1 : W']$$

(this is easily seen by intersecting all cosets of W' in W_1 by the subgroup G — the left hand side is equal to the number of nonempty intersections). Let G be the special subgroup generated by s_i, s_k. It is a finite dihedral group of order $2m_{ik}$. Moreover, $W_1 \cap G = G$ and $W' \cap G = \{1, s_k\}$ since the intersection of special subgroups is the special subgroup corresponding to the intersection of the generating sets (Theorem 4.1.6(ii) in [4]). Thus $[W_1 \cap G : W' \cap G] = m_{ik}$ and the inequality $[W_1 : W'] \geq m_{ik} > 2$ follows.

Similarly $[W_2 : W'] > 1$, just because $W' \neq W_2$: for example $s_j \in W_2 \setminus W'$.

We claim that the relations $[W_1 : W'] > 2$ and $[W_2 : W'] > 1$ can be preserved after passing to a finite quotient. Let us make this statement more precise. Since $[W_1 : W'] > 2$, there exist elements $g, h \in W_1 \setminus W'$ such that $g^{-1}h \notin W'$. By Proposition 2.2 applied to the special subgroup $W' \subset W$ there exist homomorphisms $\beta_i : W \to F_i$ $(i = 1, 2, 3, 4)$ where F_i is finite, such that

$$\beta_1(g) \notin \beta_1(W'), \quad \beta_2(h) \notin \beta_2(W'), \quad \beta_3(g^{-1}h) \notin \beta_3(W'), \quad \beta_4(s_j) \notin \beta_4(W').$$

Then the product $\beta = \beta_1 \times \beta_2 \times \beta_3 \times \beta_4 : W \to F = F_1 \times F_2 \times F_3 \times F_4$ is a homomorphism onto a finite group with the following property:

$$\beta(g), \beta(h), \beta(g^{-1}h), \beta(s_j) \notin \beta(W').$$

Since the elements $\beta(g), \beta(h)$ and $\beta(g^{-1}h) = \beta(g)^{-1}\beta(h)$ do not belong to the subgroup $\beta(W') \subset F$, the cosets $\beta(W'), \beta(g)\beta(W'), \beta(h)\beta(W')$ are pairwise distinct. As $g, h \in W_1$, these cosets are contained in $\beta(W_1)$. Thus $[\beta(W_1) : \beta(W')] > 2$. In the same way we prove that $[\beta(W_2) : \beta(W')] > 1$.

The homomorphism $\beta : W \to F$ induces a surjection

$$W = W_1 *_{W'} W_2 \to \beta(W_1) *_{\beta(W')} \beta(W_0).$$

And finally, a free product with amalgamation $G *_K H$, where G, H are finite and $[G : K] > 2, [H : K] > 1$, is a virtually free non-abelian group [11] (such a product acts properly and cocompactly on the Bass-Serre tree with vertex degrees equal to $[G : K]$ and $[H : K]$ so that the tree is not a line, hence the product is virtually free and non-elementary). This finishes the proof. □

Theorem 1.1 follows now from Proposition 2.1 and Proposition 2.3. Note that a strong rigidity is exhibited here: *every* non-affine Coxeter group has a virtual homomorphism onto a non-abelian free group, but this virtual homomorphism can be lifted to an actual homomorphism only in very specific cases.

3. The groups $\langle \{s_i\}_{i\in I} | s_i^2, (s_i s_j)^3 \rangle$

In this section we study Coxeter groups with at least 4 generators and all Coxeter exponents equal to 3. Such groups admit a surjective homomorphism onto the group

$$W = \langle r_1, r_2, r_3, r_4 | r_1^2, r_2^2, r_3^2, r_4^2, (r_1r_2)^3, (r_1r_3)^3, (r_1r_4)^3, (r_2r_3)^3, (r_2r_4)^3, (r_3r_4)^3 \rangle$$

with exactly 4 generators, indeed, any surjective map π from the set $\{s_i\}_{i\in I}$ onto $\{r_1, r_2, r_3, r_4\}$ extends to such a homomorphism, since the relations $\pi(s_i)^2 = 1$ and $(\pi(s_i)\pi(s_j))^3 = 1$ clearly hold in the image for all $i, j \in I$, whether $\pi(s_i) = \pi(s_j)$ or not.

We will construct a word-hyperbolic quotient of W as a simple tetrahedron of finite groups admitting a systolic development without flats. Groups of this type have already been studied in [12]. Then we will appeal to a result of Przytycki [10] that groups with an action on a systolic complex with no flats are word-hyperbolic.

3.1. The toroidal group.

Consider the following Coxeter group:

$$E = \langle r, s, t | r^2, s^2, t^2, (rs)^3, (rt)^3, (st)^3 \rangle.$$

This is a well-known Euclidean triangular group. It acts properly and cocompactly on $\mathbb{R}^2$ in such a way that r, s, t are reflections across the three lines L_r, L_s, L_t containing the sides of an equilateral triangle. Note that E preserves a tessellation of the plane by congruent equilateral triangles.

The elements r, s preserve the intersection point $L_r \cap L_s$ and generate a dihedral subgroup of order 6. There are exactly three reflections in this subgroup: r, s, and $rsr = srs$. This last element is a reflection across the line passing through the point $L_r \cap L_s$ and parallel to the line L_t. It follows that $rsrt = srst$ is a composition of two reflections across parallel lines, and hence is a translation (by a vector perpendicular to these lines, and of length twice the distance between these lines). In this case the vector of the translation is perpendicular to L_t, its direction is from the point $L_r \cap L_s$ to the outside of the triangle, and its length is two times the altitude of the equilateral triangles in the tessellation.

By analogy there are corresponding properties for the translations $rtrs = trts$ and $stsr = tstr$. Note that the three vectors of these translations add up to zero.

Now let ξ_r, ξ_s, ξ_t be the vectors of the translations $(stsr)^4, (rtrs)^4, (rsrt)^4$, respectively. These are the three vectors considered before multiplied by 4. Thus $\xi_r + \xi_s + \xi_t = 0$.

Let N be the subgroup of E generated by the translations by ξ_r, ξ_s, ξ_t. In view of the equality from the previous paragraph, any two of these translations already generate N. Note also that any two of these vectors are linearly independent, and an isomorphism $N \cong \mathbb{Z}^2$ follows. It is however convenient to keep all three generators due to symmetry.

Proposition 3.1. *N is a normal subgroup of E.*

Proof. It is sufficient to show that a conjugation of a generator of N by a generator of E still belongs to the subgroup N. Specifically, we have to prove that the translation by ξ_i conjugated by the reflection across the line L_j, where $i, j \in \{r, s, t\}$, is an element of N. There are two essentially distinct cases to consider: the angle between L_j and ξ_i is either right or $\frac{1}{6}\pi$.

In the right angle case, we can take ξ_r and L_r. We need to compute the composition of: the reflection across L_r, the translation by ξ_r, and the reflection across L_r. Clearly it is the translation by $-\xi_r$, which belongs to N.

In the $\frac{1}{6}\pi$ case, we can take ξ_r and L_s. Write $\xi_r = (\xi_r + \frac{1}{2}\xi_s) - \frac{1}{2}\xi_s$. Then we need to compute the composition of: the reflection across L_s, the translation by $\xi_r + \frac{1}{2}\xi_s$, the translation by $-\frac{1}{2}\xi_s$, the reflection across L_s. The vector $\xi_r + \frac{1}{2}\xi_s$ is parallel to L_s and thus the translation by this vector commutes with the reflection across L_s (and also with any translation). Hence the composition in question is: the reflection across L_s, the translation by $-\frac{1}{2}\xi_s$, the reflection across L_s, the translation by $\xi_r + \frac{1}{2}\xi_s$. The composition of the first three maps is equal to the translation by $\frac{1}{2}\xi_s$ — exactly as in the previous paragraph. Therefore the total composition is the translation by $\xi_r + \xi_s$, and so belongs to N. □

Note also that N is a finite index subgroup of E. Indeed, E acts properly and cocompactly on $\mathbb{R}^2$, so the condition $[E : N] < \infty$ is equivalent to the fact that the action of N has a compact fundamental domain. But N is generated by two translations by linearly independent vectors, and a parallelogram spanned by them is clearly such a domain.

Now we are going to explain that the finite group $G = E/N$ is a geometric reflection group on a torus. Hence the name 'the toroidal group'.

As we already mentioned, N is the group of translations by vectors from the set $\{m\xi_r + n\xi_s : m, n \in \mathbb{Z}\}$. Therefore it acts freely on $\mathbb{R}^2$ and the quotient $\mathbb{R}^2/N$ is a square with opposite sides identified in the same direction, i.e. a topological torus. Moreover, N preserves the tessellation of $\mathbb{R}^2$ by equilateral triangles, sending vertices to vertices, edges to edges and triangles to triangles. Hence N induces a triangulation of the torus $\mathbb{R}^2/N$.

It remains to show that the group $G = E/N$ acts on this torus. This is equivalent to saying that the action of E on $\mathbb{R}^2$ is compatible with the projection of $\mathbb{R}^2$ onto the torus $\mathbb{R}^2/N$, i.e. that the action of E sends N-orbits to N-orbits. But this follows from Proposition 3.1. Indeed, let Nx be an orbit of the action of N in $\mathbb{R}^2$ an $g \in E$ be any element. We claim that $g(Nx)$ is also an N-orbit (obviously the orbit of gx); we thus have to show that gnx belongs to the N-orbit of gx, or that $gnx = n'gx$ for some $n' \in N$. But the existence of an $n' \in N$ such that $gn = n'g$

is a consequence of the normality of N. This proves that $g(Nx) \subset Ngx$ and the opposite inclusion follows by considering g^{-1} in place of g.

To sum up, G is a finite group of triangulation-preserving simplicial automorphisms of a torus. We can think of generators of G as 'reflections' induced by r, s, t, although one has to keep in mind that on a torus the fixed point set of such a 'reflection' consists of two disjoint circles.

Observe also that the generators of N allow to write down the following presentation of G:

$$G = \left\langle r, s, t \;\middle|\; \begin{array}{l} r^2, s^2, t^2, (rs)^3, (rt)^3, (st)^3, \\ (rsrt)^4, (srst)^4, (rtrs)^4, (trts)^4, (stsr)^4, (tstr)^4 \end{array} \right\rangle$$

and this presentation is symmetric with respect to r, s, t.

3.2. **The systolic construction.** Let us recall a few notions from the theory of simpicial non-positive curvature [6].

(1) The *link* of a vertex v in a simplicial complex Σ is the subcomplex of Σ spanned by all simplices $\sigma \subset \Sigma$ such that the join $\sigma * v$ is a simplex of Σ.
(2) A flag simplicial complex Σ is 6-*large* if for any closed combinatorial path γ in the 1-skeleton of Σ of length 4 or 5 there exist two non-consecutive vertices of γ joined by an edge in Σ.
(3) A simplicial complex is *locally* 6-*large* if the link of each vertex is 6-large.
(4) A *systolic complex* is a connected simply-connected locally 6-large simplicial complex.

Recall also that a *simple tetrahedron of groups* is an object defined in the following way. We assign a so-called local group Γ_σ to every face σ of the tetrahedron (i.e. every vertex, edge, triangle and the tetrahedron itself), together with an injective homomorphism $\psi_{\sigma_1\sigma_2} : \Gamma_{\sigma_1} \to \Gamma_{\sigma_2}$ whenever σ_2 is a proper face of σ_1, such that the compatibility conditions are met: $\psi_{\sigma_1\sigma_2}\psi_{\sigma_2\sigma_3} = \psi_{\sigma_1\sigma_3}$ if σ_3 is a face of σ_2 which in turn is a face of σ_1.

Associated to a simple tetrahedron of groups is the *fundamental group of the tetrahedron.* This is the free product of the groups Γ_σ for all faces σ divided by the normal subgroup generated by the relations $x = \psi_{\sigma_1\sigma_2}(x)$ for all $x \in \Gamma_{\sigma_1}$ and all pairs σ_1, σ_2 such that σ_2 is a proper face of σ_1. A simple tetrahedron of groups is called *developable* if there is an action of its fundamental group on a connected 3-dimensional simplicial complex (called the *development*) such that some 3-simplex is a fundamental domain of the action, and moreover the stabilizers of faces of this 3-simplex coincide with the groups Γ_σ assigned to these faces and the inclusions of stabilizers of faces into stabilizers of smaller faces coincide with the homomorphisms $\psi_{\sigma_1\sigma_2}$.

It is a fact that a simple tetrahedron of groups is developable if and only if, for each simplex σ of the tetrahedron, the natural homomorphism from the group Γ_σ to the fundamental group of the tetrahedron is injective ([1], Theorem II.12.18).

Now we define the following simple tetrahedron of groups $\mathcal{C}$. Label the vertices by $1, 2, 3, 4$ and let the local groups be:

- for the tetrahedron — the trivial group,
- for the triangle opposite to the vertex i — the group of order 2 generated by h_i,
- for the edge with endpoints i and j — the dihedral group of order 6 generated by h_k and h_ℓ, where $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$,

- for the vertex i — the group isomorphic to the previously defined group G with generators from the set $\{h_1, h_2, h_3, h_4\} \setminus \{h_i\}$ in place of r, s, t.

Note that we have used the same generators for the face, edge and vertex groups to indicate the inclusion homomorphisms between these local groups. In this way every triangle group is a naturally a subgroup of all three relevant edge groups and every edge group is naturally a subgroup of all two relevant vertex groups, hence the compatibility conditions are evidently satisfied.

The fundamental group of the tetrahedron $\mathcal{C}$ has the following presentation

$$H = \left\langle h_1, h_2, h_3, h_4 \;\middle|\; \begin{array}{l} h_i^2 \quad \text{for } i = 1,2,3,4, \\ (h_i h_j)^3 \quad \text{for all distinct } i,j = 1,2,3,4, \\ (h_i h_j h_i h_k)^4 \text{ for all distinct } i,j,k = 1,2,3,4 \end{array} \right\rangle .$$

Clearly the group H is a quotient of W since all relations between generators of W are satisfied in H. Therefore in order to prove Theorem 1.2 we have to show that H is a non-elementary word-hyperbolic group.

To this end, we will first prove that $\mathcal{C}$ is developable. Note first that for a tetrahedron of groups we may construct *local developments*. The local development $\mathcal{D}_v$ of $\mathcal{C}$ at a vertex v is defined as follows (see [6], Section 17): it is a simple triangle of groups in which the vertex groups are the edge groups of $\mathcal{C}$ for edges containing v, the edge groups are the triangle groups of $\mathcal{C}$ for triangles containing v, and the triangle group is the tetrahedron group of $\mathcal{C}$ (and, of course, the inclusions of edges, triangles and the tetrahedron in $\mathcal{C}$ translate to the inclusions of vertices, edges and the triangle in $\mathcal{D}_v$). Similarly we can define local development of $\mathcal{C}$ at edges.

We can examine whether the local developments at vertices (which are $\mathcal{D}_v$) and at edges are developable. If they happen to be, and the developments are 6-large simplicial complexes, then we say that the tetrahedron of groups $\mathcal{C}$ is *locally* 6*-large*.

Proposition 3.2. *$\mathcal{C}$ is locally* 6*-large.*

Proof. First we have to prove that the complexes $\mathcal{D}_v$ are developable and their developments are 6-large.

Note that, from the definition, $\mathcal{D}_v$ is a triangle of groups with the following local groups: the triangle group is trivial, the edge groups are of order 2 generated by some elements g_1, g_2, g_3, and the vertex groups are dihedral of order 6 generated by pairs g_1, g_2; g_1, g_3; g_2, g_3 (here, as before, we deliberately use the same generators for edge and vertex groups to encode the inclusion homomorphisms). Then the fundamental group of $\mathcal{D}_v$ is just the Coxeter group

$$\langle g_1, g_2, g_3 | g_1^2, g_2^2, g_3^2, (g_1g_2)^3, (g_1g_3)^3, (g_2g_3)^3 \rangle .$$

It follows that $\mathcal{D}_v$ is developable, the development being the plane $\mathbb{R}^2$ and the action equal to the standard Coxeter action as described in the beginning of the section. And it is clear that the plane tessellated by equilateral triangles is 6-large.

Similarly we treat developments of $\mathcal{C}$ at edges. We get the real line with the action of the group $\mathbb{Z}_2 * \mathbb{Z}_2$, so developability and 6-largeness hold as well. □

Having shown that $\mathcal{C}$ is locally 6-large we now apply Theorem 6.1 from [6]. It states that a locally 6-large simplex of groups is developable. Denote the development of $\mathcal{C}$ by Σ. Hence there is an action of the fundamental group H on Σ and there is a fundamental 3-simplex in Σ such that the stabilizers of faces of this simplex form a tetrahedron of groups coinciding with $\mathcal{C}$.

Proposition 3.3. *The simplicial complex Σ is systolic.*

Proof. The assertion consists of two things: that Σ is locally 6-large and that is connected and simply-connected.

In order to prove that Σ is locally 6-large note first that the link at each vertex of Σ is isomorphic to the triangulation of the torus $\mathbb{R}^2/N$ associated to the group G and that the stabilizer of that vertex in H acts on this torus exactly as G does. This follows directly from the definition of the complex $\mathcal{C}$ at a vertex and faces containing that vertex.

Hence one needs to show that this triangulation of the torus $\mathbb{R}^2/N$ is locally 6-large. Consider any closed combinatorial path of length 4 or 5 in the 1-skeleton of the torus. This path lifts to the universal cover of the torus, which is $\mathbb{R}^2$. The lift is a path of the same length and its endpoints belong to the same N-orbit, so the vector between its endpoints is of the form $m\xi_r + n\xi_s$ for some $m, n \in \mathbb{Z}$. All vectors of this form determine a 'sparse' tessellation of the plane, and two vertices in this sparse tessellation are at combinatorial distance at least 8 with respect to the 1-skeleton of the original tessellation. It follows that $m = n = 0$ and the lift is also a closed path. But the structure of the tessellation of $\mathbb{R}^2$ implies that in any closed path of length 4 or 5 there are two non-consecutive vertices of the path which are joined by an edge. This remains true after passing to the torus and thus 6-largeness of the triagulation of the torus, i.e. local 6-largeness of Σ, follows.

It is also obvious that Σ is connected, and Proposition 17.12 in [6] ensures that Σ is also simply-connected. □

Observe now that H acts properly on Σ, since all vertex stabilizers are isomorphic to G and hence finite, and the action is cocompact, the fundamental domain being a 3-simplex. Thus, Theorem 1.2 in [10] indicates that H is word-hyperbolic if and only if there are no *flats* in Σ, i.e. isometrically embedded 2-dimensional subcomplexes isomorphic to the plane $\mathbb{R}^2$ tessellated by equilateral triangles.

Proposition 3.4. *The complex Σ contains no flats.*

Proof. Suppose that $F \subset \Sigma$ is a flat.

Take any vertex $v \in F$. The link $\mathrm{Lk}_v(F)$ of the vertex v in F is a regular hexagon. The link $\mathrm{Lk}_v(\Sigma)$ of v in Σ is the triangulation of the torus $\mathbb{R}^2/N$ associated to the group G. Note that the hexagon $\mathrm{Lk}_v(F)$, viewed as a subcomplex of $\mathrm{Lk}_v(\Sigma)$, is necessarily a hexagon around some vertex. Indeed, no two non-consecutive vertices of $\mathrm{Lk}_v(F)$ can be joined by an edge in $\mathrm{Lk}_v(\Sigma)$ — otherwise such two vertices would be at distance 2 in F and at distance 1 in Σ, hence the inclusion $F \subset \Sigma$ would not be isometric. Moreover, exactly as in the proof of Proposition 3.3 we see that a closed combinatorial path of length 6 in the 1-skeleton of $\mathbb{R}^2/N$ lifts to a closed path in $\mathbb{R}^2$. And the only closed combinatorial path of length 6 in the 1-skeleton of $\mathbb{R}^2$ such that no two non-consecutive vertices are joined by an edge is a regular hexagon around a vertex.

Let $w \in \mathrm{Lk}_v(\Sigma)$ be the vertex such that $\mathrm{Lk}_v(F) \subset \mathrm{Lk}_v(\Sigma)$ is the hexagon around w. Then the hexagon $\mathrm{Lk}_v(F)$ is contained also in the link $\mathrm{Lk}_w(\Sigma)$. Moreover, the vertices v and w are joined in Σ by an edge. To see this, consider a closed path of length 4 whose consecutive vertices are: a vertex of the hexagon $\mathrm{Lk}_v(F)$, v, the opposite vertex of the hexagon, w. Since Σ is 6-large ([6], Proposition 1.4), we obtain that either the two opposite edges of the hexagon or v and w are joined by an edge in Σ. The former cannot hold, however, as it would imply that these two vertices are at distance 1 in Σ while their distance in the flat F is 2. Therefore v and w have to be joined by an edge of Σ.

What we have arrived at can be formulated as follows: $\mathrm{Lk}_v(F)$ is a hexagon around the edge with endpoints v and w.

Now consider any triangle in the flat F with one vertex v; let the two other vertices be v' and v''. In the same way we see that there are vertices $w', w'' \in \Sigma$ such that $\mathrm{Lk}_{v'}(F)$ is a hexagon around the edge $v'w'$ and $\mathrm{Lk}_{v''}(F)$ is a hexagon around the edge $v''w''$.

In this situation each of the vertices w, w', w'' forms, together with the triangle $vv'v''$, a tetrahedron in Σ. But the action of a triangle stabilizer in Σ (which is a group of order 2) is transitive on the set of tetrahedra containing that triangle. It follows that there are only two tetrahedra in Σ with the triangle $vv'v''$ as a face. Thus the vertices w, w', w'' cannot be distinct. Assume, for example, that $w = w'$. Note that v and v' are neighboring vertices of F. Hence there is a vertex u in the hexagon around v and a vertex u' in the hexagon around v' such that the distance between u and u' in F equals 3. However, as $w = w'$ and $u \in \mathrm{Lk}_w(\Sigma), u' \in \mathrm{Lk}_{w'}(\Sigma)$, we see that the distance between u and u' in Σ does not exceed 2. This contradiction with the isometric nature of the inclusion $F \subset \Sigma$ concludes the proof. □

Proposition 3.5. *The group H is non-elementary.*

Proof. There are two possibilities that have to be excluded: H is finite and H is virtually infinite cyclic.

If H is finite, then the action of H on the systolic complex Σ has a fixed point ([2], Theorem C). Let x be a fixed point; there is a 3-simplex $\sigma \subset \Sigma$ such that $x \in \sigma$ as well as a triangle τ of σ with $x \notin \tau$. The stabilizer of the triangle τ is the group with 2 elements and the stabilizer of the tetrahedron σ is trivial. Therefore the nontrivial element of the former group maps σ to a 3-simplex σ' satisfying $\sigma \cap \sigma' = \tau$. But then we have $x \notin \sigma'$, contrary to the fact that x is a fixed point.

If H is virtually infinite cyclic, then we can repeat the argument from the proof of Proposition 2.1 which remains fully valid if V such a group. □

Now, Proposition 3.4 together with Theorem 1.2 from [10] implies that the group H, which is a quotient of W, is word-hyperbolic. Since it is non-elementary by Proposition 3.5, the proof of Theorem 1.2 is complete.

4. Concluding remarks

The case we have studied above is very special. It seems plausible, though, that the method of constructing word-hyperbolic quotients of Coxeter groups by defining simplices of groups where local groups are finite quotients of special subgroups of the Coxeter group might work in greater generality.

For instance, Coxeter groups with 3 generators are usually cocompact reflection groups on the hyperbolic plane $\mathbb{H}^2$. The condition on the Coxeter exponents of the generators s, t, u that has to be satisfied is

$$\frac{1}{m_{st}} + \frac{1}{m_{su}} + \frac{1}{m_{tu}} < 1$$

([4], Example 6.5.3). One can imagine the following situation. Take a Coxeter group W with the Coxeter generating set $S = \{s_i : i \in I\}$ and suppose that a 3-element subset $S' = \{s_j : i \in J\}$ generates a hyperbolic triangular group as above.

Then define a triangle of groups. The vertices of the triangle are indexed by J. And the local group assigned to the face complementary to a subset $J' \subset J$ is a finite quotient of the special subgroup $W_{I \setminus J \cup J'}$. More generally, one might ask:

Question 4.1 (Januszkiewicz). *Let W be a non-affine Coxeter group with Coxeter generating set $S = \{s_i : i \in I\}$. Suppose that, for a subset $S' = \{s_i : i \in J\}$, the special subgroup $W_{S'}$ is a cocompact reflection group on some hyperbolic space $\mathbb{H}^n$. Also, let $\rho : W \to \overline{W}$ be a surjection onto a sufficiently large finite group.*

Define an n-simplex of groups whose vertices are indexed by J and the local group assigned to the face complementary to a subset $J' \subset J$ is the group $\rho(W_{I \setminus J \cup J'})$.

Is the fundamental group of this simplex of groups word-hyperbolic?

Note that this fundamental group is a quotient of W, since if we use the groups $W_{I \setminus J \cup J'}$ in place of $\rho(W_{I \setminus J \cup J'})$, then the fundamental group is clearly W. Moreover, the simplex is developable as it clearly admits a morphism to $\overline{W}$ which is injective on the local groups $\rho(W_{I \setminus J \cup J'}) \subset \overline{W}$.

If the answer to Question 4.1 is positive, then the class of non-affine Coxeter groups admitting a non-elementary word-hyperbolic quotient is a significant portion of all non-affine Coxeter groups. Indeed, in this construction only the exponents inside the set of a few generators matter and all other exponents do not.

One might also try to use special subgroups that generate reflection groups on hyperbolic spaces which are not cocompact but of finite covolume Theorem 1.2 is a first step in that direction since the Coxeter group with 4 generators and all Coxeter exponents equal to 3 acts on $\mathbb{H}^3$ with an ideal simplex as a fundamental domain.

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