Kamil Tabiś

Uniwersytet Wrocławski

Extremes of locally self-similar Gaussian processes

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Opiekun pracy: Krzysztof Dębicki

University of Wroclaw
Faculty of Mathematics and Computer Science
Mathematical Institute

Kamil Tabiś

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dr hab. Krzysztof Dębicki

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Kamil Tabiś (Kamil.Tabis@math.uni.wroc.pl)

Abstract

We investigate exact asymptotic of $\mathbf{P}\left(\sup_{t\in[0,T]} X(t) > u\right)$ as $u \to \infty$, where $\{X(t), t \in [0,T]\}$ is a *locally self-similar* Gaussian process, i.e. the increments of X in a neighborhood of t^* (a point in which variance function of X attains its maximum) behave like increments of some self-similar Gaussian process I. In case I is the fractional Brownian motion, our result leads to well-known Piterbarg & Prisyazhnyuk's Theorem for Gaussian processes with *local stationarity* property. We also present some natural examples of *locally self-similar* processes and give applications to queueing systems and collision problem.

Introduction

One of methods useful to investigate exact asymptotic of tail distribution of a supremum of a Gaussian process is the *double–sum method* (see Pickands [9], [10] and Piterbarg [11], [12] for more details). In case of $\{Z(t), t \geq 0\}$ being a centered (zero-mean) stationary Gaussian process with covariance function $\mathbf{Cov}(Z(t), Z(s)) = 1 - |t-s|^\alpha + o(|t-s|^\alpha)$ as $t, s \to 0^+$ for some $\alpha \in (0,2]$, the double–sum method (classical Pickands' result) leads to asymptotic

$$\mathbf{P}\left(\sup_{t\in[0,T]} Z(t) > u\right) \sim \mathcal{H}_{B,\alpha}\, T u^{2/\alpha}\, \Psi(u) \qquad \text{as} \quad u \to \infty\ ,$$

where

$$\mathcal{H}_{B,\alpha} = \lim_{T\to\infty} \frac{1}{T}\, \mathbf{E} \exp\left(\sup_{t\in[0,T]} \sqrt{2} B_\alpha(t) - t^\alpha\right)$$

and $\{B_\alpha(t), t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $\alpha/2$. By $\Psi(\cdot)$ we denote the tail probability of a standard normal random variable. Constants $\mathcal{H}_{B,\alpha}$ are called *Pickands' constants* and they play a significant role in the extreme value theory of Gaussian processes.

Currently, the double–sum method is being extended for a quite wide class of Gaussian processes. Piterbarg and Prisyazhnyuk (see [11] and [12]) generalized Pickands' result into a class of Gaussian processes ξ such that variance function of ξ attains its maximum on $[0,T]$ at a unique point t^* and

$$\begin{aligned} \mathbf{Var}(\xi(t+t^*)) &= 1 - 2A_1\, |t|^\alpha + o(|t|^\alpha) \qquad \text{as} \quad t \to 0\ , \\ \mathbf{Var}(\xi(t) - \xi(s)) &= 2A_2\, \mathbf{Var}\left(B_\alpha(t) - B_\alpha(s)\right)(1 + o(1)) \qquad \text{as} \quad t, s \to t^*\ . \end{aligned}$$

With $R=\frac{A_1}{A_2}$, they obtained that

$$\mathbf{P}\left(\sup_{t\in[0,T]}\xi(t)>u\right) \sim \Psi(u)\cdot\begin{cases}\mathcal{H}^R_{B,\alpha} & \text{if} \quad t^*\in(0,T)\\ \mathcal{F}^R_{B,\alpha} & \text{if} \quad t^*=0 \quad \text{or} \quad t^*=T\end{cases} \qquad \text{as} \quad u\to\infty\ ,$$

where $\mathcal{H}^R_{B,\alpha}$ and $\mathcal{F}^R_{B,\alpha}$ are modifications of the Pickands' constants:

$$\mathcal{H}^R_{B,\alpha} = \mathbf{E}\exp\left(\sup_{t\in(-\infty,\infty)}\sqrt{2}B_\alpha(t)-(1+R)|t|^\alpha\right)\ ;$$

$$\mathcal{F}^R_{B,\alpha} = \mathbf{E}\exp\left(\sup_{t\in[0,\infty)}\sqrt{2}B_\alpha(t)-(1+R)t^\alpha\right)\ .$$

Dębicki (see [3] for more details) generalized Pickands' result into a case of processes such that

$$\mathbf{Var}\left(\xi(t)-\xi(s)\right) = 2A_2\,\mathbf{Var}\left(I(t)-I(s)\right)(1+o(1)) \qquad \text{as} \quad t,s\to t^*\ , \tag{1}$$

where $\{I(t), t\geq 0\}$ is a Gaussian process with stationary increments.

It turns out that the double–sum method can also be generalized into a case where I is a Gaussian self-similar process with not necessarily stationary increments. Processes ξ such that in (1) a Gaussian self-similar process appears, we call *locally self-similar*. The main result of the paper consists in extension of Piterbarg & Prisyazhnyuk' result.

The paper is organized as follows:

Section 1 contains definitions of fundamental notions connected to Gaussian processes and the notation. We also present theorems of Pickands and Piterbarg & Prisyazhnyuk.

In Section 2 locally self-similar Gaussian processes are defined and the main result of the paper is presented.

Section 3 gives examples of Gaussian self-similar processes and locally self-similar processes. We also investigate exact asymptotic of tail distribution of suprema for presented examples.

Section 4 contains applications of the obtained results to Gaussian fluid models theory and collision problem for Gaussian processes with differentiable sample paths.

In Section 5 we prove the main result of the paper. It relies on an essential modification of the double–sum method.

In Appendix A we present some classical theorems that are used in the paper.

1 Basic notions and notation

In the paper we deal with one-dimensional Gaussian processes on the real half line. Now we remind basic definitions.

Definition 1.1 *A stochastic process $\{\xi(t), t \geq 0\}$ is said to be Gaussian if its finite dimensional distributions are normal, i.e. for all $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$*

$$(\xi(t_1), \ldots, \xi(t_n)) \stackrel{\mathcal{D}}{=} \mathcal{N}(\mathbf{m_t}, \mathbf{\Sigma_t}) \ ,$$

where $\mathbf{m_t} = (\mathbf{E}(\xi(t_1)), \ldots, \mathbf{E}(\xi(t_n)))$ and $\mathbf{\Sigma_t} = \{\sigma_{ij}\}_{i,j=1}^n$ for $\sigma_{ij} = \mathbf{Cov}(\xi(t_i), \xi(t_j))$.

For a Gaussian process ξ and $t, s \geq 0$ we denote:

$$\begin{array}{llcl}
\textit{mean function} & a_\xi(t) & = & \mathbf{E}(\xi(t)) \ ; \\
\textit{covariance function} & R_\xi(t,s) & = & \mathbf{Cov}(\xi(t), \xi(s)) \ ; \\
\textit{variance function} & \sigma_\xi^2(t) & = & \mathbf{Var}(\xi(t)) \ = \ R_\xi(t,t) \ ; \\
\textit{variance of increments function} & V_\xi(t,s) & = & \mathbf{Var}(\xi(t) - \xi(s)) = \sigma_\xi^2(t) + \sigma_\xi^2(s) - 2R_\xi(t,s).
\end{array}$$

A process ξ is said to be *centered* if $a_\xi(t) = 0$ for all $t \geq 0$. Let $\bar{\xi}$ denote a standardized ξ process, i.e. $\bar{\xi}(t) = \frac{\xi(t) - a_\xi(t)}{\sigma_\xi(t)}$.

Definition 1.2 *A process $\{\xi(t), t \geq 0\}$ is said to be stationary if its finite dimensional distributions do not depend on shifts, i.e. for all $h, t_1, \ldots, t_n \geq 0$*

$$(\xi(t_1), \ldots, \xi(t_n)) \stackrel{\mathcal{D}}{=} (\xi(t_1 + h), \ldots, \xi(t_n + h)) \ .$$

A centered Gaussian process ξ is stationary if $\mathbf{Cov}(\xi(t), \xi(s)) = R_\xi(|t - s|)$ for $t, s \geq 0$. In the paper by a univariate function $R_\xi(\cdot)$ we denote covariance function of a stationary Gaussian process ξ, i.e. $R_\xi(t) = \mathbf{Cov}(\xi(t), \xi(0))$.

Definition 1.3 *We say that a process $\{\xi(t), t \geq 0\}$ has stationary increments if for all $h > 0$ process $\{\xi(t + h) - \xi(t), t \geq 0\}$ is stationary.*

A Gaussian process ξ such that $\xi(0) = 0$ a.s. has stationary increments if for all $t, s \geq 0$

$$\mathbf{Var}(\xi(t) - \xi(s)) \ = \ \sigma_\xi^2(|t - s|) \quad \text{or equivalent} \quad R_\xi(t,s) \ = \ \frac{\sigma_\xi^2(t) + \sigma_\xi^2(s) - \sigma_\xi^2(|t - s|)}{2} \ .$$

Definition 1.4 *A process $\{\xi(t), t \geq 0\}$ is self-similar with index H, if for all $a > 0$*

$$\{\xi(at), t \geq 0\} \stackrel{\mathcal{D}}{=} \{a^H \xi(t), t \geq 0\} \ .$$

Remark 1.1 If a process ξ is self-similar with index H, then $\mathbf{Var}(\xi(t)) = t^{2H}\mathbf{Var}(\xi(1))$. We consider only self-similar processes ξ such that $\mathbf{Var}(\xi(1)) = 1$. Then for all $t, s, a \geq 0$ we have:

$$V_\xi(t,t) = 0 \ , \quad V_\xi(t,s) = V_\xi(s,t) \ , \quad V_\xi(at,as) = a^{2H} V_\xi(t,s) \ , \quad V_\xi(t,0) = \mathbf{Var}(\xi(t)) = t^{2H} \ ,$$
$$V_\xi(t,s) \ = \ t^{2H}\, V_\xi\left(1, \frac{s}{t}\right) \ = \ t^{2H}\, V_\xi(1,x) \qquad \text{for} \quad s \leq t \quad \text{and} \quad x = \frac{s}{t} \in [0,1] \ .$$

By $\Psi(\cdot)$ we denote the tail probability of a standard normal random variable:

$$\Psi(u) \ = \ \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2} dx \ .$$

We recall (see for instance Adler [1]) that

$$\Psi(u) \ \sim \ \frac{\exp\left(-u^2/2\right)}{u\sqrt{2\pi}} \qquad \text{as} \quad u \to \infty \ . \tag{2}$$

We use the relation: $f(u) \sim g(u)$ as $u \to \infty$ iff $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1$.
Moreover by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$ we denote the gamma function.

We also use a standard notation $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$ for $k \in \mathbb{N}$ and $c \in \mathbb{R}$. Note that $\binom{c+k}{k} = \frac{\prod_{i=1}^k (c+i)}{k!}$.

Definition 1.5 *Fractional Brownian motion* (**FBm**) *with Hurst parameter* $\alpha/2 \in (0,1]$ *is a centered Gaussian process* $\{B_\alpha(t), t \geq 0\}$ *with stationary increments, continuous sample paths a.s. and variance function* $\mathbf{Var}(B_\alpha(t)) = t^\alpha$.

Remark 1.2 Let us recall that B_α is self-similar with Hurst parameter $\alpha/2$. In a special case, when $\alpha = 1$, the process B_1 is the standard Brownian motion. For $\alpha = 2$, B_2 has degenerated structure in the sense that $\{B_2(t), t \geq 0\} \stackrel{\mathcal{D}}{=} \{t\mathcal{N}, t \geq 0\}$, where $\mathcal{N}$ is a standard normal random variable.

Now we present the celebrated Pickands' Theorem for stationary Gaussian processes (see Pickands [9], [10] or Piterbarg [11] for more details).

Theorem 1.1 (Pickands) *Let* $\{Z(t), t \geq 0\}$ *be a centered stationary Gaussian process with continuous sample paths a.s. Assume that* $R_Z(t) = 1 - t^\alpha + o(t^\alpha)$ *as* $t \to 0^+$ *for some* $\alpha \in (0,2]$ *and* $R_Z(t) < 1$ *for all* $t > 0$. *Then for all* $T > 0$

$$\mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} Z(t) > u\right) \ \sim \ \mathcal{H}_{B,\alpha}(T)\, \Psi(u) \qquad \textit{as} \quad u \to \infty \ ;$$
$$\mathbf{P}\left(\sup_{t\in[0,T]} Z(t) > u\right) \ \sim \ \mathcal{H}_{B,\alpha}\, T u^{2/\alpha}\, \Psi(u) \qquad \textit{as} \quad u \to \infty \ ,$$

where

$$\mathcal{H}_{B,\alpha} \ := \ \lim_{T\to\infty} \frac{\mathcal{H}_{B,\alpha}(T)}{T} \qquad \textit{for} \qquad \mathcal{H}_{B,\alpha}(T) \ := \ \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha\right) \ .$$

We also present the Piterbarg & Prisyazhnyuk's Theorem for Gaussian processes with *local stationarity* property (see Piterbarg [11] and [12] for more details).

Theorem 1.2 (Piterbarg & Prisyazhnyuk) *Let $\{\xi(t), t \in [0,T]\}$ be a centered Gaussian process with continuous sample paths a.s. Suppose function $\sigma_\xi^2(\cdot)$ attains its maximum on $[0,T]$ at a unique point t^* and $\sigma_\xi^2(t^*) = 1$. Make following assumptions:*
a) there exist $\beta, A_1 > 0$ such that

$$\sigma_\xi(t+t^*) \;=\; 1 - A_1\,|t|^\beta + o(|t|^\beta) \qquad \text{as} \quad t \to 0\;;$$

b) ***local stationarity****: there exist $\alpha, A_2 > 0$ such that*

$$1 - \mathbf{Cov}(\bar\xi(t), \bar\xi(s)) \;=\; A_2\,|t-s|^\alpha + o(|t-s|^\alpha) \qquad \text{as} \quad t, s \to t^*\;;$$

c) ***regularity****: there exist $\gamma, A_3 > 0$ such that*

$$\mathbf{Var}(\xi(t) - \xi(s)) \;\le\; A_3|t-s|^\gamma \qquad \text{for all} \quad t, s \in [0,T]\;.$$

Then the following statements hold:
i) if $\beta < \alpha$, then

$$\mathbf{P}\left(\sup_{t\in[0,T]} \xi(t) > u\right) \;\sim\; \Psi(u) \qquad \text{as} \quad u \to \infty\;;$$

ii) if $\beta = \alpha$, with $R = \frac{A_1}{A_2}$, then

$$\mathbf{P}\left(\sup_{t\in[0,T]} \xi(t) > u\right) \;\sim\; \Psi(u) \cdot \begin{cases} \mathcal{H}_{B,\alpha}^R & \text{if} \quad t^* \in (0,T) \\ \mathcal{F}_{B,\alpha}^R & \text{if} \quad t^* = 0 \quad \text{or} \quad t^* = T \end{cases} \qquad \text{as} \quad u \to \infty\;,$$

where

$$\mathcal{H}_{B,\alpha}^R \;:=\; \lim_{T\to\infty} \mathcal{H}_{B,\alpha}^R(T)\;; \qquad \mathcal{F}_{B,\alpha}^R \;:=\; \lim_{T\to\infty} \mathcal{F}_{B,\alpha}^R(T)\;,$$

for

$$\mathcal{H}_{B,\alpha}^R(T) \;:=\; \mathbf{E}\exp\left(\sup_{t\in[-T,T]} \sqrt{2}B_\alpha(t) - (1+R)|t|^\alpha\right)\;;$$

$$\mathcal{F}_{B,\alpha}^R(T) \;:=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right)\;.$$

2 Extremes of locally self-similar Gaussian processes

2.1 Locally self-similar processes

In the rest of the paper by $\{I_\alpha(t), t \geq 0\}$ we mean a centered, self-similar with index $\alpha/2$ Gaussian process such that function $V_{I_\alpha}(t,s) = \mathbf{Var}\,(I_\alpha(t) - I_\alpha(s))$ satisfies the following conditions for all $t, s \geq 0$:

$$\mathbf{V1}: \qquad V_{I_\alpha}(t,0) \;=\; \mathbf{Var}\,(I_\alpha(t)) \;=\; t^\alpha \;;$$

$$\mathbf{V2}: \qquad V_{I_\alpha}(t,s) \;\leq\; |t-s|^\alpha \;.$$

Remark 2.1 If $V_{I_\alpha}(t,s) = |t-s|^\alpha$, then I_α is **FBm**.

Remark 2.2 Note that the Schwarz inequality implies $\mathbf{Cov}(I_\alpha(t), I_\alpha(s)) \leq (ts)^{\alpha/2}$. Therefore

$$t^\alpha + s^\alpha - 2(ts)^{\alpha/2} \;=\; |t^{\alpha/2} - s^{\alpha/2}|^2 \;\leq\; V_{I_\alpha}(t,s) \;.$$

A process $\{D_\alpha(t), t \geq 0\}$ for which $V_{D_\alpha}(t,s) = |t^{\alpha/2} - s^{\alpha/2}|^2$ is a degenerated process. Namely $\{D_\alpha(t), t \geq 0\} \stackrel{\mathcal{D}}{=} \{t^{\alpha/2}\mathcal{N}, t \geq 0\}$ where $\mathcal{N}$ is a standard normal random variable. In particular, for $\alpha = 2$, I_2 is the degenerated **FBm**.

Definition 2.1 *A centered Gaussian process $\{X(t), t \geq 0\}$ is said to be I_α–locally self-similar at a point t^* if*

$$1 - \mathbf{Cov}(\bar{X}(t), \bar{X}(s)) \;=\; A(t^*)\, V_{I_\alpha}(t,s)(1+o(1)) \qquad \textit{as} \quad t, s \to t^* \;,$$

for some process $\{I_\alpha(t), t \geq 0\}$.

For the process I_α and $R > 0$ we define the following constants:

$$\mathcal{H}^R_{I,\alpha} \;:=\; \lim_{T\to\infty} \mathcal{H}^R_{I,\alpha}(T) \;; \qquad \mathcal{F}^R_{I,\alpha} \;:=\; \lim_{T\to\infty} \mathcal{F}^R_{I,\alpha}(T) \;,$$

where

$$\mathcal{H}^R_{I,\alpha}(T) \;=\; \mathbf{E}\exp\left(\sup_{t\in[-T,T]} \sqrt{2}I_\alpha(t) - (1+R)|t|^\alpha\right) \;;$$

$$\mathcal{F}^R_{I,\alpha}(T) \;=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}I_\alpha(t) - (1+R)t^\alpha\right) \;.$$

Lemma 2.1 *For all $\alpha \in (0,2]$ and $R, T > 0$ the following inequalities hold:*

$$\mathcal{H}^R_{I,\alpha}(T) \;\leq\; \mathcal{H}^R_{B,\alpha}(T) \qquad \textit{and} \qquad \mathcal{H}^R_{I,\alpha} \;\leq\; \mathcal{H}^R_{B,\alpha} \;;$$

$$\mathcal{F}^R_{I,\alpha}(T) \;\leq\; \mathcal{F}^R_{B,\alpha}(T) \qquad \textit{and} \qquad \mathcal{F}^R_{I,\alpha} \;\leq\; \mathcal{F}^R_{B,\alpha} \;.$$

Proof:
In the proof we use Slepian's inequality (Theorem A.1). The following inequality is an immediate consequence of **V1**, **V2** and Theorem A.1, for all $\alpha \in (0,2]$, $R, T > 0$ and for all $x \in \mathbb{R}$:

$$\begin{aligned} \mathcal{F}_{I,\alpha}^R(T) &= \int_{-\infty}^{\infty} e^x \, \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2} I_\alpha(t) - (1+R)t^\alpha > x \right) dx \qquad (3) \\ &\leq \int_{-\infty}^{\infty} e^x \, \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2} B_\alpha(t) - (1+R)t^\alpha > x \right) dx \;=\; \mathcal{F}_{B,\alpha}^R(T). \end{aligned}$$

Letting $T \to \infty$ yields $\mathcal{F}_{I,\alpha}^R = \lim_{T\to\infty} \mathcal{F}_{I,\alpha}^R(T) \leq \lim_{T\to\infty} \mathcal{F}_{B,\alpha}^R(T) = \mathcal{F}_{B,\alpha}^R$. The proof for constants $\mathcal{H}_{I,\alpha}^R$ is analogous.

□

Corollary 2.2 *For constants $\mathcal{H}_{I,\alpha}^R$ and $\mathcal{F}_{I,\alpha}^R$ the following inequalities hold:*

$$\begin{aligned} \mathcal{H}_{I,2}^R &\leq \mathcal{H}_{I,\alpha}^R \,, & \mathcal{F}_{I,2}^R &\leq \mathcal{F}_{I,\alpha}^R \qquad \textit{for} \quad \alpha \in (0,2] \,; \\ \mathcal{H}_{I,2}^R &= \sqrt{1+\frac{1}{R}} \,, & \mathcal{F}_{I,2}^R &= \frac{1}{2}\left(1+\sqrt{1+\frac{1}{R}}\right) \,; \\ \mathcal{H}_{I,1}^R &\leq \frac{2(1+R)^2}{R(1+2R)} \,, & \mathcal{F}_{I,1}^R &\leq 1+\frac{1}{R} \,; \\ \mathcal{H}_{I,\alpha}^R &\leq 2\left(1+\frac{1}{R}\right) \,, & \mathcal{F}_{I,\alpha}^R &\leq 1+\frac{1}{R} \qquad \textit{for} \quad \alpha \in (1,2) \,. \end{aligned}$$

Proof:
Remark 2.2 implies $\mathcal{F}_{B,2}^R = \mathcal{F}_{I,2}^R$. Following the proof of Lemma 2.1 we have

$$\begin{aligned} \mathcal{F}_{I,\alpha}^R \;\geq\; \mathcal{F}_{D,\alpha}^R &= \mathbf{E}\exp\left(\sup_{t\in[0,\infty)} \sqrt{2} t^{\alpha/2} \mathcal{N} - (1+R)t^\alpha \right) \\ &= \mathbf{E}\exp\left(\sup_{t\in[0,\infty)} \sqrt{2} t \mathcal{N} - (1+R)t^2 \right) \;=\; \mathcal{F}_{B,2}^R \end{aligned}$$

and analogously $\mathcal{H}_{I,\alpha}^R \geq \mathcal{H}_{B,2}^R$. The remained theses of Corollary 2.2 are a consequence of Lemma 2.1 and known properties of the constants $\mathcal{H}_{B,\alpha}^R$ and $\mathcal{F}_{B,\alpha}^R$ (see [4] for more details):

$$\begin{aligned} \mathcal{H}_{B,1}^R &= \frac{2(1+R)^2}{R(1+2R)} \,, & \mathcal{H}_{B,2}^R &= \sqrt{1+\frac{1}{R}} \,; \\ \mathcal{F}_{B,1}^R &= 1+\frac{1}{R} \,, & \mathcal{F}_{B,2}^R &= \frac{1}{2}\left(1+\sqrt{1+\frac{1}{R}}\right) \,; \\ \mathcal{F}_{B,\alpha}^R &\geq 1+\frac{1}{R} \,, & 1+\frac{1}{R} \leq \mathcal{H}_{B,\alpha}^R &\leq 2\mathcal{F}_{B,\alpha}^R \qquad \text{for} \quad \alpha \in (0,1) \,; \\ \mathcal{F}_{B,\alpha}^R &\leq 1+\frac{1}{R} \,, & \mathcal{F}_{B,\alpha}^R \leq \mathcal{H}_{B,\alpha}^R &\leq 2\left(1+\frac{1}{R}\right) \qquad \text{for} \quad \alpha \in (1,2) \,. \end{aligned}$$

This completes the proof.

□

2.2 Main result

The main result of the paper is given in the following theorem.

Theorem 2.3 *Let $\{X(t), t \in [0,T]\}$ be a centered Gaussian process with continuous sample paths a.s. Suppose that function $\sigma_X^2(\cdot)$ attains its maximum on $[0,T]$ at a unique point t^* and $\sigma_X^2(t^*) = 1$. Make the following assumptions:*
a) there exist $\beta, A_1 > 0$ such that

$$\sigma_X(t+t^*) \;=\; 1 - A_1\,|t|^\beta + o(|t|^\beta) \qquad as \quad t \to 0\,; \tag{4}$$

b) the process X is I_α–locally self-similar at the point t^* and

$$1 - \mathbf{Cov}(\bar{X}(t), \bar{X}(s)) \;=\; A_2\,V_{I_\alpha}(t,s)(1+o(1)) \qquad as \quad t,s \to t^*\,. \tag{5}$$

i) If $\beta < \alpha$, then

$$\mathbf{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \;\sim\; \Psi(u) \qquad as \quad u \to \infty\,.$$

ii) If $\beta = \alpha$, with $R = \frac{A_1}{A_2}$, then

$$\mathbf{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \;\sim\; \Psi(u)\cdot \begin{cases} \mathcal{H}^R_{I,\alpha} & if \quad t^* \in (0,T) \\ \mathcal{F}^R_{I,\alpha} & if \quad t^* = 0 \quad or \quad t^* = T \end{cases} \qquad as \quad u \to \infty\,.$$

We note that Theorem 1.2 is a special case of Theorem 2.3, when I_α is **FBm**.

Remark 2.3 According to Lemma 5.1 (which we state in Section 5), another way of stating assumptions *a)* and *b)* of Theorem 2.3 is to say that:
a) there exist $\beta, A_1 > 0$ such that

$$\sigma_X^2(t+t^*) \;=\; \big(1 - A_1\,|t|^\beta + o(|t|^\beta)\big)^2 \;=\; 1 - 2A_1\,|t|^\beta + o(|t|^\beta) \qquad \text{as} \quad t \to 0\,;$$

b) the process X is I_α–locally self-similar at the point t^* and

$$\begin{aligned}\mathbf{Var}(X(t) - X(s)) \;&=\; \mathbf{Var}(\bar{X}(t) - \bar{X}(s))(1+o(1)) \;=\; 2\big(1 - \mathbf{Cov}(\bar{X}(t), \bar{X}(s))\big)\,(1+o(1)) \\ &=\; 2A_2\,V_{I_\alpha}(t,s)(1+o(1)) \qquad \text{as} \quad t,s \to t^*\,.\end{aligned}$$

The next theorem is an useful modification of Theorem 2.3.

Theorem 2.4 *Let $\{X(t), t \geq 0\}$ be a centered Gaussian process with continuous and bounded sample paths a.s. Suppose that function $\sigma_X^2(\cdot)$ attains its global maximum on $[0,T]$ at a unique point t^* and $\sigma_X^2(t^*) = 1$. Assume that assumptions a) and b) of Theorem 2.3 are fulfilled.*

i) If $\beta < \alpha$, then

$$\mathbf{P}\left(\sup_{t\geq 0} X(t) > u\right) \;\sim\; \Psi(u) \quad as \quad u \to \infty\,.$$

ii) If $\beta = \alpha$, with $R = \frac{A_1}{A_2}$, then

$$\mathbf{P}\left(\sup_{t\geq 0} X(t) > u\right) \;=\; \Psi(u)\cdot \begin{cases} \mathcal{H}^R_{I,\alpha} & if \quad t^* > 0 \\ \mathcal{F}^R_{I,\alpha} & if \quad t^* = 0 \end{cases} \qquad as \quad u \to \infty\,.$$

Proof:
Set $\Delta = [0, \delta]$ if $t^* = 0$ or $\Delta = [t^* - \delta, t^* + \delta]$ if $t^* > 0$, for any δ such that $\Delta \subset [0, \infty)$. Since $X(0) > u$ implies $\sup_{t \geq 0} X(t) > u$, then for all $u \in \mathbb{R}$

$$\Psi(u) \ \leq \ \mathbf{P}\left(\sup_{t \geq 0} X(t) > u\right) \ . \tag{6}$$

Since X has bounded sample paths a.s. therefore, assumptions of Borell's inequality (Theorem A.2) are fulfilled (with $\mathcal{T} = [0, \infty)\backslash\Delta$ and $\sigma^2 = \sup_{t \in \mathcal{T}} \sigma_X^2(t) < 1$). Hence, Corollary A.3 implies the existence of $m > 0$ such that

$$\mathbf{P}\left(\sup_{t \in \mathcal{T}} X(t) > u\right) \ \leq \ 2 \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) \qquad \text{for all} \quad u > m \ .$$

Since $\frac{1}{\sigma^2} > 1$ (using asymptotic (2)), it follows that

$$\lim_{u \to \infty} \frac{\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)}{\Psi(u)} \ = \ \sqrt{2\pi} \lim_{u \to \infty} u \exp\left(-\frac{1}{2} u^2 \left(\frac{1}{\sigma^2} - 1\right) + \frac{um}{\sigma} - \frac{m^2}{2\sigma^2}\right) \ = \ 0 \ .$$

It means that $\mathbf{P}\left(\sup_{t \in \mathcal{T}} X(t) > u\right) = o\left(\Psi(u)\right)$ as $u \to \infty$. From (6) we conclude that

$$\Psi(u) \ \leq \ \mathbf{P}\left(\sup_{t \geq 0} X(t) > u\right) \ \leq \ \mathbf{P}\left(\sup_{t \in \Delta} X(t) > u\right) + o\left(\Psi(u)\right) \qquad \text{as} \quad u \to \infty \ .$$

Finally

$$\mathbf{P}\left(\sup_{t \geq 0} X(t) > u\right) \ \sim \ \mathbf{P}\left(\sup_{t \in \Delta} X(t) > u\right) \qquad \text{as} \quad u \to \infty \ .$$

The proof is complete after using Theorem 2.3. □

The proof of Theorem 2.3 is based on the following lemma, which is a modification of Theorem 1.1.

Lemma 2.5 *Suppose the assumptions of Theorem 2.3 are fulfilled with* $\beta = \alpha$*. Then, with* $R = \frac{A_1}{A_2}$*, for all* $T > 0$

$$\mathbf{P}\left(\sup_{t \in u^{-2/\alpha}[0,T]} X(t) > u\right) \ \sim \ \mathcal{F}_{I,\alpha}^R\left(T A_2^{1/\alpha}\right) \Psi(u) \qquad \textit{as} \quad u \to \infty \qquad \textit{if} \quad t^* = 0 \ ;$$

$$\mathbf{P}\left(\sup_{t \in u^{-2/\alpha}[t^*-T, t^*+T]} X(t) > u\right) \ \sim \ \mathcal{H}_{I,\alpha}^R\left(T A_2^{1/\alpha}\right) \Psi(u) \qquad \textit{as} \quad u \to \infty \qquad \textit{if} \quad t^* > 0 \ .$$

The detailed proofs of Lemma 2.5 and Theorem 2.3 are provided in Section 5.

3 Examples

In this section we present examples of self-similar Gaussian processes I_α that satisfy conditions **V1** and **V2**. We also introduce locally self-similar processes based on them. Finally we investigate exact asymptotic of tail distribution of supremum.

3.1 Process Im-FBm of order k

Let us define a process which is k-fold integrated fractional Brownian motion.

Definition 3.1 *By $\{B_\alpha^k(t), t \geq 0\}$ we denote a centered Gaussian process such that*

$$\begin{aligned} B_\alpha^0(t) &:= B_\alpha(t) && \textit{for} \quad k=0\,; \\ B_\alpha^k(t) &:= \int_0^t B_\alpha^{k-1}(s)ds && \textit{for} \quad k>0\,. \end{aligned}$$

Denote the covariance function of the process B_α^k by:

$$\begin{aligned} C^0(\alpha;t,s) &:= \mathbf{Cov}\left(B_\alpha^0(t), B_\alpha^0(s)\right) = \frac{1}{2}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right)\;; \\ C^k(\alpha;t,s) &:= \mathbf{Cov}\left(B_\alpha^k(t), B_\alpha^k(s)\right)\;. \end{aligned}$$

Lemma 3.1 *The process B_α^k has sample paths in $C^k[0,\infty)$ a.s. The following recurrences hold for $k>0$:*

i)
$$C^k(\alpha;t,s) = \frac{1}{2}\,\frac{(ts)^k(t^\alpha+s^\alpha)}{k!\prod_{i=1}^k(\alpha+i)} - \frac{C^{k-1}(\alpha+2;t,s)}{(\alpha+1)(\alpha+2)}\;;$$

ii)
$$C^k(\alpha;t,t) = \frac{t^{\alpha+2k}}{(\alpha+2k)k!\prod_{i=1}^{k-1}(\alpha+i)} = \frac{\alpha+k}{(k!)^2(\alpha+2k)\binom{\alpha+k}{k}}\,t^{\alpha+2k}\;.$$

Proof:
The proof for $k=1$ can be found in [14]. For $k>1$ we obtain inductively:

$$\begin{aligned} C^{k+1}(\alpha;t,s) &= \mathbf{Cov}\left(\int_0^t B_\alpha^k(u)du, \int_0^s B_\alpha^k(v)dv\right) = \int_0^t\int_0^s C^k(\alpha;u,v)dudv \\ &= \int_0^t\int_0^s \frac{1}{2}\,\frac{u^{\alpha+k}v^k + v^{\alpha+k}u^k}{k!\prod_{i=1}^k(\alpha+i)} - \frac{C^{k-1}(\alpha+2;u,v)}{(\alpha+1)(\alpha+2)}\,dudv \\ &= \frac{1}{2}\,\frac{(ts)^{k+1}(t^\alpha+s^\alpha)}{(k+1)!\prod_{i=1}^{k+1}(\alpha+i)} - \frac{C^k(\alpha+2;t,s)}{(\alpha+1)(\alpha+2)}\,. \end{aligned}$$

Replacing $t=s$ in formula *i)* we get

$$C^k(\alpha;t,t) = \frac{t^{\alpha+2k}}{k!\prod_{i=1}^k(\alpha+i)} - \frac{C^{k-1}(\alpha+2;t,t)}{(\alpha+1)(\alpha+2)}$$

$$= \frac{t^{\alpha+2k}}{k!\prod_{i=1}^{k}(\alpha+i)} - \frac{t^{\alpha+2k}}{(\alpha+2k)(\alpha+1)(\alpha+2)(k-1)!\prod_{i=3}^{k}(\alpha+i)}$$

$$= \frac{(\alpha+2k-k)t^{\alpha+2k}}{(\alpha+2k)k!\prod_{i=1}^{k}(\alpha+i)} = \frac{t^{\alpha+2k}}{(\alpha+2k)k!\prod_{i=1}^{k-1}(\alpha+i)} .$$

This completes the proof. □

From now on, we denote $C_\alpha^k = (\alpha+2k)k!\prod_{i=1}^{k-1}(\alpha+i) = \frac{(k!)^2(\alpha+2k)\binom{\alpha+k}{k}}{\alpha+k}$. Note that $C_\alpha^k \sim \frac{2}{\Gamma(\alpha+1)}k^\alpha(k!)^2$ as $k \to \infty$.

Let us define a process which is k-fold integral mean of fractional Brownian motion.

Definition 3.2 *By the* **Im-FBm of order k** *with Hurst parameter* $\alpha/2 \in (0,1]$ *we denote a centered Gaussian process* $\{I_\alpha^k(t), t \geq 0\}$ *such that*

$$I_\alpha^0(t) = B_\alpha^0(t) \qquad \textit{for} \quad k=0 ;$$

$$I_\alpha^k(t) = \begin{cases} \sqrt{C_\alpha^k}\,\frac{B_\alpha^k(t)}{t^k} & \textit{if} \quad t>0 \\ 0 \ \textit{a.s.} & \textit{if} \quad t=0 \end{cases} \qquad \textit{for} \quad k>0 .$$

Denote the covariance function and the variance of increments function of the **Im-FBm of order k** by:

$$R^0(\alpha;t,s) := \mathbf{Cov}\left(I_\alpha^0(t), I_\alpha^0(s)\right) = \frac{1}{2}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right) ;$$

$$V^0(\alpha;t,s) := \mathbf{Var}\left(I_\alpha^0(t) - I_\alpha^0(s)\right) = |t-s|^\alpha ;$$

$$R^k(\alpha;t,s) := \mathbf{Cov}\left(I_\alpha^k(t), I_\alpha^k(s)\right) ;$$

$$V^k(\alpha;t,s) := \mathbf{Var}\left(I_\alpha^k(t) - I_\alpha^k(s)\right) .$$

Lemma 3.2 *The* **Im-FBm of order k** *is self-similar with Hurst parameter* $\alpha/2$ *and has sample paths in* $C^k[0,\infty)$ *a.s. The following recurrences hold for* $k>0$*:*

i) $$R^k(\alpha;t,s) = \frac{(\alpha+2k)(t^\alpha+s^\alpha)}{2(\alpha+k)} - \frac{k}{(\alpha+k)ts} R^{k-1}(\alpha+2;t,s) ;$$

ii) $$V^k(\alpha;t,s) = \frac{k\left(|t-s||t^{\alpha+1}-s^{\alpha+1}| - V^{k-1}(\alpha+2;t,s)\right)}{(\alpha+k)ts} ;$$

iii) $$\mathbf{Var}\left(I_\alpha^k(t)\right) = t^\alpha .$$

Proof:
Formula *iii)* follows from *ii)* of Lemma 3.1. To prove formula *i)*, we use *i)* of Lemma 3.1:

$$R^k(\alpha;t,s) = \frac{C_\alpha^k}{(ts)^k} C^k(\alpha;t,s) = \frac{\alpha+2k}{2(\alpha+k)}(t^\alpha+s^\alpha) - \frac{C_\alpha^k}{(ts)^k}\frac{C^{k-1}(\alpha+2;t,s)}{(\alpha+1)(\alpha+2)}$$

$$
\begin{aligned}
&= \frac{\alpha+2k}{2(\alpha+k)}(t^\alpha+s^\alpha) - \frac{k}{(\alpha+k)ts}\frac{C^{k-1}(\alpha+2;t,s)(\alpha+2k)(k-1)!\prod_{i=3}^{k}(\alpha+i)}{(ts)^{k-1}} \\
&= \frac{\alpha+2k}{2(\alpha+k)}(t^\alpha+s^\alpha) - \frac{k}{(\alpha+k)ts} R^{k-1}(\alpha+2;t,s)\ .
\end{aligned}
$$

Using formula *i)* we get

$$
\begin{aligned}
V^k(\alpha;t,s) &= t^\alpha+s^\alpha-2R^k(\alpha;t,s) = \frac{-k}{\alpha+k}(t^\alpha+s^\alpha)+\frac{k}{(\alpha+k)ts}\,2R^{k-1}(\alpha+2;t,s) \\
&= \frac{-k}{\alpha+k}(t^\alpha+s^\alpha)+\frac{k}{(\alpha+k)ts}\left(t^{\alpha+2}+s^{\alpha+2}-V^{k-1}(\alpha+2;t,s)\right) \\
&= \frac{k}{(\alpha+k)ts}\left(|t-s||t^{\alpha+1}-s^{\alpha+1}|-V^{k-1}(\alpha+2;t,s)\right)\ .
\end{aligned}
$$

Self-similarity of the process I^k_α follows from Definition 3.2 by induction:

$$
\begin{aligned}
I^k_\alpha(at) &= \frac{C^k_\alpha}{(at)^k}\int_0^{at} B^{k-1}_\alpha(s)ds = \frac{C^k_\alpha}{(at)^k}\int_0^{at}\frac{I^{k-1}_\alpha(s)s^{k-1}}{C^{k-1}_\alpha}ds \\
&= \frac{C^k_\alpha}{(at)^k}\int_0^{t}\frac{a}{C^{k-1}_\alpha}I^{k-1}_\alpha(as)(as)^{k-1}ds \overset{\mathcal{D}}{=} \frac{a^H C^k_\alpha}{t^k}\int_0^t B^{k-1}_\alpha(s)ds = a^H I^k_\alpha(t)\ .
\end{aligned}
$$

This completes the proof. □

Lemma 3.3 *The following properties hold for the* **Im-FBm of order k**. *For all $t,s\ge 0$ and $x\in[0,1]$*

i) $V_{I^k_\alpha}(t,s) \le V_{I^0_\alpha}(t,s) = |t-s|^\alpha$;

ii) $V_{I^k_\alpha}(1,x) = (1-x)\left(-x^\alpha\sum_{i=1}^{k}\frac{\binom{\alpha+2k-1}{k-i}}{\binom{\alpha+2k-1}{k}}(-x)^{i-1} + \sum_{j=0}^{\infty}\frac{\binom{\alpha+2k-1}{k+j}}{\binom{\alpha+2k-1}{k}}(-x)^j\right)$.

We omit the proof of Lemma 3.3 for $k>1$. The detailed proof for $k=1$ can be found in [14].

Remark 3.1 We note that for all $x\in[0,1]$

$$
\lim_{k\to\infty} V_{I^k_\alpha}(1,x) = (1-x)\left(-x^\alpha\sum_{i=1}^{\infty}(-x)^{i-1}+\sum_{j=0}^{\infty}(-x)^j\right) = \frac{(1-x)(1-x^\alpha)}{1+x} =: V_{I^\infty_\alpha}(1,x)\ .
$$

3.2 Process Im-G

Let $\{Z(t), t\ge 0\}$ be a centered stationary Gaussian process with continuous sample paths a.s. and covariance function $R_Z(\cdot)$ that satisfies the following assumptions:

Z1: $R_Z(\cdot)$ is continuous, nonnegative and decreasing on $[0,\infty)$;
Z2: there exist $\beta>0$ and $t_0>0$ such that $R_Z(t)<t^{-\beta}$ for $t>t_0$;
Z3: $R_Z(t)=1-\lambda t^\alpha+o(t^\alpha)$ as $t\to 0^+$ for some $\alpha\in(0,2]$.

Example 3.1 An important class of stationary Gaussian processes that satisfies assumptions **Z1**–**Z3** is a class of fractional Ornstein–Uhlenbeck processes (**FOU**). **FOU** with parameter $\alpha \in (0,2]$ is a centered stationary Gaussian process $\{U_\alpha(t), t \geq 0\}$ with continuous sample paths a.s. and covariance function $\mathbf{Cov}(U_\alpha(t), U_\alpha(s)) = \exp(-|t-s|^\alpha)$. In a special case, when $\alpha = 1$, the process U_1 is the classical Ornstein–Uhlenbeck process. The Ornstein–Uhlenbeck process plays an important role in the theory of Gaussian fluid models.

Example 3.2 Another example of stationary Gaussian process that satisfies assumptions **Z1**–**Z3** is a process which is the Lamperti transformation of $\{I_\alpha(t), t \geq 0\}$. Namely $\{Z(t) = e^{-(\alpha/2)t} I_\alpha(e^t), t \geq\}$. If I_α is **FBm** then

$$R_Z(t) \;=\; \cosh\left(\frac{\alpha}{2}t\right) - 2^{\alpha-1}\sinh^\alpha\left(\frac{t}{2}\right)\,, \quad \text{and} \quad R_Z(t) \;=\; 1 - \frac{t^\alpha}{2} + o(t^\alpha) \quad \text{as} \quad t \to 0^+\,.$$

Example 3.3 Consider a stationary Gaussian process $\{Z(t), t \geq 0\}$ with covariance function

$$\mathbf{Cov}(Z(t), Z(s)) \;=\; \frac{2\exp(-|t-s|/2)}{1+\exp(-|t-s|)}\,.$$

This process has sample paths of class $C^\infty[0,\infty)$ a.s. and satisfies assumptions **Z1**–**Z3** with $\alpha = 2$. An application of this process in the analysis of real zeros of random polynomials can be found in Shao [7].

Let us define a process which is integral mean of stationary Gaussian process.

Definition 3.3 *A centered Gaussian process* $\{X^1(t), t \geq 0\}$ *of the form*

$$X^1(t) = \begin{cases} \frac{1}{t}\int_0^t Z(s)ds & \text{if} \quad t > 0 \\ Z(0) & \text{if} \quad t = 0 \end{cases}$$

is said to be the **Im-G** *(Integral mean of Gaussian).*

Lemma 3.4 *Under assumptions* **Z1**–**Z3***, the* **Im-G** *has continuous and bounded sample paths a.s. Furthermore* $\sigma^2_{X^1}(\cdot)$ *attains its global maximum at a unique point* $t^* = 0$ *and* $\sigma^2_{X^1}(0) = 1$*. The* X^1 *is also* I^1_α*–locally self-similar at the point* $t^* = 0$ *and*

$$\sigma_{X^1}(t) \;=\; 1 - \frac{1}{(\alpha+1)(\alpha+2)}\,t^\alpha + o(t^\alpha) \qquad as \quad t \to 0^+\,;$$

$$1 - \mathbf{Cov}\left(\bar{X}^1(t), \bar{X}^1(s)\right) \;=\; \frac{1}{\alpha+2}\,V_{I^1_\alpha}(t,s)(1+o(1)) \qquad as \quad t,s \to 0^+\,.$$

A detailed proof of Lemma 3.4 can be found in [14].

The following corollary is an immediate consequence of Lemma 3.4 and Theorem 2.4.

Corollary 3.5 *Under assumptions* **Z1**–**Z3***, with* $R = \frac{1}{\alpha+1}$*, we have*

$$\mathbf{P}\left(\sup_{t\geq 0} X^1(t) > u\right) \;\sim\; \mathcal{F}^R_{I^1,\alpha}\,\Psi(u) \qquad as \quad u \to \infty\,.$$

3.3 Process Im-G of order k

Let us define a process which is k-fold integrated stationary Gaussian process.

Definition 3.4 *By* $\{Z^k(t), t \geq 0\}$ *we denote a centered Gaussian process such that*

$$\begin{aligned} Z^0(t) &:= Z(t) && \textit{for} \quad k = 0 \; ; \\ Z^k(t) &:= \int_0^t Z^{k-1}(s)ds && \textit{for} \quad k > 0 \; . \end{aligned}$$

Lemma 3.6 *Under assumptions* **Z1**–**Z3** *the process* Z^k *has sample paths in* $C^k[0,\infty)$ *a.s. and*

$$R_{Z^k}(t,s) \;=\; \frac{(ts)^k}{(k!)^2}\left(1 - \frac{t^\alpha + s^\alpha}{\binom{\alpha+k}{k}}\right) - 2C^k(\alpha; t, s)(1 + o(1)) \qquad as \quad t, s \to 0^+ \; .$$

Proof:
We have

$$\begin{aligned} R_{Z^1}(t,s) &= \mathbf{Cov}\left(\int_0^t Z^0(u)du, \int_0^s Z^0(v)dv\right) \;=\; \int_0^t \int_0^s \mathbf{Cov}\left(Z^0(u), Z^0(v)\right) dudv \\ &= \int_0^t \int_0^s 1 - |u-v|^\alpha (1 + o(1))\; dudv \;=\; ts - \frac{t^{\alpha+2} + s^{\alpha+2} - |t-s|^{\alpha+2}}{(\alpha+1)(\alpha+2)}(1 + o(1)) \\ &= ts - \frac{2C^0(\alpha+2; t, s)}{(\alpha+1)(\alpha+2)}(1 + o(1)) \qquad \text{as} \quad t, s \to 0^+ \; , \end{aligned}$$

Using formula *i)* of Lemma 3.1 we obtain inductively that

$$\begin{aligned} R_{Z^k}(t,s) &= \int_0^t \int_0^s R_{Z^{k-1}}(u,v)dudv \;=\; \frac{(ts)^k}{(k!)^2} - \frac{2C^{k-1}(\alpha+2; t, s)}{(\alpha+1)(\alpha+2)}(1 + o(1)) \\ &= \frac{(ts)^k}{(k!)^2} - \frac{(ts)^k(t^\alpha + s^\alpha)}{k! \prod_{i=1}^k (\alpha+i)} + 2\left(\frac{1}{2}\,\frac{(ts)^k(t^\alpha + s^\alpha)}{k! \prod_{i=1}^k (\alpha+i)} \; - \; \frac{C^{k-1}(\alpha+2; t, s)}{(\alpha+1)(\alpha+2)}\right)(1 + o(1)) \\ &= \frac{(ts)^k}{(k!)^2}\left(1 - \frac{t^\alpha + s^\alpha}{\binom{\alpha+k}{k}}\right) - 2C^k(\alpha; t, s)(1 + o(1)) \qquad \text{as} \quad t, s \to 0^+ \; . \end{aligned}$$

This completes the proof.

□

Definition 3.5 *A centered Gaussian process* $\{X^k(t), t \geq 0\}$ *of the form*

$$X^k(t) \;=\; \begin{cases} k!\, \frac{Z^k(t)}{t^k} & \textit{if} \quad t > 0 \\ Z(0) & \textit{if} \quad t = 0 \end{cases}$$

is said to be the **Im-G of order k**.

Lemma 3.7 *Under assumptions* **Z1**–**Z3** *the* **Im–G of order k** *has sample paths in* $C^k[0,\infty)$ *and bounded a.s. Furthermore* $\sigma^2_{X^k}(\cdot)$ *attains its global maximum at a unique point* $t^* = 0$ *and* $\sigma^2_{X^k}(0) = 1$. *The* X^k *is also* I^k_α*–locally self-similar at the point* $t^* = 0$ *and*

$$\begin{aligned}
\sigma_{X^k}(t) &= 1 - \frac{k}{(\alpha+2k)\binom{\alpha+k}{k}}\, t^\alpha + o(t^\alpha) \qquad \textit{as} \quad t \to 0^+ \; ; \\
1 - \mathbf{Cov}\left(\bar{X}^k(t), \bar{X}^k(s)\right) &= \frac{\alpha+k}{(\alpha+2k)\binom{\alpha+k}{k}}\, V_{I^k_\alpha}(t,s)(1+o(1)) \qquad \textit{as} \quad t,s \to 0^+ \; .
\end{aligned}$$

Proof:
The proof of the first part of Lemma 3.7 is analogous to the proof of Lemma 3.4. Using Lemma 3.6 we calculate

$$\begin{aligned}
R_{X^k}(t,s) &= \frac{R_{Z^k}(t,s)}{(ts)^k}(k!)^2 \;=\; 1 - \frac{t^\alpha + s^\alpha}{\binom{\alpha+k}{k}} + \frac{2(k!)^2}{(ts)^k} C^k(\alpha; t,s)(1+o(1)) \\
&= 1 - \frac{t^\alpha + s^\alpha}{\binom{\alpha+k}{k}} + \frac{2(k!)^2}{(\alpha+2k)k!\prod_{i=1}^{k-1}(\alpha+i)} R^k(\alpha; t,s)(1+o(1)) \\
&= 1 - \frac{t^\alpha + s^\alpha}{\binom{\alpha+k}{k}} + \frac{2(\alpha+k)}{(\alpha+2k)\binom{\alpha+k}{k}} R^k(\alpha; t,s)(1+o(1)) \qquad \text{as} \quad t,s \to 0^+ \; .
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma^2_{X^k}(t) &= R_{X^k}(t,t) \;=\; 1 - \frac{2t^\alpha}{\binom{\alpha+k}{k}} + \frac{2(\alpha+k)}{(\alpha+2k)\binom{\alpha+k}{k}}\, t^\alpha(1+o(1)) \\
&= 1 - \frac{2k}{(\alpha+2k)\binom{\alpha+k}{k}}\, t^\alpha + o(t^\alpha) \qquad \text{as} \quad t \to 0^+
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Var}\left(X^k(t) - X^k(s)\right) &= R_{X^k}(t,t) + R_{X^k}(s,s) - 2R_{X^k}(t,s) \\
&= \frac{2(\alpha+2k)(t^\alpha+s^\alpha) - 2k(t^\alpha+s^\alpha) - 4(\alpha+k)R^k(\alpha;t,s)}{(\alpha+2k)\binom{\alpha+k}{k}}(1+o(1)) \\
&= \frac{2(\alpha+k)}{(\alpha+2k)\binom{\alpha+k}{k}}\, V^k(\alpha; t,s)(1+o(1)) \qquad \text{as} \quad t,s \to 0^+ \; .
\end{aligned}$$

This completes the proof. □

The following corollary is an immediate consequence of Lemma 3.7 and Theorem 2.4.

Corollary 3.8 *Under assumptions* **Z1**–**Z3**, *with* $R = \frac{k}{\alpha+k}$, *we have*

$$\mathbf{P}\left(\sup_{t\geq 0} X^k(t) > u\right) \sim \mathcal{F}^R_{I^k,\alpha}\, \Psi(u) \qquad \textit{as} \quad u \to \infty \; .$$

Remark 3.2 Using Corollary 2.2 with $R = \frac{k}{\alpha+k}$ we obtain some bounds for constants that appear in Corollary 3.8:

$$\begin{aligned}
\mathcal{F}_{I^k,\alpha}^{k/(\alpha+k)} &\geq \frac{1}{2}\left(1+\sqrt{2+\frac{2}{k}}\right) = \mathcal{F}_{I^k,2}^{k/(2+k)} \qquad \text{for} \quad \alpha \in (0,2]\ ; \\
\mathcal{F}_{I^k,\alpha}^{k/(\alpha+k)} &\leq 2+\frac{\alpha}{k} \qquad \text{for} \quad \alpha \in [1,2)\ .
\end{aligned}$$

Note that $\mathcal{F}_{I^k,\alpha}^{k/(\alpha+k)}$ are directly connected to the classical Pickands' constants. Indeed, using self-similarity of the **Im-FBm of order k** (and substituting $s = t\sqrt{C_\alpha^k}^{1/\alpha}$) we obtain that

$$\begin{aligned}
\mathcal{F}_{I^k,\alpha}^{k/(\alpha+k)} &= \mathbf{E}\exp\left(\sup_{t\in[0,\infty)} \frac{\sqrt{2}}{\sqrt{C_\alpha^k}} I_\alpha\left(t(C_\alpha^k)^{1/\alpha}\right) - \frac{\alpha+2k}{\alpha+k}\, t^\alpha\right) \\
&= \mathbf{E}\exp\left(\sup_{s\in[0,\infty)} \sqrt{2}\frac{I_\alpha(s)}{\sqrt{C_\alpha^k}} - \frac{s^\alpha}{k!\prod_{i=1}^k(\alpha+i)}\right) \\
&= \mathbf{E}\exp\left(\sup_{t\in(0,\infty)} \frac{k!}{t^k}\left(\sqrt{2}\frac{B_\alpha^k(t)}{k!} - \frac{1}{(k!)^2}\frac{t^{\alpha+k}}{\prod_{i=1}^k(\alpha+i)}\right)\right) \\
&= \mathbf{E}\exp\left(\sup_{t\in(0,\infty)} \frac{k!}{t^k}\int_0^t\int_0^{s_k}\dots\int_0^{s_2}\sqrt{2}\left(\frac{B_\alpha(s)}{k!}\right) - \mathbf{Var}\left(\frac{B_\alpha(s)}{k!}\right) ds ds_2\dots ds_k\right) .
\end{aligned}$$

For $k=1$ we have an elegant formula:

$$\mathcal{F}_{I^1,\alpha}^{1/(\alpha+1)} = \mathbf{E}\exp\left(\sup_{t\in(0,\infty)} \frac{1}{t}\int_0^t \sqrt{2}B_\alpha(s) - s^\alpha\ ds\right) .$$

3.4 Process IIm-FBm

In Remark 3.1 we have seen that there exists the function $V_{I_\alpha^\infty}(1,\cdot)$ such that

$$V_{I_\alpha^\infty}(1,x) := \frac{(1-x)(1-x^\alpha)}{1+x} = \lim_{k\to\infty} V_{I_\alpha^k}(1,x) \qquad \text{for} \quad x\in[0,1]\ .$$

Thus we can define a Gaussian process $\{I_\alpha^\infty(t), t\geq 0\}$, which is self-similar with index $\alpha/2$ and such that $V_{I_\alpha^\infty}(t,s) = t^\alpha V_{I_\alpha^\infty}(1,s/t)$ for $0\leq s\leq t$.

Definition 3.6 *A Gaussian process $\{I_\alpha^\infty(t), t\geq 0\}$ with covariance function*

$$\mathbf{Cov}\left(I_\alpha^\infty(t), I_\alpha^\infty(s)\right) = \frac{t^\alpha s + s^\alpha t}{t+s}$$

is said to be the **IIm-FBm** *(Infinitely integral mean of FBm) with Hurst parameter $\alpha/2\in(0,1]$.*

Remark 3.3 The **IIm-FBm** has sample paths in $C^\infty[0,\infty)$ a.s. and satisfies assumptions **V1** and **V2**. It turns out that I_α^∞ has another interesting representation. Namely

$$\{I_\alpha^\infty(t), t \geq 0\} \stackrel{\mathcal{D}}{=} \left\{ t^{\alpha+1}\sqrt{\frac{2}{\Gamma(\alpha+1)}}\int_0^\infty B_\alpha(u)e^{-ut}du \; , \; t \geq 0 \right\} .$$

In a special case, when $\alpha = 1$, Lamperti's transformation of I_1^∞ is the process from Remark 3.3.

Consider a Gaussian process $\{X^\infty(t), t \in [0,1]\}$ of the form

$$X^\infty(t) \;=\; I_\alpha^\infty(t) - I_\alpha^\infty(1) \; .$$

Since $\mathbf{Var}\,(X^\infty(t) - X^\infty(s)) = \mathbf{Var}\,(I_\alpha^\infty(t) - I_\alpha^\infty(s))$, then X^∞ is I_α^∞–locally self-similar at each point $t \in [0,1]$ and function $\sigma^2_{X^\infty}(\cdot) = V_{I_\alpha^\infty}(1,\cdot)$ attains its maximum at a point $t^* = 0$. Furthermore

$$\lim_{t\to 0^+} \frac{1-\sigma^2_{X^\infty}(t)}{t^\beta} \;=\; \lim_{t\to 0^+} \frac{2t + t^\alpha - t^{\alpha+1}}{(t+1)t^\beta} \;=\; \begin{cases} 1 & \text{if} \quad \alpha < 1 \quad \text{and} \quad \beta = \alpha \\ 3 & \text{if} \quad \alpha = 1 \quad \text{and} \quad \beta = \alpha \\ 2 & \text{if} \quad \alpha > 1 \quad \text{and} \quad \beta = 1 < \alpha \end{cases} .$$

We can use Theorem 2.4 again to get the following corollary.

Corollary 3.9 *For the process $\{X^\infty(t), t \in [0,1]\}$ the following statement holds:*

$$\mathbf{P}\left(\sup_{t\in[0,1]} X^\infty(t) > u\right) \;\sim\; \Psi(u)\cdot \begin{cases} \mathcal{F}^1_{I^\infty,\alpha} & \textit{if} \quad \alpha < 1 \\ \mathcal{F}^3_{I^\infty,1} & \textit{if} \quad \alpha = 1 \\ 1 & \textit{if} \quad \alpha > 1 \end{cases} \qquad \textit{as} \quad u \to \infty \; .$$

Remark 3.4 Using Corollary 2.2 we can obtain some bounds for constants that appear in Corollary 3.9:

$$\mathcal{F}^1_{I^\infty,\alpha} \;\geq\; \frac{1+\sqrt{2}}{2} \qquad \text{for} \quad \alpha \in (0,1) \; , \qquad \mathcal{F}^3_{I^\infty,1} \;\leq\; \frac{4}{3} \; .$$

It is interesting to note that $\mathcal{F}^1_{I^\infty,\alpha} > 1$ for $\alpha \in (0,1)$ even though sample paths of the process X^∞ are in $C^\infty[0,\infty)$.

4 Applications

In this section we briefly present applications of our main result to the theory of Gaussian fluid models and to collision probability for Gaussian processes with differentiable sample paths. The notions are described in [14] more precisely.

4.1 Gaussian fluid models

Gaussian fluid models can approximate some telecommunication models, for example a data stream in LAN networks (*Local Area Network*) or ATM (*Asynchronous Transfer Mode*) (see [6] for more details).

Consider a buffer with infinite capacity. An input rate to the buffer is generated by n independent and statistically identical sources $\{Z_j(t), t \geq 0\}$ for $j = 1, \ldots, n$. Each source Z_j is a centered stationary Gaussian process. An output from the buffer is at rate cn. Thus $\left\{\sum_{j=1}^n Z_j(t), t \geq 0\right\} \stackrel{d}{=} \{\sqrt{n}\, Z(t), t \geq 0\}$ is a total input rate.

A stationary process $\{Q_n^*(t), t \geq 0\}$ defined by

$$Q_n^*(t) \;=\; \sup_{s \geq t} \int_t^s \left(\sum_{j=1}^n Z_j(t) - cn \right) dv$$

gives the representation for the stationary buffer content process and it is one of the important characteristics in fluid model theory. We focus on the analysis of

$$Q_n^* \;=\; Q_n^*(0) \;\stackrel{\mathcal{D}}{=}\; \sup_{t \geq 0} \left(\int_0^t \sum_{j=1}^n Z_j(s) ds - cnt \right) .$$

We are interested in a *non-emptiness probability* of the buffer under the steady state condition, i.e. $\mathbf{P}(Q_n^* > 0)$.

Remark 4.1 Asymptotical properties of Q_n^* were analyzed by Dębicki and Mandjes [5], where exact asymptotic of $\mathbf{P}(Q_n^* > bn)$ as $n \to \infty$ for fixed $b > 0$ was found. The method used by them was based on an appropriate use of Theorem 1.2. This approach does not work for the case of $\mathbf{P}(Q_n^* > 0)$.

Using Corollary 3.5 we are able to investigate exact asymptotic of the non-emptiness probability $\mathbf{P}(Q_n^* > 0)$ as $n \to \infty$.

Theorem 4.1 *If $R_{Z_1}(\cdot)$ satisfies assumptions* **Z1**–**Z3***, then*

$$\mathbf{P}(Q_n^* > 0) \;\sim\; \mathcal{F}_{I^1,\alpha}^{1/(\alpha+1)}\, \Psi\left(c\sqrt{n}\right) \qquad as \quad n \to \infty .$$

A detailed proof of Theorem 4.1 can be found in [14]. It is based on the observation that

$$\mathbf{P}(Q_n^* > 0) \;=\; \mathbf{P}\left(\sup_{t>0} \frac{1}{t} \int_0^t Z_1(s) ds > c\sqrt{n} \right) .$$

4.2 Collision probability

Consider two independent centered stationary Gaussian processes $\{Z_1(t), t \geq 0\}$ and $\{Z_2(t), t \geq 0\}$ with covariance functions $R_{Z_1}(\cdot)$ and $R_{Z_2}(\cdot)$ respectively.

For $C_1 > C_2$ we define processes $\{\eta_1^k(t), t \geq 0\}$ and $\{\eta_2^k(t), t \geq 0\}$:

$$\begin{aligned} \eta_i^0(t) &:= Z_i(t) + C_i && \text{for} \quad k = 0 \; ; \\ \eta_i^k(t) &:= \int_0^t \eta_i^{k-1}(s) ds && \text{for} \quad k > 0 \; , \qquad i = 1, 2 \; . \end{aligned}$$

We focus on a *collision probability* of the processes η_1^k and η_2^k, i.e.

$$\mathbf{P}_{col}^k(C_1, C_2) \;=\; \mathbf{P}\left(\exists_{t>0}\, \eta_2^k(t) = \eta_1^k(t)\right) \;=\; \mathbf{P}\left(\sup_{t\geq 0}\, \eta_2^k(t) - \eta_1^k(t) > 0\right) \; .$$

For $k = 1$ the problem above is a standard collision problem. In this case the processes $\eta_i^1(t) = \int_0^t Z_i(s)ds + C_i t$ are Gaussian processes with stationary increments, differentiable sample paths a.s. and linear drifts.

Remark 4.2 Collison probabilities for Gaussian processes with stationary increments were analyzed in many papers (see for example [13] and [8]). In case of **FBm** with linear drifts the collision probability is equal 1. For processes with differentiable sample paths a similar statement does not hold.

Using Corollary 3.8 we are able to investigate exact asymptotic of the collision probability $\mathbf{P}_{col}^k(C_1, C_2)$ as $C_1 - C_2 \to \infty$.

Theorem 4.2 *If R_{Z_1} and R_{Z_2} satisfy assumptions* **Z1**–**Z3** *with $\alpha_1, \alpha_2 \in (0, 2]$ respectively, then*

$$\mathbf{P}_{col}^k(C_1, C_2) \;\sim\; \mathcal{F}_{I^k,\alpha}^{k/(\alpha+k)}\; \Psi\left(\frac{C_1 - C_2}{\sqrt{2}}\right) \qquad \text{as} \quad C_1 - C_2 \to \infty \; ,$$

where $\alpha = \min(\alpha_1, \alpha_2)$.

A detailed proof for $k = 1$ can be found in [14]. For $k > 0$ the proof is analogous and it is based on the fact that

$$\mathbf{P}_{col}^k(C_1, C_2) \;=\; \mathbf{P}\left(\sup_{t\geq 0} X^k(t) > \frac{C_1 - C_2}{\sqrt{2}}\right) \; ,$$

where process $\{X^k(t), t \geq 0\}$ is **Im-G of order k** for a process $\left\{Z(t) = \frac{Z_2(t) - Z_1(t)}{\sqrt{2}}, t \geq 0\right\}$ that satisfies assumptions **Z1**–**Z3** with $\alpha = \min(\alpha_1, \alpha_2)$.

5 Proofs

In this section we present proofs of Lemma 2.5 and Theorem 2.3. We begin with the fact that X is *regular* in a neighborhood of t^* (assumption *c)* of Theorem 1.2).

Lemma 5.1 *Suppose that for a process* $\{\xi(t), t \in [0,T]\}$ *and a point* $t^* \in [0,T]$ *there exist* $\beta, A_1 > 0$ *such that*

$$\sigma_\xi(t+t^*) \ = \ 1 - A_1|t|^\beta + o(|t|^\beta) \quad \textit{as} \quad t \to 0 \ .$$

Then

$$\lim_{t,s\to 0} \frac{\mathbf{Var}(\bar{\xi}(t+t^*) - \bar{\xi}(s+t^*))}{\mathbf{Var}(\xi(t+t^*) - \xi(s+t^*))} \ = \ 1 \ .$$

An easy proof of the lemma can be found in [14].

Corollary 5.2 *Under the assumptions of Theorem 2.3, there exist* $\delta, A_3 > 0$ *such that*

$$\mathbf{Var}\left(X(t) - X(s)\right) \ \le \ A_3|t-s|^\alpha \qquad \textit{and} \qquad \mathbf{Var}\left(\bar{X}(t) - \bar{X}(s)\right) \ \le \ A_3|t-s|^\alpha$$

for all $t, s \in \Delta$*, where* $\Delta = [0,\delta]$ *if* $t^* = 0$*, or* $\Delta = [t^*-\delta, t^*+\delta]$ *if* $t^* > 0$*.*

Proof:
From **V2** and assumption *b)* of Theorem 2.3 we conclude that for all $A_3 > 2A_2$ there exist $\delta > 0$ such that

$$\mathbf{Var}\left(\bar{X}(t) - \bar{X}(s)\right) \ = \ 2A_2\, V_{I_\alpha}(t,s)(1+o(1)) \ \le \ 2A_2\ |t-s|^\alpha\,(1+o(1)) \ \le \ A_3\,|t-s|^\alpha$$

for $t, s \in \Delta$. Analogously, from Lemma 5.1 it can be shown that

$$\mathbf{Var}\left(X(t) - X(s)\right) \ \le \ A_3\ |t-s|^\alpha$$

for all $t, s \in \Delta$. This completes the proof. □

5.1 Proof of Lemma 2.5

We give the proof only for $t^* = 0$. The proof for $t^* > 0$ is analogous.
For all $u > 0$ we have

$$\begin{aligned}
&\mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u\right) \\
&= \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2}\, \mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u \,\middle|\, X(0) = v\right) dv \\
&= \ \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \int_{-\infty}^{\infty} e^{w - w^2/2u^2}\, \mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u \,\middle|\, X(0) = u - \frac{w}{u}\right) dw \ ,
\end{aligned}$$

where the last equality is a consequence of changing of variables $v = u - \frac{w}{u}$.
Note that if we define a Gaussian process

$$\left\{\chi_u(t) = \left(u\left(X\left(tu^{-2/\alpha}\right) - u\right) + w \;\middle|\; X(0) = u - \frac{w}{u}\right), t \in [0,T]\right\} ,$$

then

$$\mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) = \mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u \;\middle|\; X(0) = u - \frac{w}{u}\right) .$$

Therefore

$$\mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u\right) = \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \int_{-\infty}^{\infty} e^{w - w^2/2u^2}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) dw .$$

From (2) we conclude that in order to prove Lemma 2.5 it suffices to show that

$$\lim_{u\to\infty} \int_{-\infty}^{\infty} e^{w - w^2/2u^2}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) dw = \mathcal{F}^R_{I,\alpha}\left(TA_2^{1/\alpha}\right) . \qquad (7)$$

Before we do this we show that $\{\chi_u(t), t \in [0,T]\}$ is weakly convergent (in $C[0,T]$ with uniform metric) to $\left\{\sqrt{2}I_\alpha(s) - (1+R)s^\alpha, s \in [0, TA_2^{1/\alpha}]\right\}$ as $u \to \infty$. First we prove convergence of finite dimensional distributions and tightness of the family $\{\chi_u(t), t \in [0,T]\}$ as $u \to \infty$.

1) Convergence of finite dimensional distributions

Assumptions (4) and (5) imply the following asymptotics as $t \to 0^+$:

$$\begin{aligned}
\sigma_X^2(t) &= 1 - 2A_1 t^\alpha + o(t^\alpha) ;\\
R_X(t,0) &= \frac{1}{2}\left(\sigma_X^2(t) + \sigma_X^2(0) - \mathbf{Var}(X(t) - X(0))\right) = 1 - (A_1 + A_2)t^\alpha + o(t^\alpha) ;\\
R_X^2(t,0) &= \left(1 - (A_1 + A_2)t^\alpha + o(t^\alpha)\right)^2 = 1 - 2(A_1 + A_2)t^\alpha + o(t^\alpha) .
\end{aligned}$$

From Corollary A.6 we get that for all $t, s \in [0,T]$:

$$\begin{aligned}
\mathbf{E}(\chi_u(t)) &= -u^2\left(1 - R_X(tu^{-2/\alpha},0)\right) + w\left(1 - R_X(tu^{-2/\alpha},0)\right) ;\\
\mathbf{Var}(\chi_u(t)) &= u^2\left(\sigma_X^2(tu^{-2/\alpha}) - R_X^2(tu^{-2/\alpha},0)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{Var}(\chi_u(t) - \chi_u(s)) =\\
&= u^2\,\mathbf{Var}\left(X(tu^{-2/\alpha}) - X(su^{-2/\alpha})\right) - u^2\left(\mathbf{Cov}\left(X(tu^{-2/\alpha}) - X(su^{-2/\alpha}), X(0)\right)\right)^2\\
&= u^2\,\mathbf{Var}\left(X(tu^{-2/\alpha}) - X(su^{-2/\alpha})\right) - u^2\left(R_X(tu^{-2/\alpha},0) - R_X(su^{-2/\alpha},0)\right)^2 .
\end{aligned}$$

This gives for $t, s \in [0,T]$ and $u \to \infty$

$$\mathbf{E}(\chi_u(t)) = -(A_1 + A_2)\, t^\alpha + w\, o(1) ; \qquad (8)$$

$$\mathbf{Var}(\chi_u(t)) = 2A_2\, t^\alpha + o(1) ; \qquad (9)$$

$$\mathbf{Var}(\chi_u(t) - \chi_u(s)) = 2A_2\, V_{I_\alpha}(t,s)(1 + o(1)) . \qquad (10)$$

In equalities (9) and (10), $o(1)$ are uniform in w (since a conditional variance does not depend on partition values of the condition). Thanks to (8), (9) and (10) we conclude that finite dimensional distributions of $\{\chi_u(t), t\in[0,T]\}$ converge to finite dimensional distributions of $\left\{\chi(t)=\sqrt{2A_2}I_\alpha(t)-(A_1+A_2)t^\alpha, t\in[0,T]\right\}$ as $u\to\infty$.
Note that self-similarity of the process I_α (after substituting $s=tA_2^{1/\alpha}$) implies that

$$\left\{\sqrt{2A_2}I_\alpha(t)-(A_1+A_2)t^\alpha, t\in[0,T]\right\} \overset{\mathcal{D}}{=} \left\{\sqrt{2}I_\alpha(s)-(1+R)s^\alpha, s\in[0,TA_2^{1/\alpha}]\right\}. \quad (11)$$

2) Tightness
We use a criterion for tightness of sequence $\{\chi_u(t), t\in[0,T]\}$ given in Theorem A.7. Since $\chi_u(0)=0$ a.s. for all $u>0$, then condition *i)* of Theorem A.7 is satisfied. It remains to prove that for any $\epsilon,\rho>0$ there exist $\delta_{\epsilon,\rho}\in(0,T)$ and $u_0>0$ such that

$$\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}|\chi_u(s)-\chi_u(t)|\ge\epsilon\right) \le \rho\,\delta_{\epsilon,\rho}$$

for all $t\in[0,T]$ and $u\ge u_0$. Note that we need to check the above condition only for a centered Gaussian process $\left\{\chi_u^{(0)}(t)=\chi_u(t)-\mathbf{E}(\chi_u(t)), t\in[0,T]\right\}$. It follows from the fact that

$$\begin{aligned}
&\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}|\chi_u(s)-\chi_u(t)|\ge\epsilon\right)\\
&= \mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}\left|\chi_u^{(0)}(s)-\chi_u^{(0)}(t)+\mathbf{E}(\chi_u(s))-\mathbf{E}(\chi_u(t))\right|\ge\epsilon\right)\\
&\le \mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}\left|\chi_u^{(0)}(s)-\chi_u^{(0)}(t)\right|+\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}|\mathbf{E}(\chi_u(s))-\mathbf{E}(\chi_u(t))|\ge\epsilon\right)\\
&\le \mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}\left|\chi_u^{(0)}(s)-\chi_u^{(0)}(t)\right|\ge\frac{\epsilon}{2}\right). \qquad (12)
\end{aligned}$$

From (8) we deduce that there exists a constant $C>0$ such that

$$\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}|\mathbf{E}(\chi_u(t))-\mathbf{E}(\chi_u(s))| \le C(|t+\delta_{\epsilon,\rho}|^\alpha-t^\alpha) \le \begin{cases} C(\delta_{\epsilon,\rho})^\alpha & \text{if } \alpha\in(0,1]\\ C\delta_{\epsilon,\rho}\,\alpha T^{\alpha-1} & \text{if } \alpha\in[1,2]\end{cases}$$

for sufficiently large u. Hence we can choose $\delta_{\epsilon,\rho}$ such that $\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}|\mathbf{E}(\chi_u(t))-\mathbf{E}(\chi_u(s))|\le\frac{\epsilon}{2}$ for all $t\in[0,T]$. This proves (12).

It remains to show that for any $\epsilon,\rho>0$ there exist $\delta_{\epsilon,\rho}\in(0,T)$ and $u_0>0$ such that

$$\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]}\left|\chi_u^{(0)}(s)-\chi_u^{(0)}(t)\right|\ge\epsilon\right) \le \rho\,\delta_{\epsilon,\rho} \quad (13)$$

for all $t\in[0,T]$ and $u\ge u_0$.
Fix $\epsilon,\rho>0$. For all $t\in[0,T]$ we analyze a centered Gaussian process

$$\left\{\chi_{u,t}^{(0)}(s)=\chi_u^{(0)}(s)-\chi_u^{(0)}(t), s\in[t,t+\delta_{\epsilon,\rho}]\right\}.$$

Let δ and A_3 be such the constants that the thesis of Corollary 5.2 holds. Note that for all u such that $Tu^{-2/\alpha} \leq \delta$

$$\mathbf{Var}(\chi_u(s) - \chi_u(v)) \;\leq\; u^2\, \mathbf{Var}\left(X(su^{-2/\alpha}) - X(vu^{-2/\alpha})\right) \;\leq\; A_3\, |s-v|^{\alpha} \;\leq\; A_3 T^{\alpha} \quad (14)$$

for all $s, v \in [0,T]$. It follows the existence of u_0 such that

$$\mathbf{Var}\left(\chi_{u,t}^{(0)}(s) - \chi_{u,t}^{(0)}(v)\right) \;=\; \mathbf{Var}(\chi_u(s) - \chi_u(v)) \;\leq\; A_3\, \mathbf{Var}(B_\alpha(s) - B_\alpha(v)) \;\leq\; A_3 (\delta_{\epsilon,\rho})^{\alpha}$$

for all $s, v \in [t, t+\delta_{\epsilon,\rho}]$ and for $u \geq u_0$ independently of $t \in [0,T]$. Using Sudakov–Fernique's inequality (Theorem A.4) we obtain

$$\begin{aligned}
\mathbf{E}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \chi_{u,t}^{(0)}(s)\right) &\leq \mathbf{E}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} B_\alpha(s) - B_\alpha(t)\right) \\
&= \mathbf{E}\left(\sup_{s\in[0,\delta_{\epsilon,\rho}]} B_\alpha(s)\right) \;=\; (\delta_{\epsilon,\rho})^{\alpha/2}\, \mathbf{E}\left(\sup_{s\in[0,1]} B_\alpha(s)\right) .
\end{aligned}$$

Therefore, for sufficiently large $u \geq u_0$

$$\lim_{\delta_{\epsilon,\rho}\to 0} \mathbf{E}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \chi_{u,t}^{(0)}(s)\right) \;=\; 0\ ,$$

hence for $u \geq u_0$ there exists $\delta_{\epsilon,\rho}$ such that $\mathbf{E}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \chi_{u,t}^{(0)}(s)\right) \leq \frac{\epsilon}{2}$.

Assumptions of Borell's inequality (Theorem A.2) are satisfied for $\left\{\chi_{u,t}^{(0)}(s), s \in [t, t+\delta_{\epsilon,\rho}]\right\}$, $\mathbf{E}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \chi_{u,t}^{(0)}(s)\right) \leq \frac{\epsilon}{2} = m$ and $\sigma^2 = A_3(\delta_{\epsilon,\rho})^{\alpha}$. Corollary A.3 implies that for $u \geq u_0$

$$\begin{aligned}
\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \left|\chi_{u,t}^{(0)}(s)\right| \geq \epsilon\right) &\leq 2\,\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} \chi_{u,t}^{(0)}(s) \geq \epsilon\right) \\
&\leq 4\exp\left(-\frac{(\epsilon - \frac{\epsilon}{2})^2}{2A_3(\delta_{\epsilon,\rho})^{\alpha}}\right) \;=\; 4\exp\left(-\frac{\epsilon^2}{8A_3(\delta_{\epsilon,\rho})^{\alpha}}\right) .
\end{aligned}$$

Hence we can find $\delta_{\epsilon,\rho}$ such that

$$4\exp\left(-\frac{\epsilon^2}{8A_3(\delta_{\epsilon,\rho})^{\alpha}}\right) \;\leq\; \rho\, \delta_{\epsilon,\rho}\ .$$

It proves (13) for all $t \in [0,T]$ and, as a consequence, tightness of the family $\{\chi_u(t), t \in [0,T]\}$.

Thus from **1)** and **2)** the process $\{\chi_u(t), t \in [0,T]\}$ is weakly convergent to the process $\left\{\chi(t) = \sqrt{2A_2} I_\alpha(t) - (A_1 + A_2)t^{\alpha}, t \in [0,T]\right\}$ as $u \to \infty$.

3) Convergence of the integral

We need to show convergence of integral (7). Since $\{\chi_u(t), t \in [0,T]\}$ is weakly convergent and a functional $\Phi(f) = \sup_{t\in[0,T]} f(t)$ is continuous in uniform metric, it follows that for all $w \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) = \mathbf{P}\left(\sup_{t\in[0,T]} \chi(t) > w\right) .$$

The idea is based on an appropriate majorization of $\mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right)$ for sufficiently large u. From (14) and Sudakov–Fernique's inequality (Theorem A.4) we have

$$\mathbf{E}\left(\sup_{t\in[0,T]} \chi_u(t) - \mathbf{E}(\chi_u(t))\right) \le \mathbf{E}\left(\sup_{t\in[0,T]} \sqrt{A_3} B_\alpha(t)\right) = A_T . \tag{15}$$

Equality (8) implies that $\mathbf{E}(\chi_u(t)) \le \frac{1}{2}|w|$ for sufficiently large u. Combining it with (15) we conclude that $m = \mathbf{E}\left(\sup_{t\in[0,T]} \chi_u(t)\right) \le A_T + \frac{1}{2}|w|$. Borell's inequality (Corollary A.3) for $\sigma^2 = A_3 T^\alpha$ implies that

$$\mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) \le 2\exp\left(-\frac{(w-m)^2}{2A_3T^\alpha}\right) \le 2\exp\left(-\frac{(w - A_T - \frac{1}{2}|w|)^2}{2A_3T^\alpha}\right) .$$

Now we can apply the dominated convergence theorem to (7). We get

$$\int_{-\infty}^{\infty} e^{w-w^2/2u^2}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) dw \le \int_{-\infty}^{\infty} 2e^w \exp\left(-\frac{(w - \frac{1}{2}|w| - A_T)^2}{2A_3T^\alpha}\right) dw < \infty$$

for sufficiently large u, and as a consequence,

$$\lim_{u\to\infty} \int_{-\infty}^{\infty} e^{w-w^2/2u^2}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) > w\right) dw = \int_{-\infty}^{\infty} e^{w}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi(t) > w\right) dw .$$

Combining (11) with (3) we finally obtain

$$\begin{aligned}\int_{-\infty}^{\infty} e^{w}\, \mathbf{P}\left(\sup_{t\in[0,T]} \chi(t) > w\right) dw &= \int_{-\infty}^{\infty} e^{w}\, \mathbf{P}\left(\sup_{t\in[0,TA_2^{1/\alpha}]} \sqrt{2} I_\alpha(t) - (1+R)t^\alpha > w\right) dw \\ &= \mathcal{F}_{I,\alpha}^{R}\left(TA_2^{1/\alpha}\right) .\end{aligned}$$

This completes the proof. □

5.2 Proof of Theorem 2.3

We give the proof only for $t^* = 0$. The proof for $t^* > 0$ is analogous.

Let δ and A_3 be such the constants that the thesis of Corollary 5.2 holds. The argument similar to one in the proof of Theorem 2.4 implies that

$$\mathbf{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \sim \mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \quad \text{as} \quad u \to \infty . \tag{16}$$

Therefore, we only prove Theorem 2.3 on $[0,\delta]$.

Since $\sigma_X(t) = 1 - A_1 t^\beta + o(t^\beta)$ as $t \to 0^+$, then $\frac{1}{\sigma_X(t)} = 1 + A_1 t^\beta + o(t^\beta)$ as $t \to 0^+$ (because $\left(1 + A_1 t^\beta + o(t^\beta)\right)\left(1 - A_1 t^\beta + o(t^\beta)\right) = 1 + o(t^\beta)$ as $t \to 0^+$).
Hence there exists $A_4 > 0$ such that

$$\frac{1}{\sigma_X(t)} \geq 1 + A_4 t^\beta \qquad \text{for all} \quad t \in [0, \delta] \; . \tag{17}$$

Let $T > 0$. Denote $\Delta_k = kTu^{-2/\alpha}$ for $k = 1, \ldots, n+1$, where $n = \left[\frac{\delta}{Tu^{-2/\alpha}}\right]$. Then for $k = 1, \ldots, n$

$$\Delta_k \in [0, \delta] \qquad \text{and} \qquad [0, \delta] \subseteq [0, \Delta_1] \cup \bigcup_{k=1}^{n} [\Delta_k, \Delta_{k+1}] \; .$$

Thus we have the following inequality:

$$\mathbf{P}\left(\sup_{t \in [0, \Delta_1]} X(t) > u\right) \;\leq\; \mathbf{P}\left(\sup_{t \in [0, \delta]} X(t) > u\right) \;\leq\; \mathbf{P}\left(\sup_{t \in [0, \Delta_1]} X(t) > u\right) + S_n \; , \tag{18}$$

where $S_n = \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} X(t) > u\right)$. We will derive an upper bound for S_n. Using (17) we get

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} \frac{X(t)}{\max_{t \in [\Delta_k, \Delta_{k+1}]} \sigma_X(t)} > \frac{u}{\max_{t \in [\Delta_k, \Delta_{k+1}]} \sigma_X(t)}\right) \\
&\leq \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} \frac{X(t)}{\sigma_X(t)} > u \min_{t \in [\Delta_k, \Delta_{k+1}]} \frac{1}{\sigma_X(t)}\right) \\
&\leq \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} \bar{X}(t) > u \min_{t \in [\Delta_k, \Delta_{k+1}]} (1 + A_4 t^\beta)\right) \\
&= \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} \bar{X}(t) > u(1 + A_4 (\Delta_k)^\beta)\right) \\
&= \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t \in [\Delta_k, \Delta_{k+1}]} \bar{X}(t) > u + u^{1-2\beta/\alpha} A_4 (kT)^\beta\right) \; .
\end{aligned}$$

Now we consider a centered stationary Gaussian process $\{Z_C(t), t \in [0, \delta]\}$ with covariance function $\mathbf{Cov}(Z_C(t), Z_C(s)) = e^{-C|t-s|^\alpha}$ and $C = -\delta^{-\alpha} \log\left(1 - \frac{A_3}{2}\delta^\alpha\right)$. Note that for all $t, s \in [0, \delta]$ we have

$$\mathbf{Var}\left(\bar{X}(t) - \bar{X}(s)\right) \;\leq\; A_3 \, |t-s|^\alpha \;\leq\; 2\left(1 - e^{-C|t-s|^\alpha}\right) \;=\; \mathbf{Var}\left(Z_C(t) - Z_C(s)\right) \; . \tag{19}$$

A function $f(x) = 2\left(1 - e^{-Cx}\right)$ is concave for $x \in [0, \delta^\alpha]$. Furthermore $f(0) = 0$ and $f(\delta^\alpha) = 2\left(1 - \exp\log\left(1 - \frac{A_3}{2}\delta^\alpha\right)\right) = A_3 \delta^\alpha$. This implies that $A_3 x \leq 2\left(1 - e^{-Cx}\right)$ for all $x \in [0, \delta^\alpha]$. Substitution of $x = |t-s|^\alpha$ proves inequality (19).

Thus from Slepian's inequality (Theorem A.1) for the processes $\bar{X}$ and Z_C, we get (using stationarity of Z_C) that

$$
\begin{aligned}
S_n &\le \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t\in[\Delta_k,\Delta_{k+1}]} Z_C(t) > u + u^{1-2\beta/\alpha} A_4 (kT)^\beta \right) \\
&= \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t\in[0,\Delta_1]} Z_C(t) > u + u^{1-2\beta/\alpha} A_4 (kT)^\beta \right) \\
&= \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t\in[0,\Delta_1 C^{1/\alpha}]} \hat{Z}_C(t) > u + u^{1-2\beta/\alpha} A_4 (kT)^\beta \right) ,
\end{aligned}
$$

where $\left\{ \hat{Z}_C(t) = Z_C(tC^{-1/\alpha}), t \in [0, \Delta_1 C^{1/\alpha}] \right\}$ satisfies the assumptions of Theorem 1.1. Now observe that

$$
\begin{aligned}
S_n &\le \sum_{k=1}^{n} \mathbf{P}\left(\sup_{t\in[0,TC^{1/\alpha}u^{-2/\alpha}]} \hat{Z}_C(t) > u + u^{1-2\beta/\alpha} A_4 (kT)^\beta \right) \\
&= \sum_{k=1}^{n} \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) \, \Psi\left(u + u^{1-2\beta/\alpha} A_4 (kT)^\beta\right)(1+o(1)) \\
&= \frac{\mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right)}{\sqrt{2\pi}} \sum_{k=1}^{n} \frac{\exp\left(-\frac{1}{2}\left(u + u^{1-2\beta/\alpha} A_4 (kT)^\beta\right)^2\right)}{(u + u^{1-2\beta/\alpha} A_4 (kT)^\beta)} (1+o(1)) \\
&\le \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) \, \frac{1}{u\sqrt{2\pi}} e^{-u^2/2} (1+o(1)) \sum_{k=1}^{n} e^{-A_4 (kT)^\beta u^{2(1-\beta/\alpha)}} \\
&= \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) \Psi(u)(1+o(1)) \sum_{k=1}^{n} \left(e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \right)^{k^\beta} \\
&\le \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) \Psi(u)(1+o(1)) \, e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \sum_{k=1}^{\infty} \left(e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \right)^{k^\beta - 1} \quad \text{as} \quad u \to \infty .
\end{aligned}
$$

Using inequality $\log x \le x - 1$ for $x \ge 1$ we get

$$
\begin{aligned}
\sum_{k=1}^{\infty} \left(e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \right)^{k^\beta - 1} &\le \sum_{k=1}^{\infty} \left(e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \right)^{\log k^\beta} = \sum_{k=1}^{\infty} \left(\frac{1}{k} \right)^{\beta A_4 T^\beta u^{2(1-\beta/\alpha)}} \\
&\le 1 + \int_1^\infty \left(\frac{1}{t} \right)^{\beta A_4 T^\beta u^{2(1-\beta/\alpha)}} dt = 1 + \frac{1}{\beta A_4 T^\beta u^{2(1-\beta/\alpha)} - 1} = C_T
\end{aligned}
$$

if $\beta A_4 T^\beta u^{2(1-\beta/\alpha)} > 1$.

Consider the case $\beta = \alpha$. Then

$$
S_n \le \Psi(u)(1+o(1)) \, \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) C_T \, e^{-A_4 T^\alpha} \quad \text{as} \quad u \to \infty \tag{20}
$$

for sufficiently large $T > (\alpha A_4)^{-1/\alpha}$, where C_T is decreasing to 1. Since $\lim_{T\to\infty} \frac{\mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right)}{TC^{1/\alpha}} = \mathcal{H}_{B,\alpha}$, then $e^{-A_4 T^\alpha}$ is a dominant factor in (20).
Therefore, for every $\epsilon > 0$ there exists $T > (\alpha A_4)^{-1/\alpha}$ such that $S_n \le \epsilon \Psi(u)$ as $u \to \infty$.
Now we return to inequality (18). We conclude that for every $\epsilon > 0$ there exists sufficiently large $T > 0$ such that

$$\mathbf{P}\left(\sup_{t\in[0,\Delta_1]} X(t) > u\right) \le \mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \le \mathbf{P}\left(\sup_{t\in[0,\Delta_1]} X(t) > u\right) + \epsilon\, \Psi(u)$$

as $u \to \infty$. Applying Lemma 2.5 we get

$$\mathcal{F}^R_{I,\alpha}\left(TA_2^{1/\alpha}\right)\, \Psi(u)(1+o(1)) \le \mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \le \mathcal{F}^R_{I,\alpha}\left(TA_2^{1/\alpha}\right)\, \Psi(u)(1+\epsilon+o(1))$$

as $u \to \infty$ for every $\epsilon > 0$. Since $\lim_{T\to\infty} \mathcal{F}^R_{I,\alpha}\left(TA_2^{1/\alpha}\right) = \mathcal{F}^R_{I,\alpha}$, letting $\epsilon \to 0^+$ we obtain

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \sim \mathcal{F}^R_{I,\alpha}\, \Psi(u) \qquad \text{as} \quad u \to \infty\,.$$

In the case $\beta < \alpha$ we have

$$S_n \le \Psi(u)(1+o(1))\, \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) C_T\, e^{-A_4 T^\beta u^{2(1-\beta/\alpha)}} \qquad \text{as} \quad u \to \infty\,.$$

Hence, $S_n \le \epsilon \Psi(u)$ as $u \to \infty$ for all $\epsilon > 0$ independently of $T > 0$. Note also that

$$\begin{aligned}\mathbf{P}\left(\sup_{t\in[0,\Delta_1]} X(t) > u\right) &\le \mathbf{P}\left(\sup_{t\in[0,\Delta_1]} \bar{X}(t) > u\right) \le \mathbf{P}\left(\sup_{t\in[0,\Delta_1 C^{1/\alpha}]} \hat{Z}_C(t) > u\right)\\ &= \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right)\, \Psi\left(u\right)\left(1+o(1)\right) \qquad \text{as} \quad u \to \infty\,.\end{aligned}$$

Therefore

$$\Psi(u) \le \mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \le \Psi(u)\left(\mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) + \epsilon + o(1)\right) \qquad \text{as} \quad u \to \infty\,.$$

Since $\lim_{T\to 0^+} \mathcal{H}_{B,\alpha}\left(TC^{1/\alpha}\right) = 1$, letting $\epsilon \to 0^+$ we obtain

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \sim \Psi(u) \qquad \text{as} \quad u \to \infty\,.$$

Combining it with (16) we complete the proof of Theorem 2.3. $\square$

A Useful theorems

In the Appendix we present some classical theorems used in the paper.

A.1 Gaussian processes

From Adler [1] we cite two fundamental theorems in the theory of Gaussian processes: Slepian's inequality and Borell's inequality.

Theorem A.1 (Slepian's inequality) *Let Gaussian stochastic processes* $\{X(t), t \in \mathcal{T}\}$ *and* $\{Y(t), t \in \mathcal{T}\}$ *have bounded sample paths a.s. If*

$$\begin{aligned} \mathbf{E}\,(X(t)) &= \mathbf{E}\,(Y(t))\ ; \\ \mathbf{Var}\,(X(t)) &= \mathbf{Var}\,(Y(t))\ ; \\ \mathbf{Var}\,(X(t)-X(s)) &\geq \mathbf{Var}\,(Y(t)-Y(s)) \end{aligned}$$

for all $t, s \in \mathcal{T}$, *then for all* $u \in \mathbb{R}$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) > u\right) \geq \mathbf{P}\left(\sup_{t\in\mathcal{T}} Y(t) > u\right) .$$

Theorem A.2 (Borell's inequality) *Let* $\{X(t), t \in \mathcal{T}\}$ *be a centered Gaussian process with bounded sample paths a.s. and* $\sigma^2 = \sup_{t\in\mathcal{T}} \mathbf{Var}\,(X(t)) < \infty$.
Then $m = \mathbf{E}\,(\sup_{t\in\mathcal{T}} X(t)) < \infty$ *and for all* $u > 0$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) - m > u\right) \leq 2\exp\left(-\frac{u^2}{2\sigma^2}\right) .$$

Corollary A.3 *Under the assumptions of Theorem A.2, for* $u > m = \mathbf{E}\,(\sup_{t\in\mathcal{T}} X(t))$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) > u\right) \leq 2\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) .$$

The following inequality we state after Adler [1].

Theorem A.4 (Sudakov–Fernique) *Let* $\{X(t), t \in \mathcal{T}\}$ *and* $\{Y(t), t \in \mathcal{T}\}$ *be centered Gaussian processes with bounded sample paths a.s. If for all* $t, s \in \mathcal{T}$

$$\mathbf{Var}\,(X(t)-X(s)) \leq \mathbf{Var}\,(Y(t)-Y(s)) ,$$

then

$$\mathbf{E}\left(\sup_{t\in\mathcal{T}} X(t)\right) \leq \mathbf{E}\left(\sup_{t\in\mathcal{T}} Y(t)\right) .$$

A.2 Probability theory

We present (without proof) a useful lemma connected to conditional distributions of a normal random vector.

Lemma A.5 *Let (U, V) be a normal distributed random vector*

$$(U, V) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{E}U^2 & \mathbf{E}UV \\ \mathbf{E}UV & \mathbf{E}V^2 \end{pmatrix} \right) .$$

Then the random variable $(U \mid V = v)$ has normal distribution

$$(U \mid V = v) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(\frac{\mathbf{E}UV}{\mathbf{E}V^2} v \,,\, \frac{\mathbf{E}U^2\mathbf{E}V^2 - (\mathbf{E}UV)^2}{\mathbf{E}V^2} \right) .$$

Corollary A.6 *If $\{\xi(t), t \geq 0\}$ is a centered Gaussian process such that $\sigma_\xi^2(t^*) = 1$, then for all $t \neq t^*$*

$$(\xi(t) \mid \xi(t^*) = v) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(R_\xi(t, t^*)v \,,\, \sigma_\xi^2(t) - R_\xi^2(t, t^*) \right) .$$

From Billingsley [2] we present a criterion for tightness of sequence of stochastic processes on $C[0, T]$ (with the uniform metric given by $\rho(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)|$ for $x, y \in C[0, T]$).

Theorem A.7 *A family $\{\xi_u(t), t \in [0, T]\}$ in $C[0, T]$ is tight if and only if:*
i) for every $\rho > 0$ there exist $a, u_0 > 0$ such that

$$\mathbf{P}\left(|\xi_u(0)| \geq a\right) \leq \rho \qquad \text{for} \quad u \geq u_0$$

ii) for any $\epsilon, \rho > 0$ there exist $\delta_{\epsilon,\rho} \in (0, T)$ and $u_0 > 0$ such that

$$\mathbf{P}\left(\sup_{|s-t| \leq \delta_{\epsilon,\rho}} |\xi_u(s) - \xi_u(t)| \geq \epsilon \right) \leq \rho \qquad \text{for} \quad u \geq u_0 .$$

Moreover condition ii) holds, if for any $\epsilon, \rho > 0$ there exist $\delta_{\epsilon,\rho} \in (0, T)$ and $u_0 > 0$ such that

$$\mathbf{P}\left(\sup_{s \in [t, t+\delta_{\epsilon,\rho}]} |\xi_u(s) - \xi_u(t)| \geq \epsilon \right) \leq \rho\, \delta_{\epsilon,\rho}$$

for all $t \in [0, T]$ and $u \geq u_0$.

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