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An extension of Pickands' Lemma

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Contents

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Abstract

In this paper we extend original Pickands' Lemma, proving that if $\{X_u(t) : t \in [0,T]\}$ is a family of centered Gaussian processes such that $\lim_{u\to\infty} u^2(1-\sigma_{X_u}(t)) = d(t)$ and $\lim_{u\to\infty} u^2\mathbf{Var}(X_u(t)-X_u(s)) = 2\mathbf{Var}(Y(t)-Y(s))$, then

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u\right) = \mathcal{H}_{Y,d}(T)\cdot\Psi(u)\,(1+o(1)) \qquad \text{as} \quad u\to\infty\ ,$$

where $\mathcal{H}_{Y,d}(T) = \mathbf{E}\exp\left(\sup_{t\in[0,T]}\sqrt{2}Y(t)-\sigma_Y^2(t)-d(t)\right)$ for some centered Gaussian process $Y(\cdot)$ such that $Y(0)=0$ a.s. We also give examples of processes, for which our extension can be applied. We conclude with finding exact asymptotics of

$$\mathbf{P}\left(\sup_{t\in[A,A+T]} X\left(\frac{t}{h(u)}\right) > u\right) \qquad \text{as} \quad u\to\infty\ ,$$

for $A>0$, $\lim_{u\to\infty} h(u)=\infty$ and suitably chosen process $X(\cdot)$.

Introduction

In 1969 Pickands suggested a method of finding exact asymptotics of $\mathbf{P}\left(\sup_{t\in[0,T]} Z(t) > u\right)$ as $u\to\infty$, where $\{Z(t) : t\in[0,T]\}$ is a centered stationary Gaussian process with covariance function $\mathbf{Cov}(Z(t),Z(s)) = 1-|t-s|^\alpha + o(|t-s|^\alpha)$ as $t,s\to 0^+$, for some $\alpha\in(0,2]$. This method, called *double-sum method*, is being extended for a wide class of Gaussian processes. The core idea of finding asymptotics of $\mathbf{P}\left(\sup_{t\in[0,T]} Z(t) > u\right)$ as $u\to\infty$ is to use the following lemma, which we call Pickands' Lemma (see Pickands [5], [6] or Piterbarg [7] for more details). Let $\{B_\alpha(t) : t\geq 0\}$ be a fractional Brownian motion with Hurst parameter $\alpha/2$ and by $\Psi(\cdot)$ we denote the tail probability of a standard normal random variable.

Lemma 0.1 (Pickands' Lemma) *Let $\{Z(t) : t\geq 0\}$ be a centered stationary Gaussian process with continuous sample paths a.s. Assume that $R_Z(t,0) := \mathbf{Cov}(Z(t),Z(0)) = 1 - t^\alpha + o(t^\alpha)$ as $t\to 0^+$ for some $\alpha\in(0,2]$ and $R_Z(t,0)<1$ for all $t>0$. Then for all $T>0$*

$$\mathbf{P}\left(\sup_{t\in[0,Tu^{-2/\alpha}]} Z(t) > u\right) = \mathcal{H}_{B_\alpha}(T)\cdot\Psi(u)\,(1+o(1)) \qquad \textit{as} \quad u\to\infty\ ,$$

where

$$\mathcal{H}_{B_\alpha}(T) := \mathbf{E}\exp\left(\sup_{t\in[0,T]}\sqrt{2}B_\alpha(t)-t^\alpha\right)\ .$$

It turns out that Pickands' Lemma can be extended to non-stationary Gaussian processes. Piterbarg and Prisyazhnyuk (see [7] and [8]) generalized Pickands' Lemma into a class of so-called *locally stationary* Gaussian processes, where in the asypmtotics there appeared constants

$$\mathcal{H}_{B_\alpha}^R(T) \;=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right) \;.$$

On the other hand in the asymptotic problem considered in Dębicki [3] there appeared other constants, called *generalized Pickands' constants*:

$$\mathcal{H}_\eta(T) \;:=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}\eta(t) - \mathbf{Var}(\eta(t))\right) \;,$$

where $\eta(\cdot)$ is a centered Gaussian process with stationary increments that satisfies some regularity conditions.

We refer to [4] or [9] for other contribution in this domain, where so-called *locally self-similar* Gaussian processes were considered.

The aim of this paper is to extend the original Pickands' Lemma. The obtained asymptotics is illustrated by examples of processes, for which our extension can be apply. We conclude with finding exact asymptotic of

$$\mathbf{P}\left(\sup_{t\in[A,A+T]} X\left(\frac{t}{h(u)}\right) > u\right) \qquad \text{as} \quad u\to\infty \;,$$

for $A > 0$, $\lim_{u\to\infty} h(u) = \infty$ and suitably chosen process $X(\cdot)$.

The paper is organised as follows.
Section 1 contains notation used in the paper. In Section 2 we present main result of the paper (Theorem 2.2) and we give some examples for applications. Section 3 contains the proof of the main result of the paper. In Appendix A we present some classical theorems that are used in the paper.

1 Notation

In this section we introduce basic notation used in the paper. Definitions and classical notions connected to Gaussian processes can be found in [9].

For a Gaussian process $\{\xi(t) : t \geq 0\}$ and $t, s \geq 0$ we denote:

$$\begin{aligned} &\textit{covariance function} \qquad && R_\xi(t,s) \;=\; \mathbf{Cov}(\xi(t),\xi(s)) \;; \\ &\textit{variance function} && \sigma_\xi^2(t) \;=\; \mathbf{Var}(\xi(t)) \;=\; R_\xi(t,t) \;. \end{aligned}$$

A process $\xi(\cdot)$ is said to be *centered* if $\mathbf{E}\xi(t) = 0$ for all $t \geq 0$. Let $\bar{\xi}(\cdot)$ denote a standardized $\xi(\cdot)$ process, i.e. $\bar{\xi}(t) = \frac{\xi(t)-\mathbf{E}\xi(t)}{\sigma_\xi(t)}$.

By $\Psi(\cdot)$ we denote the tail probability of a standard normal random variable. We recall (see for instance Adler [1]) that

$$\Psi(u) \sim \frac{\exp(-u^2/2)}{u\sqrt{2\pi}} \qquad \text{as} \quad u \to \infty \, . \tag{1}$$

We use the relation: $f(u) \sim g(u)$ as $u \to \infty$ iff $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1$.

By $\{B_\alpha(t) : t \geq 0\}$ we denote a fractional Brownian motion with Hurst parameter $\alpha/2 \in (0,1]$, that is a centered Gaussian process with stationary increments, continuous sample paths a.s. and variance function $\sigma^2_{B_\alpha}(t) = t^\alpha$ for $t \geq 0$.

2 An extension of the Pickands' Lemma

Let $\{Y(t) : t \in [0,T]\}$ be a centered Gaussian process with continuous sample paths a.s. We assume that that $Y(0) = 0$ a.s. Let $d(\cdot)$ be a nonnegative continuous function on $[0,T]$. For process $Y(\cdot)$ and function $d(\cdot)$ we define the following constants:

$$\mathcal{H}_{Y,d}(T) \; := \; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t)\right) \, .$$

If $d(t) = R\sigma_Y^2(t)$ for all $t \in [0,T]$ and some constant $R > 0$, then we use notation

$$\mathcal{H}_Y^R(T) \; = \; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - (1+R)\sigma_Y^2(t)\right) \, .$$

If $d(t) = 0$ for all $t \in [0,T]$, then we use abbreviation

$$\mathcal{H}_Y(T) \; = \; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t)\right) \, .$$

Lemma 2.1 *If $\{Y(t) : t \in [0,T]\}$ is a centered Gaussian process with continuous sample paths a.s. such that $Y(0) = 0$ a.s. and $d(\cdot)$ is a nonnegative continuous function on $[0,T]$, then*

$$\mathcal{H}_{Y,d}(T) \; = \; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t)\right) \; < \; \infty.$$

Proof:
Since $Y(\cdot)$ and $d(\cdot)$ are continuous on $[0,T]$ then the process $Y(\cdot)$ has bounded sample paths a.s. and $\sigma_Y^2 = \sup_{t\in[0,T]} \sigma_Y^2(t) < \infty$. Borell's Theorem (Corollary A.3) implies that for $x > m = \mathbf{E}\left(\sup_{t\in[0,T]} Y(t)\right)$

$$\mathbf{P}\left(\sup_{t\in[0,T]} Y(t) > x\right) \; \leq \; 2\exp\left(-\frac{(x-m)^2}{2\sigma_Y^2}\right) \, .$$

Hence $\mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t) > x\right) \leq \mathbf{P}\left(\sup_{t\in[0,T]} Y(t) > x/\sqrt{2}\right)$ and, as a consequence,

$$\mathcal{H}_{Y,d}(T) \;=\; \int_{-\infty}^{\infty} e^x\, \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t) > x\right) dx \;<\; \infty\,.$$

This completes the proof.

□

In the following theorem we present main result of the paper . The detailed proof is given in Section 3.

Theorem 2.2 *Let $\{X_u(t) : t \in [0,T], u > 0\}$ be a family of centered Gaussian processes with continuous sample paths a.s. and suppose that $\sup_{t\in[0,T]} \sigma_{X_u}^2(t) = \sigma_{X_u}^2(0) = 1$.*
Let $\{Y(t) : t \in [0,T]\}$ be a centered Gaussian process with continuous sample paths a.s., $Y(0) = 0$ a.s. and $d(\cdot)$ be a function such that:

(a)

$$\lim_{u\to\infty} \sup_{t\in[0,T]} \left| u^2(1 - \sigma_{X_u}^2(t)) - 2d(t) \right| \;=\; 0\,;$$

(b)

$$\lim_{u\to\infty} \sup_{t,s\in[0,T]} \left| u^2\mathbf{Var}(X_u(t) - X_u(s)) - 2\mathbf{Var}(Y(t) - Y(s)) \right| \;=\; 0\,.$$

Then

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u\right) \;\sim\; \mathcal{H}_{Y,d}(T) \cdot \Psi(u) \qquad as \quad u \to \infty\,.$$

Remark 2.1 Combining $(1 - \sigma_{X_u}^2(t)) = (1 - \sigma_{X_u}(t))(1 + \sigma_{X_u}(t))$ with

$$\mathbf{Var}(X_u(t) - X_u(s)) = \sigma_{X_u}^2(t) + \sigma_{X_u}^2(s) - 2R_{\bar{X}_u}(t,s)\sigma_{X_u}(t)\sigma_{X_u}(s)$$

and the fact that $\lim_{u\to\infty} \sigma_{X_u}(t) = 1$ uniformly on $t \in [0,T]$ (due to (a)), we can rewrite assumptions (a) and (b) of Theorem 2.2 as follows:

(a')

$$\lim_{u\to\infty} \sup_{t\in[0,T]} \left| u^2(1 - \sigma_{X_u}(t)) - d(t) \right| \;=\; 0\,;$$

(b')

$$\lim_{u\to\infty} \sup_{t,s\in[0,T]} \left| u^2(1 - R_{\bar{X}_u}(t,s)) - \mathbf{Var}(Y(t) - Y(s)) \right| \;=\; 0\,.$$

Remark 2.2 Define family of processes $\{\chi_u(t) : t \in [0,T]\}$ as

$$\chi_u(t) = u\left(X_u(t) - R_{X_u}(t,0)X_u(0)\right)\,.$$

Then, under assumptions (a) and (b), $\{\chi_u(t) : t \in [0,T]\}$ weakly converges (in $C[0,T]$ with uniform metric) to $\left\{\sqrt{2}Y(t) : t \in [0,T]\right\}$, as $u \to \infty$. This is an essential part of the proof of

Theorem 2.2 and it is worth to note that the weak convergence is not a sufficient condition to prove the thesis of Theorem 2.2. We also note that assumptions (a) and (b) imply that $Y(\cdot)$ has uniformly continuous sample paths a.s. and $d(\cdot)$ is a nonnegative uniformly continuous function on $[0,T]$ such that $d(0)=0$. Thus, thanks to Lemma 2.1, $\mathcal{H}_{Y,d}(T)<\infty$. Furthermore $\mathbf{Var}(Y(t)-Y(s))$ is uniformly continuous on $t,s\in[0,T]$.

Remark 2.3 If $\{X_u(t):t\in[0,T]\}$ has constant variance, i.e. $\sigma^2_{X_u}(t)=1$ for all $t\in[0,T]$, then $d(t)=0$ for all $t\in[0,T]$. Thus

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t)>u\right) \sim \mathcal{H}_Y(T)\cdot\Psi(u) \qquad \text{as} \quad u\to\infty\,.$$

Example 2.1 Consider a centered stationary Gaussian process $\{Z(t):t\in[0,T]\}$ with continuous sample paths a.s. and covariance function that satisfies

$$R_Z(t,0)=1-t^\alpha+o(t^\alpha) \qquad \text{as} \quad t\to 0^+$$

for some $\alpha\in(0,2]$. Then for $\{Z_u(t)=Z(tu^{-2/\alpha}):t\in[0,T]\}$ we have

$$u^2\mathbf{Var}(Z_u(t)-Z_u(s)) \;=\; 2u^2(1-R_Z(t,s)) \;=\; 2|t-s|^\alpha+o(1) \qquad \text{as} \quad u\to\infty$$

uniformly on $t,s\in[0,T]$. Thus $Y(\cdot)=B_\alpha(\cdot)$ and $d(t)=0$ for all $t\in[0,T]$ (since $\sigma^2_{Z_u}(t)=1$ for all $t\in[0,T]$). In this case Theorem 2.2 gives asymptotics obtained in Pickands' Lemma 0.1 (see also [5], [6] or [7]).

Example 2.2 Dębicki [3] extended Pickands' result in the case where $Y(\cdot)$ is centered Gaussian process with stationary increments satisfying some regularity conditions on $\sigma^2_Y(\cdot)$. For such processes he proved the thesis of Theorem 2.2 with $d(t)=0$ for all $t\in[0,T]$. He also analyzed properties of $\mathcal{H}_Y(T)$ as $T\to\infty$. Theorem 2.2 extends result of [3] to $d(\cdot)\neq 0$.

Example 2.3 Theorem 2.2 extends also result of Dębicki and Tabiś (see [4] and [9]). They analyzed self-similar Gaussian processes $Y(\cdot)$ (with not necessarily stationary increments) with differentiable sample paths a.s. and with particular function $d(\cdot)=R\sigma^2_Y(\cdot)$. They also analyzed properties of $\lim_{T\to\infty}\mathcal{H}^R_Y(T)$.

Example 2.4 Let $\{\xi(t):t\in[0,T]\}$ be a centered Gaussian process with continuous sample paths a.s. and regularly varying variance function at 0, i.e.

$$\lim_{t\to 0^+}\frac{\sigma^2_\xi(t)}{t^\alpha}=1$$

for some $\alpha\in(0,2]$.
Consider a process $\{Z(t):t\in[0,T]\}$ such that

$$2(1-R_Z(t,s)) \;=\; \mathbf{Var}(Z(t)-Z(s)) \;=\; 2\mathbf{Var}(\xi(t)-\xi(s))(1+o(1)) \qquad \text{as} \quad t,s\to 0^+$$

(for example $R_Z(t,s)=\exp(-\mathbf{Var}(\xi(t)-\xi(s)))$ for $t,s\in[0,T]$) and let $\{X(t)=Z(t)\sigma_X(t):t\in[0,T]\}$ for some variance function $\sigma^2_X(\cdot)$. In the following lemma we consider families $\{Z_u(t)=Z(t/h(u)):t\in[0,T]\}$ and $\{X_u(t)=X(t/h(u)):t\in[0,T]\}$ for some function $h(\cdot)$ such that $\lim_{u\to\infty}h(u)=\infty$. We show necessary conditions for $\sigma_X(\cdot)$ and $h(\cdot)$ in order to apply Theorem 2.2.

Lemma 2.3 *Let $\{\xi(t) : t \in [0, T]\}$ be a centered Gaussian process with continuous sample paths a.s. and regularly varying variance function at 0, i.e.,*

$$\lim_{t \to 0^+} \frac{\sigma_\xi^2(t)}{t^\alpha} = 1 \tag{2}$$

for some $\alpha \in (0, 2]$. Suppose that

$$\lim_{u \to \infty} \sup_{t,s \in [0,T]} \Big| u^2 \mathbf{Var}(\xi(t/h(u)) - \xi(s/h(u))) - \mathbf{Var}(Y(t) - Y(s)) \Big| \;=\; 0 \tag{3}$$

for some centered Gaussian process $Y(\cdot)$ such that $Y(0) = 0$ a.s.
Let $\{Z(t) : t \in [0, T]\}$ be a centered Gaussian process such that

$$1 - R_Z(t, s) \;=\; \mathbf{Var}(\xi(t) - \xi(s))(1 + o(1)) \qquad \text{as} \quad t, s \to 0^+ \tag{4}$$

and let $\{X(t) = Z(t)\sigma_X(t) : t \in [0, T]\}$ for some variance function $\sigma_X^2(\cdot)$. Consider families of Gaussian processes $\{Z_u(t) = Z(t/h(u)) : t \in [0, T]\}$ and $\{X_u(t) = X(t/h(u)) : t \in [0, T]\}$.

Then
(i) *$Z_u(\cdot)$ satisfies assumptions of Theorem 2.2 with $Y(\cdot)$ and*

$$\lim_{u \to \infty} \frac{h(u)}{u^{2/\alpha}} = C \qquad \text{and} \qquad \sigma_Y^2(t) = (t/C)^\alpha$$

for some $C > 0$.

(ii) *$X_u(\cdot)$ satisfies assumption* (a) *of Theorem 2.2 for some function $d(\cdot)$ if and only if for some constant $R \geq 0$*

$$\lim_{t \to 0^+} \frac{1 - \sigma_X(t)}{t^\alpha} = R \qquad \text{and} \qquad d(t) = R(t/C)^\alpha \,.$$

Proof:
Ad (i) Since $\lim_{u \to \infty} h(u) = \infty$, then condition (4) implies that uniformly on $t, s \in [0, T]$,

$$u^2 \mathbf{Var}(Z_u(t) - Z_u(s)) \;=\; 2u^2 \mathbf{Var}(\xi(t/h(u)) - \xi(s/h(u)))(1 + o(1)) \qquad \text{as} \quad u \to \infty$$

with is equivalent to (b) of Theorem 2.2 (due to (3)). Besides $\sigma_{Z_u}^2(t) = 1$ for all $t \in [0, T]$, so assumption (a) is fulfilled. Since $\xi(0) = Y(0) = 0$ a.s., then from (3) we have

$$\lim_{u \to \infty} u^2 \sigma_\xi^2(t/h(u)) \;=\; \sigma_Y^2(t)$$

uniformly on $t \in [0, T]$. On the other hand, using (2) we obtain that for all $t \in [0, T]$,

$$\sigma_Y^2(t) \;=\; \lim_{u \to \infty} u^2 \sigma_\xi^2(t/h(u)) \;=\; \lim_{u \to \infty} \frac{\sigma_\xi^2(t/h(u))}{(t/h(u))^\alpha} \frac{u^2}{h^\alpha(u)} t^\alpha \;=\; t^\alpha \lim_{u \to \infty} \left(\frac{u^{2/\alpha}}{h(u)} \right)^\alpha .$$

Hence

$$\lim_{u \to \infty} \frac{h(u)}{u^{2/\alpha}} = C \qquad \text{and} \qquad \sigma_Y^2(t) = (t/C)^\alpha$$

for some $C > 0$.

Ad (ii) Assumption (a) of Theorem 2.2 states that

$$\lim_{u\to\infty} u^2(1-\sigma_X(t/h(u))) = d(t)$$

uniformly on $t\in[0,T]$. Applying the fact that $\lim_{u\to\infty}\frac{h(u)}{u^{2/\alpha}}=C$, we can write

$$d(t) \;=\; \lim_{u\to\infty}\frac{1-\sigma_X\left(tu^{-2/\alpha}\frac{u^{2/\alpha}}{h(u)}\right)}{\left(tu^{-2/\alpha}\frac{u^{2/\alpha}}{h(u)}\right)^{\alpha}}\left(\frac{u^{2/\alpha}}{h(u)}\right)^{\alpha}t^{\alpha} \;=\; (t/C)^{\alpha}\lim_{x\to 0^+}\frac{1-\sigma_X(x)}{x^{\alpha}}\;.$$

We conclude that

$$\lim_{t\to 0^+}\frac{1-\sigma_X(t)}{t^{\alpha}} = R \qquad \text{and} \qquad d(t) = R(t/C)^{\alpha}\;.$$

This completes the proof. □

Remark 2.4 If $\xi(\cdot)$ is a centered self-similar Gaussian process with index $\alpha/2\in(0,1]$ (i.e. for all $a>0$ $\{\xi(at):t\geq 0\}\stackrel{\mathcal{D}}{=}\{a^{\alpha/2}\xi(t):t\geq 0\}$), then in (3) process $Y(\cdot)$ is unique, i.e. $Y(\cdot)=\xi(\cdot)$ and in (4) we can write

$$1-R_Z(t,s) \;=\; \mathbf{Var}(Y(t)-Y(s))(1+o(1)) \qquad \text{as} \quad t,s\to 0^+\;.$$

It is not true that if

$$1-R_Z(t,s) \;=\; \mathbf{Var}(\xi(t)-\xi(s))(1+o(1)) \qquad \text{as} \quad t,s\to 0^+\;,$$

then $Y(\cdot)=\xi(\cdot)$. Indeed, consider $\{\xi(t)=B_{\alpha_1}(t)+B_{\alpha_2}(t):t\in[0,T]\}$ for $\alpha_1<\alpha_2$, where $B_{\alpha_1}(\cdot)$ and $B_{\alpha_2}(\cdot)$ are independent fractional Brownian motions. Then $\xi(\cdot)$ satisfies (2) with $\alpha=\alpha_1$ and (3) with $Y(\cdot)=B_{\alpha_1}(\cdot)$.

Remark 2.5 Family of processes $\{X_u(t)=Z(t/h(u))\sigma_X(t/h(u)):t\in[0,T]\}$ for which

$$1-R_Z(t,s) \;=\; \mathbf{Var}(\xi(t)-\xi(s))(1+o(1)) \qquad \text{as} \quad t,s\to 0^+\;,$$

plays an important role in the exact asymptotics of extreme values for Gaussian stochastic processes. Finding the exact asymptotics of $\mathbf{P}\left(\sup_{t\in[0,T]}X(t/h(u))>u\right)$ as $u\to\infty$ is a base in a celebrated *double-sum method* that allows the analysis of $\mathbf{P}\left(\sup_{t\in[0,T]}X(t)>u\right)$ as $u\to\infty$. Processes of such a form were studied for example in [3], [5], [7] or [9].

In the following theorem we continue analysis begun in Lemma 2.3 and find the asymptotics of $\mathbf{P}\left(\sup_{t\in[A,A+T]}X(t/h(u))>u\right)$ as $u\to\infty$, for $A\geq 0$.

Theorem 2.4 *Let $\{\xi(t):t\geq 0\}$ be a centered Gaussian process with continuous sample paths a.s. and regularly varying variance function at* 0, *i.e.,*

$$\lim_{t\to 0^+}\frac{\sigma_\xi^2(t)}{t^{\alpha}} = 1$$

for some $\alpha \in (0,2]$. Let $\{Z(t) : t \in [0,T]\}$ be a centered Gaussian process such that

$$1 - R_Z(t,s) \;=\; \mathbf{Var}(\xi(t) - \xi(s))(1+o(1)) \qquad as \quad t,s \to 0^+ \tag{5}$$

and let $\{X(t) = Z(t)\sigma_X(t) : t \in [0,T]\}$ for some variance function $\sigma_X^2(\cdot)$ such that

$$\lim_{t\to 0^+} \frac{1-\sigma_X(t)}{t^\alpha} = R$$

with some $R \geq 0$. Let $h(\cdot)$ be a function such that

$$\lim_{u\to\infty} \frac{h(u)}{u^{2/\alpha}} = 1 \,.$$

Suppose that for some $B \geq 0$

$$\lim_{u\to\infty} \sup_{t,s\in[0,B+T]} \left| u^2 \mathbf{Var}(\xi(t/h(u)) - \xi(s/h(u))) - \mathbf{Var}(Y(t)-Y(s)) \right| \;=\; 0 \tag{6}$$

for some centered Gaussian process $\{Y(t) : t \geq 0\}$ such that $Y(0) = 0$ a.s. and $\sigma_Y^2(t) = t^\alpha$ for $t \geq 0$.

Then for all $0 \leq A \leq B$

(i) $$\mathbf{P}\left(\sup_{t\in[A,A+T]} Z\left(\frac{t}{h(u)}\right) > u \right) \;\sim\; \Psi(u) \cdot \mathbf{E}\exp\left(\sup_{t\in[A,A+T]} \sqrt{2}Y_A(t) - \sigma_{Y_A}^2(t) \right)$$

(ii) $$\mathbf{P}\left(\sup_{t\in[A,A+T]} X\left(\frac{t}{h(u)}\right) > u \right) \;\sim\; \Psi(u)e^{-RA^\alpha} \cdot \mathbf{E}\exp\left(\sup_{t\in[A,A+T]} \sqrt{2}Y_A(t) - \sigma_{Y_A}^2(t) - R\left(t^\alpha - A^\alpha\right) \right)$$

as $u \to \infty$, where $\{Y_A(t) = Y(t) - Y(A) : t \in [A, A+T]\}$.

Proof:
Ad (i) Let $0 \leq A \leq B$. Consider $\left\{ Z_u(t) = Z\left(\frac{t+A}{h(u)}\right) : t \in [0,T] \right\}$. Then (5) and (6) implies that uniformly on $t,s \in [0,T]$

$$\lim_{u\to\infty} u^2(1 - R_{Z_u}(t,s)) \;=\; \lim_{u\to\infty} u^2 \mathbf{Var}\left(\xi\left(\frac{t+A}{h(u)}\right) - \xi\left(\frac{s+A}{h(u)}\right) \right) \;=\; \mathbf{Var}(Y(t+A) - Y(s+A)) \,.$$

Hence, the family $\{Z_u(t) : t \in [0,T]\}$ satisfies assumption (b) of Theorem 2.2 with associated process $\{Y_A(t) = Y(t+A) - Y(A) : t \in [0,T]\}$ (since $Y_A(0) = 0$ a.s.). This implies (i).

Ad (ii) Consider $\left\{ X_u(t) = Z\left(\frac{t+A}{h(u)}\right) \sigma_X\left(\frac{t+A}{h(u)}\right) : t \in [0,T] \right\}$. Since $1 - \sigma_X(\cdot)$ is regularly varying, then uniformly on $t \in [0,T]$

$$\sigma_X\left(\frac{A}{h(u)}\right) - \sigma_X\left(\frac{t+A}{h(u)}\right) \;=\; R\frac{(t+A)^\alpha - A^\alpha}{h^\alpha(u)}(1+o(1)) \;\geq\; 0 \,,$$

for sufficiency large u. Hence, $\max_{t\in[0,T]} \sigma_X\left(\frac{t+A}{h(u)}\right) = \sigma_X\left(\frac{A}{h(u)}\right)$ and

$$\mathbf{P}\left(\sup_{t\in[0,T]} X\left(\frac{t+A}{h(u)}\right) > u\right) = \mathbf{P}\left(\sup_{t\in[0,T]} \frac{X\left(\frac{t+A}{h(u)}\right)}{\sigma_X\left(\frac{A}{h(u)}\right)} > \frac{u}{\sigma_X\left(\frac{A}{h(u)}\right)}\right).$$

Now the family $\left\{\frac{X\left(\frac{t+A}{h(u)}\right)}{\sigma_X\left(\frac{A}{h(u)}\right)} : t \in [0,T]\right\}$ satisfies assumptions of Theorem 2.2 with

$$\begin{aligned} d(t) &= \lim_{u\to\infty} u^2\left(1 - \frac{\sigma_X\left(\frac{t+A}{h(u)}\right)}{\sigma_X\left(\frac{A}{h(u)}\right)}\right) = \lim_{u\to\infty} \frac{u^2}{\sigma_X\left(\frac{A}{h(u)}\right)}\left(\sigma_X\left(\frac{A}{h(u)}\right) - \sigma_X\left(\frac{t+A}{h(u)}\right)\right) \\ &= R\left((t+A)^\alpha - A^\alpha\right). \end{aligned}$$

Theorem 2.2 gives the asymptotics

$$\begin{aligned} &\mathbf{P}\left(\sup_{t\in[A,A+T]} X\left(\frac{t}{h(u)}\right) > u\right) \\ &\sim \Psi\left(\frac{u}{\sigma_X\left(\frac{A}{h(u)}\right)}\right) \cdot \mathbf{E}\exp\left(\sup_{t\in[A,A+T]} \sqrt{2}Y_A(t) - \sigma^2_{Y_A}(t) - R\left(t^\alpha - A^\alpha\right)\right) \end{aligned}$$

as $u \to \infty$. The final step of the proof is an observation that

$$\begin{aligned} \Psi\left(\frac{u}{\sigma_X\left(\frac{A}{h(u)}\right)}\right) &= \Psi\left(\frac{u}{1 - R\frac{A^\alpha}{h^\alpha(u)}(1+o(1))}\right) = \Psi\left(u\left(1 + R\frac{A^\alpha}{h^\alpha(u)}(1+o(1))\right)\right) \\ &= \Psi\left(u + R\frac{A^\alpha}{u}(1+o(1))\right) \\ &\sim \frac{1}{\sqrt{2\pi}}\frac{1}{u + R\frac{A^\alpha}{u}(1+o(1))}\exp\left(-\frac{1}{2}\left(u + R\frac{A^\alpha}{u}(1+o(1))\right)^2\right) \\ &\sim \frac{1}{\sqrt{2\pi}u}e^{-u^2/2}e^{-RA^\alpha} \sim \Psi(u)e^{-RA^\alpha} \qquad \text{as} \quad u \to \infty. \end{aligned}$$

This completes the proof of (ii).

□

Remark 2.6 If $Y(\cdot)$ is Gaussian process with stationary increments, then $\mathcal{H}_{Y_A}(T) = \mathcal{H}_Y(T)$ for all $A > 0$. But if $Y(\cdot)$ is Gaussian process with not necessarily stationary increments then $\mathcal{H}_{Y_A}(T)$ can behaves in a different way. Limiting properties of $\mathcal{H}_{Y,d}(T)$ as $T \to \infty$ play a big role in the *double-sum method* for Gaussian processes.

3 Proof of Theorem 2.2

For all $u > 0$ we have

$$\begin{aligned}\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-v^2/2}\,\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u \,\middle|\, X_u(0) = v\right) dv \\ &= \frac{1}{\sqrt{2\pi}u} e^{-u^2/2}\int_{-\infty}^{\infty} e^{w-w^2/2u^2}\,\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u \,\middle|\, X_u(0) = u - \frac{w}{u}\right) dw\ ,\end{aligned}$$

where the last equality is a consequence of changing of variables $v = u - \frac{w}{u}$.
From Corollary A.7 we conclude that $\left\{\left(X_u(t) \,\middle|\, X_u(0) = u - \frac{w}{u}\right) : t \in [0,T]\right\}$ has the same distribution as

$$\left\{X_u(t) - R_{X_u}(t,0)X_u(0) + R_{X_u}(t,0)\left(u - \frac{w}{u}\right) : t \in [0,T]\right\}\ .$$

Then

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u \,\middle|\, X_u(0) = u - \frac{w}{u}\right) = \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) - u^2(1 - R_{X_u}(t,0)) > w\right)\ ,$$

where $\{\chi_u(t) = u\left(X_u(t) - R_{X_u}(t,0)X_u(0)\right) : t \in [0,T]\}$. In the rest part of the proof, we use an abbreviation

$$\mathbf{P}_u(w) := \mathbf{P}\left(\sup_{t\in[0,T]} \chi_u(t) - u^2(1 - R_{X_u}(t,0)) > w\right)\ .$$

We have

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_u(t) > u\right) = \frac{1}{\sqrt{2\pi}u} e^{-u^2/2}\int_{-\infty}^{\infty} e^{w-w^2/2u^2}\,\mathbf{P}_u(w)dw\ .$$

In the next steps of the proof we show that $\{\chi_u(t) - u^2(1 - R_{X_u}(t,0)) : t \in [0,T]\}$ weakly converges in $C[0,T]$ to $\left\{\sqrt{2}Y(t) - \sigma_Y^2(t) - d(t) : t \in [0,T]\right\}$ as $u \to \infty$. Remark 2.2 shows that $\mathcal{H}_{Y,d}(T) < \infty$, so from (1) we conclude that in order to prove Lemma 2.2 it suffices to show that

$$\lim_{u\to\infty}\int_{-\infty}^{\infty} e^{w-w^2/2u^2}\,\mathbf{P}_u(w)dw = \mathcal{H}_{Y,d}(T)\ . \tag{7}$$

Weak convergence of $\{\chi_u(t) - u^2(1 - R_{X_u}(t,0)) : t \in [0,T]\}$ will follow from convergence of finite dimensional distributions and tightness of the family $\{\chi_u(t) : t \in [0,T]\}$ as $u \to \infty$.

1) Convergence of finite dimensional distributions

First, due to Remark 2.1, we observe that

$$\begin{aligned}\lim_{u\to\infty} u^2(1 - R_{X_u}(t,0)) &= \lim_{u\to\infty} u^2(1 - R_{\bar{X}_u}(t,0)\sigma_{X_u}(t)) \\ &= \lim_{u\to\infty} u^2(1 - \sigma_{X_u}(t)) + \lim_{u\to\infty}\sigma_{X_u}(t)u^2(1 - R_{\bar{X}_u}(t,0)) \\ &= d(t) + \sigma_Y^2(t)\end{aligned} \tag{8}$$

uniformly on $t \in [0,T]$.
Hence, $\{u^2(1-R_{X_u}(t,0)) : t \in [0,T]\}$ weakly converges to $\{\sigma_Y^2(t)+d(t) : t \in [0,T]\}$ as $u \to \infty$.

Since $\chi_u(\cdot)$ is a centered Gaussian process and

$$\begin{aligned}\lim_{u\to\infty} \mathbf{Var}(\chi_u(t)-\chi_u(s)) &= \lim_{u\to\infty} u^2\mathbf{Var}(X_u(t)-X_u(s)) - \lim_{u\to\infty} u^2(R_{X_u}(t,0)-R_{X_u}(s,0))^2 \\ &= 2\mathbf{Var}(Y(t)-Y(s))\end{aligned}$$

uniformly on $t,s \in [0,T]$, then finite dimensional distributions of $\{\chi_u(t) : t \in [0,T]\}$ converge to finite dimensional distributions of $\left\{\sqrt{2}Y(t) : t \in [0,T]\right\}$ as $u \to \infty$.

2) Tightness

We use a criterion for tightness of sequence $\{\chi_u(t) : t \in [0,T]\}$ given in Theorem A.8. Since $\chi_u(0) = 0$ a.s. for all $u > 0$, then condition (i) of Theorem A.8 is satisfied. It remains to prove that for any $\epsilon, \rho > 0$ there exist $\delta_{\epsilon,\rho} \in (0,T)$ and $u_0 > 0$ such that for $u \geq u_0$

$$\mathbf{P}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} |\chi_u(s)-\chi_u(t)| \geq \epsilon\right) \leq \rho \,.$$

Fix $\epsilon, \rho > 0$. Since $\lim_{u\to\infty} \mathbf{Var}(\chi_u(t)-\chi_u(s)) = 2\mathbf{Var}(Y(t)-Y(s))$ uniformly on $t,s \in [0,T]$, then there exists $A > 0$ such that for sufficiency large $u \geq u_0$

$$\mathbf{Var}(\chi_u(t)-\chi_u(s)) \leq A\mathbf{Var}(Y(t)-Y(s)) \tag{9}$$

for all $t,s \in [0,T]$.
Using Sudakov–Fernique's inequality (Theorem A.4), we obtain that

$$\mathbf{E}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} \chi_u(s)-\chi_u(t)\right) \leq \mathbf{E}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} \sqrt{A}(Y(s)-Y(t))\right) .$$

Since $Y(\cdot)$ is uniformly continuous, then (using Lemma A.5) for sufficiently large $u \geq u_0$

$$\lim_{\delta_{\epsilon,\rho}\to 0^+} \mathbf{E}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} |\chi_u(s)-\chi_u(t)|)\right) \leq \lim_{\delta_{\epsilon,\rho}\to 0^+} 2\mathbf{E}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} \chi_u(s)-\chi_u(t))\right) = 0 \,,$$

hence (from Chebyshev's inequality) for $u \geq u_0$ there exists $\delta_{\epsilon,\rho}$ such that

$$\mathbf{P}\left(\sup_{|t-s|\leq\delta_{\epsilon,\rho}} |\chi_u(s)-\chi_u(t)| \geq \epsilon\right) \leq \rho \,.$$

Thus the family $\{\chi_u(t) : t \in [0,T]\}$ weakly converges to $\left\{\sqrt{2}Y(t) : t \in [0,T]\right\}$ as $u \to \infty$. Since $\{u^2(1-R_{X_u}(t,0)) : t \in [0,T]\}$ converges to $\{\sigma_Y^2(t)+d(t) : t \in [0,T]\}$ as $u \to \infty$, then $\{\chi_u(t)-u^2(1-R_{X_u}(t,0)) : t \in [0,T]\}$ weakly converges to $\left\{\sqrt{2}Y(t)-\sigma_Y^2(t)-d(t) : t \in [0,T]\right\}$, as $u \to \infty$.

3) Convergence of the integral

We show the convergence of integral (7). Since $\{\chi_u(t) - u^2(1 - R_{X_u}(t,0)) : t \in [0,T]\}$ weakly converges and functional $\Phi(f) = \sup_{t\in[0,T]} f(t)$ is continuous in uniform metric, it follows that for all $w \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbf{P}_u(w) \;=\; \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t) > w\right) .$$

The idea of the proof is based on an appropriate majorization of $\mathbf{P}_u(w)$ for large u. Convergence (8) implies that $u^2(1 - R_{X_u}(t,0)) \geq B(\sigma_Y^2(t) + d(t))$ uniformly on $t \in [0,T]$ for some $B > 0$ and sufficiently large u. Combination of that and (9) with Sudakov–Fernique's inequality gives

$$\mathbf{E}\left(\sup_{t\in[0,T]} \chi_u(t) - u^2(1 - R_{X_u}(t,0))\right) \;\leq\; \mathbf{E}\left(\sup_{t\in[0,T]} \sqrt{A}Y(t)\right) - B\left(\sup_{t\in[0,T]} (\sigma_Y^2(t) + d(t))\right) \;=\; m .$$

Borell's inequality (Corollary A.3) for $\sigma^2 = A\sup_{t\in[0,T]} \sigma_Y^2(t)$ implies that

$$\mathbf{P}_u(w) \;\leq\; 2\exp\left(-\frac{(w-m)^2}{2\sigma^2}\right) .$$

Now we can apply the dominated convergence theorem to (7). We get

$$\int_{-\infty}^{\infty} e^{w - w^2/2u^2}\, \mathbf{P}_u(w) dw \;\leq\; \int_{-\infty}^{\infty} 2e^w \exp\left(-\frac{(w-m)^2}{2\sigma^2}\right) dw \;<\; \infty$$

for sufficiently large u. As a consequence,

$$\lim_{u\to\infty} \int_{-\infty}^{\infty} e^{w - w^2/2u^2} \mathbf{P}_u(w) dw \;=\; \int_{-\infty}^{\infty} e^w\, \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}Y(t) - \sigma_Y^2(t) - d(t) > w\right) dw \;=\; \mathcal{H}_{Y,d}(T) .$$

This completes the proof.

□

A Useful theorems

In the Appendix we present some classical theorems used in the paper.

A.1 Gaussian processes

All the facts are taken from Adler [1].

Theorem A.1 (Slepian's inequality) *Let Gaussian stochastic processes* $\{X(t) : t \in \mathcal{T}\}$ *and* $\{Y(t) : t \in \mathcal{T}\}$ *have bounded sample paths a.s. If*

$$\begin{aligned} \mathbf{E}\,(X(t)) &= \mathbf{E}\,(Y(t)) \;; \\ \mathbf{Var}\,(X(t)) &= \mathbf{Var}\,(Y(t)) \;; \\ \mathbf{Var}\,(X(t) - X(s)) &\geq \mathbf{Var}\,(Y(t) - Y(s)) \end{aligned}$$

for all $t, s \in \mathcal{T}$, then for all $u \in \mathbb{R}$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) > u\right) \;\geq\; \mathbf{P}\left(\sup_{t\in\mathcal{T}} Y(t) > u\right) \;.$$

Theorem A.2 (Borell's inequality) *Let $\{X(t) : t \in \mathcal{T}\}$ be a centered Gaussian process with bounded sample paths a.s. and $\sigma^2 = \sup_{t\in\mathcal{T}} \mathbf{Var}\,(X(t)) < \infty$.*
Then $m = \mathbf{E}\,(\sup_{t\in\mathcal{T}} X(t)) < \infty$ and for all $u > 0$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) - m > u\right) \;\leq\; 2\exp\left(-\frac{u^2}{2\sigma^2}\right) \;.$$

Corollary A.3 *Under the assumptions of Theorem A.2, for $u > m = \mathbf{E}\,(\sup_{t\in\mathcal{T}} X(t))$*

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) > u\right) \;\leq\; 2\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) \;.$$

Other useful facts used in the paper.

Theorem A.4 (Sudakov–Fernique) *Let $\{X(t) : t \in \mathcal{T}\}$ and $\{Y(t) : t \in \mathcal{T}\}$ be centered Gaussian processes with bounded sample paths a.s. If for all $t, s \in \mathcal{T}$*

$$\mathbf{Var}\,(X(t) - X(s)) \;\leq\; \mathbf{Var}\,(Y(t) - Y(s)) \;,$$

then

$$\mathbf{E}\left(\sup_{t\in\mathcal{T}} X(t)\right) \;\leq\; \mathbf{E}\left(\sup_{t\in\mathcal{T}} Y(t)\right) \;.$$

Lemma A.5 *Let $\{X(t) : t \in [0,T]\}$ be a Gaussian process with bounded sample paths a.s. such that $\sqrt{\mathbf{Var}\,(X(t) - X(s))}$ is uniformly continuous on $t, s \in [0,T]$. Then $X(\cdot)$ has uniformly continuous sample paths a.s. if and only if*

$$\lim_{\delta\to 0^+} \mathbf{E}\left(\sup_{|t-s|\leq\delta} X(s) - X(t)\right) \;=\; 0 \;.$$

A.2 Probability theory

We present a useful lemma concerning the conditional distributions of a normal random vector.

Lemma A.6 *Let (U, V) be a normal distributed random vector*

$$(U, V) \;\stackrel{\mathcal{D}}{=}\; \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\mathbf{E}U^2 & \mathbf{E}UV\\ \mathbf{E}UV & \mathbf{E}V^2\end{pmatrix}\right) \;.$$

Then the random variable $(U \mid V = v)$ has normal distribution

$$(U \mid V = v) \;\stackrel{\mathcal{D}}{=}\; \mathcal{N}\left(\frac{\mathbf{E}UV}{\mathbf{E}V^2}v\;,\; \frac{\mathbf{E}U^2\mathbf{E}V^2 - (\mathbf{E}UV)^2}{\mathbf{E}V^2}\right) \;.$$

Corollary A.7 *If $\{\xi(t) : t \geq 0\}$ is a centered Gaussian process such that $\sigma_\xi^2(t^*) = 1$, then for all $t \neq t^*$*

$$(\xi(t) \mid \xi(t^*) = v) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(R_\xi(t,t^*)v \, , \, \sigma_\xi^2(t) - R_\xi^2(t,t^*) \right) .$$

Following Billingsley [2] we present a criterion for tightness of sequence of stochastic processes on $C[0,T]$ (with the uniform metric given by $\rho(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|$ for $x, y \in C[0,T]$).

Theorem A.8 *A family $\{\xi_u(t) : t \in [0,T]\}$ in $C[0,T]$ is tight if and only if:*
(i) *for every $\rho > 0$ there exist $a, u_0 > 0$ such that*

$$\mathbf{P}\left(|\xi_u(0)| \geq a\right) \leq \rho \qquad \text{for} \quad u \geq u_0$$

(ii) *for any $\epsilon, \rho > 0$ there exist $\delta_{\epsilon,\rho} \in (0,T)$ and $u_0 > 0$ such that*

$$\mathbf{P}\left(\sup_{|s-t|\leq\delta_{\epsilon,\rho}} |\xi_u(s) - \xi_u(t)| \geq \epsilon \right) \leq \rho \qquad \text{for} \quad u \geq u_0 \, .$$

Moreover condition (ii) *holds, if for any $\epsilon, \rho > 0$ there exist $\delta_{\epsilon,\rho} \in (0,T)$ and $u_0 > 0$ such that*

$$\mathbf{P}\left(\sup_{s\in[t,t+\delta_{\epsilon,\rho}]} |\xi_u(s) - \xi_u(t)| \geq \epsilon \right) \leq \rho \, \delta_{\epsilon,\rho}$$

for all $t \in [0,T]$ and $u \geq u_0$.

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