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Constants in the asymptotics of suprema of Gaussian processes

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Abstract

In this paper we focus on classical *Pickands' constants* and *Piterbarg & Prisyazhnyuk's constants*. They play a crucial role in the extreme value theory of Gaussian processes. We introduce fundamental theorems connected to these constants and we present so far known properties and bounds. Furthermore we proof a few new inequalities that improve current knowledge about these constants.

1 Introduction

1.1 Notation

We introduce basic notation used in the paper. Definitions, classical notions and theorems connected to Gaussian processes can be found in [17].

For a centered (zero mean) Gaussian process $\{\xi(t) : t \geq 0\}$ and $t, s \geq 0$ we denote:

$$\begin{array}{lrcl} \textit{covariance function} & R_\xi(t,s) & = & \mathbf{Cov}(\xi(t), \xi(s)) \ ; \\ \textit{variance function} & \sigma_\xi^2(t) & = & \mathbf{Var}(\xi(t)) \ = \ R_\xi(t,t) \ . \end{array}$$

By $\Phi(\cdot)$ we denote the probability of a standard normal random variable and by $\Psi(\cdot)$ – the tail probability of a standard normal random variable, i.e. $\Psi(x) = 1 - \Phi(x)$. We recall (see for instance Adler [1]) that

$$\Psi(u) \sim \frac{\exp(-u^2/2)}{u\sqrt{2\pi}} \qquad \text{as} \quad u \to \infty \ .$$

We use the relation: $f(u) \sim g(u)$ as $u \to \infty$ iff $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1$.

By $\{B_\alpha(t) : t \geq 0\}$ we denote a *fractional Brownian motion* with Hurst parameter $\alpha/2 \in (0, 1]$, that is a centered Gaussian process with stationary increments, continuous sample paths a.s. and variance function $\sigma^2_{B_\alpha}(t) = t^\alpha$ for $t \geq 0$.

Recall that B_α is self-similar with Hurst parameter $\alpha/2$ (it means that $\{B_\alpha(at) : t \geq 0\} \stackrel{\mathcal{D}}{=} \{a^{\alpha/2} B_\alpha(t) : t \geq 0\}$ for all $a > 0$). In a special case, when $\alpha = 1$, the process B_1 is the standard Brownian motion. For $\alpha = 2$, B_2 has degenerated structure in the sense that $\{B_2(t) : t \geq 0\} \stackrel{\mathcal{D}}{=} \{t\mathcal{N} : t \geq 0\}$, where $\mathcal{N}$ is a standard normal random variable.

1.2 Pickands constants $\mathcal{H}_{B_\alpha}$

One of methods useful to investigate exact asymptotics of a tail distribution of supremum of Gaussian processes is the *double–sum method*. Currently, this idea is being extended for a wide class of Gaussian processes. The method was originally suggested by Pickands in 1969 and it allowed to find exact asymptotics of $\mathbf{P}\left(\sup_{t\in[0,\delta]} Z(t) > u\right)$ as $u \to \infty$, where $Z(\cdot)$ is a centered stationary Gaussian process (see Pickands [11], [12] or Piterbarg [14] for more details).

Lemma 1.1 (Pickands' Lemma) *Let* $\{Z(t) : t \geq 0\}$ *be a centered stationary Gaussian process with continuous sample paths a.s. Assume that* $R_Z(t,0) = 1 - t^\alpha + o(t^\alpha)$ *as* $t \to 0^+$ *for some* $\alpha \in (0,2]$ *and* $R_Z(t) < 1$ *for all* $t > 0$. *Then for all* $T > 0$

$$\mathbf{P}\left(\sup_{t\in u^{-2/\alpha}[0,T]} Z(t) > u\right) \sim \mathcal{H}_{B_\alpha}(T)\cdot\Psi(u) \qquad \text{as} \quad u \to \infty\ ,$$

where

$$\mathcal{H}_{B_\alpha}(T) \ := \ \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha\right)\ .$$

Pickands' Lemma is the core idea of finding exact asymptotics of $\mathbf{P}\left(\sup_{t\in[0,\delta]} Z(t) > u\right)$ as $u \to \infty$. In the next theorem we introduce classical *Pickands' constant* $\mathcal{H}_{B_\alpha}$.

Theorem 1.2 (Pickands' Theorem) *The limit*

$$\mathcal{H}_{B_\alpha} \ := \ \lim_{T\to\infty} \frac{\mathcal{H}_{B_\alpha}(T)}{T}$$

exists and it is positive for all $\alpha \in (0,2]$. *Values* $\mathcal{H}_{B_\alpha}$ *we call* **Pickands' constants**.
Let assumptions of Pickands' Lemma hold. Then for all $\delta > 0$

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} Z(t) > u\right) \sim \mathcal{H}_{B_\alpha}\cdot\delta u^{2/\alpha}\cdot\Psi(u) \qquad \text{as} \quad u \to \infty\ .$$

1.3 Piterbarg & Prisyazhnyuk's constants $\mathcal{H}^R_{B_\alpha}$

It 1979 Piterbarg and Prisyazhnyuk extended Pickands' method to non-stationary Gaussian processes. They analyzed a wide class of so-called *locally stationary* Gaussian processes with nonconstant variance (see Piterbarg [14], [13] and Piterbarg & Prisyazhnyuk [15] for more details – note that in [14] and [15] there is a nontrivial error, correct version of the theorems are in [13]).

Lemma 1.3 (Piterbarg & Prisyazhnyuk Lemma) *Let* $\{X(t) : t \in [0,\delta]\}$ *be a centered Gaussian process with continuous sample paths a.s. such that the variance function* $\sigma_X^2(\cdot)$ *attains its maximum on* $[0,\delta]$ *at the unique point* $t^* = 0$ *with* $\sigma_X^2(0) = 1$. *Assume that:*

a) $\sigma_X^2(\cdot)$ *is a polynomial in a neighborhood of* $t^* = 0$: *there exist* $\beta, R > 0$ *such that*

$$\sigma_X^2(t) \ = \ 1 - R\,t^\beta + o(t^\beta) \qquad \text{as} \quad t \to 0^+\ ;$$

b) $X(\cdot)$ is **locally stationary***: there exists $\alpha \in (0,2]$ such that*

$$\mathbf{Var}(X(t)-X(s)) \;=\; |t-s|^\alpha + o(|t-s|^\alpha) \qquad as \quad t,s \to 0^+ \;;$$

c) $X(\cdot)$ is regular: there exist $\gamma, C > 0$ such that

$$\mathbf{Var}(X(t)-X(s)) \;\le\; C\,|t-s|^\gamma \qquad for\ all \quad t,s \in [0,\delta]\;.$$

Then for all $T, A > 0$:

1) if $\beta = \alpha$ then

$$\mathbf{P}\left(\sup_{t\in u^{-2/\alpha}[0,T]} X(t) > u\right) \;\sim\; \mathcal{H}^R_{B_\alpha}(T)\cdot\Psi(u) \qquad as \quad u\to\infty\,,$$

where

$$\mathcal{H}^R_{B_\alpha}(T) \;:=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right)\;;$$

2) if $\beta > \alpha$ then

$$\mathbf{P}\left(\sup_{t\in u^{-2/\alpha}[A,A+T]} X(t) > u\right) \;\sim\; \mathcal{H}_{B_\alpha}(T)\cdot e^{-RA^\beta u^{2(1-\beta/\alpha)}}\cdot\Psi(u) \qquad as \quad u\to\infty\,.$$

Piterbarg & Prisyazhnyuk Lemma enables to find exact asymptotics of $\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right)$ as $u\to\infty$. In the next theorem we introduce *Piterbarg & Prisyazhnyuk's constants* $\mathcal{H}^R_{B_\alpha}$.

Theorem 1.4 (Piterbarg & Prisyazhnyuk Theorem) *The limit*

$$\mathcal{H}^R_{B_\alpha} \;:=\; \lim_{T\to\infty} \mathcal{H}^R_{B_\alpha}(T)$$

exists and it is positive for all $\alpha \in (0,2]$ and $R > 0$. Values $\mathcal{H}^R_{B_\alpha}$ we call **Piterbarg & Prisyazhnyuk constants***.*
Let assumptions of Piterbarg & Prisyazhnyuk Lemma hold. Then for all $\delta > 0$:

1) if $\beta < \alpha$ then

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \;\sim\; \Psi(u) \qquad as \quad u\to\infty\,;$$

2) if $\beta = \alpha$ then

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \;\sim\; \mathcal{H}^R_{B_\alpha}\cdot\Psi(u) \qquad as \quad u\to\infty\,;$$

3) if $\beta > \alpha$ then

$$\mathbf{P}\left(\sup_{t\in[0,\delta]} X(t) > u\right) \;\sim\; \mathcal{H}_{B_\alpha}\cdot\frac{\Gamma(1/\beta)}{\beta}\cdot\frac{1}{R^{1/\beta}}\cdot u^{2(1/\alpha-1/\beta)}\cdot\Psi(u) \qquad as \quad u\to\infty\,.$$

It is worth to note that

$$\lim_{T\to\infty} \mathbf{E} \exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right) \;<\; \infty$$

for any $R > 0$, while

$$\lim_{T\to\infty} \mathbf{E} \exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha\right) \;=\; \infty \,.$$

Suprema distributions of locally-stationary Gaussian processes were analyzed by many authors: J. Hüsler, S.M. Berman and H.U. Bräcker (see Hüsler [8] and references therein).

In [9], Hüsler and Piterbarg investigated extremes of a certain class of Gaussian processes. They obtained exact asymptotics of $\mathbf{P}\left(\sup_{t\geq 0}(X(t) - ct^\beta) > u\right)$ as $u \to \infty$, where $X(\cdot)$ is a locally-stationary self-similar Gaussian process with variance t^α (for example fractional Brownian motion $B_\alpha(\cdot)$) and $\beta > \alpha/2$, $c > 0$ are some constants. In their result appear Pickands' and Piterbarg & Prisyazhnyuk constants.

In [10], they found exact asymptotic behavior of the ruin probability for *physical fractional Brownian motion*, i.e. $\mathbf{P}\left(\sup_{t\geq 0}\left(\int_0^t Z(s)ds - ct\right) > u\right)$ as $u \to \infty$, where $Z(\cdot)$ is a.s. continuous, centered stationary Gaussian process with covariance function regularly varying at infinity with index $-a \in (-1, 0)$. In this result Pickands' constants also play a crucial role.

Similar approach to one in [9], were used by Dębicki and Mandjes in [6]. Using Piterbarg & Prisyazhnyuk Theorem, they found overflow asymptotics for queues with many Gaussian inputs. They gave detailed results on the practically important cases in which the inputs are fractional Brownian motion processes or integrated Gaussian processes.

1.4 Generalized Pickands constants $\mathcal{H}_\eta$

In [3], Dębicki also analyzed integrated Gaussian process, which was considered by Hüsler and Piterbarg in [10]. Dębicki found exact asymptotics of $\mathbf{P}\left(\sup_{t\geq 0}\left(\int_0^t Z(s)ds - ct\right) > u\right)$ as $u \to \infty$, where $Z(\cdot)$ is a.s. continuous, centered stationary Gaussian process with covariance function fulfilling some integrability conditions. To obtain his result, he introduced *generalized Pickands constants*. We present the definition and some useful properties (for more details or proofs see Dębicki [3] and [4]).

Theorem 1.5 *Let $\{\eta(t) : t \geq 0\}$ be a centered Gaussian process with stationary incements, a.s. continuous sample paths, $\eta(0) = 0$ and variance function $\sigma_\eta^2(\cdot)$ that satisfies:*

a) $\sigma_\eta^2(\cdot) \in C^1[0,\infty))$ is strictly increasing and there exist $\epsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{t(\sigma_\eta^2(t))'}{\sigma_\eta^2(t)} \;\leq\; \epsilon \;;$$

b) $\sigma_\eta^2(\cdot)$ is regularly varying at 0 with index $\alpha_0 \in (0,2]$ and $\sigma_\eta^2(\cdot)$ is regularly varying at ∞ with index $\alpha_\infty \in (0,2]$.

Let

$$\mathcal{H}_\eta(T) \ := \ \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}\eta(t) - \sigma_\eta^2(t)\right) .$$

Then the limit

$$\mathcal{H}_\eta \ := \ \lim_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T}$$

exists and it is positive. Values $\mathcal{H}_\eta$ *we call* **generalized Pickands' constants**.

Lemma 1.6 *Let* $\eta_1(\cdot)$ *and* $\eta_2(\cdot)$ *be processes that satisfy assumptions of Theorem 1.5. Then:*

1) function $\mathcal{H}_\eta(\cdot)$ *is subadditive ;*

2) if $\eta_1(t) \le \eta_2(t)$ *for all* $t \ge 0$*, then* $\mathcal{H}_{\eta_1} \le \mathcal{H}_{\eta_2}$ *.*

The above properties of generalized Pickands constants will be used in the next section.

2 Properties of Pickands constants $\mathcal{H}_{B_\alpha}$

Let us recall the definition of Pickands' constants:

$$\begin{aligned}\mathcal{H}_{B_\alpha} \ &:= \ \lim_{T\to\infty} \frac{\mathcal{H}_{B_\alpha}(T)}{T} \ = \ \lim_{T\to\infty} \frac{\mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha\right)}{T} \\ &= \ \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} e^x \cdot \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha > x\right) dx .\end{aligned}$$

Albin and Choi [2] presented a new proof of Pickands' Theorem and obtained an alternative expression for Pickands' constants, namely:

$$\mathcal{H}_{B_\alpha} \ = \ \lim_{a\to 0^+} \frac{1}{a}\mathbf{P}\left(\max_{k\ge 1} \sqrt{2}B_\alpha(ak) - (ak)^\alpha + E \le 0\right) ,$$

where E is a unit mean exponentially distributed random variable that is independent of the process $B_\alpha(\cdot)$.

2.1 Exact values and conjecture

The exact values of the Pickands' constants are known only for $\alpha = 1$ and $\alpha = 2$. The following lemma allows us to find these expressions. The proof can be found in Dębicki & Kisowski [5].

Lemma 2.1 *For all* $T, a > 0$ *we have:*

1) $\mathcal{H}_{aB_1}(T) \ = \ \mathcal{H}_{B_1}(a^2T) \ = \ (2 + a^2T)\,\Phi\left(\sqrt{\frac{a^2T}{2}}\right) + \sqrt{\frac{a^2T}{\pi}}\exp\left(-\frac{a^2T}{4}\right)$ *;*

2) $\mathcal{H}_{aB_2}(T) \ = \ \mathcal{H}_{B_2}(aT) \ = \ 1 + \frac{aT}{\sqrt{\pi}}$ *.*

After taking limit as $T \to \infty$ for $a = 1$ we get exact values $\mathcal{H}_{B_1} = \mathbf{1}$ and $\mathcal{H}_{B_2} = \frac{\mathbf{1}}{\sqrt{\pi}}$.

In general case there is only unproven conjecture that

$$\mathcal{H}_{B_\alpha} = \frac{1}{\Gamma(1/\alpha)} ,$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx$ is a standard Gamma function. In [16], Shao proved the asymptotics

$$\lim_{\alpha \to 0^+} \frac{\alpha \log(\mathcal{H}_{B_\alpha})}{\log(\alpha)} = 1 ,$$

which is consistent with the conjecture above.

2.2 Bounds

In this subsection, we present so far known lower and upper bounds for Pickands' constants and we prove a new upper bounds.

In Shao [16] it was shown that

$$\begin{aligned} \left(1-(1+1/\alpha)e^{-1/\alpha}\right)(\alpha/4)^{1/\alpha} &\leq \mathcal{H}_{B_\alpha} \\ \mathcal{H}_{B_\alpha} &\leq \left(0.77\alpha + 2.41\sqrt{8.8\alpha - \alpha^2 \log(0.4 + 2.5/\alpha)}\right)^{2/\alpha} \end{aligned} \qquad \text{for} \quad \alpha \in (0,1) ;$$

$$0.625 \cdot 5.2^{-1/\alpha} \leq \mathcal{H}_{B_\alpha} \leq (\alpha e/\sqrt{\pi})^{2/\alpha} \qquad \text{for} \quad \alpha \in (1,2) . \quad (1)$$

Other lower bounds were found in [7]:

$$\frac{\alpha/2}{2^{2+2/\alpha}\Gamma(1/\alpha)} \leq \mathcal{H}_{B_\alpha} \qquad \text{for} \quad \alpha \in (0,2) .$$

Dębicki & Kisowski [5] improved bounds (1). The proofs relied on an application of properties of generalized Pickands' constants. The estimate given in Theorem 2.2 uniformly improves (1) and gives right values for $\alpha = 1$ and $\alpha = 2$. Theorem 2.3 provides sharper bounds for α in the neighbourhood of 1.

Theorem 2.2 *Let $\alpha \in (1,2)$. Then*

$$\mathcal{H}_{B_\alpha} \leq \inf_{T>0} \frac{\left(1 + T\sqrt{\frac{\alpha-1}{\pi}}\right)\left((2+(2-\alpha)T)\,\Phi\left(\sqrt{\frac{(2-\alpha)T}{2}}\right) + \sqrt{\frac{(2-\alpha)T}{\pi}}\exp\left(-\frac{(2-\alpha)T}{4}\right)\right)}{T} .$$

Theorem 2.3 *Let $\alpha \in (1,2)$. Then*

$$\mathcal{H}_{B_\alpha} \leq \alpha \left(\frac{2 + \sqrt{\frac{2}{\pi e}}}{\alpha - 1}\right)^{1-1/\alpha} .$$

Remark 2.1 In the proof of Theorem 2.3 there is a sharper bound for $\mathcal{H}_{B_\alpha}$, namely:

$$\mathcal{H}_{B_\alpha} \leq \inf_{T>0} \frac{\mathcal{H}_{B_1}(T^\alpha)}{T} = \inf_{T>0} \frac{(2+T^\alpha)\,\Phi\left(\sqrt{\frac{T^\alpha}{2}}\right) + \sqrt{\frac{T^\alpha}{\pi}}\exp\left(-\frac{T^\alpha}{4}\right)}{T}. \tag{2}$$

After applying inequalities $\Phi\left(\sqrt{\frac{T^\alpha}{2}}\right) \leq 1$, $\sqrt{\frac{T^\alpha}{\pi}}\exp\left(-\frac{T^\alpha}{4}\right) \leq \sqrt{\frac{2}{\pi e}}$ into (2) and optimizing over $T > 0$, we get the thesis of Theorem 2.3 (see Dębicki & Kisowski [5] for more details).

The next theorem (Theorem 2.4) improves bounds given in Theorem 2.2. On Figure 1 we compare bounds obtained by Shao (1), Dębicki & Kisowski (Theorems 2.2 and 2.3) and estimate given in Theorem 2.4.

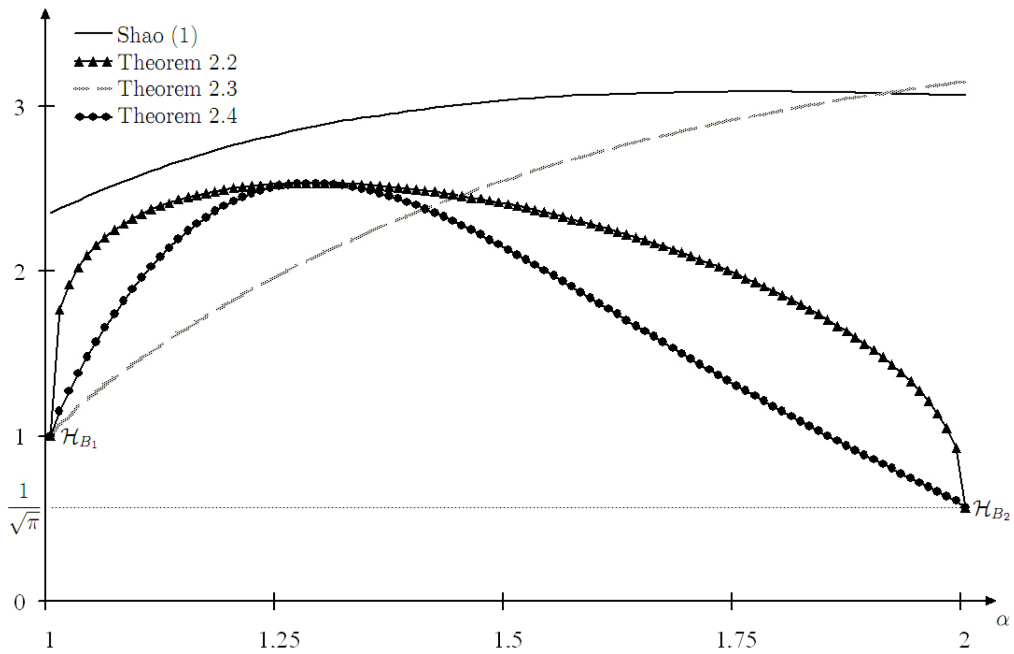

Figure 1: Upper bound (1) and estimates given in Theorems 2.2, 2.3 and 2.4.

Theorem 2.4 *Let $\alpha \in (1,2)$. Then*

$$\mathcal{H}_{B_\alpha} \leq \inf_{a>0}\inf_{T>0} \frac{\left(1+\frac{B(\alpha,a)T}{\sqrt{\pi}}\right)\left((2+A^2(\alpha,a)T)\,\Phi\left(\sqrt{\frac{A^2(\alpha,a)T}{2}}\right) + \sqrt{\frac{A^2(\alpha,a)T}{\pi}}\exp\left(-\frac{A^2(\alpha,a)T}{4}\right)\right)}{T},$$

where

$$A(\alpha,a) = \sqrt{(2-\alpha)a^{\alpha-1}} \qquad \textit{and} \qquad B(\alpha,a) = \sqrt{(\alpha-1)a^{\alpha-2}}.$$

Proof:
The proof is based on the modification of proof of Theorem 2.2. Let $\alpha \in (1,2)$ and $\{\eta_a(t) := A(\alpha,a)B_1(t) + B(\alpha,a)B_2(t) = \sqrt{(2-\alpha)a^{\alpha-1}}B_1(t) + \sqrt{(\alpha-1)a^{\alpha-2}}B_2(t) : t \geq 0\}$, where $B_1(\cdot)$, $B_2(\cdot)$ are independent fractional Brownian motions and $a > 0$.

Note that $\sigma^2_{\eta_a}(t) = (2-\alpha)a^{\alpha-1}t + (\alpha-1)a^{\alpha-2}t^2$ for $t \geq 0$. Consider function $f(t) = (2-\alpha)a^{\alpha-1} + (\alpha-1)a^{\alpha-2}t - t^{\alpha-1}$ for $t \geq 0$. Since $f'(t) = (\alpha-1)a^{\alpha-2} - (\alpha-1)t^{\alpha-2}$ is increasing for $t > 0$ and $f'(a) = 0$, then $\sigma^2_{\eta_a}(\cdot)$ attains its global minimum at $t^* = a$ and $f(a) = 0$. Hence $f(t) \geq 0$ for $t \geq 0$, which implies that $t^\alpha = \sigma^2_{B_\alpha} \leq \sigma^2_{\eta_a}(t)$ for all $t \geq 0$. From *2)* of Lemma 1.6 we obtain that $\mathcal{H}_{B_\alpha} \leq \mathcal{H}_{\eta_a}$ for all $a > 0$.

From subadditivity of generalized Pickands' constants (*1)* of Lemma 1.6) we have

$$\begin{aligned}
\mathcal{H}_{B_\alpha} &\leq \inf_{a>0} \mathcal{H}_{\eta_a} \leq \inf_{a>0}\inf_{T>0} \frac{\mathcal{H}_{\eta_a}(T)}{T} \\
&= \inf_{a>0}\inf_{T>0} \frac{\mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}\,(A(\alpha,a)B_1(t) + B(\alpha,a)B_2(t)) - A^2(\alpha,a)t - B^2(\alpha,a)t^2\right)}{T} \\
&\leq \inf_{a>0}\inf_{T>0} \frac{\mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}A(\alpha,a)B_1(t) - A^2(\alpha,a)t\right)\mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B(\alpha,a)B_2(t) - B^2(\alpha,a)t^2\right)}{T} \\
&= \inf_{a>0}\inf_{T>0} \frac{\mathcal{H}_{A(\alpha,a)B_1}(T)\,\mathcal{H}_{B(\alpha,a)B_2}(T)}{T} = \inf_{a>0}\inf_{T>0} \frac{\mathcal{H}_{B_1}(A^2(\alpha,a)T)\,\mathcal{H}_{B_2}(B(\alpha,a)T)}{T}\,.
\end{aligned}$$

Combining this with Lemma 2.1 completes the proof. □

Theorem 2.3 gives better results for α in the neighbourhood of 1 and Theorem 2.4 for α in the neighbourhood of 2. Due to numerical approximation we found a nice looking formulas for optimizers in (2) from Theorem 2.3 and Theorem 2.4. Results are given in Theorem 2.5. On Figure 2 we compare bounds from and Theorems 2.3 and 2.4 with estimates obtained by numerical approximation in Theorem 2.5.

Theorem 2.5 *Let*

$$\mathcal{H}_{B_1}(x) = (2+x)\,\Phi(\sqrt{x/2}) + \sqrt{x/\pi}\exp(-x/4) \qquad \text{for} \quad x > 0\,.$$

Then:

1) for $\alpha \in (1, 1.5]$

$$\mathcal{H}_{B_\alpha} \leq \frac{\mathcal{H}_{B_1}(T_1^\alpha(\alpha))}{T_1(\alpha)}\,, \qquad \text{where} \quad T_1(\alpha) = \frac{1}{\alpha-1}\,;$$

1) for $\alpha \in [1.4, 2)$

$$\mathcal{H}_{B_\alpha} \leq \frac{\left(1 + \frac{B(\alpha)T_2(\alpha)}{\sqrt{\pi}}\right)\mathcal{H}_{B_1}(A^2(\alpha)T_2(\alpha))}{T_2(\alpha)}\,,$$

where

$$\begin{aligned}
a(\alpha) &= \exp\left(-(e(\alpha-1))^2\right)\,, \\
T_2(\alpha) &= \exp\left(\exp(1.5 + 3.25\log(\alpha-1) + 2\log^2(\alpha-1))\right)\,, \\
A(\alpha) &= \sqrt{(2-\alpha)a^{\alpha-1}(\alpha)} \quad \text{and} \quad B(\alpha) = \sqrt{(\alpha-1)a^{\alpha-2}(\alpha)}\,.
\end{aligned}$$

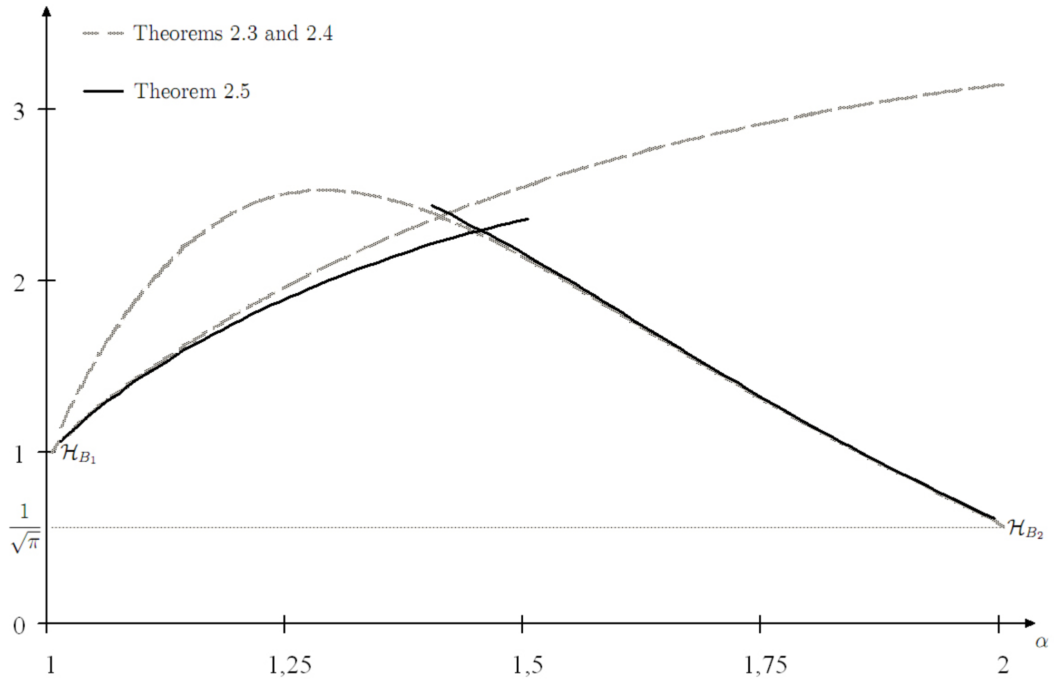

Figure 2: Upper bounds from Theorems 2.3, 2.4 and estimates given in Theorem 2.5.

3 Properties of Piterbarg & Prisyazhnyuk's constants $\mathcal{H}_{B_\alpha}^R$

Let us recall the definition of Piterbarg & Prisyazhnyuk's constants:

$$\begin{aligned} \mathcal{H}_{B_\alpha}^R &:= \lim_{T\to\infty} \mathcal{H}_{B_\alpha}^R(T) = \lim_{T\to\infty} \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right) \\ &= \lim_{T\to\infty} \int_0^\infty e^x \cdot \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha > x\right) dx \ . \end{aligned}$$

Similar to Pickands' constants, the exact values of the Piterbarg & Prisyazhnyuk's constants are known only for $\alpha = 1$ and $\alpha = 2$.

Lemma 3.1 *For all $R > 0$ we have:*

1) $\mathcal{H}_{B_1}^R = 1 + \frac{1}{R}$;

2) $\mathcal{H}_{B_2}^R = \frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}}\right)$.

Proof:
We refer to Dębicki & Mandjes [6] for computation of $\mathcal{H}_{B_1}^R$. One can check *2)* using representation $B_2(t) = t\mathcal{N}$ for $t > 0$, where $\mathcal{N}$ is a standard normal random variable. □

In the next lemma (Lemma 3.3) we show monotonicity of $\mathcal{H}_{B_\alpha}^R$ with respect to α and R. In the proof we use Slepian's Theorem (we cite it from Adler [1]).

Theorem 3.2 (Slepian's Theorem) *Let Gaussian stochastic processes $\{X(t) : t \in \mathcal{T}\}$ and $\{Y(t) : t \in \mathcal{T}\}$ have bounded sample paths a.s. If for all $t, s \in \mathcal{T}$*

$$\begin{aligned} \mathbf{E}\left(X(t)\right) &= \mathbf{E}\left(Y(t)\right) , \\ \mathbf{Var}\left(X(t)\right) &= \mathbf{Var}\left(Y(t)\right) , \\ \mathbf{Var}\left(X(t)-X(s)\right) &\geq \mathbf{Var}\left(Y(t)-Y(s)\right), \end{aligned}$$

then for all $x \in \mathbb{R}$

$$\mathbf{P}\left(\sup_{t\in\mathcal{T}} X(t) > x\right) \geq \mathbf{P}\left(\sup_{t\in\mathcal{T}} Y(t) > x\right) .$$

Lemma 3.3 *For all $\alpha_1, \alpha_2 \in (0, 2]$ and $R_1, R_2 > 0$ we have:*

1) if $\alpha_1 \leq \alpha_2$ then $\mathcal{H}_{B_{\alpha_2}}^R \leq \mathcal{H}_{B_{\alpha_1}}^R$;

2) if $R_1 \leq R_2$ then $\mathcal{H}_{B_\alpha}^{R_2} \leq \mathcal{H}_{B_\alpha}^{R_1}$.

Proof:
Ad 1) Let $0 < \alpha_1 \leq \alpha_2 \leq 2$ and $R > 0$. We will show that for all $T > 0$

$$\mathcal{H}_{B_{\alpha_2}}^R(T) \leq \mathcal{H}_{B_{\alpha_1}}^R(T^{\alpha_2/\alpha_1}) . \tag{3}$$

Consider process $\{S_{\alpha_1}(t) := B_{\alpha_1}(t^{\alpha_2/\alpha_1}) : t \geq 0\}$ and observe that for all $t \geq s \geq 0$

$$\mathbf{Var}(B_{\alpha_2}(t)-B_{\alpha_2}(s)) = t^{\alpha_2}(1-s/t)^{\alpha_2} \leq \mathbf{Var}(S_{\alpha_1}(t)-S_{\alpha_1}(s)) = t^{\alpha_2}\left(1-(s/t)^{\alpha_2/\alpha_1}\right)^{\alpha_1} . \tag{4}$$

Indeed, consider function $f(x) = (1-x)^a - 1 + x^a$ for $x \in [0, 1]$ and $a = \alpha_2/\alpha_1 \geq 1$. Since $f(0) = f(1) = 0$ and $f'(x) < 0 \Leftrightarrow x^{a-1} < (1-x)^{a-1} \Leftrightarrow x < 1/2$, then $f(x) \leq 0$ for all $x \in [0, 1]$, which implies (4).
Note that $\mathbf{Var}(S_{\alpha_1}(t)) = \mathbf{Var}(B_{\alpha_2}(t))$ for $t > 0$. Thus due to Slepian's Theorem (processes $B_{\alpha_2}(\cdot)$ and $S_{\alpha_1}(\cdot)$ have continuous and bounded sample paths on $[0, T]$), we have

$$\begin{aligned} \mathcal{H}_{B_{\alpha_2}}^R(T) &= \int_{-\infty}^{\infty} e^x \cdot \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}B_{\alpha_2}(t) - (1+R)t^{\alpha_2} > x\right) dx \\ &\leq \int_{-\infty}^{\infty} e^x \cdot \mathbf{P}\left(\sup_{t\in[0,T]} \sqrt{2}S_{\alpha_1}(t) - (1+R)t^{\alpha_2} > x\right) dx \\ &= \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_{\alpha_1}(t^{\alpha_2/\alpha_1}) - (1+R)(t^{\alpha_2/\alpha_1})^{\alpha_1}\right) \\ &= \mathbf{E}\exp\left(\sup_{t\in[0,T^{\alpha_2/\alpha_1}]} \sqrt{2}B_{\alpha_1}(t) - (1+R)t^{\alpha_1}\right) = \mathcal{H}_{B_{\alpha_1}}^R(T^{\alpha_2/\alpha_1}) . \end{aligned}$$

Thus we proofed (3). After taking limit as $T \to \infty$, we obtain $\mathcal{H}^R_{B_{\alpha_2}} \le \mathcal{H}^R_{B_{\alpha_1}}$.

Ad 2) follows from the fact that if $R_1 \le R_2$ then

$$\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R_2)t^\alpha \;\le\; \sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R_1)t^\alpha \qquad \text{a.s.}$$

This completes the proof.

□

Remark 3.1 Note that in case of Pickands' constants, there is no monotonicity with respect to α. There is a similar inequality to (3), namely $\mathcal{H}_{B_{\alpha_2}}(T) \le \mathcal{H}_{B_{\alpha_1}}(T^{\alpha_2/\alpha_1})$ for $\alpha_1 < \alpha_2$, but we can't take limit as $T \to \infty$, because $\lim_{T\to\infty} \frac{\mathcal{H}_{B_{\alpha_1}}(T^{\alpha_2/\alpha_1})}{T} = \infty$.

The following corollary is an immediate consequence of Lemma 3.1 and Lemma 3.3.

Corollary 3.4 *For all $R > 0$ we have:*

1) $\mathcal{H}^R_{B_\alpha} \;\ge\; \frac{1}{2}\left(1 + \sqrt{1+\frac{1}{R}}\right)$ *for* $\alpha \in (0,2]$ *;*

2) $\mathcal{H}^R_{B_\alpha} \;\ge\; 1 + \frac{1}{R}$ *for* $\alpha \in (0,1)$ *;*

3) $\mathcal{H}^R_{B_\alpha} \;\le\; 1 + \frac{1}{R}$ *for* $\alpha \in (1,2)$ *.*

Corollary 3.4 provides bounds for $\mathcal{H}^R_{B_\alpha}$ that don't depend on α. In the next theorem we proof sharper lower bounds which depend on α and R. Furthermore, we show connection between Piterbarg & Prisyazhnyuk's constants and Pickands' constants.

Theorem 3.5 *For all $\alpha \in (0,2]$ and $R > 0$*

$$\mathcal{H}^R_{B_\alpha} \;\ge\; \frac{\mathcal{H}_{B_\alpha}}{(e\alpha R)^{1/\alpha}} \;.$$

Proof:
Let $\alpha \in (0,2]$ and $R > 0$. For all $T > 0$ we have

$$\begin{aligned}\mathcal{H}^R_{B_\alpha} \;&\ge\; \mathcal{H}^R_{B_\alpha}(T) \;=\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - (1+R)t^\alpha\right)\\ &\ge\; \mathbf{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_\alpha(t) - t^\alpha - RT^\alpha\right) \;=\; \mathcal{H}_{B_\alpha}(T)\cdot\exp(-RT^\alpha)\;.\end{aligned}$$

Hence

$$\mathcal{H}^R_{B_\alpha} \;\ge\; \sup_{T>0}\left(\frac{\mathcal{H}_{B_\alpha}(T)}{T}\cdot Te^{-RT^\alpha}\right) \;\ge\; \inf_{T>0}\frac{\mathcal{H}_{B_\alpha}(T)}{T}\cdot\sup_{T>0} Te^{-RT^\alpha} \;\ge\; \mathcal{H}_{B_\alpha}\cdot\sup_{T>0} Te^{-RT^\alpha}\;.$$

Let $f(T) = Te^{-RT^\alpha}$. Since $f(0) = \lim_{T\to\infty} f(T) = 0$ and $f'(T) = 0 \Leftrightarrow e^{-RT^\alpha} = \alpha RT^\alpha e^{-RT^\alpha} \Leftrightarrow T = (\alpha R)^{-1/\alpha}$, then $\sup_{T>0} f(T) = f((\alpha R)^{-1/\alpha}) = (e\alpha R)^{-1/\alpha}$.
This completes the proof.

□

Remark 3.2 Note that for fixed $\alpha \in (0, 2]$

$$\mathcal{H}_{B_\alpha}^R \;\geq\; \text{Const} \cdot \left(\frac{1}{R}\right)^{1/\alpha} .$$

This gives some idea about asymptotic lower bounds for $\mathcal{H}_{B_\alpha}^R$ as $R \to 0^+$.

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