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A linear bound on the dimension in Green-Ruzsa's Theorem

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A LINEAR BOUND ON THE DIMENSION IN GREEN-RUZSA'S THEOREM

KAROL CWALINA AND TOMASZ SCHOEN

Abstract. In this paper a linear bound on the dimension in the Green-Ruzsa version of Freiman's theorem is obtained. This result is best possible up to a constant.

1. Introduction

Throughout this paper we will consider finite subsets of a (not necessarily finite) abelian group G. For any subsets $A, B \subseteq G$ we define the *sumset* $A + B = \{a + b : a \in A, b \in B\}$ and call $K(A) = |A+A|/|A|$ the *doubling* of A. In the paper, C denotes a constant which can vary from line to line.

The family of Freiman's-type theorems deals with finite sets A of small doubling, when compared with $|A|$. If that is the case then A forms a big part of a (proper) coset progression of dimension at most $d(K)$ and size at most $f(K)|A|$.

Let us recall the definition of a *coset progression*. It is any subset of G of the form $P + H$ where H is a subgroup of G and

$$P = P(x_0; x_1, \dots x_d; L_1, \dots, L_d) = \Big\{x_0 + \sum_{i=1}^{d} l_i x_i : |l_i| \leq L_i\Big\}$$

is a *generalized arithmetic progression* of *dimension* d and *size* $(2L_1+1)\cdot\ldots\cdot(2L_d+1)$. The *dimension* $d(P+H)$ of a coset progression $P+H$ is the dimension $d(P)$ of its underlying generalized arithmetic progression P and $\mathrm{size}(P+H)$ is $\mathrm{size}(P)|H|$. We say that a progression is *proper* if its cardinality equals its size.

As can be easily verified, the best possible bound for $d(P)$ is $\lfloor K-1 \rfloor$. Similarly, one cannot hope to obtain anything better than $\mathrm{size}(P) = \exp(O(K))|A|$.

Freiman's original result, which originates to the late 60s and appearance of monograph [1], concerns torsion-free groups only and is very inefficient in bound for $f(K)$. We owe to Sander's work [6], built upon Ruzsa's [2] and Chang's [3], the following formulation of Freiman's theorem.

Theorem 1 (Sanders)**.** *Let A be a finite subset of a torsion-free group G. If $|A + A| \leq K|A|$, then Freiman's theorem holds with $d(K) = CK^{7/4}\log^3 K$ and $f(K) = \exp(CK^{7/4}\log^3 K)$.*

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Chang obtains even sharper bound on the dimension, at the cost of a slightly higher degree of a polynomial in the exponent of f. Moreover, some additional conditions on $|A|$ must be imposed.

Theorem 2 (Chang). *Under the assumptions of Theorem 1, if, additionally, for some* $\epsilon > 0$, $|A| \geq \max(CK^2 \log^2 K, (K+\epsilon)^2/2\epsilon)$, *then* $A \subseteq P$ *where* P *is a proper generalized arithmetic progression of dimension* $d(P) \leq \lfloor K - 1 + \epsilon \rfloor$ *and* $\operatorname{size}(P) \leq \exp(CK^2 \log^3 K)|A|$.

Observe that in the torsion-free setting every finite coset progression is in fact a generalized arithmetic progression.

In a recent paper [5], Green and Ruzsa established a generalization of Theorem 1 for arbitrary abelian groups.

Theorem 3 (Green-Ruzsa). *Let* $A \subseteq G$ *be finite and* $|A + A| \leq K|A|$. *Then* A *is contained in a coset progression* $P + H$ *of dimension* $d(P + H) \leq CK^4 \log(K + 2)$ *and* $\operatorname{size}(P + H) \leq \exp(CK^4 \log^2(K + 2))|A|$.

In the abelian groups setting, the necessity of using coset progressions, in place of generalized ones, follows from consideration of a family of examples with $A = G = \mathbb{Z}_2^d$. In this case, the doubling of A equals 1 independently of d. Hence, the dimension cannot be bounded by any function of the doubling.

In what follows we show an analog of Theorem 2 in the general abelian groups setting, which is this.

Theorem 4. *Under the assumptions of Theorem 3, either* $A \subseteq P + H$ *for a proper coset progression such that* $d(P+H) \leq 2\lfloor K \rfloor$ *and* $\operatorname{size}(P+H) \leq \exp(CK^4 \log^2(K+2))|A|$, *or* A *is fully contained in at most* $CK^3 \log^2 K$ *cosets, whose total cardinality is bounded by* $\exp(CK^4 \log^2(K + 2))|A|$, *of some subgroup of* G .

A weaker result $d(P + H) \leq K^2$ was communicated by Green and Ruzsa and announced in [8, Exercise 6.5.18].

2. Geometry of numbers

In this section, we aim to prove the following two lemmas. Basically, they state that coset progressions are economically contained inside proper (convex) coset progressions.

Lemma 5. *Let* $X + H$ *be a convex coset progression of dimension* d. *Then, for every integer* $s \geq 1$, *there exists an* s*-proper convex coset progression* $X' + H'$ *containing* $X + H$ *of dimension* $d' \leq d$ *and* $\operatorname{size}(X' + H') \leq s^d d^{Cd} \operatorname{size}(X + H)$.

Lemma 6. *Under the assumptions of Lemma 5, there exists an* s*-proper coset progression* $P' + H'$ *containing* $X + H$ *of dimension* $d' \leq d$ *and* $\operatorname{size}(P' + H') \leq s^d d^{Cd^2} \operatorname{vol}(X)|H|$.

Here we provide some necessary definitions.

Suppose that $B \subseteq \mathbb{R}^d$ is closed, centrally symmetric and convex, $B \cap \mathbb{Z}^d$ spans $\mathbb{R}^d$ as vector space and $\phi : \mathbb{Z}^d \to G$ is a homomorphism. Then we refer to the image $X = \phi(B \cap \mathbb{Z}^d)$ as a *convex progression* of *dimension* d. The *size* of X is simply $\operatorname{size}(X) = |B \cap \mathbb{Z}^d|$, and the *volume* is $\operatorname{vol}(X) = \operatorname{vol}_d(B)$, the d-dimensional volume of B in $\mathbb{R}^d$.

Let X be a convex progression and H be a subgroup of G. Then we call $X+H$ a *convex coset progression*. By analogy with coset progressions, we define $\mathrm{size}(X+H) = \mathrm{size}(X)|H|$.

If $s \geq 1$ is an integer and if $\phi(x_1) - \phi(x_2) \in H$ implies $x_1 = x_2$ for all $x_1, x_2 \in sB \cap \mathbb{Z}^d$, then we say that $X+H$ is s-proper.

In order to relate progression's size to its volume we quote the following lemma.

Lemma 7 ([8, Lemma 3.26 and Inequality 3.14])**.** *Suppose that X is a convex progression. Then*

$$\frac{1}{2^d} \leq \frac{\mathrm{size}(X)}{\mathrm{vol}(X)} \leq \frac{3^d d!}{2^d}.$$

Proof of Lemma 5. We proceed by induction on d by reducing progression's dimension whenever it is not s-proper. Obviously, any zero-dimensional progression is so.

Fix s and let $X = \phi(B \cap \mathbb{Z}^d)$ for some $d > 0$. If $X+H$ is not s-proper then there exists a non-zero $x_h \in 2sB \cap \mathbb{Z}^d$ such that $\phi(x_h) \in H$. Consider $x_{\mathrm{irr}} \in 2sB \cap \mathbb{Z}^d$ such that $x_h = m x_{\mathrm{irr}}$ for $m \in \mathbb{N}$ as big as possible. Then, as an immediate consequence of [8, Lemma 3.4], there exists a completion $(x_1, \ldots, x_{d-1}, x_{\mathrm{irr}})$ of x_{irr} to an integral basis of $\mathbb{Z}^d$.

Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation satisfying $\psi(x_i) = e_i$, $i = 1, \ldots, d-1$ and $\psi(x_{\mathrm{irr}}) = e_d$ for (e_i) the canonical basis of $\mathbb{Z}^d$. For such transformation, $\psi(\mathbb{Z}^d) = \mathbb{Z}^d$ and $\mathrm{vol}_d(\psi(B)) = \mathrm{vol}_d(B)$.

Let $B' = \pi_{\mathbb{R}^{d-1} \times \{0\}}(\psi(B))$ and $H' = \langle H, \phi(x_{\mathrm{irr}}) \rangle$ be, respectively, the projection of $\psi(B)$ onto the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ and the subgroup of G generated by H and $\phi(x_{\mathrm{irr}})$.

Since one can treat $\phi \circ \psi^{-1}|_{\mathbb{R}^{d-1} \times \{0\}}$ as some $\phi' : \mathbb{R}^{d-1} \to G$, we have $X + H \subseteq X' + H'$ for $X' = \phi'(B' \cap \mathbb{Z}^{d-1})$. Indeed, for an arbitrary element of $X+H$ we have the following representation, with $x \in \mathbb{R}^{d-1} \equiv \mathbb{R}^{d-1} \times \{0\}$, $l \in \mathbb{Z}$ and $h \in H$:

$$\phi(\psi^{-1}(x) + l x_{\mathrm{irr}}) + h = \phi'(x) + (l\phi(x_{\mathrm{irr}}) + h) \in X' + H'.$$

Next, we estimate the size of $X' + H'$ but, for technical reasons, we prefer to consider $\mathrm{vol}(X')|H'|$ instead. These two quantities are related by Lemma 7.

Since

$$m\phi(x_{\mathrm{irr}}) = \phi(x_h) \in H,$$

it follows that

$$|H'| = |\langle H, \phi(x_{\mathrm{irr}}) \rangle| = |H + \{0, \phi(x_{\mathrm{irr}}), \ldots, (m-1)\phi(x_{\mathrm{irr}})\}| \leq m|H|.$$

In order to bound $\mathrm{vol}(X')$, consider the double-sided cone O spanned by B' and by

$$\pm\psi(x_h/2s) = \pm m\psi(x_{\mathrm{irr}})/2s = \pm m/2s \cdot e_d \in \psi(B),$$

the last stemming from $x_h \in 2sB$. From

$$\frac{2}{d}\mathrm{vol}_{d-1}(B') \cdot \frac{m}{2s} = \mathrm{vol}_d(O) \leq \mathrm{vol}_d(\psi(B)) = \mathrm{vol}_d(B)$$

we conclude that

$$\mathrm{vol}_{d-1}(B') \cdot |H'| \leq \frac{sd}{m}\mathrm{vol}_d(B) \cdot m|H| = sd\,\mathrm{vol}_d(B)|H|.$$

Notice that the inequality $\mathrm{vol}_d(O) \leq \mathrm{vol}_d(\psi(B))$ is a non-trivial one because, in general,

$$B' = \pi_{\mathbb{R}^{d-1}\times\{0\}}(\psi(B)) \not\subseteq \psi(B) \cap (\mathbb{R}^{d-1} \times \{0\})$$

and therefore $O \not\subseteq \psi(B)$. Instead, let us consider the convex set $\tau(\psi(B))$, where

$$\tau(x_1, \dots, x_d) = (x_1, \dots, x_{d-1}, x_d - CM_{\psi(B)}(x_1, ..., x_{d-1})),$$

$CM_{\psi(B)}(\cdot)$ denoting the center of mass of the corresponding fibre of $\psi(B)$. Obviously, in the spirit of Fubini's theorem, $\mathrm{vol}_d(\tau(\psi(B)) = \mathrm{vol}_d(\psi(B))$. Moreover,

$$B' \subset \tau(\psi(B)) \text{ and } \pm\psi(x_h/2s) = \pm m/2s \cdot e_d \in \psi(B) \cap \tau(\psi(B))$$

so $O \subseteq \tau(\psi(B))$ and hence $\mathrm{vol}_d(O) \leq \mathrm{vol}_d(\psi(B))$.

By inductive argument and Lemma 7 we can obtain an s-proper convex coset progression $X'' + H'' \supset X + H$ of dimension $d'' \leq d$, such that

$$\begin{aligned}
\mathrm{size}(X'')|H''| &\leq \frac{3^d d!}{2^d}\,\mathrm{vol}(X'')|H''| \\
&\leq \left(\frac{3s}{2}\right)^d (d!)^2\,\mathrm{vol}(X)|H| \\
&\leq (3s)^d (d!)^2\,\mathrm{size}(X)|H| \\
&= s^d \exp(Cd\log d)\,\mathrm{size}(X)|H|.
\end{aligned}$$

□

We prove Lemma 6 in much the same way as [4, Theorem 2.5] with an application of [4, Lemma 2.3] replaced by that of Lemma 5. One can check that both proofs result in the same asymptotic bounds on $\mathrm{size}(P' + H')$ as both [4, Lemma 2.3] and Lemma 5 establish them asymptotically the same.

3. The main argument

Let us first introduce a notion of projection. For any s-proper convex coset progression $X + H$ we define the canonical *projection* $\pi_{sX}(\cdot)$ of $sX + H$ onto sX in the following way: $\pi_{sX}(x + h) = x$ for $x \in sX$ and $h \in H$. Since $X + H$ is s-proper, this definition is unambiguous. Of course any s-proper progression is so for all $s' \leq s$ and we can consider relevant projections $\pi_{s'X}(\cdot)$ for $s' \leq s$.

We will now show an auxiliary lemma which roughly relates the doubling of a set to additive properties of its projection.

Lemma 8. *Let $A \subseteq X + H$, where $X + H$ is a 2-proper convex coset progression and $K_{\min} = \min_{Y\subseteq\pi_X(A)} |Y + \pi_X(A)|/|Y|$. Then $K(A) \geq K_{\min}$.*

Proof. Let $y_1, y_2, \ldots \in \pi_X(A)$ be all elements of $Y = \pi_X(A)$ in decreasing order with respect to the cardinality $|A_H(y_i)|$ of $A_H(y_i) = A \cap (y_i + H)$. Write $Y_i = \{y_1, \ldots, y_i\}$.

Then, by assumption, $|Y_i + Y| \geq iK_{\min}$ and there are at least $|A_H(y_i)|$ elements of $A + A$ in every H-coset of $Y_i + Y + H$. Hence

$$\begin{aligned} |A+A| &\geq \sum_i (|Y_i + Y| - |Y_{i-1} + Y|) \cdot |A_H(y_i)| \\ &= \sum_i |Y_i + Y| \cdot \big(|A_H(y_i)| - |A_H(y_{i+1})|\big) \\ &\geq \sum_i iK_{\min} \cdot \big(|A_H(y_i)| - |A_H(y_{i+1})|\big) \\ &= K_{\min} \sum_i \big(i - (i-1)\big)|A_H(y_i)| \\ &= K_{\min}|A| \end{aligned}$$

□

Notice that, as direct consequence, this lemma allows us to prove some version of the Green-Ruzsa theorem provided we can bound $K_{\min} = \min_{Y \subseteq X} |Y + X|/|Y|$ in terms of the doubling $K(X)$. While Plünnecke's inequality [8, Corollary 6.28] leads to a quadratic bound on dimension, we need some more elaborate reasoning to obtain a linear one.

Here we prove a slightly more general version of Theorem 4.

Theorem 9. *Let $A \subseteq G$ satisfy $|A+A| \leq K|A|$. Then for any integer $s \geq 1$ either there exists an s-proper coset progression $P + H$ of dimension $d(P + H) \leq 2\lfloor K \rfloor$ and $\operatorname{size}(P + H) \leq s^{2K} \exp(CK^4 \log^2(K + 2))|A|$ such that $A \subseteq P + H$, or A is fully contained in at most $CK^3 \log^2 K$ cosets, whose total cardinality is bounded by $\exp(CK^4 \log^2(K + 2))|A|$, of some subgroup of G .*

Proof. By Theorem 3 and Lemma 5, A is contained in a 2-proper convex coset progression $X+H$ of dimension $d \leq CK^4 \log(K+2)$ and $\operatorname{size}(X+H) \leq \exp(CK^4 \log^2(K+2))|A|$. Write $X = \phi(B \cap \mathbb{Z}^d)$.

Consider $Z = \phi^{-1}(\pi_X(A)) \subset \mathbb{Z}^d$. Let

$$K_{\min} = \min_{T \subseteq Z} |T + Z|/|T| = |S + Z|/|S|$$

for some $S \subseteq Z$. Obviously, $|S + S|/|S| \leq K_{\min} \leq K$, the last inequality stemming from Lemma 8. We consider two cases: either $|S| \geq CK_{\min}^2 \log^2 K_{\min}$ and therefore S satisfies the assumptions of Theorem 2, or S is too small.

In the first case, by Chang's theorem, there exists a generalized arithmetic progression Q containing S, of dimension $d(Q) \leq \lfloor K_{\min} \rfloor$ and $\operatorname{size}(Q) \leq \exp(CK_{\min}^2 \log^3 K_{\min})|S|$. By a well known Ruzsa's covering lemma [8, Lemma 2.14] there exists a subset $Z' \subseteq Z$ such that $|Z'| \leq |S + Z|/|S| = K_{\min}$ and $Z \subseteq Z' + S - S \subseteq Z' + Q - Q$. Therefore, Z is contained in a generalized arithmetic progression Q' of dimension $d(Q') \leq |Z'| + d(Q) \leq 2\lfloor K_{\min} \rfloor$ and $\operatorname{size}(Q') \leq 3^{|Z'|} 2^{d(Q')} \operatorname{size}(Q) = \exp(CK_{\min}^2 \log^3 K_{\min})|X|$.

The case concludes by moving back by ϕ to G: for $P = \phi(Q')$ we find $A \subseteq P+H$, the coset progression $P + H$ is of dimension $d(P + H) \leq 2\lfloor K_{\min} \rfloor$ and

$$\operatorname{size}(P + H) \leq \exp(CK_{\min}^2 \log^3 K_{\min})|X||H| \leq \exp(CK^4 \log^2(K + 2))|A|.$$

Application of Lemma 6 gives the desired result.

On the other hand, if $|S| < CK_{\min}^2 \log^2 K_{\min}$ then $|Z| \le |Z+S| = K_{\min}|S| < CK^3 \log^2 K$.

This concludes the proof. □

4. Remarks

A new version of Bogolyubov-Ruzsa's lemma, proved in [7], results in the following bounds in Freiman's and Green-Ruzsa's theorems.

Theorem 10 ([7, Theorems 1 and 2]). *Let A be finite and satisfy $|A+A| \le K|A|$. Then Freiman's theorem holds with $d(K) = K^{1+C(\log K)^{-1/2}}$ and $f(K) = \exp(d(K))$ if G is torsion-free and with $d(K) = (K+2)^{3+C(\log(K+2))^{-1/2}}$ and $f(K) = \exp(d(K))$ otherwise.*

These may potentially serve to obtain still better bounds in Theorem 4. To this end, we will formulate a slightly improved version of Chang's Theorem 2.

Theorem 11 (Chang). *Let $A \subseteq G$ be a finite subset of a torsion-free group G and $|A+A| \le K|A|$. If $|A| \ge K^{1+C(\log K)^{-1/2}}$ then A is contained in a proper generalized arithmetic progression P of dimension $d(P) \le \big(1+o(1)\big)K$ and $\mathrm{size}(P) \le \exp(CK^2 \log K)|A|$. If, additionally, $|A| \ge (K+\epsilon)^2/2\epsilon$, for $\epsilon > 0$, then $d(P) \le \lfloor K-1+\epsilon \rfloor$.*

Proof (sketch). This sketch will follow Green's exposition [4, proof of Theorem 3.2].

By Theorem 10 and Lemma 5, $A \subseteq$X where $X = \phi(B \cap \mathbb{Z}^d)$ is a 2-proper convex progression of dimension $d \le d(K) = K^{1+C(\log K)^{-1/2}}$ and $\mathrm{size}(X)$ bounded by $\exp(K^{1+C(\log K)^{-1/2}})|A|$.

Let us denote by d' the dimension of the linear space spanned by $\phi^{-1}(A)$. If $d' \le K-1$, we can skip the next few steps, where we establish bounds on d'.

Otherwise, by Freiman's lemma [8, Lemma 5.13],

$$K|A| \ge |A+A| \ge (d'+1)|A| - d'(d'+1)/2$$

and

$$|A| \le r(d') = \frac{d'(d'+1)}{2(d'+1-K)}.$$

Let us define d'' as the second solution to the equation $r(x) = r(d(K))$, equivalent to

$$x^2 - x\Big(\frac{d(K)\big(d(K)+1\big)}{d(K)+1-K} - 1\Big) + (K-1)\frac{d(K)\big(d(K)+1\big)}{d(K)+1-K} = 0.$$

By Viète's formula

$$d'' = \frac{d(K)\big(d(K)+1\big)}{d(K)+1-K} - 1 - d(K) = \frac{d(K)+1}{d(K)+1-K}(K-1) = \big(1+o(1)\big)K.$$

Since r is convex, $r(d') \ge |A| > r(d(K)) = K^{1+C(\log K)^{-1/2}}$ and $d' \le d(K)$, we have

$$d' \le d'' = \big(1+o(1)\big)K.$$

If $|A| \ge (K+\epsilon)^2/2\epsilon > r(\lfloor K-1+\epsilon \rfloor)$, for $\epsilon > 0$, we can conclude that $d' \le \lfloor K-1+\epsilon \rfloor$.

It remains to show that the slice of B containing $\phi^{-1}(A)$, contained itself in the hyperplane of dimension d', is reasonably small. This is done in exactly the same manner as in Green's exposition [4]. □

A literal repetition of the proof of Theorem 4 gives the following result.

Theorem 12. *Let $A \subseteq G$ satisfy $|A+A| \le K|A|$. Then for any integer $s \ge 1$ either $A \subseteq P+H$ for an s-proper coset progression $P+H$ of dimension* $d(P+H) \le \big(2+o(1)\big)K$ *and* $\mathrm{size}(P+H) \le s^{2K}\exp(C(K+2)^3\log(K+2))|A|$*, or A is fully contained in at most $K^{2+C(\log(K+2))^{-1/2}}$ cosets of some subgroup of G whose total cardinality is bounded by* $\exp(C(K+2)^3\log(K+2))|A|$.

Observe that this formulation exhibits some imperfection of characterization of unstructured sets A. Obviously, we would prefer to bound the number of cosets containing A by $K^{1+\epsilon}$ instead of $K^{2+C(\log(K+2))^{-1/2}}$. This would be near-optimal since $2K-1$ is an obvious lower bound for this problem.

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