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# The adèlic approach to the Riemann zeta function

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# THE ADÈLIC APPROACH TO THE RIEMANN ZETA FUNCTION

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## 1. Introduction

In the year 1859 Bernhard Riemann wrote his remarkable paper "Über die Anzahl der Primzahlen unter einer gegebenen Gröβe" in which he studied zeta function

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right) \text{ where } \Re(s) > 1.$$

which is now named the Riemann Zeta Function. He proved that it is meromorphic on $\mathbb{C}$ and after "completion"

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\zeta(s)$$

it satisfies functional equation of the form

$$\xi(s) = \xi(1-s).$$

In the subsequent years, thanks to the work of Richard Dedekind, it was discovered that to every algebraic number field there is attached a function, now called the Dedekind Zeta Function, of which the Riemann Zeta Function is a special case. At that point natural question arose: Does the Dedekind Zeta Function satisfy similar functional properties, i.e. could it be meromorphically continued to whole complex plane and does it satisfy a functional equation? A positive answer to this question covering all cases in general (i.e. for all algebraic number fields) was given by Erich Hecke in 1917, in the paper "Über die Zetafunktion beliebiger algebraischer

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Zahlkörper". Deeper analysis of the proof led Hecke to the discovery of the broad class of functions known nowdays as Hecke L-functions. Worth mentioning is the fact, that although there are many essentially different proofs for meromorphic continuation of the Riemann Zeta Function, it is the original method of Riemann which, after overcoming great technical difficulties, led Hecke to the proof of the meromorphic continuation for his L-functions.

In 1950, in his famous PhD thesis, written under the supervision of Emil Artin, John Tate gave a new proof of the functional equation for the Hecke L-functions, by means of abstract commutative harmonic analysis. The great advantage of Tate's method, compared to the original proof of Hecke is that, the passage from the less complicated object, such as the Riemann Zeta Function, to more complicated, such as Dedekind or Hecke L-functions, is almost instant, while in Hecke's original proof, it was the main difficulty he had to deal with.

Although the best reference to Tate's proof of the functional equation for the Hecke L-functions is the Tate Thesis itself [5], reading this paper requires facility in algebraic number theory and Fourier analysis in locally compact abelian groups. Therefore the motivation for writing this paper is to give an elementary presentation of the essence of Tate's method. Therefore we exemplify this method on the well known Riemann Zeta Function, which allows us to focus on its essential points.

## 2. Local case

In this paragraph we shall present the notion of valuation on $\mathbb{Q}$ and completion of $\mathbb{Q}$ with respect to a valuation. Having complete description of such fields, and knowing that they are locally compact, we shall study objects called *local zeta functions* defined in terms of the Fourier analysis. Then we shall pass to proving the *local functional equation.* The final form of the functional equation shall lead us to a careful study of *proportionality functions*, and to give them very explicit expressions as a meromorphic functions of one complex variable.

**Definition 2.1.** Every function

$$|\cdot|_\nu \ : \ \mathbb{Q} \longrightarrow \mathbb{R}$$

enjoying properties:

(i) $|x|_\nu \geq 0$, and $|x|_\nu = 0$ if and only if $x = 0$;
(ii) $|xy|_\nu = |x|_\nu |y|_\nu$;
(iii) $|x+y|_\nu \leq |x|_\nu + |y|_\nu$ "the triangle inequality"

for all $x$ and $y$ in $\mathbb{Q}$ is called a *valuation.*

**Example.** Here we give some basic examples of valuations.

(i)
$$|x|_0 = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \ . \end{cases}$$

This valuation is called *trivial.* Every valuation different from trivial is called *non-trivial.*

(ii) Ordinary absolute value function restricted to $\mathbb{Q}$, which will be denoted by $|\cdot|_\infty$ is a valuation.

(iii) The $p$-adic valuation:

$$|x|_p = p^{-ord_p(x)}$$

where $p$ is a prime number and $ord_p(x)$ denotes the highest exponent of the prime $p$ dividing $x$. Note that since $x$ is a rational number it can be uniquely represented, up to an order of factors, as a product of integral powers of prime numbers. Therefore it makes sense to speak about $ord_p(x)$ defined for rational numbers.

*Remark* 2.1. Every valuation defines a metric and hence also a topology on $\mathbb{Q}$ by the following formula

$$d_\nu(x, y) = |x - y|_\nu \ .$$

**Definition 2.2.** Two valuations are called *equivalent* if and only if they define the same topology on $\mathbb{Q}$.

*Remark* 2.2. From now on, wherever we speak of a valuation, we mean by this a *non-trivial valuation.*

**Definition 2.3.** Every valuation which satisfies the following *strong triangle inequality*

$$|x + y|_\nu \leq \max(|x|_\nu, |y|_\nu)$$

is called *nonarchimedean.* All other valuations are called *archimedean.*

**Example.** For every prime number $p$ the valuation $|x|_p = p^{-ord_p(x)}$ is nonarchimedean.

Valuation $|\cdot|_\infty$ is an example of archimedean valuation.

**Theorem 2.1** (Ostrowski)**.** *Every valuation on $\mathbb{Q}$ is equivalent either to $|\cdot|_p$ for some prime number p, or to $|\cdot|_\infty$.*

*Proof.* Complete proof can be found in [4] [prop. 3.7 p. 119]. □

*Remark* 2.3. Ostrowski's theorem can be formulated for an arbitrary algebraic number field $\mathbb{K}$ (i.e. a finite extension of $\mathbb{Q}$) and it states then, that up to equivalence nonarchimedean valuations come from prime ideals of the ring of integers of $\mathbb{K}$, whereas archimedean ones from embeddings of $\mathbb{K}$ into $\mathbb{C}$ (see [3] p. 89 Theorem 3.3). In the algebraic number field case archimedean valuations have to be split into two classes: coming from real and coming from complex embeddings; these two classes have to be treated slightly differently for the proof of the functional equation in the general case. For details see [5] p. 316 - 319 .

*Remark* 2.4. From now on, wherever we speak on a valuation we mean by this equivalence class of non-trivial valuation.

**Theorem 2.2.** *Let $|\cdot|_\nu$ denotes either p-adic valuation for a prime number p, or standart absolute value valuation on $\mathbb{Q}$. Completion of $\mathbb{Q}$ under the metric $d_\nu$, canonically induced by the valuation $|\cdot|_\nu$ (see Remark 2.1), is locally compact topological field and is denoted by $\mathbb{Q}_\nu$. Valuation $|\cdot|_\nu$ has unique continous extention to this completion. Moreover for $|\cdot|_\nu$ nonarchimedean i.e. $|\cdot|_\nu = |\cdot|_p$ for some prime number p, closure in p-adic topology, of the ring of integers in $\mathbb{Q}_p$, is compact and open, and denoted by $\mathbb{Z}_p$.*

*Proof.* Details of the proof in general setting of the algebraic number fields can be found in Chapter 2 sections 4 & 5 of [4] . □

*Remark* 2.5. Extension of the valuation $|\cdot|_\nu$ from $\mathbb{Q}$ to $\mathbb{Q}_\nu$ is unique, therefore by abuse of notation, we shall not introduce a new symbol for it.

2.1. **Local zeta functions.** In this section, we shall define local zeta functions, in terms of harmonic analysis on locally compact abelian groups. To this end, we shall need notions of Haar measure and Fourier transform on such groups. Also we shall give explicit description of additive *characters* and we shall refer to the general notion of multiplicative *quasicharacters* for the field $\mathbb{Q}_\nu$, giving generic examples of them.

**Definition 2.4.** Let $G$ be a locally compact abelian group, then group $Hom_{cts}(G, \mathbb{S}^1)$ of continuous group homomorphisms, endowed with compact-open topology is called *Pontryagin dual* of $G$ and is denoted by $\widehat{G}$. An element $\chi \in \widehat{G}$ is called a *character* of group $G$. A character is called *trival*, if it is a constant function equal 1 on $G$. Such charachter is the identity of $\widehat{G}$. All characters which are not trivial are called *non-trivial.*

**Theorem 2.3** (Pontryagin Duality Theorem)**.** *Let $G$ be a locally compact abelian group. Then its dual $\widehat{G}$ is locally compact abelian as well. Moreover, $G$ is topologically isomorphic to $\widehat{\widehat{G}}$ by natural isomorphism*

$$G \ni g \mapsto \psi_g \in \widehat{\widehat{G}}\ , \text{ where } \psi_g(\chi) = \chi(g)\ .$$

*Proof.* Details of proof can be found in [1] [Theorem 24.1 p. 376]. □

**Example 2.1.** Let $\mathbb{Q}_\nu^+$ denote the additive group of $\mathbb{Q}_\nu$.

(i) If valuation $|\cdot|_\nu$ is archimedean (i.e. $|\cdot|_\nu = |\cdot|_\infty$) then the following function $\chi$ is a non-trivial character:

$$\chi(\xi) = \exp(-2\pi i \lambda(\xi)), \text{ where } \lambda(x) = -x\ (mod\ 1)\ .$$

(ii) If valuation $|\cdot|_\nu$ is nonarchimedean (i.e. $|\cdot|_\nu = |\cdot|_p$ for some prime number $p$) then the following function $\chi$ is a non-trivial character:

$$\chi(\xi) = \exp(-2\pi i \lambda(\xi))$$

where $\lambda(x)$ is the unique rational number from $[0, 1)$ with a $p$-power in the denominator, such that $\lambda(x) - x \in \mathbb{Z}_p$.

**Theorem 2.4.** *For each $\eta$ in $\mathbb{Q}_\nu^+$ the mappings*

$$\xi \mapsto \chi(\xi\eta)$$

*where $\chi$ is as in Example 2.1, gives topological isomorphism between $\mathbb{Q}_\nu^+$ and $\widehat{\mathbb{Q}_\nu^+}$ .*

*Proof.* The idea of proof can be found in [5] [Lemma 2.2.1 p. 308] and details in [3] [Theorem 5.37 p. 237]. □

**Definition 2.5.** Let $G$ be a locally compact group. Every non-zero, complete, regular, invariant under left translations Borel measure on $G$, is called a *Haar measure.*

**Theorem 2.5** (Haar)**.** *If $G$ is a locally compact group, then there exists on it a Haar measure, which is unique up to a multiplicative constant.*

*Proof.* In [1] one can find Corollary 11.37 on p. 109 and Theorem 15.5 on p. 185 from which the theorem follows immediately.

□

**Definition 2.6.** Let $G$ be a locally compact abelian group and $dg$ a fixed Haar measure on it. Then we introduce the following notation:

$$L^1(G) := \left\{ f:\ G \to\ \mathbb{C} \mid \int_G |f(g)| dg\ < \infty \right\}$$

and

$$C(G) := \{ f:\ G \to\ \mathbb{C} \mid f \text{ is continous} \}\ .$$

**Definition 2.7.** Let $G$ be a locally compact abelian group and let $f \in L^1(G) \cap C(G)$. Then the function $\widehat{f}$ defined by the following formula

$$\widehat{f}(\chi) = \int_G f(\xi)\chi(\xi)\, d\xi$$

where $d\xi$ is a fixed Haar measure on $G$, is called *the Fourier transform* of $f$.

*Remark* 2.6. From now on, speaking about locally compact groups we shall always mean locally compact *abelian* groups.

**Theorem 2.6.** *With the above definition of* $\widehat{f}$ *we have*

$$\widehat{f} \in L^1(\widehat{G}) \cap C(\widehat{G}) .$$

*Proof.* A complete proof can be found in [2] [Theorem 31.5 p. 212]. □

*Remark* 2.7. By Theorem 2.3, we can identyfy $G$ and $\widehat{\widehat{G}}$ and consequently we identify

$$L^1(G) \cap C(G) \text{ and } L^1(\widehat{\widehat{G}}) \cap C(\widehat{\widehat{G}}) .$$

Hence we shall regard the double Fourier transform $\widehat{\widehat{f}}$ as a function on $G$. With this convention we have the following result.

**Theorem 2.7** (Fourier Inversion Formula)**.** *There exist unique Haar measures on* $G$ *and* $\widehat{G}$ *such that for every function* $f \in L^1(G) \cap C(G)$ *the following formula holds:*

$$f(\xi) = \widehat{\widehat{f}}(-\xi) .$$

*Proof.* This theorem is the consequence of [Theorem 31.17 p. 225] in [2]. □

*Remark* 2.8. By Theorem 2.5 and Theorem 2.7 for finding unique Haar measure one needs only to check equality from Theorem 2.7 for only one non-zero function from the class $L^1(G) \cap C(G)$ to find the normalisation constant.

**Example 2.2.** In the case of the group $\mathbb{Q}_\nu^+$, using Theorem 2.4 we have the following explicit description of the Fourier transform:

$$\widehat{f}(\eta) = \int_{\mathbb{Q}_\nu^+} f(\xi) \exp(-2\pi i \lambda(\xi\eta))\, d_\nu \xi$$

where $d_\nu$ denotes a Haar measure on $\mathbb{Q}_\nu^+$ .

**Example 2.3.** For the group $\mathbb{Q}_\nu^+$ the following normalisation of Haar measure is taken for Theorem 2.7 to hold:

(i) If $\mathbb{Q}_\nu^+$ is archimedean i.e.

$$\mathbb{Q}_\nu^+ = \mathbb{R}^+$$

then by choosing function

$$f(\xi) = \exp(-\pi|\xi|^2)$$

clearly belonging to $L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$, we obtain the well known fact from classical Fourier analysis, that the Fourier Inversion Formula holds for Haar measure equal to the standard Lebesgue measure on $\mathbb{R}^+$ i.e. $d_\nu\xi = d\xi$ on $\mathbb{R}^+$.

(ii) If $\mathbb{Q}_\nu^+$ is nonarchimedean i.e.

$$\mathbb{Q}_\nu^+ = \mathbb{Q}_p^+ \quad \textit{for some prime number } p$$

then by choosing function

$$f(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{Z}_p^+ \\ 0 & \text{if } \xi \notin \mathbb{Z}_p^+ \end{cases}$$

belonging to $L^1(\mathbb{Q}_p^+) \cap C(\mathbb{Q}_p^+)$ (since $\mathbb{Q}_p^+$ is *totally disconnected*, which means that every compact set is also open, follows that characteristic function of every compact set is continuous and by Theorem 2.2 $\mathbb{Z}_p^+$ is compact) we obtain that the Fourier Invesion Formula holds for Haar measure $d_\nu\xi$ on $\mathbb{Q}_p^+$ normalised such that $\mathbb{Z}_p^+$ gets measure one.

Now we can examine multiplicative structure of $\mathbb{Q}_\nu$, denoted by $\mathbb{Q}_\nu^*$ which shall lead us directly to the notion of *multiplicative measure* and to the notion of *quasicharacter*.

**Definition 2.8.** Each continuous homomorphism

$$c \ : \ \mathbb{Q}_\nu^* \longrightarrow \ \mathbb{C}^*$$

is called a *quasicharacter* of the group $\mathbb{Q}_\nu$. If the quasicharacter is bounded, i.e. $|c| \ = \ 1$, then we shall call it a *multiplicative character*.

**Example 2.4.** (i) By definition, the valuation $|\cdot|_\nu$ on $\mathbb{Q}_\nu^*$ is a quasicharacter.

(ii) More generally:
let $s \ \in \ \mathbb{C}$, then function

$$c(\alpha) = |\alpha|_\nu^s \equiv \exp(s \log |\alpha|_\nu)$$

is a quasicharacter.

*Remark* 2.9. Quasicharacters in Example 2.4 in [5] are called *unramified quasicharacters*. Careful examination, given in [5], of the kernel of a homomorphism $|\cdot|_\nu$, denoted usually by $\mathcal{U}_\nu$, leads to complete description of quasicharacters which are not unramified, and are called *ramified*. Moreover, one finds that each quasicharacter is either unramified or ramified, and in fact, each quasicharacter can be canonically decomposed, into the product of unramified quasicharacter and multiplicative character on the group $\mathcal{U}_\nu$. Thus considering general case of arbitrary quasicharacter (possibly ramified) leads to proof of the functional equation for the Dirichlet L-function in case of field $\mathbb{Q}$, and to functional equation for the Hecke L-function in the number field case.

*Remark* 2.10. Note that in the class of unramified quasicharachters can be identified with a Riemann surface of certain type.

(i) In the case of $|\cdot|_\nu = |\cdot|_\infty$ (archimedean) it is simply the complex plane where each complex number is representative of exactly one unramified quasicharacter.

(ii) Since $ord_p(\alpha)$ has unique continous extension from $\mathbb{Q}$ to $\mathbb{Q}_p$ by Theorem 2.2, and has discrete codomain $\mathbb{Z}$, we have:

$$|\alpha|_p^{s+\frac{2\pi i}{\log p}k} = \exp\left(\left(s + \frac{2\pi i}{\log p}k\right)\log|\alpha|_p\right) =$$
$$= \exp(s\log|\alpha|_p)\exp\left(-ord_p(\alpha)\ 2\pi i k\ \frac{\log p}{\log p}\right) = |\alpha|_p^s\ .$$

Thus in the case of $|\cdot|_\nu = |\cdot|_p$ for some prime $p$ the corresponding Riemann surface is the complex plane "glued" along the imaginary axis at the points $\frac{2\pi i}{\log p}k$ where $k \ \in \ \mathbb{Z}$. On such surface each point coresponds to exactly one quasicharacter.

*Remark* 2.11. From now on, by a quasicharacter, we shall mean by this an unramified quasicharacter.

**Definition 2.9.** For each quasicharacter $c = ||_\nu^s$, number $\sigma = \Re(s)$ is called the *exponent* of $c$.

By Theorem 2.2 $\mathbb{Q}_\nu^*$ is a locally compact abelian group. Hence there exists a Haar measure on it.

**Theorem 2.8.** *We have*

$$f(\alpha) \ \in L^1(\mathbb{Q}_\nu^*) \Longleftrightarrow \ f(\alpha)|\alpha|_\nu^{-1} \in \ L^1(\mathbb{Q}_\nu^+ \setminus \{0\}) \ .$$

*Moreover, the equality*

$$d_\nu^* \alpha \ = \ |\alpha|_\nu^{-1} d_\nu \alpha$$

*defines a Haar measure on* $\mathbb{Q}_\nu^*$.

*Proof.* Complete proof can be found in [5] [Lemma 2.3.2 p. 312]. □

**Definition 2.10.** Normalised Haar measures on $\mathbb{Q}_\nu^*$ are given by the following formulæ:

(i) If $|\cdot|_\nu$ is archimedean

$$d_\nu^* \xi \ = \ |\xi|_\nu^{-1} d_\nu \xi \ .$$

(ii) If $|\cdot|_\nu$ is nonarchimedean:

$$d_\nu^* \xi \ = \ \frac{p}{p-1} |\xi|_\nu^{-1} d_\nu \xi \ .$$

**Proposition 2.9.** *In the case of a* $|\cdot|_\nu$ *being nonarchimedean, the group* $\mathcal{U}_\nu$ *has normalised measure* $= 1$.

*Proof.* Details in [5] [Lemma 2.3.3 p. 313]. □

**Definition 2.11.** Let $S(\mathbb{Q}_\nu)$ denotes the set of all complex functions $f$ enjoying the following properties:

(i) $f$ and $\widehat{f}$ are in $L^1(\mathbb{Q}_\nu^+) \cap C(\mathbb{Q}_\nu^+)$;
(ii) $f(\alpha)|\alpha|_\nu^\sigma$ and $\widehat{f}(\alpha)|\alpha|_\nu^\sigma$ belong to $L^1(\mathbb{Q}_\nu^*)$ for $\sigma > 0$ .

**Definition 2.12** (Local Zeta Function)**.** For each $f \in S(\mathbb{Q}_\nu)$ and a quasicharacter $c$ with a positive exponent we introduce the function $\zeta_\nu(f, c)$, by the formula

$$\zeta_\nu(f, c) \ = \ \int_{\mathbb{Q}_\nu^*} f(\alpha) c(\alpha) d_\nu^* \alpha \ . \tag{1}$$

This function is called the *local zeta function*.

*Remark* 2.12. In fact, one does not need to consider such a broad class of functions as $S(\mathbb{Q}_\nu)$. For instance, in the well known case of $\mathbb{Q}_\nu = \mathbb{R}$ it is enough to take, instead of $S(\mathbb{R})$, the Schwartz class of rapidly decreasing smooth functions. In case of $\mathbb{Q}_\nu = \mathbb{Q}_p$ one can replace class $S(\mathbb{Q}_\nu)$ by the class of compactly supported continous step functions. This fact becomes completly clear in the work of André Weil [6]. Passing to the proof of the functional equation, we shall point out the step in the proof, which makes choice of class $S(\mathbb{Q}_\nu)$ less rigid. Therefore from this point on, abusing notation, by $S(\mathbb{Q}_\nu)$ we mean the Schwartz space in the archimedean case, and the space of compactly supported continous step functions in the nonarchimedean case.

*Remark* 2.13. In view of Remark 2.10 $\zeta_\nu(f, c)$ can be understood as the function of variable $s$ on Riemann surface parametrasing quasicharacters $c = ||_\nu^s$ in the following way:

$$\zeta_\nu(f, ||_\nu^s) \ = \ \int_{\mathbb{Q}_\nu^*} f(\alpha)|\alpha|_\nu^s d_\nu^* \alpha \ . \tag{2}$$

2.2. **Local functional equation.** In this section we state and prove the local functional equation. First, observe that by Remark 2.13, local zeta function $\zeta_\nu(f,||_\nu^s)$ is a complex valued function defined on a certain complex Riemann surface.

**Theorem 2.10.** *Function $\zeta_\nu(f,||_\nu^s)$ is regular (holomorphic) as a function of variable $s$ in the domain $\Re(s)=\sigma>0$.*

*Proof.* Complete proof can be found in [3] on p. 248. □

**Theorem 2.11.** *Let $\widehat{||_\nu^s}=||_\nu^{1-s}$. For any two functions $f,\ g\in S(\mathbb{Q}_\nu)$ and for $0<\sigma=\Re(s)<1$ we have:*

$$\zeta_\nu(f,||_\nu^s)\zeta_\nu(\widehat{g},\widehat{||_\nu^s})=\zeta_\nu(\widehat{f},\widehat{||_\nu^s})\zeta_\nu(g,||_\nu^s)\ . \tag{3}$$

*Proof.* Writing

$$\zeta_\nu(f,||_\nu^s)\zeta_\nu(\widehat{g},\widehat{||_\nu^s})=\int_{\mathbb{Q}_\nu^*}f(\alpha)|\alpha|_\nu^s d_\nu^*\alpha\int_{\mathbb{Q}_\nu^*}\widehat{g}(\beta)|\beta|_\nu^{1-s}d_\nu^*\beta$$

we observe that both integrals are absolutely convergent in the region $\sigma\in(0,1)$ and hypotheses of Fubini's theorem are fulfilled. Therefore we obtain

$$\zeta_\nu(f,||_\nu^s)\zeta_\nu(\widehat{g},\widehat{||_\nu^s})=\iint_{\mathbb{Q}_\nu^*\times\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\beta)|\alpha|_\nu^s|\beta|_\nu^{1-s}d_\nu^*(\alpha,\beta)=$$
$$=\iint_{\mathbb{Q}_\nu^*\times\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\beta)|\alpha\beta^{-1}|_\nu^s|\beta|_\nu d_\nu^*(\alpha,\beta)\ .$$

By the change of variables $\beta\mapsto\alpha\beta$, under which the measure $d_\nu^*(\alpha,\beta)$ is invariant, we get

$$\iint_{\mathbb{Q}_\nu^*\times\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\alpha\beta)|\beta^{-1}|_\nu^s|\alpha\beta|_\nu d_\nu^*(\alpha,\beta)\ .$$

Now, using Fubini's theorem again, we obtain

$$\int_{\mathbb{Q}_\nu^*}\left(\int_{\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\alpha\beta)|\alpha|_\nu d_\nu^*\alpha\right)|\beta|_\nu^{s-1}d_\nu^*\beta\ . \tag{4}$$

To finish the proof it is sufficient to show that the inner integral in (4) is symmetric in $f$ and $g$. To show this we consider obviously symmetric additive double integral

$$\iint_{\mathbb{Q}_\nu^+\times\mathbb{Q}_\nu^+}f(\xi)g(\eta)\exp(-2\pi i\lambda(\xi\beta\eta))d_\nu(\xi,\eta)\ . \tag{5}$$

By applying to integral (5) Fubini's theorem we obtain

$$\int_{\mathbb{Q}_\nu^+}f(\xi)\left(\int_{\mathbb{Q}_\nu^+}g(\eta)\exp(-2\pi i\lambda(\xi\beta\eta))d_\nu\eta\right)d_\nu\xi=\int_{\mathbb{Q}_\nu^+}f(\xi)\widehat{g}(\xi\beta)d_\nu\xi\ . \tag{6}$$

Now observe that the last term in the equation (6) by Definition 2.10 is equal

$$\int_{\mathbb{Q}_\nu^+}f(\xi)\widehat{g}(\xi\beta)d_\nu\xi=\begin{cases}\int_{\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\alpha\beta)|\alpha|_\nu d_\nu^*\alpha & \text{if } \mathbb{Q}_\nu \text{ archimedean}\\ \frac{p-1}{p}\int_{\mathbb{Q}_\nu^*}f(\alpha)\widehat{g}(\alpha\beta)|\alpha|_\nu d_\nu^*\alpha & \text{if } \mathbb{Q}_\nu \text{ nonarchimedean}\ .\end{cases}$$

Therefore the integral in equation (4) is symmetric.

□

**Theorem 2.12** (Local Functional Equation)**.** *The function $\zeta_\nu(f, ||_\nu^s)$ has analytic continuation from domain $\sigma > 0$ to the whole Riemann surface given by the functional equation*

$$\zeta_\nu(f, ||_\nu^s) = \rho_\nu(||_\nu^s)\zeta_\nu(\widehat{f}, ||_\nu^{1-s}) \tag{7}$$

*where the function $\rho(||_\nu^s)$ is called* proportionality function*. Proportionality function is* independent *of the function $f$, and is defined by functional equation itself in the domain $1 > \sigma > 0$, and on the whole Riemann surface by analytic continuation.*

*Proof.* By Theorem 2.11 we have

$$\zeta_\nu(f, ||_\nu^s)\zeta_\nu(\widehat{g}, \widehat{||_\nu^s}) = \zeta_\nu(\widehat{f}, \widehat{||_\nu^s})\zeta_\nu(g, ||_\nu^s) \ ,$$

and hence

$$\frac{\zeta_\nu(f, ||_\nu^s)}{\zeta_\nu(\widehat{f}, \widehat{||_\nu^s})} = \frac{\zeta_\nu(g, ||_\nu^s)}{\zeta_\nu(\widehat{g}, \widehat{||_\nu^s})} \ ,$$

unless the denominators are not identically zero.

By putting

$$\rho_\nu(||_\nu^s) = \frac{\zeta_\nu(g, ||_\nu^s)}{\zeta_\nu(\widehat{g}, \widehat{||_\nu^s})}$$

we obtain an expression which is independent of the function $f \in S(\mathbb{Q}_\nu)$ (see Remark 2.12). Therefore we can write

$$\zeta_\nu(f, ||_\nu^s) = \rho_\nu(||_\nu^s)\zeta_\nu(\widehat{f}, ||_\nu^{1-s}) \ .$$

This equality is proved in the domain $1 > \sigma > 0$ and by analytic continuation it holds on the whole Riemann surface. □

*Remark* 2.14. One has to note that we have to show that there exists at least one function $f \in S(\mathbb{Q}_\nu)$ for which $\zeta_\nu(\widehat{f}, ||_\nu^{1-s})$ is not identically equal zero. This will be done in the next section.

*Remark* 2.15. Exactly the same computations can be made in the case of general quasicharacters. In this case one makes extensive use of the knowledge on canonical decomposition of quasicharacters, as mentioned in Remark 2.9.

2.3. **Computation of the proportionality function.** In order to compute proportionality functions, one need to choose a "good" function $f$ from the class $S(\mathbb{Q}_\nu)$. This choice is different in archimedean and nonarchimedean cases, and therefore these cases have to be treated differently.

**Property 2.13.** *For $\mathbb{Q}_\nu = \mathbb{R}$ the proportionality function is given by the following formula:*

$$\rho_\infty(||_\infty^s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s) \ . \tag{8}$$

*Proof.* Let

$$f(\xi) = \exp(-\pi\xi^2) \ .$$

Obviously $f \in S(\mathbb{R})$ and it is well known from classical Fourier analysis that

$$f(\xi) = \widehat{f}(\xi) \ .$$

Therefore we have

$$\zeta_\infty(f, ||_\infty^s) = \int_{\mathbb{R}^*} f(\alpha)|\alpha|_\infty^s d^*\alpha = \int_{\mathbb{R}^+} \exp(-\pi\alpha^2)|\alpha|_\infty^s \frac{d\alpha}{|\alpha|_\infty} =$$
$$= 2\int_0^\infty \exp(-\pi\alpha^2)\alpha^{s-1}d\alpha = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) .$$

Moreover,

$$\zeta_\infty(\widehat{f}, \widehat{||_\infty^s}) = \zeta_\infty(f, ||_\infty^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right) .$$

Since neither $\zeta_\infty(f, ||_\infty^s)$ nor $\zeta_\infty(\widehat{f}, \widehat{||_\infty^s})$ are identically zero we have

$$\rho_\infty(||_\infty^s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s) .$$

□

**Property 2.14.** *For $\mathbb{Q}_\nu = \mathbb{Q}_p$ the proportionality function is given by the following formula:*

$$\rho_p(||_p^s) = \frac{1-p^{s-1}}{1-p^{-s}} . \tag{9}$$

*Proof.* Let

$$f(\xi) = \begin{cases} 1 & \text{for } \xi \in \mathbb{Z}_p \\ 0 & \text{for } \xi \notin \mathbb{Z}_p . \end{cases}$$

As follows from Theorem 2.2 this function is compactly supported and continuous, therefore it belongs to the class $S(\mathbb{Q}_p)$. Moreover,

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p^+} \chi_{\mathbb{Z}_p} \exp(-2\pi i\lambda(\xi\eta))d\eta = \int_{\mathbb{Z}_p^+} \exp(-2\pi i\lambda(\xi\eta))d\eta .$$

Thus it is integral of the additive character $\eta \mapsto \exp(-2\pi i\lambda(\xi\eta))$ over the group $\mathbb{Z}_p^+$. Therefore this integral is equal, either 1 (the measure of $\mathbb{Z}_p^+$) if $\xi \in \mathbb{Z}_p^+$ or 0 if $\xi \notin \mathbb{Z}_p^+$ which means that

$$\widehat{f}(\xi) = f(\xi) .$$

Now we can, as in the archimedean case, compute zeta functions.
Let $A_d = \{\alpha \in \mathbb{Z}_p^* \mid ord_p(\alpha) = d\}$. Note that $A_0 = \mathcal{U}_p$ by Remark 2.9 and the definition of $|\cdot|_p$ (see p. 3). Then $\mathbb{Z}_p^* = \coprod_{d=0}^\infty A_d$ and

$$\zeta_p(f, ||_p^s) = \int_{\mathbb{Z}_p^*} |\alpha|_p^s d_p^*\alpha = \int_{\coprod_{d=0}^\infty A_d} |\alpha|_p^s d_p^*\alpha =$$
$$= \sum_{d=0}^\infty \int_{A_d} |\alpha|_p^s d_p^*\alpha = \sum_{d=0}^\infty p^{-ds}\int_{A_d} d_p^*\alpha = \sum_{d=0}^\infty p^{-ds}\int_{p^{-d}A_d} d_p^* p^{-d}\alpha =$$
$$= \sum_{d=0}^\infty p^{-ds}\int_{A_0} d_p^*\alpha = \sum_{d=0}^\infty p^{-ds}\int_{\mathcal{U}_p} d_p^*\alpha = \left(\sum_{d=0}^\infty p^{-ds}\right)\int_{\mathcal{U}_p} d_p^*\alpha =$$
$$= \frac{1}{1-p^{-s}} .$$

Since $\widehat{f} = f$,

$$\zeta_p(\widehat{f}, \widehat{||_p^s}) = \zeta_p(f, ||_p^{1-s}) = \frac{1}{1-p^{s-1}} .$$

Therefore one has

$$\rho_p(s) = \frac{\zeta_p(f, ||_p^s)}{\zeta_p(\widehat{f}, \widehat{||_p^s})} = \frac{1 - p^{s-1}}{1 - p^{-s}} \ .$$

□

## 3. Passage form the local to the global case

In this seection we shall present the general notion of the restricted direct product of locally compact groups, and then define the ring of *adèles* and the group of *idèles* of $\mathbb{Q}$. Moreover, we shall describe basic relations between structures and operations on the restricted product and on its components such as: Haar measure, quasicharacters, and integration. As a consequence we shall formulate the "infinite" analogue of the Fubini theorem for the restricted direct products.

**Definition 3.1.** Let $\{G_i\}_{i\in I}$ be a family of locally compact groups such that for almost all $i \in I$ there exists an open-compact subgroup $K_i \subseteq G_i$. Then

$$\sqcap_{i\in I} G_i = \left\{ g \in \prod_{i\in I} G_i | \ g_i \in K_i \text{ for almost all } i \right\}$$

is called the *restricted direct product* of $G_i$ with respect to subgroups $K_i$.

**Theorem 3.1.** *On $\sqcap_{i\in I} G_i$ there exists canonical topology (restriction of the product topology) under which $\sqcap_{i\in I} G_i$ becomes a locally compact group.*

*Proof.* For details of construction topology on $\sqcap_{i\in I} G_i$ see [5] [Lemma 3.1.1 p. 323]. □

### 3.1. Restricted direct product.

In this section we present theorems completely characterising quasicharacters and the Haar measure on restricted direct product in terms of local data.

**Theorem 3.2.** *Let $c$ be a complex valued function on $\sqcap_{i\in I} G_i$ and let $c_i$ denote its restriction to $G_i$. The following statements are equivalent:*

(i) *$c_i$ are local quasicharacters trivial on $K_i$ for almost all $i \in I$;*

(ii) *$c$ is a quasicharacter on $\sqcap_{i\in I} G_i$ .*

*Proof.* Details in [5] [Lemma 3.2.1 and Lemma 3.2.2 p. 324]. □

**Corollary 3.3.** *Let $c$ be a quasicharacter on $\sqcap_{i\in I} G_i$. Then $c$ can be represented uniquely, up to the order of factors, in the following way:*

$$c(g) = \bigotimes_{i\in I} c_i(g_i) = \prod_{i\in I} c_i(g_i) \ .$$

*Note that if fact the last product is composed of finite number of terms $\neq 1$.*

The next theorem gives an explicit description of the Pontryagin dual of the restricted direct product of locally compact abelian groups.

**Theorem 3.4.** *Let $K_i^*$ denote the subgroup of $\widehat{G}_i$ containing only these $c_i \in \widehat{G}_i$ which are trivial on $K_i$. Then $K_i^*$ is compact-open in $\widehat{G_i}$ and $\widehat{K_i} \equiv \widehat{G}_i/K_i^*$. Moreover $\sqcap_{i\in I} \widehat{G_i}$ with respect to subgroups $K_i^*$ and $\widehat{G}$ are topologically and algebraically isomorphic.*

*Proof.* Details in [5] [Theorem 3.2.1 p. 325]. □

**Theorem 3.5.** *Let for each $i \in I$ a function $f_i \in L^1(G_i) \cap C(G_i)$ be given, such that $f_i = 1$ on $K_i$ for almost all $i \in I$. Then the function defined on $G = \prod_{i \in I} G_i$ as follows*

$$f(g) = \bigotimes_{i \in I} f_i(g_i) = \prod_{i \in I} f_i(g_i) \qquad (g = (g_i)_{i \in I})$$

*is continuous.*

*Proof.* Details in [5] [Lemma 3.3.2 p. 326]. □

**Definition 3.2.** Knowing by Theorem 2.2, that $\mathbb{Z}_p$ is a compact-open subring of locally compact ring $\mathbb{Q}_p$ for all prime numbers, we can form the ring

$$\mathbb{A} = \prod_{\nu} \mathbb{Q}_\nu$$

with respect to the subrings $\mathbb{Z}_p$. Ring $\mathbb{A}$ is called the *alèle* ring, and its elements are called *adèles.*

*Remark* 3.1. Note that, although we have been referring to restricted direct products of groups, in the case of topological rings with unity, the same holds, since component-wise multiplication is continuous on direct product of groups containing $\prod_\nu \mathbb{Q}_\nu$.

**Definition 3.3.** Restricted direct product of groups

$$\prod_{\nu} \mathbb{Q}_\nu^*$$

with respect to subgroups $\mathbb{Z}_\nu^*$ (which are compact-open by Theorem 2.2) is called the *group of idèles* and denoted by $\mathbb{A}^*$.

*Remark* 3.2. One has to note that the group of idèles is precisely the group of units of the ring of adèles, but the topology is different.

**Definition 3.4.** The embedding of $\mathbb{Q}^+$ to $\mathbb{A}$ given by

$$\xi \mapsto (\xi, \xi, \dots, \xi, \dots) \tag{10}$$

shall be called the *canonical embedding.* The same name shall be applied for the embedding of $\mathbb{Q}^*$ to $\mathbb{A}^*$ given by formula (10) .

**Theorem 3.6.** *For the ring of adèles and group of idèles we have the following characterisation of additive characters, and multiplicative quasicharachters.*

(i) *Every charachter of the ring of adéles is of the form*

$$\bigotimes_{\nu} \exp(2\pi i \lambda_\nu(\eta_\nu \xi_\nu)) = \prod_{\nu} \exp(2\pi i \lambda_\nu(\eta_\nu \xi_\nu))$$

*for some adéle $\eta$, where the last product has a finite number of terms $\neq 1$ (by Corollary 3.3). Therefore it takes the form*

$$\exp(2\pi i \sum_{\nu} \lambda_\nu(\eta_\nu \xi_\nu)) = \exp(2\pi i \lambda(\eta\xi)) \ .$$

(ii) *Every (unramified) quasicharachter of the group of idéles is of the form*

$$|\cdot|_{\mathbb{A}}^s = \bigotimes_{\nu} |\cdot|_\nu^s = \prod_{\nu} |\cdot|_\nu^s$$

*where by Corollary 3.3 the last product has finite number of terms $\neq 1$.*

**Corollary 3.7.** *For the additive group of adèle ring we have*

$$\widehat{\mathbb{A}} = \mathbb{A} \ .$$

*Proof.* This is immediate consequence of Theorem 3.6, Theorem 3.4 and Theorem 2.4. □

**Theorem 3.8.** *For all $\xi \in \mathbb{Q}$ we have:*

$$|\xi|_{\mathbb{A}} = 1$$

*and*

$$\lambda(\xi) = 0 \ .$$

*Proof.* Complete computation may be found in [5] [Theorem 4.3.1 p. 334 and Theorem 4.1.4 p. 331]. □

*Remark* 3.3. Note that by saying that $\xi \in \mathbb{Q}$ is an element of $\mathbb{A}$, we identify $\xi$ under the embeding defined by (10).

**Corollary 3.9.** *There exists 1:1 correspondence between quasicharacters of $\mathbb{A}^*$ and quasicharacters of $\mathbb{A}^*/\mathbb{Q}^*$, and characters of $\mathbb{A}$ and of $\mathbb{A}/\mathbb{Q}$ .*

Now we shall present some very useful properties of the ring of adèles and the group of idèles.

**Theorem 3.10.** *The subset $D$ of $\mathbb{A}$ given by the formula*

$$D = [0, 1) \times \prod_p \mathbb{Z}_p$$

*satisfies the following properties:*

(i)
$$\mathbb{A} = \coprod_{\xi \in \mathbb{Q}^+} (\xi + D) \ ;$$

(ii) *$D$ has measure 1.*

*Proof.* Details of proof are explained in [5] [Theorem 4.1.3 p. 330]. □

**Corollary 3.11.** *The set $D$ is nonempty, and relatively compact, therefore the group $\mathbb{A}/\mathbb{Q}^+$ is compact. Moreover, $\mathbb{Q}$ is discrete in $\mathbb{A}$.*

*Remark* 3.4. The set $D$ is often called *the additive fundamental domain* of the ring of adèles.

**Theorem 3.12.** *Let $J \subsetneq \mathbb{A}^*$ be the set of all idèles $\beta$ with $\beta \in J$ $|\beta|_{\mathbb{A}} = 1$ . Then there exists canonical factorisation of $\mathbb{A}^*$ such that*

$$\mathbb{A}^* = T \times J$$

*where $\times$ means cartesian product.*

*Proof.* Let $\nu_0$ be the archimedean valuation, and let $T$ be a subgroup of $\mathbb{A}^*$ such that $\alpha_{\nu_0} > 0$ and $\alpha_p = 1$ for all $p$. Moreover, one can write uniquely any idèle $\alpha$ in the form $\alpha = |\alpha|\beta$ where $\beta = \alpha|\alpha|^{-1} \in J$. Therefore it is clear that $\mathbb{A}^* = T \times J$. □

*Remark* 3.5. The crucial point for the existance of such decomposition is that there exists at least one archimedean valuation on $\mathbb{Q}$.

**Theorem 3.13.** *$\mathbb{Q}$ under canonical embedding into $J$ is cocompact (therefore it is cocompact in $\mathbb{A}^*$) and the* multiplicative fundamental domain *for this embedding is equal to the archimedean component for $J$ and is equal*

$$[1, e) \times \prod_p \mathbb{Z}_p^*$$

*and denoted by $E$.*

*Proof.* Let $\varphi(\beta) = \prod_p p^{ord_p \beta_p}$ for every $\beta \in J$. Therefore to every idéle $\beta \in J$ we associate in this way unique (up to sign) number $b$. Therefore by multiplicativity of the function $\varphi$ on see that the idèle $b^{-1}\beta \in E$. □

**Theorem 3.14.** *We have*

$$\int_E d^*_{\mathbb{A}}\beta = 1$$

*i.e. the measure of $E$ is equal 1.*

*Proof.* Immediate from the construction of the set $E$ by applying to its characteristic function Theorem 3.16. □

*Remark* 3.6. The general case of arbitrary algebraic number field computation of the measure of $E$ involves non-trivial arithmetic properties of the number field, and the Dirichlet Unit Theorem. Moreover, it emerges from the construction, that existance of archimedean components of idèles gives the existance of the set $E$, which encodes crucial arithmetical information.

3.2. **Fubini's theorem for restricted direct product.** In this section we shall present generalisation of the classical Fubini theorem to restricted direct products, and its consequences to the integration theory on restricted direct product of groups.

**Theorem 3.15.** *Let $G_i$ be a family of locally compact abelian groups and let $d_i$ denote a Haar measure such that*

$$\int_{K_i} d_i g_i = 1$$

*for almost all $i \in I$. Then there exists a* unique *Haar measure $\prod_{i\in I} d_i$ on $\prod_{i\in I} G_i$ such that putting*

$$G_j \times \prod_{i\neq j} G_j$$

*for almost every $j \in I$ we obtain that*

$$\prod_{i\in I} d_i = d_j \times \prod_{i\neq j} d_i$$

*in the sense of Fubini theorem.*

*Proof.* Details of proof can be found in [5] [Section 3.3 p. 325]. □

**Theorem 3.16** (Fubini theorem for restricted direct product)**.** *Let $f_i \in L^1(G_i) \cap C(G_i)$ and let $f = \bigotimes_{i\in I} f_i$ be as in Theorem 3.5. If*

$$\bigotimes_{i\in I}\left[\int_{G_i} |f_i(g_i)| d_i g_i\right] = \prod_{i\in I}\left[\int_{G_i} |f_i(g_i)| d_i g_i\right]$$

*is convergent (note that the product not need to be finite), then*

$$f \in L^1(G) \cap C(G)$$

*and*

$$\int_G f(g)dg = \prod_{i\in I}\int_{G_i} f_i(g_i) d_i g_i \ .$$

*Proof.* Proof can be found both in [1] and in a somewhat less generality in [5] [Theorem 3.3.1 p. 326]. □

*Remark* 3.7. Note that part of the thesis of the theorem 3.16 is convergance of the product $\prod_{i\in I}$.

**Corollary 3.17.** *Let $f_i \in L^1(G_i) \cap C(G_i)$ for all $i \in I$ and $f_i$ is characteristic function of $K_i$ for almost all $i \in I$. Then we have the following representation of the Fourier transform of the function $f$*

$$\widehat{f}(g) = \bigotimes_{i \in I} \widehat{f}_i(g_i) = \prod_{i \in I} \widehat{f}_i(g_i) \ ,$$

*where the last product is in fact finite.*

*Remark* 3.8. For $f_i$ as above, $f_i \in L^1(G_i) \cap C(G_i)$ and hence its Fourier transform is well defined.

**Definition 3.5.** The space of functions satysfying the following properties:

(i)
$$f = \bigotimes_{\nu} f_\nu = \prod_{\nu} f_\nu \text{ where } f_\nu \in S(\mathbb{Q}_\nu) \ ;$$

(ii) for almost all valuations $\nu$, $f_\nu$ is of characteristic function of $\mathbb{Z}_p$ ;

(iii)
$$f||_{\mathbb{A}}^{\sigma} \in L^1(\mathbb{A}^*) \text{ for } \sigma > 1 \ ;$$

shall be denoted $S(\mathbb{A})$.

*Remark* 3.9. Note that since for all valuations $\nu$, $S(\mathbb{Q}_\nu) \subsetneq L^1(\mathbb{Q}_\nu) \cap C(\mathbb{Q}_\nu)$ by Remark 2.12 hence every function from the class $S(\mathbb{A})$ satysfies the hypothesis of the Theorem 3.5 and therefore it satysfies the thesis of Theorem 3.16.

*Remark* 3.10. Moreover, one has to point out that by Corollary 3.7 and Corollary 3.17 one has that

$$f \in S(\mathbb{A}) \Longleftrightarrow \widehat{f} \in S(\mathbb{A}) \ .$$

*Remark* 3.11. In view of the work of André Weil [6] one can understand that the most "natural" class of of functions which can replace class $S(\mathbb{A})$ is the *Schwartz-Bruhat* space of $\mathbb{A}$.

## 4. Global case

In the first part of this section we shall discuss analytic properties of functions defined on adèles and idèles, which does not have its counterpart in analytic properties in local case. Next, we shall define global zeta function in the same way as in the local case, then using consequences of Poisson summation formula we shall establish its functional equation.

### 4.1. Poisson summation formula.

In this section we introduce the key tool needed in the proof of the Global Functional Equation which is the Poisson Summation Formula for locally compact abelian groups.

**Theorem 4.1** (Poisson summation formula)**.** *Let $G$ be a locally compact abelian group and let $\Lambda < G$ be a discrete, cocompact subgroup. Let $F$ denote the fundamental domain of the group $G$ with respect to the subgroup $\Lambda$. Then for every function $f \in L^1(G) \cap C(G)$ satisfying conditions:*

(i) $\sum_{\lambda \in \Lambda} f(\lambda + g)$ *is uniformly convergent for every $g \in F$ ;*

(ii) $\sum_{\lambda \in \Lambda} |\widehat{f}(\lambda)|$ *is convergent ;*

*we have*

$$\sum_{\lambda \in \Lambda} f(\lambda) = \sum_{\lambda \in \Lambda} \widehat{f}(\lambda) \ . \tag{11}$$

*Remark* 4.1. Observe that by Corollary 3.11, $\mathbb{Q}^+$ canonically embedded into $\mathbb{A}$ satisfies the hypotheses of Theorem 4.1.

*Remark* 4.2. Having multiplication in $\mathbb{A}$, and therefore coexisting group of idéles, one can formulate the following theorem, which shall be used directly in the proof of the global functional equation.

**Theorem 4.2.** *If the function $f$ satisfies the following conditions:*

(i)
$$f \in L^1(\mathbb{A}) \cap C(\mathbb{A}) \ ;$$

(ii)
$$\sum_{\xi\in\mathbb{Q}} f(\alpha(\mathfrak{a}+\xi))$$
*is convergent for all idèles $\alpha$ and adèles $\mathfrak{a}$, uniformly for $\mathfrak{a} \in D$ ;*

(iii)
$$\sum_{\xi\in\mathbb{Q}} |\widehat{f}(\alpha\xi)|$$
*is convergent for all idèles $\alpha$ ;*

*then*
$$\frac{1}{|\alpha|_{\mathbb{A}}}\sum_{\xi\in\mathbb{Q}} \widehat{f}(\xi/\alpha) = \sum_{\xi\in\mathbb{Q}} f(\alpha\xi) \ .$$

*Proof.* Complete proof of this theorem is presented in [5] [Theorem 4.2.1 p. 333]. □

*Remark* 4.3. We refer to [5] for the proof that every function from the class $S(\mathbb{A})$ satisfies properties (ii) and (iii) of the above theorem.

4.2. **Global zeta function.** In this section we introduce the notion of the Global Zeta Function, and show its relation with the local zeta functions.

**Theorem 4.3.** *For every function $f \in S(\mathbb{A})$ and for every complex number $s$ such that $\Re(s) = \sigma > 1$ the integral*
$$\int_{\mathbb{A}^*} f(\alpha)|\alpha|^s_{\mathbb{A}} d^*_{\mathbb{A}}\alpha$$
*is convergent.*

*Proof.* This is an immediate consequence of Definition 3.5. □

**Definition 4.1.** For $\Re(s) = \sigma > 1$ and $f \in S(\mathbb{A})$ we define the *Global Zeta Function* as follows
$$\zeta(f, ||^s_{\mathbb{A}}) = \int_{\mathbb{A}^*} f(\alpha)|\alpha|^s_{\mathbb{A}} d^*_{\mathbb{A}}\alpha \ .$$

**Theorem 4.4.** *For $\sigma > 1$ the Global Zeta Function admits the following canonical decomposition*
$$\zeta(f, ||^s_{\mathbb{A}}) = \bigotimes_{\nu} \zeta_\nu(f_\nu, ||^s_\nu) = \prod_{\nu} \zeta_\nu(f_\nu, ||^s_\nu)$$
*where the last product is infinite and convergent for $\sigma > 1$.*

*Proof.* This fact is an immediate consequence of Theorem 3.16 because Definition 3.5 and Theorem 3.6 states that the hypotheses of Theorem 3.16 are fulfilled. Note that convergence of the product of integrals was discussed in Remark 3.7. □

*Remark* 4.4. The argument in [6] shows, that zeta functions are in fact tempered distributions on the Schwartz-Bruhat space of $\mathbb{A}$. Use of the symbol $\bigotimes_{i\in I}$ refers to the fact that all mentioned constructions can be understood as a tensor product in suitably defined category.

4.3. **Global functional equation.** In this section we shall prove the functional equation for the global zeta function. First we shall prove some auxiliary lemmas and then we shall use them to prove the functional equation itself.

**Lemma 4.5.** *The integral*

$$\zeta(f, ||_{\mathbb{A}}^s, t) = \int_J f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta$$

*is absolutely convergent for all* $s \in \mathbb{C}$ *and almost all parameters* $t \in \mathbb{R}_{>0}^*$.

*Proof.* By Theorem 3.13 we know that $J/\mathbb{Q}^*$ is compact. Moreover, we know that since $|\xi|_{\mathbb{A}} = 1$ for $\xi \in \mathbb{Q}^*$, therefore for each continuous homomorphism to $\mathbb{C}^*$ (quasicharacter) $||_{\mathbb{A}}^s$ restricted to $J$ $||_{\mathbb{A}}^s = 1$, by the construction of $J$, therefore it is a character. Thus we have the following estimation of the integral

$$\zeta(f, ||_{\mathbb{A}}^s, t) = \int_J f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta \ll$$
$$\ll \int_J |f(t\beta)||t\beta|_{\mathbb{A}}^\sigma d_{\mathbb{A}}^*\beta = \int_J |f(t\beta)|t^\sigma \; d_{\mathbb{A}}^*\beta =$$
$$= t^\sigma \int_J |f(t\beta)| d_{\mathbb{A}}^*\beta \; .$$

Since $f \in S(\mathbb{A})$ the last integral converges for all $s \in \mathbb{C}$. □

**Lemma 4.6** (Automorphic relation)**.** *For* $\Re(s) = \sigma > 1$ *we have*

$$\zeta(f, ||_{\mathbb{A}}^s, t) + f(0) \int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta = \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}, 1/t) + \widehat{f}(0) \int_E |t\beta|_{\mathbb{A}}^{1-s} d_{\mathbb{A}}^*\beta \; . \tag{12}$$

*Proof.*

$$\zeta(f, ||_{\mathbb{A}}^s, t) + f(0) \int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta = \int_J f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta + f(0) \int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta \; .$$

Since $E$ is fundamental domain for $J$, therefore we have

$$\int_{\coprod_{\xi \in \mathbb{Q}^*} \xi E} f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta$$
$$= \sum_{\xi \in \mathbb{Q}^*} \int_{\xi E} f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta$$
$$= \sum_{\xi \in \mathbb{Q}^*} \int_E f(\xi t\beta)|\xi t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*(\xi\beta).$$

Since $|\xi t\beta|_{\mathbb{A}}^s = |t\beta|_{\mathbb{A}}^s$ and $d_{\mathbb{A}}^*(\xi\beta) = d_{\mathbb{A}}^*\beta$ the last integral equals to

$$\sum_{\xi \in \mathbb{Q}^*} \int_E f(\xi t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta.$$

Observe that by Remark 4.3 the sum $\sum_{\xi \in \mathbb{Q}^+} f(\xi t\beta)$ is uniformly convergent for $\beta \in E$ and therefore we can change the order of summation and integration

$$\sum_{\xi \in \mathbb{Q}^*} \int_E f(\xi t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta + f(0) \int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta =$$
$$= \sum_{\xi \in \mathbb{Q}^+} \int_E f(\xi t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta =$$
$$= \int_E \Big[ \sum_{\xi \in \mathbb{Q}^+} f(\xi t\beta) \Big] |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta \; .$$

By Theorem 4.2 (see Remark 4.3) we obtain

$$\int_E \Big[\sum_{\xi\in\mathbb{Q}^+} f(\xi t\beta)\Big]|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta = \ = \int_E \Big[\sum_{\xi\in\mathbb{Q}^+} \widehat{f}(\xi/t\beta)\Big]|t\beta|_{\mathbb{A}}^{1-s} d_{\mathbb{A}}^*\beta \ .$$

Measure $d_{\mathbb{A}}^*\beta$ is invariant under the change of variables $(\beta \mapsto 1/\beta)$ and hence

$$\int_E \Big[\sum_{\xi\in\mathbb{Q}^+} \widehat{f}(\xi\beta/t)\Big]|t\beta|_{\mathbb{A}}^{1-s} d_{\mathbb{A}}^*\beta \ . \tag{13}$$

Repeating the same computations, except for the last step with application of Theorem 4.2, to the right had side of equation (12) we also obtain formula (13). Therefore the proof is completed. □

**Lemma 4.7.** *We have*

$$\int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta = t^s \ . \tag{14}$$

*Proof.*

$$\int_E |t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta = t^s \int_E |\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta \ .$$

Here the second integral is taken over the factor group $J/\mathbb{Q}^*$. The second integral is integral over the character of the group $J/\mathbb{Q}^*$. Since $||_{\mathbb{A}}^s = 1$ on $J/\mathbb{Q}$ therefore the integral is equal to the measure of $E$ which $= 1$. □

*Remark* 4.5. In view of the Corollary 3.9, one should note that we have not lost generality by assuming that composition of canonical quotient projection and quasicharacter is still quasicharacter.

**Theorem 4.8** (Global functional equation)**.** *The function*

$$\zeta(f, ||_{\mathbb{A}}^s)$$

*has meromorphic continuation to the whole complex plane, with simple pools at points* $s = 0$ *and* $s = 1$ *with residuum* $-f(0)$ *and* $f(1)$ *respectively, and satisfies the following functional equation*

$$\zeta(f, ||_{\mathbb{A}}^s) = \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}) \ .$$

*Proof.* By Theorem 3.12 and Theorem 3.15 for $\sigma = \Re(s) > 1$ we have

$$\zeta(f, ||_{\mathbb{A}}^s) = \int_{\mathbb{A}^*} f(\alpha)|\alpha|_{\mathbb{A}}^s d_{\mathbb{A}}^*\alpha = \int_0^\infty \left(\int_J f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta\right)\frac{dt}{t} = \int_0^\infty \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t}$$

where integral

$$\zeta(f, ||_{\mathbb{A}}^s, t) = \int_J f(t\beta)|t\beta|_{\mathbb{A}}^s d_{\mathbb{A}}^*\beta$$

is absolutely convergent for all $s \in \mathbb{C}$ by Lemma 4.5 .

For $\sigma > 1$ we can write

$$\zeta(f, ||_{\mathbb{A}}^s) = \int_0^\infty \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} = \int_0^1 \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} + \int_1^\infty \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} \ . \tag{15}$$

The integral $\int_1^\infty \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t}$ is in fact the integral of the function $f(\alpha)|\alpha|_{\mathbb{A}}^s$ over the idéles $|\alpha|_{\mathbb{A}} > 1$ denoted by $\mathbb{A}_{>1}^*$, therefore for $\sigma < 1$ we have

$$\int_{\mathbb{A}_{>1}^*} f(\alpha)|\alpha|_{\mathbb{A}}^s d_{\mathbb{A}}^*\alpha \ll \int_{\mathbb{A}_{>1}^*} |f(\alpha)||\alpha|_{\mathbb{A}}^{1+\varepsilon} d_{\mathbb{A}}^*\alpha \ll \int_{\mathbb{A}^*} |f(\alpha)||\alpha|_{\mathbb{A}}^{1+\varepsilon} d_{\mathbb{A}}^*\alpha \tag{16}$$

for all $\varepsilon > 0$, therefore the third integral is convergent by the definition of the class $S(\mathbb{A})$.

Applying Lemma 4.6 and Lemma 4.7 to the integral

$$\int_0^1 \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t}$$

we obtain the following formula

$$\int_0^1 \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} = \int_0^1 \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}, 1/t)\frac{dt}{t} + \int_0^1 \widehat{f}(0)\Big(\frac{1}{t}\Big)^{1-s}\frac{dt}{t} - \int_0^1 f(0)t^s\frac{dt}{t}$$

valid for $\sigma > 1$. Evaluating the second and third integrals we obtain

$$\int_0^1 \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} = \int_0^1 \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}, 1/t)\frac{dt}{t} + \frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s} \ .$$

After change of variables ($t \mapsto 1/t$) we obtain the following expression (see Remark 4.6)

$$\int_0^1 \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} = \int_1^\infty \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}, t)\frac{dt}{t} + \frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s}$$

and therefore

$$\zeta(f, ||_{\mathbb{A}}^s) = \int_1^\infty \zeta(f, ||_{\mathbb{A}}^s, t)\frac{dt}{t} + \int_1^\infty \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}, t)\frac{dt}{t} + \frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s} \ . \qquad (17)$$

Both integrals are absolutely convergent by the same argument as in (16), therefore they represent analytic functions for all $s \in \mathbb{C}$. Moreover, from the shape of the equation (17) one can see that it is invariant under substitution $(f, ||_{\mathbb{A}}^s) \mapsto (\widehat{f}, ||_{\mathbb{A}}^{1-s})$ . □

*Remark* 4.6. The behaviour of the function $\zeta(f, ||_{\mathbb{A}}^s, t)$ under the change of variables $t \to 1/t$ is precisely the place which reflects the automorphic formula for the Theta function in Hecke's original proof.

## 5. Comparison

In this section we shall show the way to obtain classical expresion for the functional equation of the Riemann Zeta Function from Tate's zeta function.

By Definition 3.5 we know that every function $f \in S(\mathbb{A})$ can be viewed as

$$f = \bigotimes_\nu f_\nu = \prod_\nu f_\nu \text{ where } f_\nu \in S(\mathbb{Q}_\nu)$$

where the $\prod_\nu$ is in fact finite product (see Corollary 3.17).
Corollary 3.17 relates the correspondence between Fourier transforms of $f \in S(\mathbb{A})$ and the Fourier transforms of the local factors in the following way

$$\widehat{f}(\alpha) = \bigotimes_\nu \widehat{f}_\nu(\alpha_\nu) = \prod_\nu \widehat{f}_\nu(\alpha_\nu)$$

where the last product is in fact a finite product.

Moreover, we know by Theorem 3.6 that a quasicharacter of idèles $||_{\mathbb{A}}^s$ is canonically decomposable in the following way:

$$|\alpha|_{\mathbb{A}}^s = \bigotimes_\nu |\alpha_\nu|_\nu^s \ .$$

Combining these two facts, we obtain the following canonical decomposition

$$|f(\alpha)||\alpha|_{\mathbb{A}}^\sigma = \bigotimes_\nu |f_\nu(\alpha)||\alpha_\nu|_\nu^\sigma$$

where almost all factors are equal 1 on $\mathcal{U}_\nu$. (One should note that $\mathcal{U}_\nu$ is well defined just in nonarchimedean case). Therefore by Definition 3.5 such function is integrable for $\sigma > 1$.

Theorem 4.4 provides the following canonical decomposition of the Global Zeta Function into product of local zeta functions

$$\zeta(f, ||_{\mathbb{A}}^s) = \bigotimes_\nu \zeta_\nu(f, ||_\nu^s) \ .$$

By choosing the local functions as in Property 2.13 and Property 2.14 for $\sigma > 1$ we obtain:

$$\zeta(f, ||_{\mathbb{A}}^s) = \zeta_\infty(f_\infty, ||_\infty^s) \prod_p \zeta_p(f_p, ||_p^s) =$$

$$= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \prod_p \left(\frac{1}{1-p^{-s}}\right) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(s) \quad (18)$$

which is precisely the completed Riemann Zeta Function!
By Corollary 3.17 and because of our choice of local functions we have

$$f = \widehat{f} \ .$$

Therefore for $\widehat{f}$ by (18) we obtain that

$$\zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s}) = \zeta(f, ||_{\mathbb{A}}^{1-s}) = \xi(1-s) \ . \quad (19)$$

Consequently by Theorem 4.8 we obtain two equivalent formulæ:

$$\zeta(f, ||_{\mathbb{A}}^s) = \zeta(\widehat{f}, ||_{\mathbb{A}}^{1-s})$$

and

$$\xi(s) = \xi(1-s) \ .$$

In this way we have reproved functional equation for $\zeta(s)$ from Riemann's paper "Über die Anzahl der Primzahlen unter einer gegebenen Grö$\beta$e".

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