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The Linnik's method of analytic continuation of the Dirichlet L-functions

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KAROL GIERSZEWSKI

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1. Introduction

In [4] Yuri Vladimirovich Linnik studied the following formula

$$L(s,\chi,X)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}e\left(-n/X\right)\ . \tag{1}$$

He observed, not only that this formula converges to the well known Dirichlet L-function as $X\to\infty$ i.e.

$$\lim_{X\to\infty}L(s,\chi,X)=L(s,\chi)\ ,$$

but also, that it can be expressed in the terms of the Riemann zeta function i.e.

$$L(s,\chi,X)=\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)\left(Z_X\left(\frac{a}{q}\right)\right)^{-w}dw\ .$$

Apparently the idea from Linnik's paper was used in [5] and [3] chronologically.

This explicit approximation was used by V. G. Sprindžuk in [5] to show that the Riemann Hypothesis is equivalent to the Generalized Riemann Hypothesis with the additional condition imposed on the imaginary parts of the zeros of the Dirichlet L-functions. This additional condition is independent of GRH.

The idea of approximating one L-function by its *additive twists* was used by J. Kaczorowski and A. Perelli in [3] to classify the Selberg class of degree 1.

One of results of the deeper investigation of the Linnik's method introduced in [4] was observation that, one does not actually need to know neither that L-function, which is approximated by the Riemann zeta function, has an analytic continuation to the entire function, nor that it satisfies the functional equation. This leads to the new proof of the functional equation for the Dirichlet L-functions (cf. [2]). This

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proof is especially interesting because it is showing, that crucial and most desired analytic properties of L-functions, such as analycic continuation and fulfilling certain functional equations, are hereditary properties in the case of the Selberg class of degree 1.

While the original Linnik's paper [4] is unfortunately crowded with typographical mistakes, the monograph [2] presents this method with all thous mistakes corrected, and with the proof of the functional equation explicitly written. The aim of this paper is to give self contained and down to earth (1) presentation of the Linnik's method, which could be useful for both scholars and researchers. (2)

The organisation of the paper is as follows: in the the following section we introduce standard number theoretic notation. The third section is devoted to show that the expression (1) indeed approximates the Dirichlet L-function, and is expressed in the terms of the Riemann zeta function. Therefore it contains what was introduced in [4]. The fourth section presents properties of certain Barnes integral, defining hypergeometric function. More general study of such integrals and functions can be found in [3] [cf. sections 4-6 paginæ 215-226]. The fifth section contains the proof of the functional equation for the Dirichlet L-functions, which can be found writen explicitly .

2. Basic notion

First of all we shall introduce the notation, which we shall be using through whole paper. Let χ be a primitive, non-principal character (mod q), where $q \geq 2$, $X > 0$ and $Z_X(\frac{a}{q}) = \frac{1}{X} + 2\pi i \frac{a}{q}$. We shall also be using number theoretic convention for the exponential function, contour integration and summation

$$e\left(x\right) := \exp\left(2\pi i x\right) \ ,$$

$$\int_{(d)} f(z)dz := \int_{d-i\infty}^{d+i\infty} f(z)dz \ ,$$

$$\sum_{a=1}^{q}{}^{*} := \sum_{a=1,\ (a,q)=1}^{q} \ .$$

Customary for complex variables we shall be writing $s = \sigma + it$ and $w = u + iv$. Moreover, we shall use the following formulæ

$$\chi(n) = \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) e\left(-\frac{an}{q}\right) \tag{2}$$

and

$$e^{-z} = \frac{1}{2\pi i} \int_{(2)} \Gamma(w) z^{-w} dw \quad \text{ for } \Re(z) > 0 \ , \tag{3}$$

where

$$z^{-w} = e^{-wl(z)} \ ,$$

and $l(z)$ denotes the branch of $\log(z)$ on $\mathbb{C} \setminus (-\infty, 0]$, satisfying $|\Im(l(z))| < \pi$. Then the integral (3) is uniformly convergent on compact sets in parameter z.

(1) The monograph [2] while presenting the Linnik's method leaves all the technical details to the reader.

(2) While writing this paper the monograph [2] is unpublished.

3. Functional equation I

In this section we shall present computations showing, that formula (1) on the one hand converges to the Dirichlet L-function, and on the other can be expressed in terms of the Riemann zeta function.

Theorem 3.1. *Let $X > 0$, $\delta > 0$ and $\sigma > -1$. Then the following asymptotic formula holds*

$$L(s, \chi, X) = L(s, \chi) + O\left(X^{-\delta} \int_{(-\delta)} |\Gamma(w) L(s + w, \chi)| \, |dw|\right) . \tag{4}$$

Proof. Observe that for $X > 0$ series (1) is absolutely convergent for *any complex parameter s*. Since $n/X > 0$ substituting $e(-n/X)$ by integral representation (3) we obtain

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e(-n/X) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \frac{1}{2\pi i} \int_{(2)} \Gamma(w) \left(\frac{n}{X}\right)^{-w} dw . \tag{5}$$

Because the series in (5) is absolutely convergent for any complex parameter s and the integral is uniformly convergent for any n, we change the order of summation and integration obtaining

$$\frac{1}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s+w}} \Gamma(w) X^w dw . \tag{6}$$

Using the assumption that $\sigma > -1$, we observe that $\Re(s+w) > 1$ (since $\Re(w) = 2$), therefore the series $\sum_{n=1}^{\infty} \chi(n)/n^{s+w}$ in the formula (6) is absolutely convergent, representing the Dirichlet L-function $L(s+w, \chi)$, thus the formula (6) can be rewritten as follows

$$\frac{1}{2\pi i} \int_{(2)} L(s + w, \chi) \Gamma(w) X^w dw . \tag{7}$$

We shall now shift the line of integration in (7) to $\Re(w) = -\delta$. We obtain this by computing the integral

$$\frac{1}{2\pi i} \int_{\mathcal{C}} L(s + w, \chi) \Gamma(w) X^w dw , \tag{8}$$

where $\mathcal{C}$ is closed, positively oriented, piecewise smooth curve, consisting of four line segments: $[2 - iT, 2 + iT], [2 + iT, -\delta + iT], [-\delta + iT, -\delta - iT], [-\delta - iT, 2 - iT]$. Note first, that we know (by partial summation) that the function $L(s, \chi)$ has analytic continuation to the holomprphic function in the half plane $\sigma - \delta > -1$, with polynominal growth there [cf. [1] p. 120 Ex. 9] i.e.

$$|L(\sigma + it, \chi)| = O_\chi(|t|^A) \text{ when } t \to \infty \text{ for some fixed A} > 0 .$$

We restrict now σ to be such that $\sigma - \delta > -1$. The function $L(s + w, \chi)\Gamma(w)X^w$ is therefore meromorphic for $\Re(w) + \sigma > -1$ and has one simple pole at $w = 0$, and by Cauchy Integral Theorem we have

$$\frac{1}{2\pi i} \int_{\mathcal{C}} L(s + w, \chi) \Gamma(w) X^w dw = \mathrm{Res}_{w=0} L(s + w, \chi) \Gamma(w) X^w = L(s, \chi) .$$

By Stirling formula

$$|\Gamma(u + iv)| = (2\pi)^{\frac{1}{2}} |v|^{u-\frac{1}{2}} e^{-\frac{\pi|v|}{2}} (1 + O_{u_1,u_2}(1/|v|)) \text{ where } u_1 \le u \le u_2, \ |v| \to \infty ,$$

we know, that on two horizontal segments, the integral (8) is estimated as follows

$$\left|\int_{2\pm iT}^{-\delta\pm iT} L(s+w,\chi)\Gamma(w)X^w dw\right| \leq$$
$$\leq \int_{2\pm iT}^{-\delta\pm iT} |L(s+w,\chi)||\Gamma(w)||X^w||dw| =$$
$$= \int_{2\pm iT}^{-\delta\pm iT} |L(s+u\pm iT,\chi)||\Gamma(u\pm iT)||X^{u\pm iT}||du| \ll_\chi$$
$$\ll_\chi \int_{2\pm iT}^{-\delta\pm iT} |s+u\pm iT|^A (2\pi)^{\frac{1}{2}}|T|^{u-\frac{1}{2}}e^{-\frac{\pi|T|}{2}}(1+O_{-\delta,2}(1/|T|))X^u|du| \ . \quad (9)$$

For every T the last integral in (9) is bounded, length of line segments $[2+iT,-\delta+iT],[-\delta-iT,2-iT]$ is finite and independent of T, and for $|T|\to\infty$ the function under integral (9) decays exponentially to zero, therefore as $|T|\to\infty$ the last integral in (9) tends to zero.

In this way we have obtained that

$$\frac{1}{2\pi i}\int_{(2)} L(s+w,\chi)\Gamma(w)X^w dw +$$
$$+\frac{1}{2\pi i}\int_{(-\delta)} L(s+w,\chi)\Gamma(w)X^w dw =$$
$$= \mathrm{Res}_{w=0} L(s+w,\chi)\Gamma(w)X^w = L(s,\chi) \ .$$

Therefore we have that

$$\frac{1}{2\pi i}\int_{(2)} L(s+w,\chi)\Gamma(w)X^w dw = L(s,\chi) - \frac{1}{2\pi i}\int_{(-\delta)} L(s+w,\chi)\Gamma(w)X^w dw \ . \quad (10)$$

Since

$$-\frac{1}{2\pi i}\int_{(-\delta)} L(s+w,\chi)\Gamma(w)X^w dw = O\left(X^{-\delta}\int_{(-\delta)} |\Gamma(w)L(s+w,\chi)|\,|dw|\right) \ ,$$

we get

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}e\left(-n/X\right) = L(s,\chi) + O\left(X^{-\delta}\int_{(-\delta)} |\Gamma(w)L(s+w,\chi)|\,|dw|\right) \ .$$

□

Corollary 3.2. *When $X\to\infty$ and $\sigma>-1$ then*

$$L(s,\chi,X) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}e\left(-n/X\right) \to L\left(s,\chi\right) \ .$$

Lemma 3.3. *Let $X>0$, and let $\sigma>-1$. Then the following formula holds*

$$L(s,\chi,X) = \frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)\left(Z_X\left(\frac{a}{q}\right)\right)^{-w} dw \ , \quad (11)$$

where $\zeta(s+w)$ is the Riemann zeta function.

Proof. Observe that for $X > 0$ series (1) is absolutely convergent for *any complex parameter* s. Substituting in (1) $\chi(n)$ by its expansion written in (2) we have

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e\left(-n/X\right) = \sum_{n=1}^{\infty} \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) e\left(-\frac{an}{q}\right) \frac{1}{n^s} e\left(-n/X\right) =$$

$$= \sum_{n=1}^{\infty} \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) e\left(-nZ_X\left(\frac{a}{q}\right)\right) \frac{1}{n^s} .$$

Changing the order of summation we obtain

$$\sum_{n=1}^{\infty} \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) e\left(-nZ_X\left(\frac{a}{q}\right)\right) \frac{1}{n^s} =$$

$$= \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n^s} e\left(-nZ_X\left(\frac{a}{q}\right)\right) .$$

Since $\Re\left(nZ_X\left(\frac{a}{q}\right)\right) > 0$, therefore using integral expansion (3) of $e\left(z\right)$, we have

$$\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{1}{2\pi i} \int_{(2)} \Gamma(w) \left(nZ_X\left(\frac{a}{q}\right)\right)^{-w} dw .$$

We observe, that the integral is uniformly convergent on compact sets and that the series is convergent absolutely, therefore we change the order of summation and integration obtaining

$$\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{1}{n^{s+w}} \Gamma(w) \left(Z_X\left(\frac{a}{q}\right)\right)^{-w} dw . \tag{12}$$

Using the assumption that $\sigma > -1$ we observe that $\Re(s+w) > 1$ (since $\Re(w) = 2$), therefore the series $\sum_{n=1}^{\infty} 1/n^{s+w}$ in formula (12) is absolutely convergent, representing the Riemann zeta function $\zeta(s+w)$. Therefore (12) can be rewritten as follows

$$\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \zeta(s+w)\Gamma(w) \left(Z_X\left(\frac{a}{q}\right)\right)^{-w} dw .$$

Finally we have obtained the following formula

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e\left(-n/X\right) = \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \zeta(s+w)\Gamma(w) \left(Z_X\left(\frac{a}{q}\right)\right)^{-w} dw \text{ for } \sigma > -1 . \tag{13}$$

□

Remark 3.1. Observe, that in the expression (13), when $X \to \infty$ then $Z_X(a/q)^{-w} \to e^{-\pi i w a/q}$. For $\Im(w) \to -\infty$, the function $e^{-\pi i w a/q}$ tends exponentially to infinity, causing divergence in the integral. Therefore in the expresion (11) one can not instantly pass to the limit.

4. Hypergeometric functions

In this section we shall present the notion and properties of certain Barnes integral, defining elementary hypergeometric function, which shall be used, as a mean, to deal with convergence problems mentioned in Remark 3.1. Till the end of this section, we shall use the following notation

$$(c)_k = c(c+1)(c+2)\dots(c+k-1) \text{ for } k \geq 1$$

and

$$A = \{z \in \mathbb{C} \mid \Re(z) > 0\} \ ,$$
$$B = \{z \in \mathbb{C} \mid |z| < 1\} \setminus (-1, 0] \ ,$$
$$C = \{z \in \mathbb{C} \mid |z| > 1\} \ ,$$
$$D = A \cup B \cup C \ .$$

Moreover let $\Omega \subsetneq \mathbb{C} \setminus \{1\}$ be a bounded domain in the right half-plane.

Remark 4.1. The notation of this whole section, as well as the technique of the proof of Theorem 4.1, is consistent with notation and technique used in [3] [cf. section 4 paginæ 215-217].

Definition 4.1. The Barnes integral is defined as follows

$$\frac{1}{2\pi i}\int_{(d)} \Gamma(c-w)\Gamma(w)z^{-w}dw \ , \tag{14}$$

where $\Re(c) > d > 0$ and $|\arg z| < \pi$. Moreover z^{-w} is defined as in (3).

Theorem 4.1 (cf. Theorem 4.1 in [3])**.** *Let $d < \sup_{s\in\Omega}|\Re(s)|$ be a positive number. Then the Barnes integral*

$$H_{(d)}(z,s) = \frac{1}{2\pi i}\int_{(d)} \Gamma(1-s-w)\Gamma(w)z^{-w}dw \tag{15}$$

is absolutely and uniformly convergent on compact subsets of $A \times \Omega$, and $H_{(d)}(z,s)$ has holomorphic continuation to $A \times \Omega$ as a single-valued function. Moreover, for $(z,s) \in B \times \Omega$ we have

$$H_{(d)}(z,s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Gamma(1-s+k)z^k = \Gamma(1-s)(1+z)^{s-1} \ . \tag{16}$$

The proof of Theorem 4.1 will immediately follow from the next five lemmas.

Lemma 4.2. *For any compact set $\mathcal{K} \subsetneq A \times \Omega$, we have the the following estimation*

$$\Gamma(1-s-w)\Gamma(w)z^{-w} \ll_{d,\mathcal{K}} |v|^{-\sigma}|z|^d e^{|v|(|\Im(l(z))|-\pi)} \tag{17}$$

as $|v| \to \infty$.

Proof. First we observe that

$$|z^{-w}| = |z|^{-u}e^{v\Im(l(z))} \ . \tag{18}$$

By Stirling formula

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}}|v|^{u-\frac{1}{2}}e^{-\frac{\pi|v|}{2}}(1+O_{u_1,u_2}(1/|v|)) \quad \text{where } u_1 \leq u \leq u_2,\ |v| \to \infty \ , \tag{19}$$

we have that

$$|\Gamma(c-w)| = (2\pi)^{\frac{1}{2}}|\Im(1-s)-v|^{\Re(1-s)-u-\frac{1}{2}}e^{-\frac{\pi|\Im(1-s)-v|}{2}}(1+O(1/|\Im(1-s)-v|)) \ . \tag{20}$$

Combining (18), (19) and (20) we obtain the following estimate

$$|\Gamma(1-s-w)\Gamma(w)z^{-w}| =$$
$$= |z|^{-u}e^{v\Im(l(z))}(2\pi)|v|^{u-\frac{1}{2}}|\Im(1-s)-v|^{\Re(1-s)-u-\frac{1}{2}}e^{-\frac{\pi|v|}{2}}e^{-\frac{\pi|\Im(1-s)-v|}{2}}\times$$
$$\times(1+O_{u_1,u_2}(1/|v|))(1+O_{u_1,u_2}(1/|\Im(1-s)-v|)) \ll_{u_1,u_2}$$
$$\ll_{u_1,u_2} |z|^{-u}e^{v\Im(l(z))}|v^{-\sigma}|e^{-\pi|v|} \ . \quad (21)$$

Observe that (21) is $o(1/|v|)$ for $v\to-\infty$. For $v\to\infty$ we have

$$|z|^{-u}e^{v\Im(l(z))}|v^{-\sigma}|e^{-\pi|v|} = |z|^{-u}e^{v(\Im(l(z))-\pi)}|v^{-\sigma}| \ .$$

□

Corollary 4.3. *The integral (15) is absolutely and uniformly convergent on compact sets $\mathcal{K}\subsetneq A\times\Omega$.*

Proof. Since $(\Im(l(z))-\pi)=K<0$ by assumption in (3), by Lemma 4.2 we have that for $v\to\infty$ the expression (21) is $o(1/|v|)$.

Thus we estimate the integral

$$\frac{1}{2\pi i}\int_{(d)}\Gamma(1-s-w)\Gamma(w)z^{-w}dw \ll_{d,\mathcal{K}} |z|^{-d}\int_{(d)}o(1/|v|)dv<\infty \ . \quad (22)$$

Estimate (22) is uniform in any compact subset of $A\times\Omega$ since estimate (17) is such. □

Remark 4.2. Observe that presence of two, symmetric to each other, Γ factors in integral (14), is crucial for its convergence. Lack of one of them would cause the same problems as mentioned in Remark 3.1.

Lemma 4.4. *Let $(z,s)\in B\times\Omega$. Let $\mathcal{C}$ be the contour consisting of the vertical line $[d-iV,d+iV]$ and two horizontal segments $[d\pm iV,-\infty\pm iV]$, where V is any fixed positive number. Then for $(z,s,w)\in B\times\Omega\times\mathcal{C}$ we have the following estimate*

$$\Gamma(1-s-w)\Gamma(w)z^{-w} \ll_{\mathcal{C},\mathcal{K}} |u|^{-\sigma}|z|^{-u} \ ,$$

uniformly for (z,s) in any compact set $\mathcal{K}\subsetneq B\times\Omega$.

Proof. On the line $[d-iV,d+iV]$ for any $(z,s)\in B\times\Omega$ the function $\Gamma(1-s-w)\Gamma(w)z^{-w}$ is holomorphic.

Using the well known formula

$$\Gamma(w)\Gamma(1-w)=\frac{\pi}{\sin\pi w}$$

we have

$$\Gamma(1-s-w)\Gamma(w)z^{-w}=\Gamma(1-s-w)\frac{\pi}{\sin(\pi w)\Gamma(1-w)}z^{-w} \ . \quad (23)$$

We have that for $w\in\mathcal{C}$, on two horizontal segments $[d\pm iV,-\infty\pm iV]$ $|\arg(-w)|\le\pi-\varepsilon$, for any arbitrary fixed ε. Since $(z,s)\in\mathcal{K}\subsetneq B\times\Omega$, $\mathcal{K}$ compact, we can apply the Stirling formula

$$\log\Gamma(w+a)=(w+a+1/2)\log(w)-w+1/2\log(2\pi)+O_\varepsilon(1/|w|)$$

for $|w|\to\infty$, $a\in K\subsetneq\mathbb{C}$, K-compact and $|\arg(w)|\le\pi-\varepsilon$, to the formula

$$\frac{\Gamma(1-s-w)}{\Gamma(1-w)} \ ,$$

obtaining

$$\left|\log\frac{\Gamma(1-s-w)}{\Gamma(1-w)}\right| = |-s\log(-w)+O_\varepsilon(1/|w|)| \ .$$

Therefore

$$\left|\frac{\Gamma(1-s-w)}{\Gamma(1-w)}\right| \ll_{\varepsilon,\mathcal{K}} |-w|^{-\sigma} \ .$$

But since $w \in [d \pm iV, -\infty \pm iV]$, we have $|-w|^{-\sigma} \sim u^{-\sigma}$. We also have

$$\left|\frac{\pi}{\sin \pi w}\right| \ll \frac{1}{|e^{\pi v} - e^{-\pi v}|} \ .$$

Therefore on $\mathcal{C}$ we obtain

$$\left|\frac{\pi}{\sin \pi w}\right| \ll_{\mathcal{C}} 1 \ .$$

Since $|z^{-w}| = |z|^{-u}$ one has on $\mathcal{C}$

$$\Gamma(1-s-w)\Gamma(w)z^{-w} \ll_{\mathcal{C},\mathcal{K}} |u|^{-\sigma}|z|^{-u} \ .$$

□

Corollary 4.5. *The integral*

$$H_{\mathcal{C}}(z,s) = \frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(1-s-w)\Gamma(w)z^{-w}dw \tag{24}$$

is absolutely and uniformly convergent on compact sets $\mathcal{K} \subsetneq B \times \Omega$.

Proof. Fact that $(z,s) \in B \times \Omega$, implies, that $|z| < 1$. Thus using the estimation from Lemma 4.4 we observe that the function $\Gamma(1-s-w)\Gamma(w)z^{-w}$ decays exponentially on $\mathcal{C}$ as $|w| \to \infty$. Therefore the integral (24) is absolutely and uniformly convergent on compact sets $\mathcal{K} \subsetneq B \times \Omega$. □

Lemma 4.6. *For* $(z,s) \in (A \cap B) \times \Omega$ *we have that*

$$H_{(d)}(z,s) = H_{\mathcal{C}}(z,s) \ .$$

Proof. Let $(z,s) \in \mathcal{K} \subsetneq (A \cap B) \times \Omega$ where $\mathcal{K}$ is compact. By Corollary 4.3 and Corollary 4.5 we know that on $(A \cap B) \times \Omega$ both integrals $H_{(d)}(z,s)$ and $H_{\mathcal{C}}(z,s)$ are absolutely and uniformly convergent on compact sets. To obtain the equality we shall join half-lines $[d \pm iV, d \pm i\infty)$ and $[d \pm iV, \infty \pm iV)$ by two arcs of the circle $|w| = R$. We denote the resulting contour by $\mathcal{C}_R^{\pm}$. Since in the region $(A \cap B) \times \Omega \times E$, where $\partial E = \mathcal{C}_R^{\pm}$ and E consists of two connected components, the function $\Gamma(1-s-w)\Gamma(w)z^{-w}$ is holomorphic, and by the Cauchy Integral Theorem we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}_R^{\pm}} \Gamma(1-s-w)\Gamma(w)z^{-w}dw = 0 \ . \tag{25}$$

Using exactly the same technique as in the proof of Lemma 4.4, one shows that

$$|\Gamma(1-s-w)\Gamma(w)z^{-w}| \ll_{\varepsilon,\mathcal{K}} |-w|^{-\sigma}|z|^{-u}\left|\frac{1}{e^{\pi v} - e^{-\pi v}}\right|$$

for $\Re(w) < d$ and $|\arg(w)| < \pi$.

For $\Re(w) < d$ and $|\arg(w)| \leq \frac{3}{4}\pi$, we have $|w| \ll 2|u|$. Therefore

$$|\Gamma(1-s-w)\Gamma(w)z^{-w}| \ll_{\varepsilon,\mathcal{K}} |-w|^{-\sigma}|z|^{-u} \ll 2|u|^{-\sigma}|z|^{-u} \ ,$$

and, since $|z| < 1$, it implies that $|\Gamma(1-s-w)\Gamma(w)z^{-w}|$ tends exponentially to zero as $|u| \to \infty$.

For $\Re(w) < d$ and $\frac{3}{4}\pi < |\arg(w)| < \pi$ we have $|w| \ll 2|\pm v|$. Therefore

$$|\Gamma(1-s-w)\Gamma(w)z^{-w}| \ll_{\varepsilon,\mathcal{K}} \left|\frac{1}{e^{\pi v} - e^{-\pi v}}\right| |-w|^{-\sigma}|z|^{-u} \ll \left|\frac{1}{e^{\pi v} - e^{-\pi v}}\right| 2|v|^{-\sigma}|z|^{-u} \ .$$

Since $|z| < 1$, the function tends exponentially to zero as $|w| \to \infty$. Therefore the integral over two circle arcs $|w| = R$, from the function $\Gamma(1-s-w)\Gamma(w)z^{-w}$, tends to zero, as $R \to \infty$. Therefore by formula (25) we have that

$$\frac{1}{2\pi i}\int_{(d)} \Gamma(1-s-w)\Gamma(w)z^{-w}dw = \frac{1}{2\pi i}\int_{\mathcal{C}} \Gamma(1-s-w)\Gamma(w)z^{-w}dw\ .$$

□

Corollary 4.7. *The function $H_{(d)}(z,s)$ has holomorphic continuation to a single-valued function on $(A \cup B) \times \Omega$.*

Proof. Since $H_{\mathcal{C}}(z,s)$ is holomorphic on $B \times \Omega$, $H_{(d)}(z,s)$ is holomorphic on $A \times \Omega$, and $H_{\mathcal{C}}(z,s) = H_{(d)}(z,s)$ for $(z,s) \in (A \cap B) \times \Omega$ by Lemma 4.6, we obtain continuation. □

Lemma 4.8. *For $(z,s) \in \mathcal{K} \subsetneq (B \cup \{0\}) \times \Omega$, $\mathcal{K}$ compact, the series*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Gamma(1-s+k)z^k$$

is absolutely and uniformly convergent, and equal to

$$\Gamma(1-s)(1+z)^{s-1}\ .$$

Proof. Observe that for any compact $\mathcal{K} \subsetneq (B \cup \{0\}) \times \Omega$ the function

$$\Gamma(1-s)(1+z)^{s-1}$$

is holomorphic. Moreover, it has the Taylor expansion at $z = 0$ equal

$$\Gamma(1-s)(1+z)^{s-1} = \Gamma(1-s)\sum_{k=0}^{\infty}\binom{s-1}{k}z^k =$$
$$= \Gamma(1-s)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(s-1)_k z^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(1-s+k)z^k\ .$$

By the uniqueness of the Taylor expansion, the Lemma folows. □

Lemma 4.9. *For any compact set $\mathcal{K} \subsetneq B \times \Omega$, we have the following equality*

$$H_{\mathcal{C}}(z,s) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(1-s+k)z^k\ .$$

Proof. Consider the vertical line $[-k-1/2+iV, -k-1/2-iV]$, where $k \in \mathbb{N}$. Using the same technique as in the proof of Lemma 4.4, since neither $\Gamma(1-s-w)$ nor $\Gamma(1-w)$ has any w-poles at $-k+1/2$, one has the following estimation

$$|\Gamma(1-s-w)\Gamma(w)z^{-w}| \ll_{\varepsilon,\mathcal{K}} |u|^{-\sigma}|z|^{-u} = |-k+1/2|^{-\sigma}|z|^{k-1/2}\ .$$

Therefore, since $|z| < 1$ for $k \to \infty$, the integral

$$\int_{-k-1/2+iV}^{-k-1/2-iV} \Gamma(1-s-w)\Gamma(w)z^{-w}dw \tag{26}$$

decays to zero as $k \to \infty$. Consider the following contour $\mathcal{C}_k$, consisting of two horizontal line segments $[-k-1/2 \pm iV, d \pm iV]$ and two vertical segments $[-k-1/2+iV, -k-1/2-iV]$, and $[d+iV, d-iV]$. The contour $\mathcal{C}_k$ is rounding the

w-poles of the function $\Gamma(1-s-w)\Gamma(w)z^{-w}$. Therefore by the Residue Theorem we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}_k}\Gamma(1-s-w)\Gamma(w)z^{-w}dw=\sum_{l=0}^{k}\mathrm{Res}_{w=-l}\Gamma(1-s-w)\Gamma(w)z^{-w}=$$

$$=\sum_{l=0}^{k}\Gamma(1-s+l)\frac{(-1)^l}{l!}z^l\ .$$

Since integral (26) decay to zero as $k\to\infty$, as well as $\mathcal{C}_k\to\mathcal{C}$, therefore

$$H_{\mathcal{C}}(z,s)=\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma(1-s-w)\Gamma(w)z^{-w}dw=\sum_{k=0}^{\infty}\Gamma(1-s+k)\frac{(-1)^k}{k!}z^k\ .$$

□

Corollary 4.10. *For $(z,s)\in(A\cap(B\cup\{0\}))\times\Omega$ we have the following equality*

$$H_{(d)}(z,s)=\Gamma(1-s)(1+z)^{s-1}\ .$$

Proof. By Lemma 4.6

$$H_{(d)}(z,s)=H_{\mathcal{C}}(z,s)\quad\text{on}\quad(A\cap B)\times\Omega\ .$$

By Lemma 4.9

$$H_{\mathcal{C}}(z,s)=\sum_{k=0}^{\infty}\Gamma(1-s+k)\frac{(-1)^k}{k!}z^k\quad\text{for}\quad(z,s)\in(A\cap B)\times\Omega\ .$$

By Lemma 4.8

$$\sum_{k=0}^{\infty}\Gamma(1-s+k)\frac{(-1)^k}{k!}z^k=\Gamma(1-s)(1+z)^{s-1}\quad\text{on}\quad(A\cap(B\cup\{1\}))\times\Omega\ .$$

Therefore

$$H_{(d)}(z,s)=\Gamma(1-s)(1+z)^{s-1}\quad\text{on}\quad(A\cap B)\times\Omega\ .$$

□

Corollary 4.11. *The function $H_{(d)}(z,s)$ has holomorphic continuation to $D\times\Omega$.*

Proof. Since $\Gamma(1-s)(1+z)^{s-1}$ is holomorphic and single valued function on $D\times\Omega$ and $(A\cap B)\times\Omega\subsetneq D\times\Omega$ has convergence point, one obtains holomorphic continuation of $H_{(d)}(z,s)$ to $D\times\Omega$. □

5. Functional equation II

In this section, using the hypergeometric function introduced previously, we shall obtain limit in formula (13), avoiding divergence mentioned in Remark 3.1. To this end, in formula (13), we shall first change the line of integration from (2) to $(-\sigma/2)$, where $-1<\sigma<0$, then use the asymmetric form of the functional equation for the Riemann zeta function, and express the resulting formula in terms of the hypergeometric function introduced in the previous section.

Lemma 5.1. *Assume that $-1<\sigma<0$, then the following equality of integrals holds*

$$\frac{1}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw=\varkappa+\frac{1}{2\pi i}\int_{(-\sigma/2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw\ , \tag{27}$$

where

$$\varkappa=\mathrm{Res}_{w=1-s}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}=\Gamma(1-s)Z_X(a/q)^{s-1}\ .$$

Proof. Consider integral

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw\ , \tag{28}$$

where $\mathcal{C}$ consist of four line segments $[2-iT,2+iT]$, $[-\sigma/2+iT,-\sigma/2-iT]$ and $[-\sigma/2\pm iT,2\pm iT]$.

Since the function $\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}$ is meromorphic in the region $\Re(w)>0$ (we know this, because the Riemann zeta function $\zeta(s+w)$ is meromorphic in this region), we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw = \mathrm{Res}_{w=1-s}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w} = \varkappa\ .$$

By Stirling formula

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}}|v|^{u-\frac{1}{2}}e^{-\frac{\pi|v|}{2}}(1+O_{u_1,u_2}(1/|v|))\ \ \text{where } u_1\le u\le u_2,\ |v|\to\infty\ ,$$

and the well known estimation

$$\zeta(\sigma+it) = O(t^{3/2}) \text{ uniformly for } 0\le\sigma\le 2\ ,$$

(cf. [6] p. 81 formula 5.1.1) we know, that on two horizontal segments, the integral (28) is estimated as follows

$$\begin{aligned}\left|\int_{2\pm iT}^{-\sigma/2\pm iT}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw\right| &\le \\ \le \int_{2\pm iT}^{-\sigma/2\pm iT}|\zeta(s+w)||\Gamma(w)||Z_X(a/q)^{-w}||dw| &= \\ = \int_{2\pm iT}^{-\sigma/2\pm iT}|\zeta(s+u\pm iT)||\Gamma(u\pm iT)||Z_X(a/q)^{-u\mp iT}||du| &\ll \\ \ll \int_{2\pm iT}^{-\sigma/2\pm iT}|s+u\pm iT|^{3/2}(2\pi)^{\frac{1}{2}}|T|^{u-\frac{1}{2}}e^{-\frac{\pi|T|}{2}}(1+O_{-\sigma/2,2}(1/|T|))e^{-ul(Z_X(a/q))}|du|\ . \end{aligned} \tag{29}$$

For every T the last integral in (29) is bounded, and for $|T|\to\infty$ the function under the integral decays exponentially to zero. Therefore as $|T|\to\infty$ the last integral in (29) tends to zero.

In this way we have obtained

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw - \\ -\frac{1}{2\pi i}\int_{(-\sigma/2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw &= \\ = \mathrm{Res}_{w=0}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w} &= \varkappa\ .\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw &= \\ = \varkappa + \frac{1}{2\pi i}\int_{(-\sigma/2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw\ .\end{aligned}$$

□

Lemma 5.2. *Assume that $-1 < \sigma < 0$, then the following equality holds*

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)\left(Z_X\left(a/q\right)\right)^{-w}dw =$$
$$= \frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\overline{\chi}(a)\left(\varkappa+\frac{(2\pi)^{s-1}}{i}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\times\right.$$
$$\left.\times\left(e^{i\frac{\pi s}{2}}H_{(-\sigma/2)}\left(-\frac{Z_X(a/q)}{2\pi n}i,\ s\right)-e^{-i\frac{\pi s}{2}}H_{(-\sigma/2)}\left(\frac{Z_X(a/q)}{2\pi n}i,\ s\right)\right)\right)\ . \quad (30)$$

Proof. First observe that by Lemma 5.1 one has

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\int_{(2)}\zeta(s+w)\Gamma(w)\left(Z_X\left(a/q\right)\right)^{-w}dw =$$
$$= \frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\left(\int_{(-\sigma/2)}\zeta(s+w)\Gamma(w)Z_X(a/q)^{-w}dw+2\pi i\varkappa\right)\ . \quad (31)$$

Applying to the right hand side of (31), the functional equation for the Riemann zeta function

$$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\ ,$$

one obtains

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\left(\int_{(-\sigma/2)}\zeta(1-s-w)2(2\pi)^{s+w-1}\sin\left(\frac{\pi(s+w)}{2}\right)\times\right.$$
$$\left.\times\ \Gamma(1-s-w)\Gamma(w)Z_X(a/q)^{-w}dw+2\pi i\varkappa\right)\ .$$

Because $\Re(1-s-w) = 1+\sigma/2 > 1$, therefore $\zeta(1-s-w)$ has the Dirichlet series expansion, and one has

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\left(\int_{(-\sigma/2)}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1-s-w}}\right)2(2\pi)^{s+w-1}\sin\left(\frac{\pi(s+w)}{2}\right)\times\right.$$
$$\left.\times\ \Gamma(1-s-w)\Gamma(w)Z_X(a/q)^{-w}dw+2\pi i\varkappa\right) =$$
$$= \frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\left(\int_{(-\sigma/2)}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1-s-w}}2(2\pi)^{s+w-1}\sin\left(\frac{\pi(s+w)}{2}\right)\times\right.\right.$$
$$\left.\left.\times\ \Gamma(1-s-w)\Gamma(w)Z_X(a/q)^{-w}dw\right)+2\pi i\varkappa\right)\ .$$

Since the Dirichlet series is convergent and the integral is uniformly convergent on compact sets, one changes the order of summation and integration obtaining

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\frac{\overline{\chi}(a)}{2\pi i}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}(2\pi)^{s-1}\left(\int_{(-\sigma/2)}2\sin\left(\frac{\pi(s+w)}{2}\right)\times\right.\right.$$
$$\left.\left.\times\ \Gamma(1-s-w)\Gamma(w)\left(\frac{Z_X(a/q)}{2\pi n}\right)^{-w}dw\right)+2\pi i\varkappa\right)\ .$$

Previous formula is equivalent to the following

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\left(\varkappa+\frac{(2\pi)^{s-1}}{i}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\left(\frac{1}{2\pi i}\int_{(-\sigma/2)}2i\sin\left(\frac{\pi(s+w)}{2}\right)\times\right.\right.$$
$$\left.\left.\times\ \Gamma(1-s-w)\Gamma(w)\left(\frac{Z_X(a/q)}{2\pi n}\right)^{-w}dw\right)\right)\ .$$

Using the Euler identity

$$2i\sin\left(\frac{\pi(s+w)}{2}\right)=e^{i\frac{\pi(s+w)}{2}}-e^{-i\frac{\pi(s+w)}{2}}=-i^{-w}e^{i\frac{\pi s}{2}}-i^{-w}e^{-i\frac{\pi s}{2}}\ ,$$

we obtain

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\left(\varkappa+\frac{(2\pi)^{s-1}}{i}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\left(\frac{1}{2\pi i}\int_{(-\sigma/2)}\left(-i^{-w}e^{i\frac{\pi s}{2}}-i^{-w}e^{-i\frac{\pi s}{2}}\right)\times\right.\right.$$
$$\left.\left.\times\ \Gamma(1-s-w)\Gamma(w)\left(\frac{Z_X(a/q)}{2\pi n}\right)^{-w}dw\right)\right)\ .$$

Rewriting above formula we obtain

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\left(\varkappa+\frac{(2\pi)^{s-1}}{i}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\times\right.$$
$$\times\left(e^{i\frac{\pi s}{2}}\frac{1}{2\pi i}\int_{(-\sigma/2)}\Gamma(1-s-w)\Gamma(w)\left(-\frac{Z_X(a/q)}{2\pi n}i\right)^{-w}dw\ -\right.$$
$$\left.\left.-\ e^{-i\frac{\pi s}{2}}\frac{1}{2\pi i}\int_{(-\sigma/2)}\Gamma(1-s-w)\Gamma(w)\left(\frac{Z_X(a/q)}{2\pi n}i\right)^{-w}dw\right)\right)\ . \tag{32}$$

One expresses formula (32) in terms of the function $H_{(-\sigma/2)}(z,s)$ obtaining

$$\frac{\tau(\chi)}{q}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\left(\varkappa+\frac{(2\pi)^{s-1}}{i}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\times\right.$$
$$\left.\times\left(e^{i\frac{\pi s}{2}}H_{(-\sigma/2)}\left(-\frac{Z_X(a/q)}{2\pi n}i,\ s\right)-e^{-i\frac{\pi s}{2}}H_{(-\sigma/2)}\left(\frac{Z_X(a/q)}{2\pi n}i,\ s\right)\right)\right)\ . \tag{33}$$

□

Theorem (Main Theorem)**.** *Let $L(s,\chi)$ be the Dirichlet L-function associated to the primitive, non-principal character χ. Assume that $L(s,\chi)$ has analytic continuation to the half plane $\sigma>-1$ with polynominal growth there and that the Riemann zeta function has meromorphic continuation to the whole complex plane and satisfies functional equation*

$$\zeta(s)=2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\ ,$$

then $L(s,\chi)$ satisfy the functional equation

$$L(s,\chi)=\frac{2\omega_\chi}{\sqrt{q}}\left(\frac{2\pi}{q}\right)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi(s+a(\chi))}{2}\right)L(1-s,\overline{\chi})\ , \tag{34}$$

and has analytic continuation to the whole complex plane.

Proof. By Corollary 3.2 we have

$$L(s,\chi,X) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} e\left(-n/X\right) \to L\left(s,\chi\right) \ ,$$

when $X \to \infty$. On the other hand, by Lemma 3.3 and Lemma 5.2, we have

$$\begin{aligned} L(s,\chi,X) &= \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \frac{\overline{\chi}(a)}{2\pi i} \int_{(2)} \zeta(s+w)\Gamma(w) \left(Z_X\left(a/q\right)\right)^{-w} dw = \\ &= \frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \left(\varkappa + \frac{(2\pi)^{s-1}}{i} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \times \right. \\ &\times \left. \left(e^{i\frac{\pi s}{2}} H_{(-\sigma/2)} \left(-\frac{Z_X(a/q)}{2\pi n} i,\ s \right) - e^{-i\frac{\pi s}{2}} H_{(-\sigma/2)} \left(\frac{Z_X(a/q)}{2\pi n} i,\ s \right) \right) \right) \ . \end{aligned}$$

Applying identity (16) to formula (33), one obtains

$$\begin{aligned} &\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \left(\varkappa + \frac{(2\pi)^{s-1}}{i} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \times \right. \\ &\qquad \times \left(e^{i\frac{\pi s}{2}} \Gamma(1-s) \left(1 + \frac{Z_X(a/q)}{2\pi n} i \right)^{s-1} - \right. \\ &\qquad\qquad \left. \left. - \ e^{-i\frac{\pi s}{2}} \Gamma(1-s) \left(1 - \frac{Z_X(a/q)}{2\pi n} i \right)^{s-1} \right) \right) \ , \end{aligned}$$

or in more explicit terms

$$\begin{aligned} &\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \left(\Gamma(1-s) \left(\frac{1}{X} + \frac{2\pi i a}{q} \right)^{s-1} + \frac{(2\pi)^{s-1}}{i} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \times \right. \\ &\qquad \times \left(e^{i\frac{\pi s}{2}} \Gamma(1-s) \left(1 + \frac{1/X + a/q}{n} \right)^{s-1} - \right. \\ &\qquad\qquad \left. \left. - \ e^{-i\frac{\pi s}{2}} \Gamma(1-s) \left(1 - \frac{1/X - a/q}{n} \right)^{s-1} \right) \right) \ . \quad (35) \end{aligned}$$

Observe, that now one can pass to the limit $X \to \infty$, since the obsicle mentioned in Remark 3.1 has been omited!

$$\begin{aligned} &\frac{\tau(\chi)}{q} \sum_{a=1}^{q}{}^{*} \overline{\chi}(a) \left(\Gamma(1-s) \left(\frac{2\pi i a}{q} \right)^{s-1} + \frac{(2\pi)^{s-1}}{i} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \times \right. \\ &\quad \times \left. \left(e^{i\frac{\pi s}{2}} \Gamma(1-s) \left(1 + \frac{a}{qn} \right)^{s-1} - e^{-i\frac{\pi s}{2}} \Gamma(1-s) \left(1 - \frac{a}{qn} \right)^{s-1} \right) \right) \ . \quad (36) \end{aligned}$$

Computing and grouping terms in (36) we obtain

$$\frac{\tau(\chi)}{q}\left(\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\left(\frac{2\pi i a}{q}\right)^{s-1}+\frac{(2\pi)^{s-1}}{i}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\times\right.$$
$$\left.\times\left(\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\frac{1}{(qn)^{s-1}}\left(e^{i\frac{\pi s}{2}}(qn+a)^{s-1}-e^{-i\frac{\pi s}{2}}(qn-a)^{s-1}\right)\right)\right) .$$

Because $\frac{1}{n^{1-s}}\frac{1}{n^{s-1}}=1$ and $i^{s-1}=\frac{1}{i}e^{i\frac{\pi s}{2}}$ one has

$$\frac{\tau(\chi)}{q}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(\frac{1}{i}e^{i\frac{\pi s}{2}}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{a^{1-s}}+\frac{1}{i}\sum_{a=1}^{q}{}^{*}\,\overline{\chi}(a)\sum_{n=1}^{\infty}\left(\frac{e^{i\frac{\pi s}{2}}}{(qn+a)^{1-s}}-\frac{e^{-i\frac{\pi s}{2}}}{(qn-a)^{1-s}}\right)\right) .$$

Changing the order of summation we obtain

$$\frac{\tau(\chi)}{q}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(\frac{1}{i}e^{i\frac{\pi s}{2}}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{a^{1-s}}+\right.$$
$$\left.+\frac{1}{i}\left(e^{i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{(qn+a)^{1-s}}-e^{-i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{(qn-a)^{1-s}}\right)\right) .$$

Using the obvious identity $(qn-a)=(q(n-1)+q-a)$, and by chaging the order of summands in the sum from $\sum^{*q}_{a=1}$ to $\sum^{*1}_{a=q}$, we have

$$\frac{\tau(\chi)}{q}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(\frac{1}{i}e^{i\frac{\pi s}{2}}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{a^{1-s}}+\right.$$
$$\left.+\frac{1}{i}\left(e^{i\frac{\pi s}{2}}\sum_{n\geq q+1}\frac{\overline{\chi}(n)}{n^{1-s}}-e^{-i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\sum_{a=q}^{1}{}^{*}\,\frac{\overline{\chi}(-a+q)}{(q(n-1)+q-a)^{1-s}}\right)\right) .$$

Hence

$$\frac{\tau(\chi)}{q}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(\frac{1}{i}e^{i\frac{\pi s}{2}}\sum_{a=1}^{q}{}^{*}\,\frac{\overline{\chi}(a)}{a^{1-s}}+\right.$$
$$\left.\frac{1}{i}\left(e^{i\frac{\pi s}{2}}\sum_{n\geq q+1}\frac{\overline{\chi}(n)}{n^{1-s}}-e^{-i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\frac{\overline{\chi}(-n)}{n^{1-s}}\right)\right) .$$

Since for $(a,q)\neq 1$, $\overline{\chi}(a)=0$ and $\overline{\chi}(-n)=\overline{\chi}(-1)\overline{\chi}(n)$, we have

$$\frac{\tau(\chi)}{q}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(\frac{1}{i}\left(e^{i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{1-s}}-\overline{\chi}(-1)e^{-i\frac{\pi s}{2}}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^{1-s}}\right)\right) . \tag{37}$$

Because $\Re(1-s)=1-\sigma$, where $-1<\sigma<0$, the Dirichlet series in (37) is absolutely and uniformly convergent, and the formula takes form

$$\frac{\tau(\chi)}{qi}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(e^{i\frac{\pi s}{2}}-\overline{\chi}(-1)e^{-i\frac{\pi s}{2}}\right)L\left(1-s,\overline{\chi}\right) . \tag{38}$$

Now we have to consider two cases: $\chi(-1)=\chi(1)=1$ or $\chi(-1)=-\chi(1)=-1$ i.e. the *parity* of the character.

First consider the case when $\chi(-1)=\chi(1)=1$, than formula (38) is of the form

$$\frac{\tau(\chi)}{qi}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(e^{i\frac{\pi s}{2}}-e^{-i\frac{\pi s}{2}}\right)L\left(1-s,\overline{\chi}\right)=$$
$$=\frac{\tau(\chi)}{qi}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}2i\sin\left(\frac{\pi s}{2}\right)L\left(1-s,\overline{\chi}\right)\ ,$$

or equivalently

$$\frac{2\omega_\chi}{\sqrt{q}}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\sin\left(\frac{\pi s}{2}\right)L\left(1-s,\overline{\chi}\right)\ , \tag{39}$$

where

$$\omega_\chi=\frac{\tau(\chi)}{\sqrt{q}}\ .$$

When $\chi(-1)=-\chi(1)=-1$, than formula (38) is of the form

$$\frac{\tau(\chi)}{qi}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\left(e^{i\frac{\pi s}{2}}+e^{-i\frac{\pi s}{2}}\right)L\left(1-s,\overline{\chi}\right)=$$
$$=\frac{\tau(\chi)}{qi}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}2\cos\left(\frac{\pi s}{2}\right)L\left(1-s,\overline{\chi}\right)\ . \tag{40}$$

But since

$$\cos\left(\frac{\pi s}{2}\right)=\sin\left(\frac{\pi(s+1)}{2}\right)\ ,$$

by putting

$$\omega_\chi=\frac{\tau(\chi)}{i\sqrt{q}}\ ,$$

formula (40) takes form

$$\frac{2\omega_\chi}{\sqrt{q}}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\sin\left(\frac{\pi(s+1)}{2}\right)L\left(1-s,\overline{\chi}\right)\ . \tag{41}$$

Gathering together (39) and (41), one has

$$\frac{2\omega_\chi}{\sqrt{q}}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\sin\left(\frac{\pi(s+a(\chi))}{2}\right)L\left(1-s,\overline{\chi}\right)\ ,$$

where

$$a(\chi)=\begin{cases}1 & \text{if } \chi(-1)=-1\\ 0 & \text{if } \chi(-1)=-1\ .\end{cases}$$

Therefore, after passing to the limit $X\to\infty$, we have

$$L\left(s,\chi\right)=\frac{2\omega_\chi}{\sqrt{q}}\Gamma(1-s)\left(\frac{2\pi}{q}\right)^{s-1}\sin\left(\frac{\pi(s+a(\chi))}{2}\right)L\left(1-s,\overline{\chi}\right)\ .$$

□

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