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ON SOME COMPLEX EXPLICIT FORMULÆ CONNECTED WITH DIRICHLET COEFFICIENTS OF INVERSES OF SPECIAL TYPE L-FUNCTIONS FROM THE SELBERG CLASS

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1. Introduction

Let S^{Γ} denote the subset of the Selberg class [9] consisting of the functions with a functional equation expressible with exactly one Γ function. That is a function $F \in \mathrm{S}^{\Gamma}$ satisfies the following five axioms ($s = \sigma + it$ here and further on).

(1) (Dirichlet series) For $\sigma > 1$, F is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}.$$

(2) (Analytic continuation) For some $m \geqslant 0$, $(s-1)^m F(s)$ is an entire function of finite order.

(3) (Functional equation) F satisfies a functional equation of the form

$$\Phi_F(s) = \omega \overline{\Phi}_F(1-s)$$

where

$$\Phi_F(s) = \mathrm{Q}^s \Gamma(\lambda s + \mu) F(s)$$

with $\mathrm{Q} > 0$, $\lambda > 0$, $\Re\mu \geqslant 0$ and $|\omega| = 1$.

(4) (Ramanujan hypothesis) For every $\varepsilon > 0$, $a_F(n) \ll_{\varepsilon} n^{\varepsilon}$.

(5) (Euler product) For $\sigma > 1$

$$F(s) = \prod_p F_p(s)$$

where

$$\log F_p(s) \coloneqq \sum_{m=1}^{\infty} \frac{b(p^m)}{p^{ms}}$$

and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

The known invariants of functions from the Selberg class S, the degree, the ξ-invariant, the parity and the shift, may be written as

$$d_F = 2\lambda\ , \qquad \xi_F + 1 = 2\mu\ , \qquad \eta_F + 1 = 2\Re\mu \qquad \text{and} \qquad \theta_F = 2\Im\mu$$

for such F.

We note that, although the data in the functional equation in S are, in general, not unique, see for example Section 4 of Vignéras [16], Section 2 of Conrey-Ghosh [4], Section 3 of Kaczorowski [9] and Kaczorowski-Perelli [13], they are unique in the special case of the functional equation from S^{Γ} as a immediate consequence of a

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simple form of the given invariants. Throughout this paper we fix $F \in \mathrm{S}^{\Gamma}$ and data Q, λ, μ, ω.

We denote by μ_F the Dirichlet convolution inverse of a_F, i.e. we formally have

$$\frac{1}{F(\sigma+it)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^{\sigma+it}}. \tag{1.1}$$

For brevity of notation we put

$$\varkappa_F := \begin{cases} -\frac{\eta_F+1}{2d_F} & \text{if } \eta_F > -1 \\ -\frac{1}{d_F} & \text{if } \eta_F = -1. \end{cases}$$

For z from the upper half-plane $\mathfrak{h} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $m(F, z)$ is defined as follows:

$$m(F,z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^{sz}}{F(s)} ds, \tag{1.2}$$

where $F \in \mathrm{S}^{\Gamma}$. The path of integration consists of the half-line $s = \varkappa_F + it$, $\infty > t \geqslant 0$, the smooth arc $\mathcal{A}$ on the upper half-plane joining points $\varkappa_F$ and $3/2$ separating possible real zeros of $F\overline{F}$ from the zeros above the real line, and the half-line $s = 3/2 + it$, $0 \leqslant t < \infty$. By Lemma 4 the integral defining (1.2) converges uniformly on compact subsets of $\mathfrak{h}$, therefore for $z \in \mathfrak{h}$ the function $m(F, \cdot)$ is holomorphic.

To formulate the main result of this paper we need two auxiliary functions

$$R(F,z) = \sum_{\substack{F(\beta)=0 \\ 0 \leqslant \beta \leqslant 1}} \operatorname*{Res}_{s=\beta} \frac{e^{sz}}{F(s)} \ ,$$

$$J_\nu(w) = \sum_{k=0}^{\infty} \frac{(-1)^k (w/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},$$

where J_ν denotes the familiar Bessel function of the first kind of order $\nu \in \mathbb{R}$ that we briefly discuss in Section 2 of this paper. As usual, δ_a^b denotes the Kronecker delta. We also use the notation $\overline{m}(F,z) := \overline{m(F,\overline{z})}$.

Theorem 1. *Let $F \in \mathrm{S}^{\Gamma}$. Then $m(F, \cdot)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at the points $z = \log n$, $\mu_F(n) \neq 0$, $n \in \mathbb{N}$, and residues*

$$\operatorname*{Res}_{z=\log n} m(F,z) = -\frac{\mu_F(n)}{2\pi i} \ .$$

Moreover, it satisfies the following functional equation

$$\begin{aligned} m(F,z) + \overline{m}(\overline{F},z) = &-\frac{2\overline{\omega}}{d_F \mathrm{Q}^{1+2i\frac{\theta_F}{d_F}}} e^{-i\frac{\theta_F}{d_F}z} \sum_{n=1}^{\infty} \frac{\overline{\mu_F(n)}}{n^{1+i\frac{\theta_F}{d_F}}} \cdot \\ &\cdot \left(\left(\mathrm{Q}^2 n e^z\right)^{\frac{1}{2}-\frac{1}{d_F}} J_{\frac{1}{2}d_F+\eta_F}\left(2\left(\mathrm{Q}^2 n e^z\right)^{-\frac{1}{d_F}}\right) - \delta_{-1}^{\eta_F} \frac{1}{\Gamma\left(\frac{1}{2}d_F\right)} \right) - R(F,z). \end{aligned} \tag{1.3}$$

In this theorem the order of the Bessel function depends explicitly on the degree and the parity of F in the following way

$$\nu = \frac{1}{2}d_F + \eta_F, \text{ where } d_F > 0 \text{ and } \eta_F \geqslant -1,$$

hence it is greater than -1. By [12, Theorem 1] and [14, Theorem] we know that if F is in the Selberg class and has a positive degree then either $d_F = 1$ or $d_F \geqslant 2$. Moreover, in the case of $d_F = 1$ by [12, Theorem 2] we know that either $\eta_F = -1$ or $\eta_F = 0$. Therefore in the case of $d_F = 1$ the order ν is equal $\pm\frac{1}{2}$ while in the case of $d_F \geqslant 2$ the order is estimated $\nu \geqslant 0$. By [8, formulæ (14), (15) p. 79] we have

$$J_{-\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}} \cos w \quad \text{and} \quad J_{\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}} \sin w.$$

In the case of $d_F = 1$ the above formulæ give much simpler form of the functional equation (1.3). In particular one easily obtain the result of K. Bartz [2] since the Riemann zeta function belongs to S^{Γ} with $d_F = 1$ and $\eta_F = -1$. Therefore Theorem 1 generalises this result. It also generalises a result of A. Łydka [15, Theorem 1.3] since by results contained in [3, 5, 6] the function $L(s+\frac{1}{2}, E)$ belongs to S^{Γ} with $d_{L(s+\frac{1}{2},E)} = 2$ and $\eta_{L(s+\frac{1}{2},E)} = 0$, where $L(s,E)$ denotes the global L-function of an elliptic curve over $\mathbb{Q}$.

In fact the class S^{Γ} contains many more functions. Let χ be a primitive, non principal Dirichlet character. Then for every $\theta \in \mathbb{R}$ the Dirichlet L-function $L(s + i\theta, \chi)$ belongs to S^{Γ}. Let f be a normalised newform of weight k and level N, i.e. $f \in \mathbf{S}_k^{new}(N)$, such that f is a common eigenvector for all Hecke operators T_p. Then the associated L-function $L(f, s + \frac{k}{2})$ belongs to S^{Γ} [5, 6, 9].

Neither the complete structure of the Selberg class S, nor even the structure of S^{Γ} is known, although many conjectures are formulated [9, 13]. We note here that our result is completely independent of those conjectures.

Let us explicitly state here that the function $m(F, \cdot)$ is just a tool aimed at proving Ω and $\Omega_{\pm}$ results for the summatory functions of the function μ_F. So far this aim was achieved for the summatory function of the function μ_ζ i.e. the classical arithmetic Möbius function [10, Theorem 1]. Therefore our research is primarily motivated by the arithmetical nature of the elements of the Selberg class and the main result of this paper is just a step towards obtaining Ω results for the summatory function of μ_F where $F \in \mathrm{S}^{\Gamma}$.

2. Useful facts

In the formulation of Theorem 1 the Bessel function of the first kind appear. One has to note that the series defining $w^{-\nu} J_\nu(w)$ converges absolutely, and uniformly in any bounded domain of w and ν, and, apart the cases when ν is an integer, function $J_\nu(w)$ has a branch point at $w = 0$. Since for all positive integers n and z in the complex plane we have

$$\left| 2\left(\mathrm{Q}^2 n e^z\right)^{-\frac{1}{d_F}} \right| = 2\left(\mathrm{Q}^2 n\right)^{-\frac{1}{d_F}} e^{-\frac{\Re z}{d_F}} > 0,$$

we avoid the branch point at $w = 0$. For $z = x$, $x \neq \log n$, and for $F \in \mathrm{S}^{\Gamma}$ such that $F = \overline{F}$ (i.e when coefficients $a_F(n)$ are real numbers) function $m(F, x)$ is a real valued function. Therefore for the function $J_\nu(w)$ we implicitly choose the standard real branch on the positive part of the real axis.

In order to compute integral (4.5) we use the following simple lemma

Lemma 1. [7, Chapter 5.3., pp. 203-204 & formulæ (9) p. 205 & (3) p. 211] *Let*

$$G(w|a,b) := \frac{1}{2\pi i} \int_{\mathcal{M}} \frac{\Gamma(s+a)}{\Gamma(b-s)} w^s ds$$

where $\mathcal{M}$ is a smooth curve, except finite number of points, beginning and ending in $-\infty$, encircling clockwise all poles of $\Gamma(s+a)$ exactly once, while a and b are arbitrary complex numbers. Then this integral is convergent for all $|w|>1$. Moreover, for such w we have

$$G\left(w|a,b\right)=w^{-\frac{1}{2}(a-b+1)}J_{a+b-1}\left(2w^{-\frac{1}{2}}\right).$$

To prove the convergence of the integral (1.2) we use the following lemma.

Lemma 2. [11, Lemma 1] *Let $F\in S$. Then for every $\varepsilon>0$ there exists $M=M(\varepsilon)$ such that $\mu_F(n)\ll_\varepsilon n^\varepsilon$ for $(n,M)=1$.*

We also use the following Stirling formulæ [7, (3) & (6) p. 62]

$$\log\Gamma(s+\alpha)=\left(s+\alpha-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log 2\pi+\mathrm{O}\left(|s|^{-1}\right) \tag{2.1}$$

as $|s|\to\infty$, uniformly for $|\arg(s-\alpha)|\leqslant\pi-\varepsilon$ and α in $\mathbb{C}$, where ε is fixed, and

$$|\Gamma(\sigma+it)|=e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\frac{1}{2}}\left(2\pi\right)^{\frac{1}{2}}+\mathrm{O}_{a,b}\left(|t|^{-1}\right) \tag{2.2}$$

as $|t|\to\infty$ uniformly for $a\leqslant\sigma\leqslant b$.

3. Auxiliary results

First we state some technical lemmas.

Lemma 3. *Let $F\in\mathrm{S}^\Gamma$. Then for every $\varepsilon>0$ series* (1.1) *is absolutely and uniformly convergent in the half-plane $\sigma\geqslant 1+\varepsilon$.*

Proof. By Lemma 2 it follows that

$$\prod_{(p,M)>1}F_p(s)\frac{1}{F(s)}=\sum_{\substack{n=1\\(n,M)=1}}^{\infty}\frac{\mu_F(n)}{n^s}$$

converges absolutely and uniformly for $\sigma\geqslant 1+\varepsilon$ for every $\varepsilon>0$. Using axiom (5) we have

$$\begin{aligned}\sum_{m=1}^{\infty}a_F(p^m)p^{-ms}&=F_p(s)=e^{\sum_{m=1}^{\infty}b(p^m)p^{-ms}}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=1}^{\infty}b(p^m)p^{-ms}\right)^k=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m=1}^{\infty}c_k(p^m)p^{-ms},\ \sigma>\theta,\end{aligned} \tag{3.1}$$

where

$$c_k(p^m)=\sum_{\substack{l_1+\dots+l_k=m\\l_i>0}}b(p^{l_1})\dots b(p^{l_k})$$

for $k\geqslant 1$, $c_0(1)=1$ and $c_0(p^m)=0$ for $m\geqslant 1$. Thanks to the estimation $b(p^m)\ll p^{\theta m}$ with $\theta<\frac{1}{2}$ we have

$$c_k(p^m)\ll p^{\theta m}\sum_{\substack{l_1+\dots+l_k=m\\l_i>0}}1=p^{\theta m}\binom{m-1}{k-1} \tag{3.2}$$

and, consequently, the inner sum in (3.1) is uniformly and absolutely convergent for $\sigma \geqslant \frac{1}{2} > \theta$. For such σ we interchange the order of summation in k and m obtaining

$$\sum_{m=1}^{\infty} a_F(p^m)p^{-ms} = \sum_{m=1}^{\infty} p^{-ms} \sum_{k=0}^{\infty} \frac{c_k(p^m)}{k!}$$

and by uniqueness of the Dirichlet series expansion we have

$$a_F(p^m) = \sum_{k=0}^{\infty} \frac{c_k(p^m)}{k!}.$$

Since by the definition of c_k we have that $c_k(p^m) = 0$ for $k > m$ we have in fact

$$a_F(p^m) = \sum_{k=0}^{m} \frac{c_k(p^m)}{k!}.$$

Using estimation (3.2) and observing that expression $\frac{1}{k!}\binom{m-1}{k-1}$ attains the maximal value for $k \approx \sqrt{m}$, applying the following inequalities

$$\frac{1}{n!} \leqslant \left(\frac{e}{n}\right)^n, \quad \binom{n}{q} \leqslant \left(\frac{ne}{q}\right)^q$$

one obtains

$$a_F(p^m) \ll p^{m\theta} \sum_{k=1}^{m} \frac{1}{k!}\binom{m-1}{k-1} \ll p^{m\theta} \frac{m}{\sqrt{m}!}\binom{m-1}{\sqrt{m}-1} \ll$$
$$p^{m\theta}\left(\frac{e}{\sqrt{m}}\right)^{\sqrt{m}}\left(\frac{me}{\sqrt{m}}\right)^{\sqrt{m}} = p^{m\theta}e^{2\sqrt{m}}.$$

Hence the Dirichlet series

$$F_p(s) = \sum_{m=1}^{\infty} a_F(p^m)p^{-ms}$$

converges absolutely and uniformly for $\sigma > \frac{1}{2} > \theta$ and hence the lemma follows. □

In order to simplify notation, for $F \in \mathrm{S}^{\Gamma}$ we put

$$h_F(s) = \mathrm{Q}^{2s-1}\frac{\Gamma(\lambda s + \mu)}{\Gamma(\lambda(1-s) + \overline{\mu})}.$$

Then the functional equation from axiom (3) takes form

$$F(s) = \omega \frac{\overline{F}(1-s)}{h_F(s)}, \tag{3.3}$$

called the asymmetric form of the functional equation.

Lemma 4. *The integral* (1.2) *converges uniformly for z in a compact subset of* $\mathfrak{h}$.

Proof. We consider the three parts of the contour $\mathcal{C}$ in (1.2) separetly. Since

$$\frac{1}{F\left(\frac{3}{2}+it\right)} = \sum_{n=1}^{\infty} \mu_F(n)n^{-\frac{3}{2}-it} \ll \sum_{n=1}^{\infty} |\mu_F(n)|\, n^{-\frac{3}{2}} \ll 1$$

and $\left|e^{\left(\frac{3}{2}+it\right)z}\right| = e^{\frac{3}{2}x-yt} \ll_x e^{-yt}$ ($z = x+iy$ here and further on), the integral over the vertical line segment $s = 3/2+it$, $0 \leqslant t < \infty$ converges uniformly for z in a compact subset of $\mathfrak{h}$. For z being in a compact subset of $\mathbb{C}$, the integral is also uniformly convergent on the arc $\mathcal{A}$, since the function $e^{\cdot z}F(\cdot)$ is holomorphic there.

To obtain the uniform convergence of the integral over the vertical half-line $s = \varkappa_F + it$, $\infty > t \geqslant 0$, one has to proceed as follows. First by the functional equation (3.3) we obtain

$$\frac{e^{sz}}{F(s)} = \overline{\omega}\frac{h_F(s)e^{sz}}{\overline{F}(1-s)}.$$

Since $\Re(1-\varkappa_F - it) = 1 + |\varkappa_F|$ we have

$$\frac{1}{\overline{F}(1-\varkappa_F-it)} = \sum_{n=1}^{\infty}\overline{\mu_F}(n)n^{\varkappa_F-1-it} \ll \sum_{n=1}^{\infty}|\mu_F(n)|\, n^{\varkappa_F-1} \ll 1.$$

Then, by applying to h_F the Stirling formula (2.2) we obtain

$$|h_F(\varkappa_F+it)| = \left|\frac{d_F}{2}t\right|^{-\frac{d_F}{2}(1+2|\varkappa_F|)} + \mathrm{O}\left(|t|^{-1}\right) .$$

Hence for $s = \varkappa_F + it$ we have

$$\frac{1}{F(s)} \ll |t|^{-\frac{d_F}{2}(1+2|\varkappa_F|)}$$

as $t \to \infty$. Since $|e^{sz}| = e^{-|\varkappa_F|x-yt} \ll_x e^{-yt}$, consequently the integrand of (1.2) is estimated by e^{-ty}, hence the integral is uniformly convergent over the vertical line segment $s = \varkappa_F + it$, $\infty > t \geqslant 0$ for z being in a compact subset of $\mathfrak{h}$. Therefore the integral (1.2) is uniformly convergent over $\mathcal{C}$ for z in a compact subset of $\mathfrak{h}$, and the lemma follows. □

Lemma 5. *Let $F \in \mathrm{S}^{\Gamma}$ and let $\rho = \beta + i\gamma$ run through non-trivial zeros of the function F. Then for $|t| > 2$ we have the following formulæ*

$$\frac{F'}{F}(s) = \sum_{|t-\gamma|\leqslant 1}\frac{1}{s-\rho} + \mathrm{O}_F(\log t) \tag{3.4}$$

and

$$\log F(s) = \sum_{|t-\gamma|\leqslant 1}\log(s-\rho) + \mathrm{O}_F(\log t), \tag{3.5}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$, where the implied constants depend only on F and $-\pi < \Im \log(s-\rho) < \pi$.

Proof. Formula (3.4) follows immediately from [1, Lemma 2.4]. Therefore we have to deal with the second formula.

Integrating equation (3.4) on the line segment joining $2+it$ and s, and supposing that t is not equal to the ordinate of any zero, we obtain

$$\log F(s) - \log F(2+it) = \sum_{|t-\gamma|\leqslant 1}(\log(s-\rho) - \log(2+it-\rho)) + \mathrm{O}_F(\log t).$$

By axiom (5) we have

$$|\log F(2+it)| \leqslant \sum_p\sum_{m=1}^{\infty}\left|\frac{b(p^m)}{p^{m(2+it)}}\right| \leqslant \sum_p\sum_{m=1}^{\infty}\frac{|b(p^m)|}{p^{2m}}.$$

Since $|b(p^m)| \ll p^{m\theta} \leqslant p^{\frac{m}{2}}$ we have

$$\frac{|b(p^m)|}{p^{2m}} \ll \frac{1}{p^{\frac{3}{2}m}}$$

and consequently

$$|\log F(2+it)| \ll \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \ll 1.$$

Moreover, since $|t-\gamma| \leqslant 1$, terms $\log(2+it-\rho) = \log|2+it-\rho| + i\arg(2+it-\rho)$ are bounded, and by [1, Lemma 2.3] their number is estimated by $\mathrm{O}_F(\log t)$. Thus the lemma follows for $t>2$ being not an ordinate of a zero of F, and by continuity for all s in the strip $-1 \leqslant \sigma \leqslant 2$. □

Because the series (1.1) is absolutely and uniformly convergent for $\sigma \geqslant 1+\varepsilon$, for every $\varepsilon>0$, it follows that for such σ the function F from S^{Γ} does not have any zeros. Thus as a corollary from (3.5) we have

$$\log F(\sigma+it) \ll_{\varepsilon,F} \log\left(|t|+2\right), \text{ as } |t| \to \infty \tag{3.6}$$

for every $\varepsilon>0$, in the strip $1+\varepsilon \leqslant \sigma \leqslant 2$.

For brevity of notation we put

$$\upsilon_F := \frac{|\theta_F|}{d_F} + 1 \ .$$

Then we have

Lemma 6. *Let $z \in \mathfrak{h}$, $s = Re^{i\varphi}$, $R\sin\varphi \geqslant \upsilon_F$, $R|\cos\varphi| \geqslant \frac{1}{2}|\varkappa_F|$, where $\frac{\pi}{2} < \varphi < \pi$ and let $F \in \mathrm{S}^{\Gamma}$. Then for $R \geqslant R_0(x,y)$ we have*

$$\left|\frac{e^{sz}}{F(s)}\right| \leqslant e^{-y\frac{R}{2}}.$$

Proof. Using the asymmetric form of the functional equation for $F \in \mathrm{S}^{\Gamma}$ given by (3.3) we obtain

$$\log\left|\frac{e^{sz}}{F(s)}\right| = \Re(sz) - \log\left|\overline{F}(1-s)\right| + \log|h_F(s)|\,.$$

Since $\Re(1-s) = 1+R|\cos\varphi| \geqslant 1+\frac{1}{2}|\varkappa_F|$ we have by (3.6) $\log\left|\overline{F}(1-s)\right| \ll_{\varkappa_F} \log R$. Since $R\sin\varphi \geqslant \upsilon_F$, we have

$$\log|\sin(\lambda s+\mu)| = \frac{d_F}{2}\pi R\sin\varphi + \mathrm{O}(1).$$

Using the well-known formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and the Stirling formula (2.1) we estimate

$$\begin{aligned}\log|h_F(s)| = (2R\cos\varphi)\log \mathrm{Q} + (d_F R\cos\varphi)\log\left(\frac{1}{2}d_F R\right)\\ -d_F R\left(\varphi-\frac{\pi}{2}\right)\sin\varphi - d_F R\cos\varphi + \mathrm{O}(\log R).\end{aligned} \tag{3.7}$$

Consequently

$$\log\left|\frac{e^{sz}}{F(s)}\right| = d_F R\log\left(\frac{d_F}{2}R\right)\cos\varphi + Rf(\varphi,x,y) + \mathrm{O}(\log R), \tag{3.8}$$

where

$$f(\varphi,x,y) \coloneqq (x+2\log\mathrm{Q}-d_F)\cos\varphi - \left(y+d_F\left(\varphi-\frac{\pi}{2}\right)\right)\sin\varphi.$$

Since

$$f\left(\frac{\pi}{2},x,y\right) = -\left(y+d_F 2\pi\right)$$

and

$$\frac{\partial f}{\partial\varphi}(\varphi,x,y) \ll_{x,y} 1, \quad \frac{\pi}{2}<\varphi<\pi,$$

we have for $\frac{\pi}{2}<\varphi\leqslant\pi+1/\sqrt{\log R}$

$$f(\varphi,x,y) = -\left(y+d_F 2\pi\right) + \mathrm{O}_{x,y}\left(\frac{1}{\sqrt{\log R}}\right).$$

Hence, for such φ and sufficiently large R, we have

$$\log\left|\frac{e^{sz}}{F(s)}\right| \leqslant -y\frac{R}{2}.$$

For $\pi+1/\sqrt{\log R}\leqslant\varphi\leqslant\pi$ we have $|\cos\varphi|\gg 1/\sqrt{\log R}$ and hence using (3.8) we have

$$\log\left|\frac{e^{sz}}{F(s)}\right| = -d_F R\log\left(\frac{d_F}{2}R\right)|\cos\varphi| + \mathrm{O}_{x,y}(R) \leqslant -y\frac{R}{2}$$

for sufficiently large R, and the lemma follows. □

4. Proof of Theorem 1

We split the proof of the theorem into two parts. First we prove that function $m(F,\cdot)$ has a meromorphic continuation to the whole complex plane, then we show the functional equation.

By $\mathcal{A}_T$, $T>0$, we denote an arc joining points $\varkappa_F+iT$ and $-T+i\upsilon_F$ centred at the point $\varkappa_F+i\upsilon_F$. In the region bounded by the contour $\mathcal{D}_T$ consisting of the arc $\mathcal{A}_T$ and line segments $[-T+i\upsilon_F, \varkappa_F+i\upsilon_F]$, $[\varkappa_F+i\upsilon_F, \varkappa_F+iT]$, function $e^{z\cdot}/F(\cdot)$ has no singularities. Therefore

$$\frac{1}{2\pi i}\int_{\mathcal{D}_T}\frac{e^{sz}}{F(s)}ds = 0.$$

For $\Im(z)>0$ and T sufficiently large, by Lemma 6 we have

$$\frac{1}{2\pi i}\int_{\mathcal{A}_T}\frac{e^{sz}}{F(s)}ds \ll \frac{1}{2\pi}\int_{\mathcal{A}_T}e^{-\frac{yt}{2}}|ds| \ll Te^{-\frac{yT}{2}} \to 0 \text{ as } T\to\infty.$$

Therefore we can shift the line of integration $(\varkappa_F+i\infty,\varkappa_F]$ to $\mathcal{D}$ consisting of the half-line $s=\sigma+i\upsilon_F$, $-\infty<\sigma\leqslant\varkappa_F$ and the vertical line segment $[\varkappa_F+i\upsilon_F,\ \varkappa_F]$. Therefore we obtain

$$m(F,z) = \frac{1}{2\pi i}\left(\int_{\mathcal{D}}+\int_{\mathcal{A}}+\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty}\right)\frac{e^{sz}}{F(s)}ds =: m_{\mathcal{D}}(F,z)+m_{\mathcal{A}}(F,z)+m_{\mathcal{L}}(F,z) \tag{4.1}$$

where $\mathcal{A}$ is the arc part of $\mathcal{C}$ and $\mathcal{L} = [3/2, 3/2 + i\infty)$. For $s = Re^{i\varphi} = \sigma + i\upsilon_F$ with $\sigma \leqslant \varkappa_F$ we have

$$|e^{sz}| = e^{\sigma x - \upsilon_F y}.$$

Using (3.8) we have

$$\begin{aligned}\log\left|\frac{1}{F(\sigma+i\upsilon_F)}\right| &= -d_F R \log\left(\frac{d_F}{2}R\right)|\cos\varphi| \\ &- R\left((2\log \mathrm{Q} - d_F)|\cos\varphi| + \left(d_F\left(\varphi - \frac{\pi}{2}\right)\right)\sin\varphi\right) + \mathrm{O}(\log R) = \\ &- d_F R \log\left(\frac{d_F}{2}R\right)|\cos\varphi| + \mathrm{O}(R).\end{aligned}$$

Therefore

$$\frac{1}{F(\sigma+i\upsilon_F)} \ll e^{-c|\sigma|\log(|\sigma|+2)}$$

for $c > 0$ depending only on F. Hence $m_{\mathcal{D}}(F,\cdot)$ is an entire function. As we have seen in the proof of Lemma 4 function $m_{\mathcal{A}}(F,\cdot)$ is also entire. Let $\Im(z) > 0$. Since the series $1/F\left(\frac{3}{2}+it\right) = \sum_{n=1}^{\infty}\mu_F(n)n^{-\frac{3}{2}-it}$ converges absolutely and uniformly on the vertical line $\mathcal{L}$, and, for $T > 0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\mu_F(n)\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}+i\infty} e^{(z-\log n)s}ds\right| &\leqslant \\ e^{\frac{3}{2}x}\sum_{n=1}^{\infty}|\mu_F(n)|\,n^{-\frac{3}{2}}\frac{1}{2\pi}\int_T^{\infty} e^{-yt}dt &\ll_{F,x} \frac{1}{y}e^{-yT},\end{aligned}$$

therefore in $m_{\mathcal{L}}(F,\cdot)$ we can interchange the order of summation and integration obtaining

$$m_{\mathcal{L}}(F,z) = \sum_{n=1}^{\infty}\mu_F(n)\frac{1}{2\pi i}\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} e^{(z-\log n)s}ds.$$

Computing the integral we obtain

$$\frac{1}{2\pi i}\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} e^{(z-\log n)s}ds = \frac{1}{z-\log n}\; e^{(z-\log n)s}\Big|_{\frac{3}{2}}^{\frac{3}{2}+i\infty}.$$

Because $\Im(z) > 0$ we have

$$\left|e^{(z-\log n)\frac{3}{2}+it}\right| = e^{\frac{3}{2}(x-\log n)-ty} \ll_{x,n} e^{-ty} \to 0 \text{ as } t \to \infty.$$

Therefore we obtain

$$\frac{1}{2\pi i}\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} e^{(z-\log n)s}ds = \frac{e^{\frac{3}{2}z}}{n^{\frac{3}{2}}}\frac{1}{z-\log n}$$

and consequently

$$m_{\mathcal{L}}(F,z) = -\frac{e^{\frac{3}{2}}z}{2\pi i}m_0(F,z),$$

where

$$m_0(F,z) = \sum_{n=1}^{\infty}\frac{\mu_F(n)}{n^{3/2}}\frac{1}{z-\log n}. \tag{4.2}$$

Because (4.2) is uniformly convergent on any compact subset of $\mathbb{C}\setminus\{z = \log n \mid \mu_F(n) \neq 0,\ n \in \mathbb{N}\}$ we obtain a meromorphic continuation of $m_{\mathcal{L}}(F,\cdot)$ and, consequently,

$m(F,\cdot)$ to the whole complex plane. The only singularities are those generated by $m_0(F,\cdot)$ i.e. simple poles at $\log n$, $n\in\mathbb{N}$, $\mu_F(n)\neq 0$, with residues

$$\operatorname*{Res}_{z=\log n} m(F,z) = -\frac{\mu_F(n)}{2\pi i}.$$

Let us now consider $\overline{m}(\overline{F},z)$, where $\Im(z)<0$. Changing the variable $s\mapsto \overline{s}$ in (1.2), we have

$$\overline{m}(\overline{F},z) = \frac{1}{2\pi i}\int_{-\overline{\mathcal{C}}}\frac{e^{sz}}{F(s)}ds,$$

where $\overline{\mathcal{C}}$ denotes the contour conjugate to $\mathcal{C}$ and the minus sign indicates the reversed orientation. As in the first part of the proof, we replace the half-line $[\varkappa_F,\varkappa_F+i\infty)$, by the contour $-\overline{\mathcal{D}}$ consisting of the vertical line segment $[\varkappa_F,\varkappa_F-i\upsilon_F]$ and the half line $s=\sigma-i\upsilon_F$, $0\geqslant\sigma>-\infty$. Therefore we have as in (4.1) that

$$\begin{aligned}\overline{m}(\overline{F},z) = \frac{1}{2\pi i}\left(\int_{-\overline{\mathcal{D}}}+\int_{-\overline{\mathcal{A}}}+\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}}\right)\frac{e^{sz}}{F(s)}ds = \\ m_{-\overline{\mathcal{D}}}(F,z)+m_{-\overline{\mathcal{A}}}(F,z)+\frac{e^{\frac{3}{2}z}}{2\pi i}m_0(F,z)\end{aligned}\tag{4.3}$$

and the equality extends to $z\in\mathbb{C}$ by analytic continuation. From (4.1) and (4.3) we obtain for $z\in\mathbb{C}\setminus\{\log n \mid \mu_F(n)\neq 0,\ n\in\mathbb{N}\}$

$$m(F,z)+\overline{m}(\overline{F},z) = \frac{1}{2\pi i}\int_{\mathcal{E}}\frac{e^{sz}}{F(s)}ds+\frac{1}{2\pi i}\int_{\mathcal{A}_2}\frac{e^{sz}}{F(s)}ds,$$

where $\mathcal{E}$ is the path consisting of $(-\infty+i\upsilon_F,\varkappa_F+i\upsilon_F]$, $[\varkappa_F+i\upsilon_F,\varkappa_F-i\upsilon_F]$ and $[\varkappa_F-i\upsilon_F,-\infty-i\upsilon_F)$ and $\mathcal{A}_2=\mathcal{A}\cup-\overline{\mathcal{A}}$ is a closed loop. Since $\mathcal{A}$ separates the real zeros of $F\overline{F}$ from the zeros above the real line, there are no points inside the loop $\mathcal{A}_2$, apart from the interval $[0,1]$, where $e^{z\cdot}/F(\cdot)$ could have a singularity. Computing residues and noting that the orientation of $\mathcal{A}_2$ is clockwise, we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{A}_2}\frac{e^{sz}}{F(s)}ds = -\sum_{\substack{F(\beta)=0\\ 0\leqslant\beta\leqslant 1}}\operatorname*{Res}_{s=\beta}\frac{e^{sz}}{F(s)} = -R(F,z).$$

By (3.7) we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\mu_F(n)|}{n}\frac{1}{2\pi}\int_{\varkappa_F}^{-\infty}|h_F(\sigma\pm i\upsilon_F)|\,n^{-|\sigma|}e^{-|\sigma|x-y\upsilon_F}\,|d\sigma| \ll \\ \sum_{n=1}^{\infty}\frac{|\mu_F(n)|}{n}\frac{1}{2\pi}\int_{\varkappa_F}^{-\infty}e^{-c_1|\sigma|}n^{-|\sigma|}e^{-|\sigma|x-y\upsilon_F}\,|d\sigma| \ll \\ \sum_{n=1}^{\infty}\frac{|\mu_F(n)|}{n^{1+|\varkappa_F|}}\ll_{x,y} 1.\end{aligned}\tag{4.4}$$

where $c_1>0$. By the functional equation (3.3), the expansion of $1/\overline{F}(1-s)$ into the absolutely and uniformly convergent Dirichlet series, we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{e^{sz}}{F(s)}ds = \frac{\overline{\omega}}{\mathrm{Q}}\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{\Gamma(\lambda s+\mu)}{\Gamma(\lambda(1-s)+\overline{\mu})}\left(\mathrm{Q}^2e^z\right)^s\left(\sum_{n=1}^{\infty}\frac{\overline{\mu_F}(n)}{n^{1-s}}\right)ds.$$

By the estimation (4.4) in the latter formula we can interchange the order of summation and integration obtaining

$$\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{e^{sz}}{F(s)}ds=\frac{\overline{\omega}}{\mathrm{Q}}\sum_{n=1}^{\infty}\frac{\overline{\mu_F(n)}}{n}\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{\Gamma(\lambda s+\mu)}{\Gamma(\lambda(1-s)+\overline{\mu})}\left(\mathrm{Q}^2ne^z\right)^s ds.$$

Under the substitution $\lambda s\mapsto s$, we have

$$\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{\Gamma(\lambda s+\mu)}{\Gamma(\lambda(1-s)+\overline{\mu})}\left(\mathrm{Q}^2ne^z\right)^s ds = \frac{2}{d_F}\frac{1}{2\pi i}\int_{\lambda\mathcal{E}}\frac{\Gamma(s+\mu)}{\Gamma(\lambda+\overline{\mu}-s)}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac{2}{d_F}}\right)^s ds. \tag{4.5}$$

Observe now that if $\eta_F>-1$ then all poles of $\Gamma(s+\mu)$ are encircled by the contour $\lambda\mathcal{E}$ clockwise. Since $\left|\mathrm{Q}^2ne^z\right|=\mathrm{Q}^2ne^x>1$ for all n, Q and for $x>-\log\left(\mathrm{Q}^2n\right)$, by Lemma 1 we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{\Gamma(\lambda s+\mu)}{\Gamma(\lambda(1-s)+\overline{\mu})}\left(\mathrm{Q}^2ne^z\right)^s ds = -\frac{2}{d_F}\left(\mathrm{Q}^2ne^z\right)^{-i\frac{\theta_F}{d_F}}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac12-\frac{1}{d_F}}J_{\frac12 d_F+\eta_F}\left(2\left(\mathrm{Q}^2ne^z\right)^{-\frac{1}{d_F}}\right)\right)$$

and by analytic continuation for all x.

If $\eta_F=-1$ then the only pole of $\Gamma(s+\mu)$ to the right of the contour $\lambda\mathcal{E}$ is at the point $s=-i\frac{\theta_F}{2}$. Therefore we move the contour $\lambda\mathcal{E}$ to encircle point $s=-i\frac{\theta_F}{2}$ and denote this contour by $\lambda\mathcal{E}'$. By the residue theorem we have that

$$\frac{1}{2\pi i}\int_{\lambda\mathcal{E}}\frac{\Gamma(s+\mu)}{\Gamma(\lambda+\overline{\mu}-s)}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac{2}{d_F}}\right)^s ds = \frac{1}{2\pi i}\int_{\lambda\mathcal{E}'}\frac{\Gamma(s+\mu)}{\Gamma(\lambda+\overline{\mu}-s)}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac{2}{d_F}}\right)^s ds-\operatorname*{Res}_{s=-i\frac{\theta_F}{2}}\frac{\Gamma(s+\mu)}{\Gamma(\lambda+\overline{\mu}-s)}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac{2}{d_F}}\right)^s.$$

Since

$$\operatorname*{Res}_{s=-i\frac{\theta_F}{2}}\frac{\Gamma(s+\mu)}{\Gamma(\lambda+\overline{\mu}-s)}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac{2}{d_F}}\right)^s=\frac{1}{\Gamma\left(\frac12 d_F\right)}\left(\mathrm{Q}^2ne^z\right)^{-i\frac{\theta_F}{d_F}}$$

by Lemma 1 we obtain

$$\frac{1}{2\pi i}\int_{\mathcal{E}}\frac{\Gamma(\lambda s+\mu)}{\Gamma(\lambda(1-s)+\overline{\mu})}\left(\mathrm{Q}^2ne^z\right)^s ds = -\frac{2}{d_F}\left(\mathrm{Q}^2ne^z\right)^{-i\frac{\theta_F}{d_F}}\left(\left(\mathrm{Q}^2ne^z\right)^{\frac12-\frac{1}{d_F}}J_{\frac12 d_F-1}\left(2\left(\mathrm{Q}^2ne^z\right)^{-\frac{1}{d_F}}\right)-\frac{1}{\Gamma\left(\frac12 d_F\right)}\right)$$

for $x>-\log\left(\mathrm{Q}^2n\right)$, and by analytic continuation for all x, hence the theorem follows.

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