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# Stability of Lagrangian Points in Restricted Three Body Problem

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**Abstract**
Lagrangian Points are well known from their usage in the process of exploring our universe. Main reason of their application is their character of stability. In this paper we will discuss an issue of stability of Lagrangian points in Restricted Three Body Problem. We shall present the way that does not require explicit location of $L_1, L_2$ and $L_3$ points, which in fact cannot be derived, and often creates confusion on many papers investigating these points.

*Keywords*: Stability, Lagrangian Points, Three Body Problem

## 1. Introduction - Equations of Celestial Mechanics

N-body problem is a problem of classical mechanics based on determining trajectories of motions of N bodies with given masses $m_i > 0$, initial velocities $v_i > 0$ and starting locations (i.e. initial conditions). If a vector of position of $i$-th body is denoted by $q_i \in \mathbb{R}^3$, then the second law of dynamics tells us, that a force acting on a given body is equal to $m_i\ddot{q}_i$. On the other hand this force is equal to a net gravitational force, which is then equal to

$$m_i\ddot{q}_i = \sum_{j=1, j\neq i}^{N} \frac{Gm_im_j(q_j - q_i)}{\|q_j - q_i\|^3} = \frac{\partial U}{\partial q_i}, \qquad i = 1, \ldots, N \tag{1}$$

where

$$U = \sum_{1\leq i<j\leq N} \frac{Gm_im_j}{\|q_j - q_i\|}$$

is called negative potential. Let $q = (q_1, \ldots, q_N) \in \mathbb{R}^{3N}$ be a column vector consisting of all coordinates of all position vectors, and let

$$M = \operatorname{diag}(m_1, m_1, m_1, \ldots, m_N, m_N, m_N)$$

be a diagonal matrix of order $3N$. Then equation (1) can be rewritten as follows

$$M\ddot{q} - \frac{\partial U}{\partial q} = 0.$$

We define $p = (p_1, \dots, p_N) \in \mathbb{R}^{3N}$ as $p = M\dot{q}$. Then $p_i = m_i\dot{q}_i$ is a momentum of $i$-th component of our system. Equation (1) and can be rearranged to a Hamiltonian equation of the form:

$$\begin{aligned} \dot{q}_i &= \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= \sum_{j=1, j\neq i}^{N} \frac{G m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} = -\frac{\partial H}{\partial q_i}, \end{aligned} \qquad i = 1, \dots, N. \tag{2}$$

where $H = T - U$ is the Hamiltonian, and $T$ is a kinetic energy of a system:

$$T = \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} = \sum_{i=1}^{N} m_i \frac{\|\dot{q}_i\|^2}{2}.$$

We see that $N$-body problem is a Hamiltonian system of $6N$ differential equations.

We will further consider case $N = 3$. Our goal is to determine stability of equilibria in Restricted Three Body Problem (R3CP). In [5] and [4] we can find a lot more information about general problem of Three Bodies.

## 2. Preliminaries - rotating frame

Analysis of Lagrangian points we will present in further chapters uses a notion of rotating frame[1]. Let $\mathcal{A} = \{O_1, \hat{x}_1, \hat{y}_1, \hat{z}_1\}$ and $\mathcal{B} = \{O_2, \hat{x}_2, \hat{y}_2, \hat{z}_2\}$ be two coordinates systems of a phase of space $\mathbb{R}^3$, where $O_i$ denotes origin, and $\hat{x}_i, \hat{y}_i, \hat{z}_i$ are unit vectors of axes. In both frames we consider euclidean inner product. We will assume, that the frame $\mathcal{A}$ is fixed, where the frame $\mathcal{B}$ will be changing in time. We will use the following definition:

**Definition.** Transformation $D : \mathcal{B} \to \mathcal{A}$ is called a rotation, if $D(O_2) = O_1$ and $D$ preserves distance.

Let us recall theorem classifying isometries. Note, that each translation is a composition of 2 symmetries with respect to parallel planes, where each rotation is a composition of at most 3 symmetries with respect to planes intersecting at some point.

**Theorem.** If $f : X \to X$ is an isometry of $n$-dimensional euclidean space, then $f$ can be represented as a composition of at most $n + 1$ symmetries with respect to hyperplanes of space $X$.

Based on that theorem applied to $\mathbb{R}^3$ we see, that a rotation defined above is a composition of classical rotation (a rotation around a line coming through origin) and a translation. System of coordinates can now be identified with an isometry of the space of coordinates $\mathbb{R}^3$ into the phase of space where the motion of our bodies is considered. Each vector of the phase of space can be written in a versor base using coordinates from the coordinates space. Choose of a base determines an isometry. Assuming, that for each $t \in \mathbb{R}$ we picked system of coordinates $\mathcal{B}_t$, and so we defined an isometry $D_{\mathcal{B}_t} : \mathbb{R}^3 \to \mathcal{B}_t$, we can now consider a situation, where our system changes in time.

**Definition.** Moving coordinates system is a smooth function $D : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, such that for each $t \in \mathbb{R}$ a function $D(t, \cdot) = D_{\mathcal{B}_t}(\cdot)$ is a rotation.

[1] This idea was presented in books [1], [8] and [6] and adapted to author's Master Thesis [3]

We will recall notion of angular velocity and its basic property.

**Definition.** Angular velocity is a vector describing rotational movement, given by

$$\Omega = \frac{r \times v}{\|r\|^2},$$

where $r$ is a vector joining origin with a rotating body and $v$ is its linear velocity. Let $\omega := \|\Omega\|$. For a body rotating around fixed axis a vector $\Omega$ lies on the same axis and his sense is determined by a right-hand rule. We also have

$$\|v\| = \omega\|r\| \sin\theta, \tag{3}$$

where $\theta$ is an angle between a position vector $r$ and a velocity vector $v$. We will further consider motion on a circle which means that both vectors are perpendicular. It simplifies latter formula to the form

$$\|v\| = \omega\|r\|.$$

**Theorem.** (rotating frame of reference) Let $D : \mathcal{B}_t \to \mathcal{A}$ be a rotation of a coordinates system $\mathcal{B}_t$ with respect to a given fixed coordinates system $\mathcal{A}$. Denote $r = r(t)$ as a vector of a point in moving coordinates system and $q = q(t)$ as a vector corresponding to a fixed system. Let $\Omega$ be an angular velocity of a rotation of a system. Assume that in a frame $\mathcal{A}$ a motion is given by Netwon's law $F = m\ddot{q}$. Then in rotating frame a motion is as for each point in position $r$ and a mass $m$ there were three additional fictitious forces:

Euler force $-m\dot{\Omega} \times r$,

Coriolis force $-2m\Omega \times \dot{r}$,

centripelar force $-m\Omega \times (\Omega \times r)$.

Hence

$$m\ddot{r} = F - m\dot{\Omega} \times r - 2m\Omega \times \dot{r} - m(\Omega \times (\Omega \times r)).$$

Proof of this theorem can be found here [1].

## 3. Lagrangian Points

Classical problem of Three Body Problem has no stationary solutions. As a consequence we will now consider a situation, where one of the bodies has mass so small comparing to remaining bodies that we can omit it. This situation does not seem to be proper from the mathematical point of view, however it simplifies our model and helps finding solutions. As an example of such system we can consider a set Sun-Earth-Space Probe. With such additional assumption this problem is called Restricted Three Body Problem (R3BP).

From now on, we will assume that the distance between two massive bodies is constant. It turns out that there are five points such that our space probe located in any of those points can stay in constant position with respect to two remaining bodies. What is more, stability of that probe will depends on the choice of a point and a ratio of masses of two big bodies.

We will also assume that two bodies have masses $m_1$ and $m_2$, where without loose of generality we can assume that $m_1 \geq m_2$. Mass of the light body will be denoted by $m$. Distance between bodies $m_1$ and $m_2$ will be denoted by $R$. We assume that center of mass of that system is fixed to the origin, $r_1$ is a vector of position of $m_1$, and $r_2$ is a vector of position of $m_2$ (so $|r_1 - r_2| = R$). Coordinates of third body will be defined by the vector $r$. Bodies $m_1$ and $m_2$ will always move one the plane $OXY$, and by the Kepler's Law their trajectories are circles. We will use the following definition:

**Definition.** If a body of mass $m$ is placed in a position determined by the vector $r$ and it will remain in stable position with respect to both massive bodies, then this point is called a Lagrangian point or a libration point.

In considered situation we now introduce moving frame: origin of our system will be placed in the center of mass, a segment joining bodies $m_1$ and $m_2$ will lay on axis $OX$, axis $OY$ will lay on the plane of motion of the two massive bodies, it will be perpendicular to the axis $OX$ and will intersect center of mass. Axis $OZ$ will have direction perpendicular to the plane of motion. Whole frame will be oriented using right-hand rule. If we position the body $m$ in the Lagrangian point and consider equations of motion for all three bodies, which is set in moving frame, such configuration will be a stationary solution of our equations. Since in moving frame all massive bodies have fixed placement, it is sufficient to investigate equations of motions for the body of mass $m$.

## 4. Location

We search for the solutions of equations

$$m\ddot{r} = -\frac{Gm_1 m}{\|r - r_1\|^3}(r - r_1) - \frac{Gm_2 m}{\|r - r_2\|^3}(r - r_2) = F. \tag{4}$$

In order to solve them, we will use rotating frame of reference. If $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors of corresponding axes $OX, OY, OZ$ of rotating frame of reference, and $\Omega$ is a angular velocity along axis $OZ$, then

$$\begin{aligned} \Omega &= \omega\hat{k}, \\ r &= x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \\ r_1 &= -\alpha R\hat{\imath}, \\ r_2 &= \beta R\hat{\imath}, \end{aligned}$$

where

$$\alpha = \frac{m_2}{m_1 + m_2}, \qquad \beta = \frac{m_1}{m_1 + m_2}.$$

Using this notation we have the following lemma:

**Lemma.**

$$\omega^2 = \frac{G(m_1 + m_2)}{R^3}.$$

Using theorem on rotating frame of reference we know, that equation (4) in rotating frame of reference changes to the following form:

$$m\ddot{r} = F - 2m\Omega \times r - m\Omega \times (\Omega \times r) - m\dot{\Omega} \times r =: F_\Omega,$$

but $\Omega = const$ hence the equation above simplifies to the form:

$$\ddot{r} + 2\Omega \times r = -\frac{Gm_1}{\|r-r_1\|^3}(r-r_1) - \frac{Gm_2}{\|r-r_2\|^3}(r-r_2) - \Omega \times (\Omega \times r). \tag{5}$$

We also have $\Omega = (0, 0, \omega)^T$, hence

$$\Omega \times \dot{r} = \begin{pmatrix} -\omega\dot{y} \\ \omega\dot{x} \\ 0 \end{pmatrix}, \qquad \Omega \times (\Omega \times r) = \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix}.$$

Let us derive (5) calculating each coordinate by substituting values from above:

$$\begin{cases} \ddot{x} - 2\omega\dot{y} &= -\frac{Gm_1}{\|r-r_1\|^3}(x+\alpha R) - \frac{Gm_2}{\|r-r_2\|^3}(x-\beta R) + \omega^2 x, \\ \ddot{y} + 2\omega\dot{x} &= -\frac{Gm_1}{\|r-r_1\|^3}y - \frac{Gm_2}{\|r-r_2\|^3}y + \omega^2 y, \\ \ddot{z} &= -\frac{Gm_1}{\|r-r_1\|^3}z - \frac{Gm_2}{\|r-r_2\|^3}z. \end{cases} \tag{6}$$

Taking

$$U := -\frac{Gm_1}{\|r-r_1\|} - \frac{Gm_2}{\|r-r_2\|} - \frac{\omega^2}{2}(x^2+y^2)$$

equation (6) can be briefly written as:

$$\begin{cases} \ddot{x} - 2\omega\dot{y} &= -\frac{\partial U}{\partial x}, \\ \ddot{y} + 2\omega\dot{x} &= -\frac{\partial U}{\partial y}, \\ \ddot{z} &= -\frac{\partial U}{\partial z}. \end{cases} \tag{7}$$

Let us notice that if we multiply equations (7) by $\dot{x}, \dot{y}, \dot{z}$ respectively, a system we obtain from this

$$\begin{cases} \ddot{x}\dot{x} - 2\omega\dot{y}\dot{x} &= -\dot{x}\frac{\partial U}{\partial x}, \\ \ddot{y}\dot{y} + 2\omega\dot{x}\dot{y} &= -\dot{y}\frac{\partial U}{\partial y}, \\ \ddot{z}\dot{z} &= -\dot{z}\frac{\partial U}{\partial z}, \end{cases}$$

after adding up all equations gives

$$\frac{d}{dt}\left(\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + U\right) = 0.$$

Expression

$$C := \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + U = \frac{1}{2}v^2 + U$$

is called Jacobi integral of equation (6). Because $v \geq 0$, then

$$C \leq -2U$$

limits positions of bodies, which locations and momenta must satisfy this inequality.

Our attempts on solving (5) were translated to the cartesian coordinates (6) and to investigating a function of potential $U$. For the start we will notice:

**Lemma.** Libration points lay on the plane $OXY$ of motion of bodies $m_1$ and $m_2$.

*Proof.* Using third equation from (7) a stationary point must satisfy the following

$$-\frac{\partial U}{\partial z} = 0.$$

On the other side

$$\frac{\partial U}{\partial z} = \left( \frac{Gm_1}{\|r - r_1\|^3} + \frac{Gm_2}{\|r - r_2\|^3} \right) z = 0,$$

hence it must be $z = 0$. □

Using that lemma we can skip third equation from (7) and focus on remaining two. Because we are looking for stationary solutions we enforce condition $\dot{r} = 0$ (body of mass $m$ remains steady with respect to remaining two). We will use this additional information in system (6).

$$\begin{cases} 0 &= \omega^2 x - \frac{Gm_1(x+\alpha R)}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{Gm_2(x-\beta R)}{((x-\beta R)^2+y^2)^{3/2}}, \\ 0 &= \omega^2 y - \frac{Gm_1 y}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{Gm_2 y}{((x-\beta R)^2+y^2)^{3/2}}. \end{cases}$$

We multiply both side of last equation by $1 = \frac{\omega^2}{G(m_1+m_2)/R^3}$ and we obtain

$$\begin{cases} 0 &= \omega^2 x - \frac{\beta(x+\alpha R)R^3\omega^2}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha(x-\beta R)R^3\omega^2}{((x-\beta R)^2+y^2)^{3/2}}, \\ 0 &= \omega^2 y - \frac{\beta y\omega^2}{((x+\alpha R)^2+y^2)^{3/2}} - \frac{\alpha y\omega^2}{((x-\beta R)^2+y^2)^{3/2}}. \end{cases} \tag{8}$$

Let us notice that second equation of (8) is satisfied when $y = 0$. Hence we will start from searching solutions laying on axis $OX$. Using this assumption it is sufficient to search for solutions satisfying the following identity

$$0 = \omega^2 x - \frac{\beta(x+\alpha R)R^3\omega^2}{|x+\alpha R|^3} - \frac{\alpha(x-\beta R)R^3\omega^2}{|x-\beta R|^3}. \tag{9}$$

Let us substitute $x = R(u+\beta)$ (such substitution places the body $m_1$ at a point $-1$, the body $m_2$ at a point 0, and the center of mass at a point $-\beta$, and unit distance after substitution is distance $R$ before):

$$\begin{aligned} 0 &= Ru + R\beta - \frac{\beta(Ru+R\beta+\alpha R)R^3\omega^2}{|Ru+R\beta+\alpha R|^3} - \frac{\alpha(Ru+R\beta-\beta R)R^3\omega^2}{|Ru+R\beta-\beta R|^3} \\ &= Ru + R\beta - \frac{\beta(u+1)R^4}{|u+1|^3} - \frac{\alpha u R^4}{|u|^3} \\ &= Ru + R\beta - \frac{\beta R}{\operatorname{sgn}(u+1)(u+1)^2} - \frac{\alpha R}{\operatorname{sgn}(u)u^2} \\ &= Ru + R\beta - \frac{\beta R}{s_1(u+1)^2} - \frac{\alpha R}{s_o u^2}. \end{aligned} \tag{10}$$

where we denoted $s_0 := \operatorname{sgn}(u+1), s_1 := \operatorname{sgn}(u)$. Dividing (10) by $R$ we obtain

$$u + \beta - \frac{\beta}{s_1(u+1)^2} - \frac{\alpha}{s_0 u^2} = 0.$$

Let us define $f : \mathbb{R} \setminus \{-1, 0\} \to \mathbb{R}$ as follows:

$$f(u) := u + \beta - \frac{\beta}{s_1(u+1)^2} - \frac{\alpha}{s_0 u^2}.$$

Our problem is simplified to finding roots of $f$.

**Theorem.** Exists exactly three Langrangian points on the line connecting bodies $m_1$ and $m_2$.

Proof can be found here: [3]. However, this theorem and its proof gives only qualitative result. Because we do not know explicit form of equilibria we often use approximations, assuming that $\alpha \ll 1$. This assumption makes sense for most investigating systems, for instance $\alpha \approx 3 \cdot 10^{-6}$ if we take Sun-Earth system. For given $\alpha$ we can approximate location of Lagrangian points as follows:

$$L_1 = \left(R\left(1 - (\alpha/3)^{1/3}\right), 0\right), \tag{11}$$

$$L_2 = \left(R\left(1 + (\alpha/3)^{1/3}\right), 0\right), \tag{12}$$

$$L_3 = \left(-R\left(1 + \frac{5}{12}\alpha\right), 0\right). \tag{13}$$

where given coordinates are coordinates on the plane $OXY$. Point $L_1$ described by (11) lays between $m_1$ and $m_2$ bodies, close to $m_2$. Point $L_2$ given by (12) lays opposite to $m_2$ with respect to the point $L_1$. Last collinear point can be found on the opposite side of $m_1$, close to $m_2$ (13). Better evaluation can be found on site [9].

**Example.** Let us consider Earth-Moon system. In this case $\alpha = \frac{1}{82.45}$ and $R = 384.000$ km. Because $\alpha$ is big, we can expect that formulas (11) – (13) will not give satisfactory result. Using (11) – (13) we obtain the following:

$L_1 = (0.840695R, 0),$
$L_2 = (1.159305R, 0),$
$L_3 = (-1.005054R, 0).$

If we take a center of mass into account, we obtain that:

$L_1$ is located $0.852881R$ from Earth,
$L_2$ is located $1.171491R$ from Earth,
$L_3$ is located $0.992868R$ from Earth.

We will now compare these results with ones obtained from solving equation using Maple (to be more precise, roots of $f$ were searched). Approximated solutions are as follows:

$L_1$ is located $0.8491521R$ from Earth,
$L_2$ is located $1.1677259R$ from Earth,
$L_3$ is located $0.992929R$ from Earth.

Results presented in [7] are: $0.849R, 1.168R, -0.993R$ respectively. This example shows an advantage of solving equation. We obtained values consistent with expected and published. However applying formulas (11) $-$ (13) gives us result different from expected. This difference becomes larger when $\alpha$ is closer to $1/2$.

Let us back to (8). This time we will search for non-trivial solutions for second equation.

**Theorem.** Exists two equilibria, that are not collinear with two massive bodies. These points lay on vertices of equilateral triangles. One side of each of two possible triangle is a segment between two massive bodies. They are given by the following coordinates:

$$L_4 = \left(\frac{1}{2}R\left(\frac{m_1-m_2}{m_1+m_2}\right), \frac{\sqrt{3}}{2}R\right), \tag{14}$$

$$L_5 = \left(\frac{1}{2}R\left(\frac{m_1-m_2}{m_1+m_2}\right), -\frac{\sqrt{3}}{2}R\right). \tag{15}$$

Both triangles lay on the plane of motion.

We skip proof, sketch can be found here [2], and details here [3].

## 5. Stability

From a practice point of view knowledge about location is not enough. We often want to know a type of stability for given stationary point. Let us consider any libration point. We will say that it is stable if and light body placed in that point has stable position. In this chapter we will use the following criteria:

**Lemma.** Let $f : U \to \mathbb{R}^n$ be of class $\mathcal{C}^1$, where $U$ is an open and non-empty subset of $\mathbb{R}^n$. Let $p$ be a stationary point of local dynamical system $\varphi$ generated by the equation $x' = f(x)$. Then existance of positive eigenvalue of $Df(p)$ implies instability of stationary point $p$.

Let us back to (7). Notice, that stability in the direction of $OZ$ axis perpendicular to the plane of motion is determined by third equation only, where stability on the plane $OXY$ is determined by remaining equations. This is why we will seperate our problem into two stages. We will firstly check stability of points in the direction perpendicular to the plane $OXY$. Secondly we will check stability on the plane $OXY$. Obtaining positive result in both cases will give us complete stability of points in any direction.

**Lemma.** All Lagrangian Points are stable in the direction perpendicular to the plane $z = 0$.

*Proof.*

$$-\frac{\partial^2 U}{\partial z^2} = -\frac{3Gm_1}{\|r-r_1\|^5}z - \frac{Gm_1}{\|r-r_1\|^3} - \frac{3Gm_2}{\|r-r_2\|^5}z - \frac{Gm_2}{\|r-r_2\|^3}$$

and so

$$-\frac{\partial^2 U}{\partial z^2}(z)\bigg|_{z=0} = -\frac{Gm_1}{\|r-r_1\|^3} - \frac{Gm_2}{\|r-_2\|^3} < 0.$$

□

In the following step we will focus on first and second equations of a system (7):

$$\begin{cases} \ddot{x} = & -\frac{\partial U}{\partial x} + 2\omega\dot{y}, \\ \ddot{y} = & -\frac{\partial U}{\partial y} - 2\omega\dot{x}. \end{cases} \tag{16}$$

Let us introduce the following notation:

$$\mu_1 = Gm_1, \qquad \mu_2 = Gm_2,$$
$$\rho_1 = \|r - r_1\|, \quad \rho_2 = \|r - r_2\|.$$

We have $\mu_1 + \mu_2 = \omega^2 R^3$. Let us now substitute

$$\begin{cases} v_x = \dot{x}, \\ v_y = \dot{y}, \end{cases}$$

to change (16) into a system of ordinary differential equations:

$$\begin{cases} \dot{x} & = v_x, \\ \dot{y} & = v_y, \\ \dot{v_x} & = -\frac{\partial U}{\partial x} + 2\omega\dot{y}, \\ \dot{v_y} & = -\frac{\partial U}{\partial y} - 2\omega\dot{x}. \end{cases} \tag{17}$$

In order to determine stability we will use linearization methods. Let $\mathcal{L}$ be a Jacobi matrix of (17). Then

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\partial^2 U}{\partial x^2} & -\frac{\partial^2 U}{\partial y \partial x} & 0 & 2\omega \\ -\frac{\partial^2 U}{\partial x \partial y} & -\frac{\partial^2 U}{\partial y^2} & -2\omega & 0 \end{pmatrix},$$

Let us calculate explicitly coefficients of a matrix $\mathcal{L}$. Using our notation

$$-\frac{\partial U}{\partial x} = -\frac{\mu_1(x+\alpha R)}{\rho_1^3} - \frac{\mu_2(x-\beta R)}{\rho_2^3} + \omega^2 x,$$
$$-\frac{\partial U}{\partial y} = -\frac{\mu_1 y}{\rho_1^3} - \frac{\mu_2 y}{\rho_2^3} + \omega^2 y.$$

Then

$$-\frac{\partial^2 U}{\partial x^2} = \frac{3\mu_1(x+\alpha R)^2}{\rho_1^5} + \frac{3\mu_2(x-\beta R)^2}{\rho_2^5} + \omega^2 - \frac{\mu_1}{\rho_1^3} - \frac{\mu_2}{\rho_2^3},$$
$$-\frac{\partial^2 U}{\partial y^2} = \frac{3\mu_1 y^2}{\rho_1^5} + \frac{3\mu_2 y^2}{\rho_2^5} + \omega^2 - \frac{\mu_1}{\rho_1^3} - \frac{\mu_2}{\rho_2^3},$$
$$-\frac{\partial^2 U}{\partial x \partial y} = -\frac{\partial^2 U}{\partial y \partial x} = \frac{3\mu_1(x+\alpha R)y}{\rho_1^5} + \frac{3\mu_2(x-\beta R)y}{\rho_2^5}.$$

Let us denote:

$$\begin{aligned} A &= \frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3}, \\ B &= 3y^2\left(\frac{\mu_1}{\rho_1^5} + \frac{\mu_2}{\rho_2^5}\right), \\ C &= 3y\left(\frac{\mu_1(x+\alpha R)}{\rho_1^5} + \frac{\mu_2(x-\beta R)}{\rho_2^5}\right), \\ D &= 3\left(\frac{\mu_1(x+\alpha R)^2}{\rho_1^5} + \frac{\mu_2(x-\beta R)^2}{\rho_2^5}\right). \end{aligned}$$

Under this notation we will receive:

$$\begin{aligned} -\frac{\partial^2 U}{\partial x^2} &= \omega^2 - A + D, \\ -\frac{\partial^2 U}{\partial y^2} &= \omega^2 - A + B, \\ -\frac{\partial^2 U}{\partial x \partial y} = -\frac{\partial^2 U}{\partial y \partial x} &= C. \end{aligned}$$

A matrix $\mathcal{L}$ has the following simple form:

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 - A + D & C & 0 & 2\omega \\ C & \omega^2 - A + B & -2\omega & 0 \end{pmatrix}. \tag{18}$$

We will now derive characteristic polynomial of a matrix (18)

$$\begin{aligned} p_{\mathcal{L}}(\lambda) &= det(\mathcal{L} - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ \omega^2 - A + D & C & -\lambda & 2\omega \\ C & \omega^2 - A + B & -2\omega & -\lambda \end{vmatrix} \\ &= \lambda(-\lambda^3 - 2\omega C + \lambda(\omega^2 - A + B) - 4\omega^2\lambda) \\ &\quad +((\omega^2 - A + D)(\omega^2 - A + B) - 2\lambda\omega C - C^2 - \lambda^2(\omega^2 - A + D)) \\ &= \lambda^4 - \lambda^2[(\omega^2 - A + B) + (\omega^2 - A + D) - 4\omega^2)] + [(\omega^2 - A + D)(\omega^2 - A + B) - C^2]. \end{aligned}$$

We will now consider $L_1$, $L_2$ and $L_3$ points.

**Theorem.** Lagrangian points $L_1, L_2$ and $L_3$ are unstable on the plane $OXY$.

*Proof.* Based on $(11)-(13)$ we see, that because $y = 0$, then $B = C = 0$. Moreover $\rho_1^2 = (x+\alpha R)^2$, $\rho_2^2 = (x - \beta R)^2$. Let us further notice, that

$$D = 3\left(\frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3}\right) = 3A.$$

Characteristic polynomial simplifies to the form

$$p_{L_i}(\lambda) = \lambda^4 + \lambda^2(2\omega^2 - A) + (\omega^2 - A)(\omega^2 + 2A), \qquad i = 1, 2, 3, \tag{19}$$

where $p_{L_i}$ denotes characteristic polynomial for $i$-th Lagrangian Point. In order to show instability, it is sufficient to show that characteristic polynomial (19) has real positive eigenvalue. For this, it is sufficient to show, that equation

$$\Lambda^2 + \Lambda(2\omega^2 - A) + (\omega^2 - A)(\omega^2 + 2A) = 0, \tag{20}$$

where $\Lambda := \lambda^2$, has two real solutions, and either of them is positive. Let us determine discriminant of equation (20):

$$\Delta = 4\omega^4 - 4\omega^2 A + A^2 - 4(\omega^4 + \omega^2 A - 2A^2) = A(9A - 8\omega^2).$$

Hence $\Delta > 0$ if and only if, when

$$A > \frac{8}{9}\omega^2 \text{ lub } \ A < 0.$$

Based on formulas (11) and (12) we have estimations:

$$|l_1 - \beta R| < R, \tag{21}$$

$$|l_2 - \beta R| < R, \tag{22}$$

where $l_i$ denotes first coordinate of $i$-th Lagrangian Point. We know, that $l_1$ is a solution of equation (9), so:

$$0 = -\frac{\mu_1(l_1 + \alpha R)}{|l_1 + \alpha R|^3} - \frac{\mu_2(l_1 - \beta R)}{|l_1 - \beta R|^3} + \omega^2 l_1. \tag{23}$$

In addition,

$$A = \frac{\mu_1}{|l_1 + \alpha R|^3} + \frac{\mu_2}{|l_1 - \beta R|^3}.$$

Using that and relation $\alpha + \beta = 1$ in equation (23) we obtain:

$$\begin{aligned} 0 &= l_1\omega^2 - \frac{\mu_1 l_1}{|l_1 + \alpha R|^3} - \frac{\mu_2 l_1}{|l_1 - \beta R|^3} - \frac{\mu_1 \alpha R}{|l_1 + \alpha R|^3} - \frac{\mu_2 \alpha R}{|l_1 - \beta R|^3} + \frac{\mu_2 R}{|l_1 - \beta R|^3} \\ &= l_1\omega^2 - l_1 A - \alpha R A + \frac{\mu_2 R}{|l_1 - \beta R|^3}. \end{aligned} \tag{24}$$

Let us note, that

$$\alpha = \frac{m_2}{m_1 + m_2} = \frac{\mu_2}{\mu_1 + \mu_2} = \frac{\mu_2}{\omega^2 R^3}$$

and based on (21) we see, that:

$$\frac{1}{|l_1 - \beta R|^3} > \frac{1}{R^3}.$$

We determine from equation (24) value of $A$ and use last two relations to receive the following estimation:

$$\begin{aligned} A &= \frac{1}{l_1 + \alpha R}\left(l_1\omega^2 + \frac{\mu_2 R}{|l_1 - \beta R|^3}\right) \\ &> \frac{1}{l_1 + \frac{\mu_2}{\omega^2 R^3} R}\left(l_1\omega^2 + \frac{\mu_2 R}{R^3}\right) \\ &= \frac{1}{\frac{1}{\omega^2}\left(l_1\omega^2 + \frac{\mu_2}{R^2}\right)}\left(l_1\omega^2 + \frac{\mu_2}{R^2}\right) \\ &= \omega^2. \end{aligned}$$

That ends proof for $L_1$. Because (22) holds, the same idea gives us instability of $L_2$.

In order to determine stability of $L_3$ we first note, that from formula (13) we can have the following inequality:

$$|l_3 + \alpha R| < R. \tag{25}$$

We now do analogous calculations as for $L_1$, that give us

$$0 = l_3\omega^2 - l_3 A + \beta R A - \frac{\mu_1 R}{|l_3 + \alpha R|^3}.$$

Because

$$\beta = \frac{\mu_1}{\omega^2 R^3}$$

and from inequality (25) we have

$$-\frac{R}{|l_3 + \alpha R|^3} < -\frac{R}{R^3},$$

determining value of $A$ leads us to the following estimations:

$$\begin{aligned} A &= \frac{1}{\beta R - l_3}\left(l_3\omega^2 - \frac{\mu_1 R}{|l_3 + \alpha R|^3}\right) \\ &< \frac{1}{\frac{\mu_1}{\omega^2 R^3} R - l_3}\left(l_3\omega^2 - \frac{\mu_1 R}{R^3}\right) \\ &= \frac{1}{\frac{1}{\omega^2}\left(\frac{\mu_1}{R^2} - l_3\omega^2\right)}\left(l_3\omega^2 - \frac{\mu_1}{R^2}\right) \\ &= -\omega^2 < 0 \end{aligned}$$

that completes proof for $L_3$. □

**Theorem.** If

$$m_1 \geq 25 m_2 \left(\frac{1 + \sqrt{1 - \frac{4}{625}}}{2}\right),$$

then $L_4$ and $L_5$ are stable on the plane $OXY$.

*Proof.* First we notice, than from (14) and (15) we obtain

$$\rho_1 = \rho_2 = R$$

and

$$x + \alpha R = \frac{1}{2}R, \quad x - \beta R = -\frac{1}{2}R.$$

For $L_4$ coefficients $A, \ldots, D$ are given by the following formulas:

$$\begin{aligned} A &= \omega^2, \\ B &= \frac{9}{4}\omega^2, \\ C &= \frac{3\sqrt{3}}{4}\omega^2\tau, \qquad \text{where } \tau := \frac{m_1 - m_2}{m_1 + m_2}, \\ D &= \frac{3}{4}\omega^2. \end{aligned}$$

Characteristic polynomial for $L_4$ has the form:

$$p_{L_4}(\lambda) = \lambda^4 + \lambda^2\omega^2 - \frac{27}{16}\omega^4(1 - \tau^2). \tag{26}$$

For $L_5$ we obtain the same results, hence we now focus on $L_4$ only. Substituting $\Gamma := \lambda^2$, and calculating discriminant

$$\Delta = \omega^4 - \frac{27}{4}\omega^4(1 - \tau^2)$$

we have the following solutions:

$$\begin{aligned} \Gamma_1 &= \frac{-\omega^2 + \omega^2\sqrt{1 - \frac{27}{4}(1 - \tau^2)}}{2}, \\ \Gamma_2 &= \frac{-\omega^2 - \omega^2\sqrt{1 - \frac{27}{4}(1 - \tau^2)}}{2}. \end{aligned}$$

Hence roots of polynomial (26) are:

$$\begin{aligned} \lambda_1 &= i\omega\sqrt{\frac{1 - \sqrt{1 - \frac{27}{4}(1 - \tau^2)}}{2}} \\ &= i\frac{\omega}{2}\sqrt{2 - \sqrt{27\tau^2 - 23}}, \\ \lambda_2 &= -i\frac{\omega}{2}\sqrt{2 - \sqrt{27\tau^2 - 23}}, \\ \lambda_3 &= i\frac{\omega}{2}\sqrt{2 + \sqrt{27\tau^2 - 23}}, \\ \lambda_4 &= -i\frac{\omega}{2}\sqrt{2 + \sqrt{27\tau^2 - 23}}. \end{aligned}$$

Based on [9] for stability of $L_4$ it is sufficient, that all the eigenvalues are complex numbers with no real part. It is so, if

$$2 - \sqrt{27\tau^2 - 23} \geq 0, \tag{27}$$

$$27\tau^2 - 23 \geq 0. \tag{28}$$

Because $\tau \leq 1$, then inequality (27) always holds. Solving inequality (28) we have:

$$\begin{aligned}
\frac{m_1 - m_2}{m_1 + m_2} &\geq \sqrt{\frac{23}{27}} \\
m_1 &\geq \frac{m_2\left(1 + \sqrt{\frac{23}{27}}\right)}{1 - \sqrt{\frac{23}{27}}} \\
m_1 &\geq m_2\left(\frac{27 + 23 + 2\sqrt{27 \cdot 23}}{4}\right) \\
m_1 &\geq 25m_2\left(\frac{1 + \sqrt{1 - \frac{4}{625}}}{2}\right).
\end{aligned}$$

Hence, if

$$m_1 \geq 25m_2\left(\frac{1 + \sqrt{1 - \frac{4}{625}}}{2}\right) \approx 24.9599m_2$$

then $L_4$ is stable, and si $L_5$ is. □

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