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On Centers of Curves

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Abstract
There are several ways of defining a center of a curve. In this paper we will recall three basic such definitions, based on calculus, and also propose a different way which is interesting from the dynamical point of view.
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1. Introduction - how to measure a middle of a curve? Three different approaches

Let us try to answer a question given in the name of this section. What we can measure is for instance a length of a curve, a mass, or a time required to travel from the beginning to the end of it. This section covers three different approaches to this topic - all of them are well known in calculus (see [2]).

In this paper by bounded curve we understand a curve which is bounded in topological sense, that is it is contained in some open ball. Note that there are bounded curves of infinite length (for instance fractal curves), hence there are such curves with infinite mass.

Center of a mass

Let Γ be a curve in $\mathbb{R}^n$. It is given by continuous mapping $\gamma : [a,b] \ni t \mapsto \gamma(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$, so the image of γ is Γ. Denote $f(x) = f(x_1, \ldots, x_n)$ to be a function defining linear density of Γ. Then total mass is given by the integral

$$M(\Gamma) = \int_\Gamma f dl.$$

The center of a mass is given by the following rule. Denote $s = (s_1, \ldots, s_n)$, where

$$s_k = \frac{1}{M(\Gamma)} \int_\Gamma x_k f(x) dl.$$

s_k is a center of a mass with respect to $k-$th coordinate. Then s is a center of mass of a curve Γ. Note, that if f and γ are bounded, then the center is well defined.

Example 1.1. Consider an arc of a cycloid given by the equations

$$\begin{aligned} x(t) &= a(t - \sin t), \\ y(t) &= a(1 - \cos t), \end{aligned}$$

where $t \in [0, 2\pi]$. We assume that $f \equiv 1$.

Note, that an arc is symmetric with respect to the line $x = \pi \cdot a$. Therefore $s_x = \pi \cdot a$. Moreover,

$$\begin{aligned} s_y &= \frac{\int_\Gamma y dl}{\int_\Gamma dl} = \frac{\int_0^{2\pi} a(1-\cos t)\sqrt{(a(1-\cos t))^2 + (a \sin t)^2)} dt}{\int_0^{2\pi} \sqrt{(a(1-\cos t))^2 + (a \sin t)^2)} dt} \\ &= \frac{4}{3} a. \end{aligned}$$

Hence, $s = \left(\pi \cdot a, \frac{4}{3} a\right)$.

Example 1.2. Consider simple curve given by the mapping

$$\gamma : [0, 1] \ni t \mapsto (t, 0) \in \mathbb{R}^2.$$

We now assume that a linear density is proportional to the distance from the origin, so $f(x, y) = x$. Immediately $s_y = 0$, and

$$\begin{aligned} M(\Gamma) &= \int_0^1 t\sqrt{1^2 + 0^2} dt = \frac{1}{2}, \\ \int_\Gamma x f(x, y) dl &= \int_0^1 t^2 \sqrt{1^2 + 0^2} dt = \frac{1}{3}. \end{aligned}$$

Therefore a center of mass of this segment is in $\frac{2}{3}$ of its length, measuring from the origin:

$$s = \left(\frac{2}{3}, 0\right).$$

Notice, that a distribution of a mass different from uniform and symmetric dislocates a center from the point which is not in the center of a length. Concentration of a mass close to one of vertexes (of a Γ considered as a segment) moves the center closer to this vertex.

We can also try to find a center of mass of a curve which is of infinite length. This requires reasonable distribution of a mass so the center does not ,,goes to infinity", that is we can derive both coordinates and they are finite.

Example 1.3. Consider $\Gamma = (0, +\infty) \times \{0\} \subset \mathbb{R}^2$ and $\gamma(t) = (t, 0)$ for $t \in (0, +\infty)$. Assume that $f(x, y) = \frac{1}{1+x^3}$. Let us now calculate a center of mass.

$$\begin{aligned} y_s &= 0, \\ dt &= dl, \\ M(\Gamma) &= \int_0^\infty \frac{dt}{1+t^3} = \frac{1}{6}\left(\ln\frac{(t+1)^2}{t^2-t+1} + 2\sqrt{3}\arctan\left(\frac{2t-1}{\sqrt{3}}\right)\right)_{t=0}^{t=\infty} \\ &= \frac{2\pi}{3\sqrt{3}}, \\ x_s &= \int_0^\infty \frac{tdt}{1+t^3} = \frac{1}{6}\left(\ln\frac{t^2-t+1}{(t+1)^2} + 2\sqrt{3}\arctan\left(\frac{2t-1}{\sqrt{3}}\right)\right)_{t=0}^{t=\infty} \\ &= \frac{2\pi}{3\sqrt{3}}. \end{aligned}$$

Hence

$$s = (1, 0).$$

In last example we saw that despite a curve is not bounded, proper distribution of a mass can lead to calculable center of a mass. Notice that if we take $\Gamma = \mathbb{R}$ and uniformly distribute mass along it, the center of a mass is not well defined. Giving one more example, a bounded curve of infinite length (such as Koch Curve) can also have well defined center of a mass, but not the center of its length as it can has infinite length. We will go back to this example in later section.

Partition into two equally length segments

Consider $\Gamma \subset \mathbb{R}^n$ and its parametrization $\gamma : [a, b] \to \mathbb{R}^n$. Let us pick $t_0 \in (a, b)$ and define

$$\begin{aligned} \gamma_1 &: [a, t_0] \to \mathbb{R}^n, \quad \gamma_1(t) = \gamma(t), \\ \gamma_2 &: [t_0, b] \to \mathbb{R}^n, \quad \gamma_2(t) = \gamma(t). \end{aligned}$$

Denote $\Gamma_1 = \gamma([a, t_0])$ and $\Gamma_2 = \gamma_2([t_0, b])$. Note that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \{\gamma(t_0)\}$.

Our goal is to find t_0 such that

$$\int_{\Gamma_1} dl = \int_{\Gamma_2} dl = \frac{1}{2}\int_\Gamma dl.$$

In other words, t_0 divides a set of times in such a way, that it halves a length of a curve Γ. A point $s = \gamma(t_0)$ is a center of a length of a curve Γ.

Example 1.4. Consider a cycloid from example 1.1. We have

$$\begin{aligned}\int_{\Gamma_t} tdl &= \int_0^t \sqrt{(a(1-\cos u))^2 + (a\sin u)^2}du = \\ &= 2a\int_0^t \sin\frac{u}{2}dt = 4a\left(1-\cos\frac{t}{2}\right). \\ \int_\Gamma dl &= 8a,\end{aligned}$$

where Γ_t is an image of $\gamma|_{[a,t]}$. Therefore we are looking for t such that

$$1 - \cos\frac{t}{2} = 1.$$

Hence $t = \pi$ and

$$\gamma(\pi) = (\pi \cdot a, 2a)$$

is a middle of a length of a single cycloid arc.

In the latter example it was crucial that a curve is of finite length. In case of curves that are not bounded this center is not well defined.

Partition into two equally time-length segments

Consider $\Gamma \subset \mathbb{R}^n$ and its parametrization $\gamma : [a,b] \to \mathbb{R}^n$. The length of it is given by

$$|\Gamma| = \int_\Gamma dl = \int_a^b |\gamma'|dt.$$

We partition a domain into two equal segments $I_1 = [a,c]$ and $I_2 = [c,b]$, where $c = \frac{a+b}{2}$. Define

$$\gamma_i : I_i \to \mathbb{R}^n$$

as a parametrization of Γ_i gives as an image of γ_i. Note, that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \{\gamma(c)\}$.

A point $\gamma(c)$ is called time-center of a curve. It divided a curve into two segments such that a time required to travel along these segments from the center to either of ends is equal.

Example 1.5. Consider $\Gamma = [0,1] \subset \mathbb{R}$. Take two parametrizations

$$\begin{aligned}\gamma &: [0,1] \ni t \mapsto t \in \mathbb{R}, \\ \mu &: [0,1] \ni t \mapsto t^2 \in \mathbb{R}.\end{aligned}$$

We have $\gamma'(t) = 1$ and $\mu'(t) = 2t$, so velocities are different, and so[1]

$$\begin{aligned} |\gamma_1| &= \frac{1}{2}, \\ |\gamma_2| &= \frac{1}{2}, \\ |\mu_1| &= \frac{1}{4}, \\ |\mu_2| &= \frac{3}{4}. \end{aligned}$$

In first case, for γ, we are in the middle of a segment after time $t = \frac{1}{2}$. In second case, for μ, we are in 1/4th of a curve, meaning we move faster as we move away from the origin.

2. Asymptotic center

In previous section we saw many examples of curves with different location of their centers depending on what type of center we are looking for, what type of parametrization we chose, or what type of mass distribution we take. In most cases we assumed that a curve has to be bounded, so respective integrals are convergent.

We will expand these definitions to the case of a curve defined on an open interval (a, b) (without any restriction, that is either of ends can be infinite). Motivation for this definition comes from dynamical systems. Considering heteroclinic orbit (that is: an orbit connecting two fixed points) we would like to answer the following question: assuming we forget about a part of an orbit which is close to both fixed points, where is a center of the remaining part? In other words - where do we expect our point to be on such piece of an orbit?

All three definitions we recalled in section 1. cannot be used here, because in fact a heteroclinic orbit in $\mathbb{R}^n$ is a curve[2] in $\mathbb{R}^{n+1}$ which is (mostly) not bounded. We will propose different approach that let us derive (in some cases) a proper center.

Let Γ be a curve in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, where first coordinate describes a time and remaining ones describe spatial position of points. Let $\gamma : A \to \mathbb{R}^n$ be its modified parametrization, where $A = (t_0, t_1)$ is an open interval. This parametrization is not arbitrary - in case of dynamical system is given by the dynamics. Considering a solution of a system of n differential equations, γ describes a solution in a phase of

[1] We abuse (only for this example) notation of the length by substituting parametrization in place of an actual curve.

[2] For a given dynamical system $\phi : \mathbb{R} \times X \to X$ an image is in X, but the domain is $\mathbb{R} \times X$.

space. So in general the image of γ is a projection of Γ into spacial coordinates. For simplicity we will denote that image as well by Γ.

Denote $a = \gamma(t_0)$ and $b = \gamma(t_1)$. Pick $\varepsilon > 0$ and denote B_ε to be an interval contained in A such that γ_ε is a map of which image is a curve $\Gamma_\varepsilon := \Gamma \setminus (B(a,\varepsilon) \cup B(b,\varepsilon))$. Then $\gamma_\varepsilon : B_\varepsilon \to \Gamma_\varepsilon$ and a choice of ε determines uniquely γ_ε (see Remark 2.2. for some cases we have to proceed differently). Denote $a_\varepsilon = \gamma_\varepsilon(\inf B_\varepsilon)$ and $b_\varepsilon = \gamma_\varepsilon(\sup B_\varepsilon)$.

On each curve Γ_ε, which is now by the definition a bounded curve, we are looking for a point x_ε such a time required to reach either of ends of Γ_ε is equal, that is we are looking for t_ε such that it halves the interval B_ε. This is easy since we are working with bounded interval B_ε which middle is in

$$t_\varepsilon := \frac{\inf B_\varepsilon + \sup B_\varepsilon}{2}.$$

Note that this definition is proper only for finite values of ends. t_ε and x_ε are well defined since B_ε is bounded. Note that each $x_\varepsilon \subset B(0,R)$ and $\overline{\Gamma}$ is compact. Time required to reach either of points a_ε and b_ε from the point x_ε is equal, hence x_ε is what we presented in third approach in previous section.

It is important that when cutting of a part of an orbit, we always obtain a piece which requires finite amount of time to be passed (regardless of original parametrization).

Definition 2.1. Assume $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\varepsilon_n \searrow 0$. If for any such sequence there exists a common limit

$$\beta := \lim_{n\to+\infty} x_{\varepsilon_n}$$

then this limit is called an *asymptotic center* of a curve.

Remark 2.2. The set Γ_ε may not be connected. If so, denote

$$\Delta_\varepsilon := \{A \subset \mathbb{R}^n : A \text{ is a component of } \Gamma_\varepsilon\}.$$

Then we choose $A \in \Delta_\varepsilon$ such that $\operatorname{dist}(A,a) = \operatorname{dist}(A,b) = \varepsilon$ and define $\Gamma_\varepsilon := A$. If there are many of such sets A, then we set $\Gamma_\varepsilon = \emptyset$ and $x_\varepsilon = a$.

In the definition of asymptotic center we would also like to avoid a situation where for some sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ we obtain that for infinitely many $n \in \mathbb{N}$ we have $\#\Delta_{\varepsilon_n} \geq 2$. In other words, we want to make sure that for $\varepsilon' s$ sufficiently small we have $\#\Delta_\varepsilon = 1$ so the limit β exists. This assumption is in fact very natural - curves that are not of such behaviour and shape are unseen in dynamical systems.

For given definition it is likely to obtain either positive or negative result of the existance of β. In the following example we briefly describe the case where such β does not exist.

Example 2.3. Consider $\gamma : \mathbb{R} \to (-1, 1)$ defined as follows:

(1) Set $\gamma(0) = 0$.
(2) On the interval $\left(-\frac{1}{2}, \frac{1}{2}\right) \subset (-1, 1)$ velocity is constant.
(3) On the interval $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ velocity is constant. On the interval $\left(\frac{1}{2}, \frac{3}{4}\right)$ velocity is such that that the center lays at point $x = \frac{1}{2}$ (so a point moves faster here).
(4) On the intervals $\left(-\frac{7}{8}, -\frac{3}{4}\right)$ and $\left(\frac{3}{4}, \frac{7}{8}\right)$ velocities are such that that the center lays at point $x = -\frac{1}{2}$ (point moves slower and faster respectively).

We continue this process up to infinity, obtaining non-converging sequence of points. Each of the intervals in this construction is an image of the interval $(k, k + 1)$ for some $(k \in \mathbb{Z})$. Gluing those parametrizations on each segment we obtain γ. Because by taking $\varepsilon_n = 2^{-n}$ points x_ε jumps between two nonequal points, hence our curve has no asymptotic center. This example can be obtained from careful modification of one period of a tangent function.

A careful modification of last example can give us $x_{\varepsilon_n} = \frac{(-1)^n}{2^n}$ for the same sequence of ε_n as above. Thus we can obtain a non-symmetric curve with well defined β. We will now present an example of dynamical system generated by a differential equation, that produces heteroclinic orbit with proper asymptotic center.

Example 2.4. Consider the following Cauchy problem

$$\begin{cases} \dot{x} = (x-2)^2(x+2)^2, \\ x(0) = 0. \end{cases}$$

The solution of this equation is a function

$$t = \frac{1}{32}\left(-\frac{4x}{x^2-4} + \ln\left|\frac{x+2}{x-2}\right|\right).$$

Note that it is antisymmetric with respect to the origin. Consider heteroclinic orbit between two fixed points $a = -2$ and $b = 2$. Then, thanks for the symmetry, the origin is an asymptotic center of that orbit, and so the curve being a solution of the equation on the interval $(-2, 2)$.

This can be generalized to equations of the form

$$\dot{x} = (x - x_1)^2(x - x_2)^2$$

with the same initial condition. It follows from earlier equation that $\frac{x_1+x_2}{2}$ describes an asymptotic center.

Example 2.5. Let us go back to the Koch Curve (we will denote by K). In section 1. we noted that it has infinite length although it is bounded. Considering both ends as fixed points we can properly define mapping $\gamma : (0,1) \to K \setminus \{\text{ends of } K\}$ to be a function such that the asymptotic center is a point $\gamma\left(\frac{1}{2}\right)$. Brief description is as follows (in every step we omit ends):

Step 1. Take 1st iteration of the Koch Curve, that is a segment $(0,1) \times \{0\}$. It can be parametrized uniformly by an inclusion. By this it follows that an asymptotic center is just a geometric center of a segment

Step 2. Take 2nd iteration of a curve, partition it to 4 segments using vertices as points of partition. Each segment is now uniformly parametrized by corresponding piece of $(0,1)$ of the length $1/4$.

Step k. Take k-th iteration, partition it to 4^k segments, each one parametrized by a piece of $(0,1)$ of the lenght $1/4^k$.

It is easy to check that from 2nd step an asymptotic center is always a point $\left(0, \frac{\sqrt{3}}{6}\right)$. If we denote a parametrization of each step as γ_n then we have

$$|\gamma_{n+1} - \gamma_n| \leq \frac{C}{3^n}$$

for some $C > 0$. Each γ_n is continuous. Denote $\gamma := \lim\limits_{n\to\infty} \gamma_n$. It is easy to check that

$$|\gamma(t) - \gamma_n(t)| \leq \sum_{k=n}^{\infty} \frac{C}{3^k}$$

which tends to 0 as n goes to infinity. Hence γ is a limit of uniformly converging sequence of parametrizations γ_n that are continuous. Therefore γ is continuous mapping $(0,1) \to K \setminus \{ends\}$.

As a result an asymptotic center of a Koch Curve is a point $\left(0, \frac{\sqrt{3}}{6}\right)$.

3. Further applications

We will start from recalling some definitions and theorems from dynamical systems (see [1]).

Definition 3.1. Assume φ is a dynamical system generated by the linear equation $x' = Ax$ for some matrix $A \in M(n, \mathbb{R})$. φ is called a hyperbolic if a spectrum $\sigma(A)$ is disjoint with imaginary axis, that is there are no purely imaginary eigenvalues of A. The set of $n-$dimensional hyperbolic dynamical systems is denoted by $H(n)$.

For further applications we will now recall one of basic theorems from qualitative theory of differential equations.

Theorem 3.2 (Grobman, Hartman)**.** *Assume $n \geq 1$ and U is an open set in $\mathbb{R}^n$. Assume that $f \in C^1(U, \mathbb{R}^n)$ is generating a local dynamical system φ. Pick $p \in U$ such that $f(p) = 0$ and assume that $Df(p)$ generates a hyperbolic dynamical system ψ. Then there exists an open neighbourhood V of p contained in U, an open neighbourhood W of 0 in $\mathbb{R}^n$ and a homeomorphism $h : V \to W$ such that if $\varphi(t, x) \in V$, then $h(\varphi(t, x)) = \psi(t, h(x))$.*

Using Grobnam-Hartman theorem we can obtain greater class of dynamical systems of which existing heteroclinic orbits have asymptotic center.

We will now describe such class. For simplicity, consider two-dimensional dynamical system φ with two fixed points p and q. Assume, that $Df(p)$ and $Df(q)$ are hyperbolic matrices. Denote Γ to be a trajectory connecting two fixed points p and q. Without loose of generality we can assume, that eigenspace of p containing Γ is repelling, and eigenspace corresponding to a point q is attracting. If two eigenvalues corresponding to those eigenspaces are opposite, then φ and $-\varphi$ [3] restricted to some neighbourhoods of fixed points and this trajectory describes the same dynamics, hence they move with the same velocity along Γ in those neighbourhoods. We can now find an asymptotic center on that trajectory - remaining part is bounded and requires finite time so it has well defined time-center.

Now by analogy this argument follows for higher dimensions.

[3] By $(-\varphi)$ we denote a dynamical system that has reversed time comparing to φ.

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