ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Karol Gryszka

Uniwersytet Jagielloński

Extreme Properties of Equivalent Flows

Praca semestralna nr 3

(semestr letni 2012/13)

Opiekun pracy: Dr hab. Klaudiusz Wójcik

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

Extreme properties of equivalent flows

Abstract
This paper contains brief collection of some properties of equivalent flows with or without fixed points, most of which are considered on compact metric spaces. Such properties as conservation of entropy via equivalent flows in case of both smooth and arbitrary class of equivalence, growth rate of periodic orbits and its connections with enropy are considered.

Keywords: Dynamical systems, topological equivalence, periodic orbits, entropy.

---

1. Preliminaries

We introduce notation and basic definitions used later. Let (X, d) be a metric space.

Definition 1.1. A dynamical system (a flow) is a continuous function $\phi\colon \mathbb{R} \times X \to X$ such that $\phi(0, x) = x$ and for any x, s, and t, $\phi(t, (\phi(s, x))) = \phi(t + s, x)$. We will denote $\phi_t := \phi(t, \cdot)$.

Definition 1.2. The orbit $o(x)$ of a point x is a set

$$o(x) = \{\phi(t, x) \mid t \in \mathbb{R}\}.$$

Positive orbit $o^+(x)$ of a point x is a set

$$o^+(x) = \{\phi(t, x) \mid t \geq 0\}.$$

A point x is T-periodic if $\phi(T, x) = x$ for some $T > 0$.

We will also denote

$$\mathrm{Per}(x) := \inf\{T > 0 \mid \phi(T, x) = x\}.$$

Definition 1.3. Two flows $\phi\colon \mathbb{R} \times X \to X$ and $\psi\colon \mathbb{R} \times Y \to Y$ are equivalent if there exists a homeomorphism $h\colon X \to Y$ such that $h(o_\phi(x)) = o_\psi(h(x))$ for all $x \in X$ while time orientation is preserved, that is a homeomorphism sends orbits onto orbits without changing the direction of time.

The following lemma comes from [6] and it is important for further topics.

Lemma 1.4. *For a pair of equivalent flows ϕ and ψ on compact metric spaces M and W, respectively, let $\alpha\colon M \to W$ be a homeomorphism that sends each orbit of ϕ onto an orbit of ψ preserving the time orientation. If there are no fixed points of ϕ or ψ, then there is a continuous function $\theta\colon W \times \mathbb{R} \to \mathbb{R}$ such that for each $x \in M$ and $s, t \in \mathbb{R}$, the following holds:*

1. *$\theta(x, 0) = 0$ and $\theta(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing;*
2. *$\theta(x, s+t) = \theta(x, s) + \theta(\phi(x, s), t)$;*
3. *$\alpha(\phi(x, t)) = \psi(\alpha(x), \theta(x, t))$.*

Remark 1.5. The same result holds for entropy, see [4].

2. Topological entropy of equivalent flows

We will investigate the topological entropy of equivalent flows. Namely, the property of topological entropy that is zero, positive or infinite, is preserved in case of flows without fixed points.

Let us assume, that ϕ and ψ are flows on X and Y, respectively, and have the same orbits with the same directions. Denote X_0 to be a set of all fixed points of ϕ. Then we can obtain a function θ given in lemma 1.4 by removing the set X_0. A function θ is additive (condition 2.) and ψ is a flow obtained by changing time in ψ by θ.

We will present various facts (see [2]) based on relations of invariant measures. Denote $M(\phi)$ to be the family of all ϕ-invariant ergodic Borel probability measures on X.

Proposition 2.1. *Assume that the flows ϕ and ψ have the same orbits with the same directions and they have no fixed points. Let θ be the additive functional of ϕ obtained from lemma 1.4. Then, for each $m \in M(\phi)$,*

$$E_{\hat{m}}(f) = \frac{1}{E_m(\theta(1,x))} E_m\left(\int_0^{\theta(1,x)} f(\psi_t(x))dt\right),$$

where

$$E_m(f) = \int_X f(x)dm(x)$$

defines $\hat{m} \in M(\psi)$. The map $\Gamma : M(\psi) \ni m \mapsto \hat{m} \in M(\psi)$ is bijective and $G(\phi, m) = G(\psi, \hat{m})$, where

$$G(\psi, m) := \left\{x \in X : \lim_{T\to\infty} \int_0^T f(\phi_t(x))dt = E_m(f) \text{ for all continuous } f\right\}.$$

Corollary 2.2. *Under the same assumptions, let $\hat{\theta}$ be the additive function of ψ such that $\psi_t(x) = \phi_{\hat{\theta}(t,x)}(x)$. Then*

$$\frac{1}{E_{\hat{m}}\theta(1,x)} E_{\hat{m}}\left(\int_0^{\hat{\theta}(1,x)} f(\phi_t(x))dt\right) = E_m(f)$$

for any $m \in M(\psi)$.

The following lemma is a classic result concerning topological entropy via measure theory.

Lemma 2.3. *We have*

$$h(\phi_q) = \sup_{\mu \in M(\phi_1)} h_\mu(\phi_1) = \sup_{m \in M(\phi)} h_m(\phi_1),$$

where $h_\mu(\phi_1)$ *denotes the metrical entropy of* ϕ_1 *with respect to the measure* μ.

We are ready to state the main theorem of this section. In sums up the entropy of equivalent flows without fixed points.

Theorem 2.4. *Let* ϕ *and* ψ *be topological flows on compact metric spaces* X *and* Y*, respectively. If they are equivalent and they have no fixed points, then* $h(\phi) = C(\phi, \psi)h(\psi)$*, where* $C(\phi, \psi)$ *is a finite positive number, and* $h(\phi), h(\psi)$ *are topological entropies.*

Proof. Let

$$X^+ := \{x \in X : \phi \text{ and } \psi \text{ have the same direction at } x\},$$

and

$$X^- := \{x \in X : \phi \text{ and } \psi \text{ have the opposite direction at } x\}.$$

Then both X^+ and X^- are closed invariant sets, so we have

$$h(\phi) = \max\{h(\phi|X^+), h(\phi|X^-)\}.$$

On the other hand we have $h(\phi_t) = h(\phi_{-t})$. Therefore we may assume that ϕ and ψ have the same orbits with the same directions. Then we obtain an additive functional $\theta(t, x)$ of ϕ such that $\phi_t(x) = \psi_{\theta(t,x)}(x)$. Thus for any $m \in M(\phi)$ we have

$$h_{\hat{m}}(\psi_1) = \frac{1}{E_m(\theta(1, x))} h_m(\psi_1),$$

where $\hat{m} \in M(\psi)$ is defined in proposition 2.1. (see [3] for more details). We have

$$h(\phi) = \sup_{m \in M(\phi)} h_m(\psi_1) = \sup_{m \in M(\phi)} E_m(\theta(1, x)) h_{\hat{m}}(\psi_1),$$

and so putting $c_1 = \min_{x \in X} \theta(1, x)$ and $c_2 = \max_{x \in X} \theta(1, x)$

$$\begin{aligned} c_1 h(\psi) &= c_1 \sup_{\hat{m} \in M(\psi)} h_{\hat{m}}(\psi_1) = c_1 \sup_{m \in M(\phi)} h_{\hat{m}}(\psi_1) \leq \\ &\leq h(\phi) \leq c_2 \sup_{m \in M(\phi)} h_{\hat{m}}(\psi_1) \\ &= c_2 h(\psi). \end{aligned}$$

So we have proven our theorem. □

Finally, we present a theorem that describes equivalent flows with fixed points.

Theorem 2.5. *There exists a pair of weakly equivalent flows with fixed points one of which has a positive topological entropy and the other has zero entropy.*

Detailed construction can be found in [2].

Similar result can be proven for the case of finite positive entropy and infinite entropy. It was long unknown whether there exists such examples in the differentiable case. This problem is presented in next section.

3. Topological entropy of equivalent smooth flows

We saw that finite non-zero topological entropy for a flow cannot be an invariant because its value is affected by time reparametrization. However, in case of 0 and ∞ topological entropy are invariant for equivalent flows without fixed points.

In last section we noted that there is a counterexample in equivalent flows with fixed points. Construction presented in [2] is a suspension of a transitive subshift and thus it is not differentiable. Ohno in his paper asked the following question:

Is zero-topological entropy an invariant for equivalent **differentiable** flows?

The answer is negative and it was firstly given by Sun, Young and Zhou in their paper (see [5]). We will present basic steps leading to the result.

Suppose, that $f\colon M \to M$ is a C^∞ diffeomorphism of a smooth manifold M of dimension $\dim M = m \geq 2$ with the following properties:

1. f has positive topological entropy.
2. f is minimal in the sense that all positive orbits are dense in M.

Such f can be constructed (see [1]).

Definition 3.1. Consider the space

$$\Omega := M \times [0,1]/\sim,$$

where $\sim$ is an equivalence relation gluing $(y,1)$ with $(f(y),0)$. The standard suspension of f is the flow ψ_t on Ω defined by $\psi_t(y,s) = (y, t+s)$ for $0 \leq t+s < 1$.

Let X denote the smooth vector field associated with suspension flows $\psi\colon \Omega\times\mathbb{R} \to \Omega$. For any $\alpha \in C^\infty(\Omega,[0,1])$ a vector field αX is also C^∞ and thus induce a differentiable flow.

Theorem 3.2 (Main theorem)**.** *There exists two functions $\alpha, \hat{\alpha} \in C^\infty(\Omega,[0,1])$ satisfying the following:*

(1) αX and $\hat{\alpha} X$ induce equivalent flows ϕ and $\hat{\phi}$;

(2) ϕ has hero topological entropy and $\hat{\phi}$ has positive topological entropy.

Main theorem is based on several results from [5] we will present now.

For convenience we fix one point $p = (x_0, 0) \in \Omega$ and consider α and $\hat{\alpha}$ satisfying the following conditions (we will refer to them as **(H)**):

1. $\alpha(p) = \hat{\alpha}(p) = 0, \alpha(q) > 0, \hat{\alpha}(q) > 0$ for $q \neq p$,

2. there exists a small neighbourhood $V$of p in Ω such that $\alpha(q) = \hat{\alpha}(q) \equiv 1, q \in \Omega \setminus V$.

Second condition means that both α and $\hat{\alpha}$ are flat on $\Omega \setminus V$ meaning the construction leads to non-analytic function.

Proposition 3.3. *If α and $\hat{\alpha}$ are $C^\infty(\Omega, [0,1])$ and satisfy **(H)** and X is the suspension vector field on Ω described above, then αX and $\hat{\alpha} X$ induce equivalent differential flows on Ω with one singularity p.*

For this section we will introduce slightly different definition of an additive function.

Suppose ψ_t is a measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$ and Ω is divided into disjoint invariant measurable sets A and N such that $\mu(A) = 1$ and $\mu(N) = 0$. Further suppose that $\theta(t, x)$ is a real measurable function defined on $\mathbb{R} \times (\Omega \setminus N) = \mathbb{R} \times A$ with the following properties for every fixed $x \in A$:

1. $\theta(t, x)$ is continuous and non-decreasing in t;
2. $\theta(t + s, x) = \theta(s, x) + \theta(t, \phi_s(x))$ for all t and s;
3. $\theta(0, x) = 0, \lim_{t \to \infty} \theta(t, x) = \infty, \lim_{t \to -\infty} \theta(t, x) = -\infty$.

Then θ is called additive function of ψ_t with carrier A. We also say, that θ is integrable if it is integrable in Ω for every fixed t.

Lemma 3.4. *If ψ_t is a measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$ and $\alpha(x)$ is a non-negative, integrable functional satisfying*

$$E_\mu(\alpha) = \int_\Omega \alpha(x) d\mu(x) > 0,$$

then the function

$$\theta(t, x) = \int_0^t \alpha(\psi_s, x) dx$$

is an integrable additive function.

Theorem 3.5. *Let ψ_t be an arbitrary measurable flow on a Borel probability space $(\Omega, \mathcal{B}, \mu)$, and let θ be any integrable additive function of ψ_t. If the flow ϕ_t on $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{\mu})$ is the time changed flow of ψ_t by θ, then we have the inequality*

$$h_{\hat{\mu}}(\phi_t) \hat{\mu}(\hat{\Omega}) \leq h_\mu(\psi_t) \mu(\Omega)$$

for all fixed t, where $h_{\hat{\mu}}(\phi_t)$ and $h_\mu(\psi_t)$ denote the measurable-theoretic entropies of a the homeomorphisms ϕ_t and ψ_t respectively.

Definition 3.6. An ergodic measure is atomic if it is supported on a periodic orbit.

Corollary 3.7. *If $E_\mu(\gamma) = +\infty$ for all non-atomic ergodic measures μ of f, then ϕ_t has only atomic invariant Borel probability measures.*

Proposition 3.8. *Let μ be an ergodic measure of f on M. Then we have*

$$h_{\overline{\mu}}(\psi) = h_\mu(f),$$

where $\overline{\mu}$ is defined as follows:

$$\int_\Omega \xi d\overline{\mu} := \int_M \int_0^1 \xi(x,t)dtd\mu, \qquad \forall\ \xi \in C^(0)(\Omega).$$

Now we will establish main theorems of this section that implies existance of suggested flows.

Theorem 3.9. *There exists a function $\alpha \in C^\infty(\Omega, [0,1])$, satisfying **(H)**, such that the flow defined by the vector field $Y = \alpha X$ has zero topological entropy.*

Theorem 3.10. *There exists a function $\hat{\alpha} \in C^\infty(\Omega, [0,1])$, satisfying **(H)**, such that $\hat{Y} = \hat{\alpha} X$ has positive topological entropy.*

Proof of main theorem. The flows induced by the C^∞ vector fields $Y = \alpha X$ and $\hat{Y} = \hat{\alpha} X$ have zero topological entropy and positive topological entropy. They are equivalent by proposition 3.3. □

The construction given in [5] is not analytic. It seems likely that zero entropy is an invariant of equivalent analytic flows.

4. Extreme growth rate of periodic orbits

We give the definition of growth rate of periodic orbits (see [6]). Let $\phi\colon \mathbb{R} \times X \to X$ be a flow on a metric space (X, d). For given $A \in \mathbb{R}^+$, we define the number of periodic orbits of at most A-period by

$$\pi(\phi, A) := \max\{1, \sharp\{o(x) \subset X \mid \phi(x,a) = x, \text{for some } 0 < a \le A\}\}.$$

Note, that if the set of orbits with given period A is infinite, we write $\pi(\phi, A) = +\infty$. Let $p(\phi, A) := \frac{1}{A}\log \pi(\phi, A)$ and $p(\phi) := \limsup_{A\to+\infty} p(\phi, A)$. $p(\phi)$ is called the growth rate of periodic orbits for ϕ. Note that $p(\phi) \in [0, +\infty]$.

We state main theorems.

Theorem 4.1. *There exists a pair of equivalent flows $\phi\colon \mathbb{R}\times M \to M$ and $\psi\colon \mathbb{R}\times W \to W$ with fixed points on compact metric spaces M and W, respectively, such that*

1. $p(\phi) = \infty$;
2. $p(\psi) = 0$.

There is a huge difference in the flows with fixed points. In case of flows without fixed points extreme growth rates are invariant in equivalent flows.

Theorem 4.2. *For a pair of equivalent flows without fixed points, $\phi\colon \mathbb{R}\times M \to M$ and $\psi\colon \mathbb{R}\times W \to W$ on compact metric spaces M and W, respectively, the following two items hold:*

1. $p(\phi) = 0 \iff p(\psi) = 0$;
2. $p(\phi) = \infty \iff p(\psi) = \infty$.

Going for more extreme cases, Sun and Zhang shown the following theorems.

Theorem 4.3. *There is a flow ψ on a compact space W such that*

1. $h(\psi) = \infty$;
2. $p(\psi) = 0$.

Theorem 4.4. *There is a flow ϕ on a compact space M such that*

1. $h(\phi) = 0$;
2. $p(\phi) = \infty$.

Theorem 4.1 can be proven using the following proposition. Different proof was presented in [6] and it is based on constructing discrete dynamical system such that its suspension has desired conditions (previously we saw that suspended flows preserve some properties of original flows).

Proposition 4.5. *There exists compact metric spaces X and Y, a pair of equivalent flows $\phi\colon \mathbb{R}\times X \to X$ and $\psi\colon \mathbb{R}\times Y \to Y$ with fixed points, such that*

1. $p(\phi) = a$,
2. $p(\psi) = b$,

where a, b are any numbers from closed interval $[0, +\infty]$.

Proof. We will construct two flows on compact metric space such that they have arbitrarily given growth rate of periodic orbits. It was already proven in [6] in simple case $a = 0$ and $b = +\infty$.

Assume $\lambda \in (0, +\infty)$. We will construct such flow (Z, ξ_λ) on the subset of $\mathbb{R}^2$ that $p(\xi_\lambda) = \lambda$. For simplicity we write ξ instead of ξ_λ.

Note, that:

$$\frac{1}{A}\log 2^{\frac{\lambda}{\log 2}A} = \lambda.$$

If we define

$$\mu(A,\lambda) := \left\lfloor 2^{\frac{\lambda}{\log 2}A} + 1 \right\rfloor,$$

then

$$2^{\frac{\lambda}{\log 2}A} \le \mu(A,\lambda) \le 2^{\frac{\lambda}{\log 2}A} + 1.$$

Let $\mathcal{X} := \{X_n : x \in \mathbb{N}\}$, where

$$X_0 := \{(0,0)\}, \qquad X_n := \partial B\left((0,0), \frac{1}{n}\right),$$

where $B((a,b),r)$ denotes open ball in $(\mathbb{R}^2, d_2)$. We now define dynamic (and so ξ) on the set $Z := \bigcup \mathcal{X}$. Let X_0 be a fixed point of ξ and X_n are periodic orbits of the following period:

(1^0) For $n = 1, 2, \ldots, \mu(1,\lambda)$ circles X_n are 1-periodic.
(2^0) For $n = \mu(1,\lambda)+1, \ldots, \mu(2,\lambda)$ circles X_n are 2-periodic.
(k^0) For $n = \mu(k-1,\lambda)+1, \ldots, \mu(k,\lambda)$ circles X_n are k-periodic.

Here and later we assume that each periodic orbit on any circle is an orbit where move occurs with constant velocity. We also assume that the direction of motion on all orbits (both periodic and asymptotically periodic) is identical and consistent with positive orientation on the plane.

In case of $\mu(k,\lambda) = \mu(k+1,\lambda)$ we take no k-periodic orbits.

It follows from the construction, that for (Z,ξ) we have $p(\xi) = \lambda$.

Assume $\lambda = 0$. We now define (Z,ξ) on the same Z as previously, but the dynamic on circles is different:

For given $n \in \mathbb{N} \setminus \{0\}$ the set X_n is a n-periodic orbit.

For X_0 it remains unchanged.

In this case for (Z,ξ) we have $p(\xi) = 0$.

Finally, assume $\lambda = +\infty$. Again, X_0 is a fixed point. X_1 is a 1-periodic orbit. Sets X_n for $n \ge 2$ have the following dynamic:

$X_{2^{(k-1)^2}+1}, \ldots X_{2^{k^2}}$ are k-periodic orbits, for $k = 1, 2, \ldots$

In this case $p(\xi) = +\infty$.

To finish our proposition it is sufficient to take $X = Y = Z$ and choose one of three cases for given a and b in order to set flows ϕ and ψ on X and Y, respectively and notice, that identity is searched homeomorphism $\square$

The latter proposition is in fact stronger than the theorem 4.1, but the theorem itself points out the most extreme situation.

5. Asymptotic period and a growth rate

We will extend definitions and theorems from the latter section. They are ideas presented by an author in this paper, thus not yet published.

Let (X, d) be a metric space and ϕ be a flow on X. Given $x \in X$ and $\varepsilon > 0$ we define

$$A(x, \varepsilon) := \{t \geq 0 \mid d(\phi(t, x), x) > \varepsilon\}.$$

$A(x, \varepsilon)$ is a set of times for which a positive orbit of x travels outside the ball $\overline{B}(x, \varepsilon)$. This set is a sum of at most countably many pairwise disjoint and open intervals (q_i, r_i). It is also acceptable that $r_i = +\infty$ for some i. Define

$$w_t := \begin{cases} 0, & t \notin A(x, \varepsilon), \\ \operatorname{diam}(q_i, r_i), & t \in (q_i, r_i). \end{cases}$$

$(w_t)_{t \in \mathbb{R}}$ contains at most countably many different non-negative real numbers including $+\infty$ if necessary (which are the lengths of corresponding intervals (q_i, r_i)). Hence these numbers form a at most countable sequence, namely $(v_n)_{n \in \mathbb{N}}$. We will always identify $(w_t)_{t \in \mathbb{R}}$ with $(v_n)_{n \in \mathbb{N}}$ and use first symbol for this sequence. We will also refer to the sequence $(w_t)_{t \in \mathbb{R}}$ as a travelling time(s) of a point outside the ball. Note, that if we decrease ε, we do not decrease elements of $(w_t)_{t \in \mathbb{R}}$.

Take

$$W(x, \varepsilon) := \limsup_{t \to +\infty} w_t$$

which traces asymptotic time of travel of a point x outside the ball $\overline{B}(x, \varepsilon)$. Note, that $W(x, \varepsilon) = 0$ if $o^+(x) \subset \overline{B}(x, \varepsilon)$, and $W(x, \varepsilon) = +\infty$ if there exists i_0 such that $r_{i_0} = +\infty$.

Definition 5.1. Asymptotic period of a point x is defined as

$$AP(x) := \lim_{\varepsilon \to 0} \limsup_{t \to +\infty} W(\phi(t, x), \varepsilon).$$

By this definition each point x has its asymptotic period since we allow $+\infty$ to be a part of the sequence $(w_t)_{t \in \mathbb{R}}$. If $AP(x) = 0$ then a point x is called asymptotically fixed. If a point x has finite asymptotic period, then it is called asymptotically periodic. If $AP(x) = +\infty$ then a point x is called asymptotically non-periodic.

Remark 5.2. If x is a $T-$periodic orbit, then it is asymptotically periodic and $AP(x) = T$. If x is a fixed point, then it is asymptotically fixed.

We will now introduce growth rate for asymptotically periodic orbits.

Definition 5.3. For given $A \in \mathbb{R}^+$ we define the number of asymptotically periodic orbits of at most A-period by

$$\pi_{AP}(\phi, A) := \max\{1, \sharp\{o(x) \subset X \mid 0 < AP(x) \leq A\}\}.$$

And similarly

$$p_{AP}(\phi, A) := \frac{1}{A} \log \pi_{AP}(\phi, A),$$
$$p_{AP}(\phi) := \limsup_{A \to +\infty} p_{AP}(\phi, A).$$

In the same way we define growth rate of essentially asymptotically periodic orbits.

Definition 5.4. For given $A \in \mathbb{R}^+$ we define the number of essentially asymptotically periodic orbits of at most A-period by

$$\pi_{EAP}(\phi, A) := \max\left\{1, \#\left(\{o(x) \subset X \mid 0 < AP(x) \leq A\} \setminus P\right)\right\},$$

where $P := \{o(x) \subset X \mid 0 < \mathrm{Per}(x) \leq A\}$. Similarly:

$$p_{EAP}(\phi, A) := \frac{1}{A} \log \pi_{EAP}(\phi, A),$$
$$p_{EAP}(\phi) := \limsup_{A \to +\infty} p_{EAP}(\phi, A).$$

Finally, the strongest theorem of this section, also containing results from section 4.

Theorem 5.5. *Let a, b, c, d be arbitrarily given numbers from the interval $[0, +\infty]$. There exists compact metric spaces X and Y, a pair of equivalent flows $\phi\colon \mathbb{R} \times X \to X$ and $\psi\colon \mathbb{R} \times Y \to Y$ with fixed points, such that*

1. $p_{EAP}(\phi) = a$,
2. $p_{EAP}(\psi) = b$,
3. $p(\phi) = c$,
4. $p(\psi) = d$.

This theorem can be proven using a construction which is similar to the one given in proposition 4.5.

6. Final remarks

As we saw fixed points play important role in the complexity and behaviour of the flow. The extreme values of entropy and growth rate of periodic orbits are two great example of this statement. These properties are both invariant in case of equivalent flows without fixed points where fixed point can disturb those values extremely. However they are not going in pairs in sense of theorems 4.3 and 4.4. We can thus measure a complexity of a dynamical system in various ways.

References

[1] *Construction d'un diffeomorphisme minimal d'entropie topologique non nulle*, M.R. Herman, Erg. Th. Dyn. Systems, **1** (1981), 67-76. MR627787 (83c:58046).

[2] *A weak equivalence and topological entropy*, T. Ohno, Plub. RIMS, Kyoto Univ. **16** (1980), 289-298.

[3] *Time changes of flows*, H. Totoki, Mem. Fac. Sci Kyushu Univ. Ser. Am **20** (1966), 27-55.

[4] *Entropy on flows, revisited*, W. Sun, E. Vargas, Bol. Soc. Bra. Mat. **30** (1999, 313-333. MR1726916 (2001d:37007)

[5] *Topological entropies of equivalent smooth flows*, W. Sun, T. Young, Y. Zhou, Transactions of the AMS. **361** (6) (June 2009), 3071-3082.

[6] *Extreme growth rate of periodic orbits in flows*, W. Sun, Ch. Zhang, Proceedings of the A.M.S. **140** (4) (2012), 1387-1392.