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Erdős conjecture on matchings in hypergraphs

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ERDŐS CONJECTURE ON MATCHINGS IN HYPERGRAPHS

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Abstract. In 1959 Erdős and Gallai determined the minimum number of edges in a graph, which guarantees the existence of a matching of a given size. A few years later Erdős conjectured that the maximum possible number of edges of a k-uniform hypergraph H on n vertices with matching number $\nu(H)$ is achieved if either the hypergraph is a clique or the complement of a clique. Until today only some special cases of this conjecture have been confirmed. The aim of this paper is to survey recent results on this long-standing open problem and possible ways to attack it. Based on the idea similar to that employed by Figaj and Łuczak we state a new conjecture on a structure of k-uniform hypergraphs not containing a matching of a prescribed size which, if true, would imply Erdős Conjecture in its asymptotic version.

1. Erdős Conjecture: the statement and basic results

Let us recall first some basic definitions of hypergraph theory. A *k-uniform hypergraph* is a pair $H = (V, E)$, where $V := V(H)$ is a finite set of vertices and $E := E(H) \subset \binom{V}{k}$ is a family of k-element subsets of V, called edges of H. A *matching* in H is a set of disjoint edges of H. By *the size* of a matching we mean the number of edges contained in it. The size of the largest matching in H is called the *matching number* and is denoted by $\nu(H)$. We say that a matching is *perfect* if it is of size $|V|/k$. For integers n and k, *a clique* $K_n^{(k)}$ is a hypergraph on n vertices such that every k-tuples of its vertices is an edge, i.e. $E(K_n^{(k)}) = \binom{V}{k}$. Usually we identify hypergraph H with its set of edges $E(H)$, therefore whenever we write $|H|$, we mean $|E(H)|$.

Let us now define a parameter, which is the main subject of our studies on Erdős Conjecture, and which describes the relation between $|E(H)|$ and $\nu(H)$.

Definition. *Let k, s and n be positive integers such that $0 \leq s \leq n/k$. Then $m^s(k,n)$ is defined as the smallest positive integer m such that every k-uniform hypergraph H on n vertices with at least m edges contains a matching of size s, i.e.*

$$m^s(k,n) = \min\{m : |E(H)| \geq m \Longrightarrow \nu(H) \geq s\}.$$

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Although the function $m^s(k,n)$ has been extensively studied for the last few years, its behaviour has not been completely understood yet. However, it is easy to find it for some special choices of parameters s, k, and n. One of them is the case of perfect matchings, when $s = n/k$. It is easy to see that in a graph case when $k = 2$, we have $m^{n/2}(2,n) = \binom{n-1}{2} + 1$. A similar formula holds in general, i.e.

$$m^{n/k}(k,n) = \binom{n-1}{k} + 1. \tag{1}$$

Indeed, to see that the lower bound for $m^{n/k}(k,n)$ yields it is enough to consider a hypergraph consisting of a clique $K^{(k)}_{n-1}$ and one isolated vertex. To prove upper bound, let n be divisible by k and consider an arbitrary k-uniform hypergraph H on n vertices with at least $\binom{n-1}{k} + 1$ edges. The complement H^c of such a hypergraph H has fewer than $\binom{n-1}{k-1}$ edges and, since there are $\binom{n-1}{k-1}$ edge disjoint perfect matchings in $K^{(k)}_n$, H^c misses at least one of them. Therefore, there exists a perfect matching in H and so $m^{n/k}(k,n) \le \binom{n-1}{k} + 1$.

The formula for $m^s(2,n)$ for general s was obtained by Erdős and Gallai [5], who proved that in the graph case when $k = 2$ and $1 \le s \le n/2$ we have

$$m^s(2,n) = \max\left\{\binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2}\right\} + 1. \tag{2}$$

A proof of this formula and its asymptotic version are discussed in Section 3.

A few years later Erdős asked if a similar formula holds for hypergraphs as well, more precisely, he stated the following conjecture [4].

Erdős Conjecture. *For all $k \ge 2$ and $1 \le s \le n/k$*

$$m^s(k,n) = \max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\} + 1. \tag{3}$$

Erdős Conjecture implies that the extremal graphs for this problem are a clique or the complement of a clique. Being more precise, the lower bound in (3) is due to the following two hypergraphs. The first bound is achieved by

$$K^{(k)}_{ks-1} \cup (n-ks+1)K_1,$$

that is, the clique on $ks-1$ vertices with $n-ks+1$ isolated vertices. The second critical value is due to

$$K^{(k)}_n - K^{(k)}_{n-s+1},$$

a graph obtained from the clique $K^{(k)}_n$ by deleting all edges of some fixed clique $K^{(k)}_{n-s+1}$. Equivalently, one can view it as a k-hypergraph consisting of all k-elements sets intersecting a given subset of $s-1$ vertices. The graphs defined above clearly do not contain s independent k-tuples and they are maximal with this property if $n \geq sk$.

In its full generality, Erdős Conjecture is still wide open. Let us briefly discuss some special cases for which it has been verified (more results related to it are discussed in Section 2).

The formula (3) is trivially true for $s = 1$, since every single edge is a matching. For $s = 2$, the conjecture is equivalent to the Erdős-Ko-Rado theorem and was proved in 1961 in [6]. This celebrated result can be stated as follows.

Theorem 1. *Let $n \geq 2k$. If $\mathcal{F} \subset \binom{V}{k}$ is a family of k-element subsets of n-element set V such that every two sets in $\mathcal{F}$ have non empty intersection, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, equality holds if and only if $\mathcal{F}$ consists of k-element subsets that contain i, for some $i \in V$.

Hence, for $n \geq 2k$, we have

$$m^2(k,n) = \binom{n-1}{k-1} + 1.$$

Observe, that in the case $s = 2$ the maximum in (3) is achieved by the second term and the only extremal graph for this case is $K_n^{(k)} - K_{n-1}^{(k)}$. Thus, the conjecture for $s = 2$ and $n \geq 2k$ follows from Erdős-Ko-Rado theorem.

In 1965 Erdős [4] confirmed the conjecture for larger values of s, but only if n is sufficiently large with respect to s and k.

Theorem 2. *For every $k \geq 2$ there exists c_k such that for $n > c_k s$*

$$m^s(k,n) = \sum_{i=1}^{\min\{k,s-1\}} \binom{s-1}{i}\binom{n-s+1}{k-i} + 1. \tag{4}$$

Erdős's proof of the above result starts with Erdős-Ko-Rado theorem for $s = 2$ and uses induction on s. Note that if $k \leq s-1$, then the value on the right-hand side of (4) is equal to $\binom{n}{k} - \binom{n-s+1}{k} + 1$, and for $k > s-1$ it is exactly $\binom{ks-1}{k} + 1$, and so formula (3) holds for sufficiently large n.

There are several more results on Erdős Conjecture which confirm it for some n depending on s and k. In 1976 Bollobás, Daykin and Erdős [3] proved the conjecture for $n \geq 2k^3 s$. A few years later, Frankl and Füredi improved the bound on n to $100ks^2$ (see [8]). Recently, Huang, Loh and Sudakov [10] have announced to solve the problem for every $n \geq ck^2 s$, for some $c > 0$.

2. New results and techniques

In this section we present two partial results on Erdős Conjecture, which have been obtained recently. We also discuss the techniques which were used to prove them hoping they could be instrumental in solving Erdős Conjecture in general.

The first result, due to Frankl, Rödl and Ruciński [9], makes use of so called *shifting technique*. This method is described in detail in Frankl's survey [8]. Here we sketch only its main idea and some results which the method is based on.

For an arbitrary hypergraph H let order its vertex set linearly, say $V(H) = \{1, 2 \ldots, n\}$. Given $1 \le i < j \le n$ and an edge $e \in E(H)$, the (i, j)-shift $S_{ij}(e)$ of e is defined as follows:

$$S_{ij}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } j \in e, i \notin e, (e \setminus \{j\}) \cup \{i\} \notin H, \\ e \text{ otherwise.} \end{cases}$$

We define $S_{ij}(H) = \{S_{ij}(e) : e \in H\}$. Clearly, if we keep on shifting then finally we end up with a *shifted* hypergraph, i.e. a hypergraph H such that $S_{ij}(H) = H$ for all $1 \le i < j \le n$. One can also observe, that $\binom{n}{2}$ shifts are sufficient to make a hypergraph shifted, if we do them in the right order.

Another useful observation on shifted hypergraphs is as follows. If an edge $\{i_1, i_2, \ldots, i_k\}$ belongs to a shifted hypergraph, then for every $i'_r \le i_r$ for $r = 1, 2, \ldots, k$, the edge $\{i'_1, i'_2, \ldots, i'_k\}$ belongs to this hypergraph as well. Clearly, this property tells us a lot on the structure of a shifted hypergraph, even though we do not know a structure of the starting hypergraph.

The most important property of shifting is that it preserves the size of a hypergraph and that it does not increase the size of the largest matching. We express the last statement formally in the following theorem.

Theorem 3. *Let H be hypergraph on n vertices and $1 \le i < j \le n$. Then,*

$$\nu(S_{ij}(H)) \le \nu(H).$$

Proof. Assume that the assertion does not hold, that is $\nu(S_{ij}(H)) > \nu(H)$ for some $1 \le i < j \le n$. Let consider the largest matching in $S_{ij}(H)$ and denote it by M. Clearly, there exists an edge $e \in M$ such that $e \notin H$. Then $i \in e$, $j \notin e$, $(e \setminus \{i\}) \cup \{j\} = e' \in H$ and $(M \setminus e) \subset H$. Note that there exists another edge f in M such that $j \in f$. If it is not the case, then $M' = (M \setminus e) \cup e'$ is a matching in H and $|M'| > \nu(H)$, a contradiction. Since $f \in H$, also $(e' \setminus \{j\}) \cup \{i\} = f' \in H$. Then $(M \setminus \{e, f\}) \cup \{e', f'\}$ is a matching in H of size larger then $\nu(H)$, a contradiction. □

Using shifting technique, Frankl, Rödl and Ruciński [9] proved Erdős Conjecture for $k = 3$, and $n \ge 4s$. Note that in this range the maximum in (3) is achieved by the second term.

Theorem 4. *For all $s \geq 2$ and $n \geq 4s$, if H is a 3-uniform hypergraph with $|V(H)| = n$ and $\nu(H) \leq s-1$, then $|H| \leq \binom{n}{3} - \binom{n-s+1}{3}$.*

Although the proof is very technical the main idea of the argument is similar to that applied by Erdős in [4]; the authors use an induction on s, with the beginning case $s = 2$ following from the Erdős-Ko-Rado theorem.

Another approach for dealing with hypergraphs matchings was presented recently by Alon, Huang and Sudakov [1]. They obtained some new partial results on Erdős Conjecture while working on conjecture of Manickam-Miklós-Singhi on nonnegative k-sums problem.

Using simple counting methods, they proved the following bound on the number of edges in a hypergraph with a given matching number.

Theorem 5. *If $n+1 > (k+1)^3$, any k-uniform hypergraph on n vertices with matching number at most $(n+1)/(k+1)$ has at least $\frac{1}{k+2}\binom{n}{k}$ edges missing from it.*

Recall that if Erdős Conjecture is true, then in the extremal case, when a graph is a clique or a complement of a clique, the number of edges missing from H must be at least $\frac{1}{2}\binom{n}{k}$. It is far larger than the bound $\frac{1}{k+2}\binom{n}{k}$ in Theorem 5, so there is still some room for improvement.

And indeed, in the same paper Alon, Huang, and Sudakov improved the bound from Theorem 5 by using more sophisticated probabilistic tools. For their applications it was sufficient to prove a weaker version of Erdős Conjecture which bounds the number of edges as a function of *the fractional matching number* $\nu^*(H)$ instead of $\nu(H)$. Let us recall that

$$\begin{aligned} \nu^*(H) &= \max \sum_{e \in E(H)} w(e) \qquad w : E(H) \to [0,1] \\ &\quad \text{subject to} \sum_{i \in e} w(e) \leq 1 \quad \text{for every vertex } i. \end{aligned}$$

Note that $\nu^*(H)$ is always greater or equal than $\nu(H)$.

Since finding the fractional matching number is a linear programming problem. Therefore one can consider its dual problem, which gives the fractional covering number $\tau^*(H)$.

$$\begin{aligned} \tau^*(H) &= \min \sum_{i} v(i) \qquad v : V(H) \to [0,1] \\ &\quad \text{subject to} \sum_{i \in e} v(i) \geq 1 \quad \text{for every edge } e. \end{aligned}$$

By the duality we have $\tau^*(H) = \nu^*(H)$.

To get an upper bound for the number of edges of a hypergraph not containing a matching of size s it is enough to find a function $v : V(H) \to [0,1]$, satisfying $\sum_{i\in V(H)} v(i) \le s$ that maximizes the number of k-tuples e, where $\sum_{i\in e} v(i) \ge 1$. Since this number is monotone increasing in every $v(i)$, one can assume that it is maximized by a function v with $\sum_{i\in V(H)} v(i) = s$.

Combining this fact with a clever application of the probabilistic method, Alon, Huang, and Sudakov [1] proved the following strengthening of Theorem 5.

Theorem 6. *If $n \ge C(k+1)^2$ with $C \ge 1$, and H is a k-uniform hypergraph on n vertices with fractional covering number $\tau^*(H) = (n+1)/(k+1)$, then there are at least $\left(\frac{1}{13} - \frac{1}{2C}\right)\frac{n^k}{k!}$ k-sets which are not edges in H.*

One of the main tools used in the proof of Theorem 6 is Feige's inequality. It bounds the probability that sum of nonnegative independent random variables exceeds its expectation by a given amount. We present it below.

Lemma 7. *Given m independent nonnegative random variables $X_1, \ldots, X_m$ each of expectation at most 1, then*

$$Pr\left(\sum_{i=1}^{m} X_i < m + \delta\right) \ge \min\left\{\delta/(1+\delta), \frac{1}{13}\right\}.$$

Observe that this inequality is in a way stronger than Markov's inequality since the number of variables m does not appear explicitly in the lower bound.

3. Graph case – proof and asymptotic version

As mentioned in Section 1, Erdős Conjecture for graphs was proven first by Erdős and Gallai in [5]. Let us restate this result again.

Theorem 8. *Let $G = (V, E)$ be a graph with a vertex set $V = \{1, \ldots, n\}$ and $4 \le 2s \le n$. If G contains no matching of size s, then*

$$|E| \le \max\left\{\binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2}\right\}.$$

Below we present Akiyama-Frankl's [2] proof of this theorem which is based on the shifting technique. An application of the shift operation makes the argument much simpler and shorter than previous proofs of this result.

Proof. Let G be a graph with vertex set $V = \{1, \ldots, n\}$ and suppose that $\nu(G) < s$. According to the remarks made about shifting technique in Section 2, we can assume that G is shifted. Note that since G does not contain s pairwise disjoint edges, one of the following s subsets of V:

$$e_i = \{i, 2s+1-i\}, \quad i = 1, \ldots, s,$$

is not an edge in G. However, using the fact that G is shifted and $e_i \notin E(G)$ for some $i \in \{1,\dots,s\}$, we can deduce that

$$E(G) \subset G_i := \{e \in \binom{V}{2} : e \cap [1, i-1] \neq \emptyset \ \text{ or } \ e \subset [1, 2s-i]\}.$$

Clearly G_i contains no s pairwise disjoint edges. Hence,

$$\begin{aligned} |E(G)| &\leq \max_{1\leq i\leq s} |G_i| = \max\{|G_1|, |G_s|\} \\ &= \max\left\{\binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2}\right\}, \end{aligned}$$

and the assertion follows. □

Let us remark that if G contains no s pairwise disjoint edges, and $S_{ij}(G)$ is isomorphic to G_i for some $1 \leq i \leq s$, then G is isomorphic to G_i as well. Hence, equality in Theorem 8 holds if and only if either $G = K_{2s-1} \cup (n-2s+1)K_1$ or $G = K_n - K_{n-s+1}$.

Although we have already showed Erdős Conjecture for graphs, we shall state and prove an asymptotic version of Theorem 8 by another method which, perhaps, can be also used in the hypergraph case (see Section 4).

Let us start with the following result on the structure of graphs without large matchings introduced by Figaj and Łuczak in [7].

Lemma 9. *If a graph $G = (V, E)$ contains no matching saturating at least d vertices, then there exists a partition $\{S, T, U\}$ of V such that:*

(i) *the induced subgraph $G[U]$ has maximum degree at most $\sqrt{|V|} - 1$,*
(ii) *there are no edges between the sets T and U,*
(iii) $2|S| + |T| < d + \sqrt{|V|}$.

Since we believe that Lemma 9 can be generalized to obtain similar result for hypergraphs, it is crucial for our approach. For the completeness of the argument, we present its original proof here, which is a direct corollary of the following version of Tutte's theorem. Here and below for a graph G by $q(G)$ we denote the number of all components of G which contain an odd number of vertices.

Theorem 10. *If a graf $G = (V, E)$ contains no matching saturating at least d vertices, then there exists a set $S \subset V$ such that $q(G - S) > |S| + |V| - d$.*

Proof of Lemma 9. The proof of Lemma 9 is an easy consequence of Tutte's theorem. Suppose that a graph $G = (V, E)$ contains no matching saturating at least d vertices. By Theorem 10, there exists a subset $S \subset V$ such that $q(G - S) > |S| + |V| - d$. Denote the odd components of a graph $G[V \setminus S]$ with at most $\sqrt{|V|}$ vertices by $U_1, U_2, \dots, U_k$ and let $U = \bigcup_{i=1}^{k} U_i$ and $T = V \setminus (S \cup U)$. Then, for such a partition $V = S \cup U \cup T$, each vertex of U has at most $\sqrt{|V|} - 1$ neighbors and there are no edges between sets T and U. Hence, conditions (i) and (ii) clearly hold. Since there are fewer than $\sqrt{|V|}$ components in $G[V \setminus S]$ of size larger than $\sqrt{|V|}$, from Tutte's condition we get

$$|U| \geq k > q(G[V \setminus S]) - \sqrt{|V|} > |S| + |V| - d - \sqrt{|V|}.$$

Therefore,

$$|T| = |V| - |S| - |U| < d + \sqrt{|V|} - 2|S|$$

and (iii) holds. □

Now we can state and prove an asymptotic version of Erdős Conjecture for graphs. It is a simple consequence of Lemma 9.

Theorem 11. *If a graph $G = (V, E)$ with n vertices contains no matching of size s, then*

$$|E| \leq \max\left\{2s^2, ns - \frac{s^2}{2}\right\} + o(n^2).$$

Proof. Let $G = (V, E)$ be a graph with n vertices containing no matching of size s. By Lemma 9, there exists a partition S, T, U of V such that each vertex of U has at most $\sqrt{|V|} - 1$ neighbors in $G[U]$, there are no edges between the sets T and U, and

$$2|S| + |T| < 2s + \sqrt{|V|}.$$

Then,

$$|T| \leq 2s + 2|S| + o(n) \quad \text{and} \quad |S| \leq s + o(n).$$

The number of edges of a graph G can be bounded from above in the following way.

$$\begin{aligned} |E| &\leq \binom{|S|}{2} + \binom{|T|}{2} + \frac{1}{2}|U|\Delta(G[U]) + |S||T| + |S||U| \\ &\leq \frac{|S|^2}{2} + \frac{(2s - 2|S| + o(n))^2}{2} + \frac{1}{2}|U|\sqrt{n} + |S|(n - |S|) \\ &= \frac{3}{2}|S|^2 + (n - 4s)|S| + 2s^2 + o(n^2). \end{aligned}$$

Since the last expression is a quadratic function of $|S|$, it achieves maximum value for either $|S| = 0$, or $|S| = s$. Hence,

$$|E| \leq \max\left\{2s^2, ns - \frac{s^2}{2}\right\} + o(n^2),$$

and the assertion follows. □

4. The structural conjecture

In the previous section we presented a structural lemma for graphs, which do not contain a matching of a given size (Lemma 9), and used it to verify an asymptotic version of Erdős Conjecture in a graph case. The proof of Lemma 9 was based on Tutte's theorem, but this structural lemma can be also proven in a different way. Although the new proof is neither simpler nor shorter, it is crucial for our approach. We believe that its main idea might be generalized and used to obtain analogous structural result for hypergraphs. Not having proved the result in a general case completely, we state its asymptotic version as a new conjecture.

Conjecture 1. *Let $k \geq 2$, $1 \leq s \leq n/k$ and let $H = (V, E)$ be a k-uniform hypergraph with n vertices. If H contains no matching of size s, then there exists a partition $\{S_1, S_2, \ldots, S_k, S_{k+1}\}$ of V such that*

$$k|S_1| + \frac{k}{2}|S_2| + \ldots + \frac{k}{k-1}|S_{k-1}| + |S_k| \leq ks + o(n^k).$$

Moreover, if for $e \in E$ we set $k_i(e) = |e \cap S_i|$ for $i = 1, \ldots, k+1$, then, for all i, j, where $1 \leq j < i \leq k+1$, there are at most $o(n^k)$ edges e of H such that $k_i(e) < i$ and $k_j(e) = 0$.

In Section 3 we have already proved Conjecture 1 for $k = 2$. Here, we show how the asymptotic version of Erdős Conjecture for 3-uniform hypergraphs might be deduced from Conjecture 1. Let us state it below for an arbitrary k.

Conjecture 2. *For all $k \geq 2$ and $1 \leq s \leq n/k$*

$$m^s(k, n) = \max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\} + o(n^k).$$

Theorem 12. *For $k = 3$ Conjecture 2 follows from Conjecture 1.*

Proof. Let $H = (V, E)$ be 3-uniform hypergraph with n vertices not containing a matching of size s and assume that Conjecture 1 is true. Observe that if $s = o(n)$ then the statement clearly holds. Assume now that $s = \alpha n$ for some constant $0 < \alpha < 1/3$. The existence of a partition $S_1 \cup S_2 \cup S_3 \cup S_4$ implied by Conjecture 1 leads to the following upper bound on the number of edges of H:

$$\begin{aligned}
|H| \;&\leq\; \binom{|S_1|}{3} + \binom{|S_1|}{2}\binom{n-|S_1|}{1} + \binom{|S_1|}{1}\binom{n-|S_1|}{2} \\
&+\; \binom{|S_2|}{3} + \binom{|S_2|}{2}\binom{n-|S_1|-|S_2|}{1} + \binom{|S_3|}{3} \\
&\leq\; \frac{|S_1|^3}{6} + \frac{|S_1|^2}{2}(n-|S_1|) + |S_1|\frac{(n-|S_1|)^2}{2} \\
&+\; \frac{|S_2|^3}{6} + \frac{|S_2|^2}{2}(n-|S_1|-|S_2|) + \frac{|S_3|^3}{6} + o(n^3).
\end{aligned}$$

Thus, to prove the theorem it is enough to maximize the function

$$f(x,y,z) := \frac{x^3}{6} + \frac{x^2}{2}(1-x) + x\frac{(1-x)^2}{2} + \frac{y^3}{6} + \frac{y^2}{2}(1-x-y) + \frac{z^3}{6}$$

over the polytope

$$P = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0,\ 3x + \frac{3}{2}y + z \leq 3\alpha\}.$$

Standard but somewhat technical calculations show that f achieves its maximum value in a non-zero vertex which belongs to the face defined by the equation

$$3x + \frac{3}{2}y + z = 3\alpha.$$

Hence,

$$\begin{aligned} |H| &\leq \max\{f(\alpha,0,0), f(0,2\alpha,0), f(0,0,3\alpha)\}n^3 + o(n^3) \\ &= \max\{f(\alpha,0,0), f(0,0,3\alpha)\}n^3 + o(n^3) \\ &= \max\{\frac{s^3}{6} - \frac{ns^2}{2} + \frac{n^2 s}{2}, \frac{9s^3}{2}\} + o(n^3), \end{aligned}$$

and the assertion follows. □

The proof that Conjecture 1 implies Erdős Conjecture in a general case seems to be much more technical. Note however that it is certainly the case when Erdős Conjecture holds. Indeed, then the problem is to maximize the sum

$$\sum_{j=1}^{k} \left(\sum_{i=j}^{k} \binom{|S_j|}{i} \binom{\sum_{l=j+1}^{k+1} |S_l|}{k-i} \right),$$

which bounds the number of edges of H, under the condition

(i) $k|S_1| + \frac{k}{2}|S_2| + \frac{k}{3}|S_3| + \cdots + \frac{k}{k-1}|S_{k-1}| + |S_k| \leq ks$,
(ii) $|S_1| + |S_2| + \cdots + |S_{k+1}| = n$,
(iii) $|S_i| \geq 0$ for $i = 1, \ldots, k+1$.

If Erdős Conjecture holds then it is maximized only if either $|S_1| = s$, or $|S_k| = ks$.

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