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Constructions of closed symplectic manifolds

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Constructions of closed symplectic manifolds

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1. Introduction

A symplectic manifold is a pair (M, ω) consisting of a manifold M and nondegenerate, closed 2-form ω on M, called a symplectic form or a symplectic structure. In particular, M is of even dimension. The study of such structures came from physics, but since they were introduced in mathematics, they gained a lot of independent interest, and now they form a constantly growing field of research. There are now several manuals offering a comprehensive introduction to the subject, among them the book of McDuff and Salamon [28] or Cannas da Silva [8].

Probably the most important theorem about symplectic manifolds is the Darboux Theorem below

Theorem 1.1 (cf. [28], Theorem 3.15)**.** *Any symplectic manifold (M, ω) of dimension $2n$ is locally symplectomorphic to $\mathbb{R}^{2n}$ with standard symplectic form ω_0.*

This theorem implies that there are no local phenomena for symplectic manifolds, only the global ones. In the beginning, one of the obstructions in studying these phenomena in the beginning, was the small number of known examples. Up to 1975 there were essentially only two classes of examples known. In the noncompact case there are cotangent bundles (for any manifold X the bundle T^*X admits the canonical symplectic structure) and in the compact case there are Kähler manifolds. Then, starting with Thurston's article [32], a lot of constructions of symplectic manifolds has been discovered. In this note we will try to present some of them, but this list will be far from being complete. We will concentrate on the compact case, because of the following result, due to Gromov

Theorem 1.2 (cf. [20])**.** *Let M be an open manifold. Then for every cohomology class $a \in H^2(M)$ and every nondegenerate 2-form α, there exist a closed nondegenerate 2-form (i.e. symplectic form) on M, such that $[\omega] = a$ and ω is homotopic to α through nondegenerate 2-forms.*

It is most natural to ask which compact even-dimensional manifolds admit a symplectic structure. There are some natural necessary conditions. The first one comes from the property that any symplectic structure ω on M is of the form $\omega(X, Y) = g(JX, Y)$ for some Riemanniam metric g and almost complex structure J on M, where g and J are not determined uniquely, but rather up to homotopy class (cf. [28], Proposition 4.1). Hence, in particular, there must exist a homotopy class of almost complex structures on M. The second one follows from $[\omega]^k \neq 0$ for each $k = 1, \ldots, n$, which in turn is an easy consequence of Stokes' Theorem. Observe that the latter condition implies, in particular, that M is orientable. We can ask the following

Question 1.3.

(1) Does every compact even-dimensional manifold M admitting an almost complex structure and cohomologically symplectic (i.e. having a candidate for the class $[\omega]$) admits a symplectic structure?
(2) For a fixed homotopy class of almost complex structures and fixed cohomology class a with $a^k \neq 0$ for $k = 1, \ldots, n$, does there exist a symplectic structure ω with associated almost complex structure in given homotopy class and $[\omega] = a$?

The answer to this question proved to be surprisingly hard. In dimension 2 obviously every orientable surface is symplectic for any volume form as symplectic structure. In dimension 4 the question is much harder, but thanks to some special methods which we will not cover here (Donaldson invariants, Seiberg–Witten invariants), the answer is already known. For example, $\mathbb{C}P^2\sharp\mathbb{C}P^2\sharp\mathbb{C}P^2$ (where $\sharp$ denotes the connected sum) admits an almost complex structure and is cohomologically symplectic, but does not admit any symplectic structure. In higher dimensions little is known.

Since Question 1.3 is hard in general, it seems that mathematicians are still far away from any classification results. But the simplier question, whether the class of closed symplectic manifolds coincides with the class of closed Kähler manifolds, has been answered and, maybe more importantly, resulted in many interesting constructions of symplectic manifolds. That is why we will use this question as a pretext for introducing some of those constructions.

The paper is organized as follows. In sections 2 and 3 we recall basic properties of Kähler manifolds and obtain some topological properties of manifolds admitting Kähler metrics, namely restrictions on Betti numbers, Hard Lefschetz Property and formality, which we will later use as criteria for the nonexistence of such a metric. In further sections we present examples of closed symplectic non-Kähler manifolds. Section 4 is devoted to nilmanifolds, section 5 to a family of examples obtained by blowing up in the symplectic category and in section 6 a family of examples obtained by symplectic connected sum. Section 7 discusses shortly some other constructions, not treated in the previous sections.

2. Kähler manifolds

In this section we gather some of the results concerning topological consequences of the existence of a Kähler metric on a given manifold. This way we can formulate some necessary conditions of topological nature for the manifold to be of Kähler type. Good references for this section are [19, 26]. In particular, all of the results stated below without proof can be found in one of these. We will start by recalling some basic definitions.

Definitions. Let M be a complex manifold of real dimension $2n$. A complex structure on M is equivalent to some integrable almost complex structure J on TM. For any real vector bundle E over M we can consider its complexification $E_{\mathbb{C}} = E\otimes\mathbb{C}$ with the complex structure given by $i(X\otimes\mu) = X\otimes i\mu$. In particular, the complexification $T_{\mathbb{C}}M$ of TM carries two complex structures, the other one being the extension of J to $T_{\mathbb{C}}M$ by $J(X\otimes\mu) = JX\otimes\mu$. They do not coincide, in fact, $T_{\mathbb{C}}M$ admits a decomposition

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

into subspaces such that J coincides with i on $T^{1,0}M$ and with $-i$ on $T^{0,1}M$, and the maps

$$TM \ni X \mapsto X\otimes 1 - JX\otimes i \in T^{1,0}M$$
$$TM \ni X \mapsto X\otimes 1 + JX\otimes i \in T^{0,1}M$$

are holomorphic and antyholomorphic isomorphisms respectively. Endomorphism J induces an endomorphism on the cotangent bundle, which we will also denote by J, given as $(J\xi)(X) = \xi(JX)$. Again, it extends to the complexified cotangent bundle $T^*_{\mathbb{C}}M = (T_{\mathbb{C}}M)^*$ and the analogous decomposition of $T^*_{\mathbb{C}}M$ follows,

$$T^*_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^*.$$

Finally, J can be extended to the complexified exterior bundle $(\bigwedge^* T^*M)_{\mathbb{C}} = \bigwedge^* T^*_{\mathbb{C}}M$ by $(J\xi)(X_1,\dots,X_k) = \xi(JX_1,\dots,JX_k)$ and $(\bigwedge^k T^*M)_{\mathbb{C}}$ decomposes as

$$(\bigwedge^k T^*M)_{\mathbb{C}} = \bigoplus_{p+q=k} T_{p,q}M = \bigoplus_{p+q=k} \left(\bigwedge^p T_{1,0}M\right) \otimes \left(\bigwedge^q T_{0,1}M\right).$$

Notice that $J\xi = i^{p-q}\xi$ for any $\xi \in T_{p,q}M$.

Complex differential k-forms on M are just sections of $(\bigwedge^k T^*M)_{\mathbb{C}}$. In particular, the projections $(\bigwedge^k T^*M)_{\mathbb{C}} \to T_{p,q}M$ give us the projection

$$\Omega^k(M;\mathbb{C}) \to \Omega^{p,q}(M)$$

into the space of forms of type (p,q). Naturally,

$$\Omega^{p_1,q_1}(M) \wedge \Omega^{p_2,q_2}(M) \subset \Omega^{p_1+p_2,q_1+q_2}(M).$$

We will now define a number of operators on the space of forms, which, in general setting, might seem to be unrelated, but which will be related in the case of Kähler manifolds, and these relations, together with the elliptic theory, will imply interesting results about topology of compact Kähler manifolds. The first operator is an ordinary exterior derivative d. While in general, for an arbitrary almost complex structure J, we have $\mathrm{d}(\Omega^{p,q}(M)) \subset \Omega^{p+q+1}(M;\mathbb{C})$, the integrability of J implies a stronger restriction, namely $\mathrm{d}(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. It is now natural to write $\mathrm{d} = \partial + \overline{\partial}$, where

$$\partial(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M),$$
$$\overline{\partial}(\Omega^{p,q}(M)) \subset \Omega^{p,q+1}(M).$$

The relation $d^2 = 0$ implies immediately $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. It means that both ∂ and $\overline{\partial}$ are differentials and thus define cohomology rings $H^{p,q}_{\partial}$ and $H^{p,q}_{\overline{\partial}}$, respectively. The latter is known as the Dolbeaut cohomology.

Let now $\mathrm{d}^c = J^{-1}\,\mathrm{d}\,J$. A straightforward calculation yields $\mathrm{d}^c = -i(\partial - \overline{\partial})$ and $\mathrm{d}\,\mathrm{d}^c = -\mathrm{d}^c\,\mathrm{d} = 2i\partial\overline{\partial}$. Naturally, d^c is also a differential and we can consider the cohomology $H^*_{\mathrm{d}^c}(M)$.

We also need a notion of a Hermitian metric on M. Such a metric is just a map $h : T_{\mathbb{C}}M \times T_{\mathbb{C}}M \to \mathbb{C}$ such that h is anti-$\mathbb{C}$-linear in the first coordinate, $\mathbb{C}$-linear in the second, $h(Y,X) = \overline{h(X,Y)}$ and $h(X,X) \in (0,+\infty)$ for any $X \neq 0$. Choosing a Hermitian metric h on M is equivalent to choosing a J-compatible Riemannian metric on TM, where J-compatibility means, that $g(JX,JY) = g(X,Y)$. Indeed, for a J-compatible g we can define Hermitian metric h by $h(X \otimes \mu, Y \otimes \lambda) = \overline{\mu}\lambda g(X,Y)$ and, conversely, for a Hermitian metric h its real part $\frac{1}{2}\mathfrak{Re}\,h(X,Y) = \frac{1}{4}(h(X,Y) + h(Y,X))$ composed with injection $TM \cong T_{1,0}M \subset T_{\mathbb{C}}M$ is a Riemannian metric on TM.

Since all the information about the Hermitian metric h is encoded in its associated J-compatible Riemannian metric g, we will concentrate on the latter, and refer to it as a Hermitian metric as well. Observe that for Hermitian g, the form ω given as $\omega(X,Y) = g(JX,Y)$ is a skew-symmetric 2-form on TM. It is called the fundamental form of a Hermitian metric g. It is immediate from the J-compatibility of g that $\omega(JX, JY) = \omega(X, Y)$, so the extension of ω to $T_{\mathbb{C}}M$ is of type $(1,1)$. In particular, the operator

$$L : \Omega^k(M) \ni \xi \mapsto \omega \wedge \xi \in \Omega^{k+2}(M)$$

extends to $\Omega^k(M;\mathbb{C})$ as an operator of type $(1,1)$.

From this moment assume, that M is compact. With a Hermitian metric g we can associate in a canonical way a volume form ν and a Hodge-$\star$ operator. First, choose an arbitrary orthonormal frame $\{e_1, \ldots, e_{2n}\}$ of T_pM at some point $p \in M$. The induced metric on T_p^*M is determined by the condition that the dual basis of T_p^*M is orthonormal. It is easy to see that this notion does not depend on the choice of the orthonormal frame at p. Now, orthonormal frames can be chosen locally (by parallel transport), so the induced metric is smooth. For an orthonormal frame $\{e_1^*, \ldots, e_{2n}^*\}$ of the cotangent space, we can define volume element, $v \in \bigwedge^{2n} T_p^*M$, as $v = e_1^* \wedge \ldots \wedge e_{2n}^*$. This is again independent of the choice of an orthonormal frame and smooth, so we obtain a volume form $v \in \Omega^{2n}(M)$.

The Hodge-$\star$ operator is obtained by endowing the exterior bundle with a metric. We do it by demanding that for an orthonormal frame $\{e_1^*, \ldots, e_{2n}^*\}$ of T^*M the family $\left\{e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* : 1 \le i_1 < \ldots < i_k \le 2n,\ k \ge 0\right\}$ is an orthonormal frame. As previously, this is well-defined and smooth. The Hodge-$\star$ operator is the unique operator $\star : \Omega^*(M) \to \Omega^{2n-*}(M)$ satisfying

$$\xi \wedge \star\eta = g(\xi, \eta)v.$$

For a k-form ξ we have

$$\star^2\xi = (-1)^{k(2n-k)}\xi = (-1)^k\xi.$$

Since $\Omega^*(M)$ is now equipped with a (function valued) scalar product, we can now consider adjoints of operators on this space. Let Λ be an adjoint of L that is the unique operator satisfying $g(L\xi, \eta) = g(\xi, \Lambda\eta)$. Then $\Lambda = \star^{-1}L\star$. Indeed,

$$g(L\xi, \eta)v = L\xi \wedge \star\eta = \omega \wedge \xi \wedge \star\eta = \xi \wedge L \star \eta = g(\xi, \star^{-1}L \star \eta)v.$$

In a similar way, we construct the adjoints of other operators, namely

$$\begin{aligned} \partial^\star &= - \star \overline{\partial} \star, \\ \overline{\partial}^\star &= - \star \partial \star, \\ \mathrm{d}^\star &= - \star \mathrm{d} \star = \partial^\star + \overline{\partial}^\star, \\ \mathrm{d}^{c\star} &= - \star \mathrm{d}^c \star. \end{aligned}$$

Now, all these operators can be, in a natural way, extended to the space of complex differential forms. Hodge-$\star$ operator acts as $\star : \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$. As a result we have $\partial^\star(\Omega^{p,q}(M)) \subset \Omega^{p-1,q}(M)$ and $\overline{\partial}^\star(\Omega^{p,q}(M)) \subset \Omega^{p,q-1}(M)$.

The existence of the adjoint operators makes it possible to consider Laplacians of the differentials. And so, let $\Delta_{\mathrm{d}} = \mathrm{d}^\star \mathrm{d} + \mathrm{d}\, \mathrm{d}^\star$, $\Delta_\partial = \partial^\star\partial + \partial\partial^\star$ and $\Delta_{\overline{\partial}} = \overline{\partial}^\star\overline{\partial} + \overline{\partial}\overline{\partial}^\star$. We say that a k-form ξ is d-harmonic iff $\Delta_{\mathrm{d}}\xi = 0$. Space of d-harmonic k-forms is denoted by $\mathcal{H}_d^k(M)$. In the same way, we define $\mathcal{H}_\partial^{p,q}(M)$ and $\mathcal{H}_{\overline{\partial}}^{p,q}(M)$. Observe that ξ is d-harmonic iff $\mathrm{d}\,\xi = \mathrm{d}^\star\,\xi = 0$ and similarly for ∂ and $\overline{\partial}$.

The last notion we will use in this section is that of primitive forms. A differential k-form ξ is said to be primitive iff $\Lambda\xi = 0$. We write $\xi \in \mathcal{P}^k(M)$. A theory of

Lie algebra representations (see the next subsection) yields that $\Omega^*(M)$ admits a decomposition, known as Lefschetz decomposition

$$\Omega^*(M) = \mathcal{P}^*(M) \oplus L\mathcal{P}^*(M) \oplus L^2\mathcal{P}^*(M) + \dots,$$

and the sum is finite.

We will now apply all of these notions to compact Kähler manifolds.

Definition 2.1. A Hermitian manifold (M, g) is said to be Kähler if the fundamental form of g is closed, that is $\mathrm{d}\,\omega = 0$.

Remark 2.2. Every Kähler manifold is a symplectic manifold with the fundamental form as symplectic structure.

Topology of compact Kähler manifolds. For Kähler manifolds we have the following set of equalities

Theorem 2.3 (Kähler identities, cf. [26], Proposition 3.1.12)**.** *For a Kähler manifold (M,g)*

(1) $[\overline{\partial}, L] = [\partial, L] = 0$ *and* $[\overline{\partial}^\star, \Lambda] = [\partial^\star, \Lambda] = 0$*,*
(2) $[\overline{\partial}^\star, L] = i\partial$*,* $[\partial^\star, L] = -i\overline{\partial}$ *and* $[\Lambda, \overline{\partial}] = -i\partial^\star$*,* $[\Lambda, \partial] = i\overline{\partial}^\star$*,*
(3) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_{\mathrm{d}}$ *and* Δ_{d} *commutes with* $\star$*,* ∂*,* $\overline{\partial}$*,* $\partial^\star$*,* $\overline{\partial}^\star$*,* L *and* Λ*.*

Now, if moreover M is Kähler, we can use the elliptic theory and obtain the Hodge decomposition theorem.

Theorem 2.4 (Hodge decomposition)**.** *For a compact Kähler manifold M we have*

$$\Omega^k(M;\mathbb{C}) = \mathcal{H}^k_{\mathrm{d}}(M;\mathbb{C}) \oplus \mathrm{d}\,\Omega^{k-1}(M;\mathbb{C}) \oplus \mathrm{d}^\star\,\Omega^{k+1}(M;\mathbb{C}),$$
$$\Omega^{p,q}(M) = \mathcal{H}^{p,q}_{\partial}(M) \oplus \partial\Omega^{p-1,q}(M) \oplus \partial^\star\Omega^{p+1,q}(M),$$
$$\Omega^{p,q}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M) \oplus \overline{\partial}\Omega^{p-1,q}(M) \oplus \overline{\partial}^\star\Omega^{p+1,q}(M),$$

and the spaces of harmonic forms $\mathcal{H}^k_{\mathrm{d}}(M;\mathbb{C})$, $\mathcal{H}^{p,q}_{\partial}(M)$ and $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$ are finite dimensional. In particular

$$H^k(M;\mathbb{C}) \cong \mathcal{H}^k_{\mathrm{d}}(M;\mathbb{C}),$$
$$H^{p,q}_{\partial}(M) \cong \mathcal{H}^{p,q}_{\partial}(M),$$
$$H^{p,q}_{\overline{\partial}}(M) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(M),$$

and all the cohomology spaces are of finite dimension.

The equality of Laplacians means that the space of harmonic forms agree, i.e. $\mathcal{H}^k_{\mathrm{d}}(M;\mathbb{C}) = \bigoplus_{p+q=k}\mathcal{H}^{p,q}_{\overline{\partial}}(M) = \bigoplus_{p+q=k}\mathcal{H}^{p,q}_{\partial}(M)$. In particular, we can see that a form ξ is ∂-harmonic iff it is $\overline{\partial}$-harmonic. Let ξ be $\overline{\partial}$-harmonic and consider $\overline{\xi}$.

$$\overline{\partial}\,\overline{\xi} = \overline{\partial\xi} = 0 \quad \overline{\partial}^\star\overline{\xi} = \overline{\partial^\star\xi} = 0.$$

We may conclude that ξ is $\overline{\partial}$-harmonic iff $\overline{\xi}$ is. So, $\overline{\mathcal{H}^{p,q}_{\overline{\partial}}(M)} = \mathcal{H}^{q,p}_{\overline{\partial}}(M)$. Now it is easy to observe that for an odd natural k, the space $\mathcal{H}^k_{\mathrm{d}}(M;\mathbb{C})$ must be even dimensional. Since $H^k_{\mathrm{d}}(M;\mathbb{C}) \cong \mathcal{H}^k(M;\mathbb{C})$ we can state the following

Proposition 2.5. *The odd Betti numbers $b^k = \dim H^k(M;\mathbb{C}) = \dim H^k(M)$ of compact Kähler manifold are even.*

Let H be an operator $H = [\Lambda, L]$. It can be computed that $H(\xi) = (n-k)\xi$ for a k-form ξ. Hence $[H, L] = -2L$ and $[H, \Lambda] = 2\Lambda$, and the operators H, L, Λ constitute an $\mathfrak{sl}(2;\mathbb{C})$ representation on the space of forms. The algebra $\mathfrak{sl}(2;\mathbb{C})$ is semisimple, so we can use the general theory of representations of semisimple Lie algebras. For more information, see for example [25].

First, any representation of semisimple Lie algebra is completely reducible, so we can write $\Omega^*(M, \mathbb{C})$ as a direct sum of irreducible $\mathfrak{sl}(2;\mathbb{C})$ modules. All irreducible $\mathfrak{sl}(2;\mathbb{C})$-modules are of the form $\mathbb{C}\{a_{-k}, a_{-k+2}\dots, a_k\}$ where a_i is an eigenvector of H of eigenvalue i, i.e. an $(n-k)$-form, a_k is primitive and $La_i = a_{i-2}$. In particular, $L^i : \mathbb{C}\{a_i\} \to \mathbb{C}\{a_{-i}\}$ is isomorphism. Since the natural grading of each irreducible module by eigenvalues of H is symmetric with respect to zero, this isomorphism is an isomorphism on every completely reducible representation. We can conclude that

$$L^k : \Omega^{n-k}(M;\mathbb{C}) \to \Omega^{n+k}(M;\mathbb{C})$$

is an isomorphism for each $k = 0, \dots, n$. This result clearly holds in the real case as well,

$$L^k : \Omega^{n-k}(M) \cong \Omega^{n+k}(M).$$

Laplacian Δ_{d} commutes with L and Λ, therefore the space of harmonic forms $\mathcal{H}^*_{\mathrm{d}}(M)$ is an $\mathfrak{sl}(2)$-submodule of $\Omega^n(M)$, and so

$$L^k : \mathcal{H}^{n-k}(M) \cong \mathcal{H}^{n+k}(M).$$

Moreover, L commutes with d (since ω is closed), so it induces a map in cohomology, which we denote by the same letter.

From the commutative diagram

$$\begin{array}{ccc} H^{n-k}(M) & \xrightarrow{L^k} & H^{n+k}(M) \\ \cong \uparrow & & \uparrow \cong \\ \mathcal{H}^{n-k}_{\mathrm{d}}(M) & \xrightarrow[\cong]{L^k} & \mathcal{H}^{n-k}_{\mathrm{d}}(M) \end{array}$$

induced map in cohomology $L^k : H^{n-k}(M) \to H^{n+k}(M)$ is an isomorphism for each $k = 0, \dots, n$. This property of L is known as Hard Lefschetz Property. Therefore, we obtain

Proposition 2.6. *Compact Kähler manifold M satisfies Hard Lefschetz Property.*

While Proposition 2.5 gives a criterion based solely on the topology of the underlying manifold, Proposition 2.6 concerns the actual cohomology class $[\omega]$. Therefore, if manifold M have some odd Betti number b^k for k odd, it means that it cannot admit any Kähler metric, i.e. it is not of Kähler type, while if for some closed 2-form ω on M class $[\omega]$ does not give an isomorphism in Hard Lefschetz Property, it means only that ω is not cohomologous to the fundamental form of any Kähler metric. To show that M is not of Kähler type, we have to show that Hard Lefschetz Property fails for each class $a \in H^2(M)$.

To obtain further cohomological criteria for the existence of Kähler metric, we will look into the notion of formality of a manifold.

3. Formality

In the considerations below we fix the field $\mathbb{K}$. All of the following results (if not stated otherwise stated otherwise) can be found in [13].

Definition 3.1. A differential graded algebra is a graded algebra $\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}^k$ together with a map $\mathrm{d} : \mathcal{A} \to \mathcal{A}$ of degree $+1$, such that

(1) $\mathcal{A}$ is graded commutative, $\forall_{x \in \mathcal{A}^k,\ y \in \mathcal{A}^l} : xy = (-1)^{kl} yx$,
(2) d is a differential, $\mathrm{d}^2 = 0$,
(3) d is a derivation, $\forall_{x \in \mathcal{A}^k,\ y \in \mathcal{A}^l} : \mathrm{d}(xy) = (\mathrm{d}\, x)y + (-1)^k x(\mathrm{d}\, y)$.

With each differential graded algebra (or DGA in short) we can associate its cohomology ring $H^*(\mathcal{A})$ defined in the usual manner. The most natural example of a DGA is a De Rham complex of (real or complex) differential forms on a manifold. Also, any graded algebra with graded commutative multiplication is a DGA with $\mathrm{d} = 0$. In particular, the cohomology ring of DGA is also a DGA. The augumentation ideal of a DGA, $A(\mathcal{A})$, is an ideal consisting of elements of nonzero degree, $A(\mathcal{A}) = \bigoplus_{k=1}^{\infty} \mathcal{A}^k$. A differential d is said to be decomposable if $\mathrm{d}(\mathcal{A}) \subset A(\mathcal{A})A(\mathcal{A})$.

For any natural number k and any vector space V we define the algebra $\bigwedge(V, k)$ as a free graded algebra over V, where elements of V are assumed to be of degree k. If V comes with a grading of its own, we just write $\bigwedge(V)$.

Definition 3.2. We say that $(\mathcal{B}, \mathrm{d}_{\mathcal{B}})$ is an elementary extension of $(\mathcal{A}, \mathrm{d}_{\mathcal{A}})$ iff

(1) there exists a finite-dimensional vector space V and a natural number $k > 0$ such that $\mathcal{B} = \mathcal{A} \otimes \bigwedge(V, k)$,
(2) $\mathrm{d}_{\mathcal{B}}|_{\mathcal{A}} = \mathrm{d}_{\mathcal{A}}$,
(3) $\mathrm{d}_{\mathcal{B}}(V) \subset \mathcal{A}$.

Definition 3.3. A differential graded algebra $(\mathcal{M}, \mathrm{d})$ is called minimal iff there exist a sequence of subalgebras

$$\mathbb{K} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}$$

such that $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$, d is decomposable and $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ is an elementary extension for each $i \geq 1$.

A morphism of differential graded algebras is just a map $\rho : \mathcal{A} \to \mathcal{B}$, which is a morphism of degree zero in the category of graded algebras and commutes with differentials. We use the notation $\rho : (\mathcal{A}, \mathrm{d}_{\mathcal{A}}) \to (\mathcal{B}, \mathrm{d}_{\mathcal{B}})$

Definition 3.4. A morphism $\rho : (\mathcal{M}_{\mathcal{A}}, \mathrm{d}_{\mathcal{M}_{\mathcal{A}}}) \to (\mathcal{A}, \mathrm{d}_{\mathcal{A}})$ is a minimal model for $(\mathcal{A}, \mathrm{d}_{\mathcal{A}})$ iff $(\mathcal{M}_{\mathcal{A}}, \mathrm{d}_{\mathcal{M}_{\mathcal{A}}})$ is a minimal differential graded algebra and ρ induces an isomorphism in cohomology.

We won't address the question of existence or uniqueness of minimal models for a given DGA. In the case of $H^1(\mathcal{A}) = 0$ there exist a simple, inductive construction of unique $\mathcal{M}_{\mathcal{A}} \to \mathcal{A}$ (cf. [13, 31]). In the following sections, for all examples the minimal model will be described directly, and will be unique (up to isomorphism).

Important class of DGA's consists of formal ones.

Definition 3.5.

- A differential graded algebra $\mathcal{A}$ is called formal iff there exists a homomorphism $\rho : (\mathcal{A}, \mathrm{d}_{\mathcal{A}}) \to (H^*(\mathcal{A}), \mathrm{d}_{H^*(\mathcal{A})} = 0)$ inducing isomorphism in cohomology.

- A manifold M is said to be formal iff there exist a minimal model of its cohomology ring which is formal.

The main result of [13] is the following

Theorem 3.6. *A compact manifold of Kähler type is formal.*

A useful criterion for the formality of DGA is given by Massey products. Let $[a] \in H^p(\mathcal{A})$, $[b] \in H^q(\mathcal{A})$ and $[c] \in H^r(\mathcal{A})$ be such that $[a][b] = [b][c] = 0$. Then we can find $x \in \mathcal{A}^{p+q-1}$ and $y \in \mathcal{A}^{q+r-1}$ satisfying $ab = \mathrm{d}_{\mathcal{A}}\, x$ and $bc = \mathrm{d}_{\mathcal{A}}\, y$. The element $z = xc + (-1)^{p+1}ay$ is closed, by a simple calculation. Its cohomology class $[z] \in H^{p+q+r-1}(\mathcal{A})$ is our first candidate for a Massey product of $[a]$, $[b]$ and $[c]$. But $[z]$ is not independent of the choices of the representants for $[a]$, $[b]$, $[c]$. For $a' = a + \mathrm{d}_{\mathcal{A}}\, \alpha$ we obtain $a'b = ab + \mathrm{d}_{\mathcal{A}}\, \alpha b$, so we can choose $x' = x + \alpha b$. Finally,

$$z' = x'c + (-1)^{p+1}a'y = z + (-1)^{p+1}\,\mathrm{d}_{\mathcal{A}}(\alpha y) + 2\alpha bc$$

and $[z'] \in [z] + H^*(\mathcal{A})[c] \subset [z] + [a]H^*(\mathcal{A}) + H^*(\mathcal{A})[c]$. Similarly, we can show that for any change of representatives $[z'] - [z] \in [a]H^*(\mathcal{A}) + H^*(\mathcal{A})[c]$. On the other hand, we can always take $x + \hat{x}$, for some closed $\hat{x}$ instead of x. Then still $\mathrm{d}_{\mathcal{A}}(x + \hat{x}) = \mathrm{d}_{\mathcal{A}}\, x = ab$, but $[(x + \hat{x})c + (-1)^{p+1}ay] = [z] + [\hat{x}][c]$. And similarly, modyfing y by a closed $\hat{y}$ we can modify $[z]$ by $[a][\hat{y}]$. This leads us to the following definition

Definition 3.7. A (triple) Massey product of classes $[a] \in H^p(\mathcal{A})$, $[b] \in H^q(\mathcal{A})$ and $[c] \in H^r(\mathcal{A})$ satisfying $[a][b] = [b][c] = 0$, denoted as $\langle [a], [b], [c] \rangle$, is the class $[z]$, for z defined as above, modulo the ideal $[a]H^*(\mathcal{A}) + H^*(\mathcal{A})[c] \subset H^*(\mathcal{A})$.

The criterion we announced above is the following

Proposition 3.8. *All Massey products in $H^*(\mathcal{A})$ vanish for a formal $\mathcal{A}$.*

Proof. We will need the following lemma

Lemma 3.9. *Let $(\bigwedge(V), \mathrm{d})$ be a free differential graded algebra for some graded vector space V. Let $C_i \subset V_i$ consist of d-closed elements. If $(\bigwedge(V), \mathrm{d})$ is formal then there is a graded vector subspace $N \subset V$ complementary to C and such that every closed element in the ideal (N) is exact.*

The converse is also true, but we will not use it. If we assume that this lemma holds, then the proof follows easily. In fact, we can always choose x and y in the construction of Massey product such that $x, y \in N$. But then $xc + (-1)^{p+1}ay \in (N)$ is closed, and hence exact. □

Proof of Lemma 3.9. $(\bigwedge(V), \mathrm{d})$ is formal, so there is a map

$$\rho : \left(\bigwedge(V), \mathrm{d}\right) \to \left(H^*\left(\bigwedge(V), \mathrm{d}\right), 0\right)$$

inducing isomorphism in cohomology. Now, the decomposability of d implies that no nonzero elements of C are exact, and so $\ker \rho \cap C = \{0\}$. In particular, $V = C \oplus N$ where $N = \ker \rho \cap V$. Since $\ker \rho$ is an ideal in $\bigwedge(V)$ and $N \subset \ker \rho$, there must be $(N) \subset \ker \rho$ which concludes the proof. □

4. Nilmanifolds

We can now present the first (also chronologically) example of a closed symplectic manifold not of Kähler type. It is usually assigned to Kodaira and Thurston.

Example 4.1 (cf. [32])**.** We will present a construction a 4-dimensional symplectic manifold M_{KT}, which will be a fiber bundle over the torus, with tori as fibers. A torus is just the quotient of an Euclidean space by a lattice, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Define an action ρ of $\mathbb{Z}^2$ on $\mathbb{T}^2$ by $\rho(1,0) = \mathrm{id}$ and $\rho(0,1)$ induced on $\mathbb{T}^2$ by an operator A on $\mathbb{R}^2$ with matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is well-defined, because $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$. Consider the product $\mathbb{R}^2 \times \mathbb{T}^2$ with the action $\widetilde{\rho}$ of $\mathbb{Z}^2$ given in the natural way on $\mathbb{R}^2$ and by ρ on $\mathbb{T}^2$. Fibers of $\mathbb{R}^2 \times \mathbb{T}^2 \to \mathbb{R}^2$ are preserved by this action, therefore Kodaira–Thurston's manifold

$$M_{KT} = \left(\mathbb{R}^2 \times \mathbb{T}^2\right)/\widetilde{\rho}$$

is a fibre bundle $\mathbb{T}^2 \to M \to \mathbb{T}^2$. Differential forms

$$\xi_1' = \mathrm{d}\,x_1,\ \ \xi_2' = \mathrm{d}\,x_2,$$
$$\xi_3' = \mathrm{d}\,x_3 - x_2\,\mathrm{d}\,x_4,\ \ \xi_4' = \mathrm{d}\,x_4$$

on $\mathbb{R}^4$ descend to $\mathbb{R}^2 \times \mathbb{T}^2$ and then, being invariant with respect to the action $\widetilde{\rho}$, to the forms ξ_1, ξ_2, ξ_3, ξ_4 on M_{KT}. The form $\omega_{M_{KT}} = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4$ is clearly closed and nondegenerate, and hence $(M_{KT}, \omega_{M_{KT}})$ is symplectic. We will now check that it is not Kähler.

Observe that the universal cover of M_{KT} is $\mathbb{R}^4$ and the group of deck transformations, $\mathrm{Aut}_{M_{KT}}(\mathbb{R}^4)$, is generated by

$$\phi_1 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \mapsto (x_1 + 1, x_2, x_3, x_4) \in \mathbb{R}^4$$
$$\phi_2 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + 1, x_3 + x_4, x_4) \in \mathbb{R}^4$$
$$\phi_3 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + 1, x_4) \in \mathbb{R}^4$$
$$\phi_4 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4 + 1) \in \mathbb{R}^4$$

Using the duality, universal coeffitient theorem and Hurewicz theorem we obtain

$$\begin{aligned} H^1(M_{KT}) &\cong (\pi_1(M_{KT})/\left[\pi_1(M_{KT}), \pi_1(M_{KT})\right]) \otimes \mathbb{R} \\ &\cong \left(\mathrm{Aut}_{M_{KT}}(\mathbb{R}^4)/\left[\mathrm{Aut}_{M_{KT}}(\mathbb{R}^4), \mathrm{Aut}_{M_{KT}}(\mathbb{R}^4)\right]\right) \otimes \mathbb{R} \end{aligned}$$

There is only one nonzero commutator in $\mathrm{Aut}_{M_{KT}}(\mathbb{R}^4)$, that is $\phi_2^{-1}\phi_4^{-1}\phi_2\phi_4 = \phi_3$, so $H^1(M_{KT}) \cong \mathbb{R}^3$ and $b^1(M_{KT}) = 3$.

Kodaira–Thuston's manifold is an example of a compact nilmanifold, that is a compact quotient of a nilpotent Lie group (in this case $\mathbb{R}^4$). This class of manifolds is very convenient for the construction of numerous examples, because its cohomology ring can be obtained in a strictly algebraic way. This is due to the following

Theorem 4.2 (Nomizu, cf. [30])**.** *Let M be a compact nilmanifold. Then there exist the unique simple connected nilpotent Lie group G and the lattice $\Gamma \subset G$, such that $M \cong G/\Gamma$ and the cohomology ring of the manifold M is isomorphic to the cohomology $H^*(\mathfrak{g})$ of $\mathfrak{g}$ (here $H^*(\mathfrak{g})$ denotes the cohomology of Chevalley and Eilenberg, cf.* [11]*)..*

Applying this result to M_{KT} we can compute its cohomology.

Cohomology	Generators
$H^0(M_{KT})$	1
$H^1(M_{KT})$	$\xi_1,\ \xi_2,\ \xi_4$
$H^2(M_{KT})$	$\xi_1\wedge\xi_2,\ \xi_1\wedge\xi_4,\ \xi_2\wedge\xi_3,\ \xi_3\wedge\xi_4$
$H^3(M_{KT})$	$\xi_1\wedge\xi_2\wedge\xi_3,\ \xi_1\wedge\xi_3\wedge\xi_4,\ \xi_2\wedge\xi_3\wedge\xi_4$
$H^4(M_{KT})$	$\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_4$

It is immediate that $\omega_{M_{KT}}\wedge\xi_4 = \mathrm{d}(\xi_1\wedge\xi_3)$, and so the Lefschetz property fails for M_{KT}. A minimal model of $H^*(M_{KT})$ is just

$$\mathbb{R}\subset \bigwedge\{\zeta_1,\zeta_2,\zeta_4\}\subset\bigwedge\{\zeta_1,\zeta_2,\zeta_4\}\otimes\bigwedge\{\zeta_3\}$$

with all generators ζ_1, ζ_2, ζ_3, ζ_4 of degree one and the differential d defined on generators as $\mathrm{d}\,\zeta_1 = \mathrm{d}\,\zeta_2 = \mathrm{d}\,\zeta_4 = 0$, $\mathrm{d}\,\zeta_3 = -\zeta_2\zeta_4$. It cannot be formal, since $[\zeta_2][\zeta_4]\neq 0$.

The following family of symplectic manifolds is the generalization of Kodaira–Thurston example.

Example 4.3 (cf. [12])**.** A generalized Heisenberg group $H(1,p)$ is a subgroup of $\mathrm{GL}(p+2;\mathbb{R})$ consisting of matrices of the form

$$H(1,p)=\left\{H\in\mathrm{GL}(p+2;\mathbb{R}):\ H=\begin{bmatrix} I_p & A & C\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\right\}$$

where A and C are $1\times p$ matrices and I_p is the $p\times p$ identity matrix. We can impose system of coordinates on $H(1,p)$ by setting $x_i(H)=a_i$, $y(H)=b$ and $z_i(H)=c_i$ where $A=\begin{bmatrix} a_1 & \dots & a_p\end{bmatrix}^T$, $C=\begin{bmatrix} c_1 & \dots & c_p\end{bmatrix}^T$ and $i=1,\dots,p$. Then it is a simple calculation that invariant forms on $H(1,p)$ are

$$\mathrm{d}\,x_1,\dots,\mathrm{d}\,x_p,\mathrm{d}\,y,\mathrm{d}\,z_1-x_1\,\mathrm{d}\,y,\dots,\mathrm{d}\,z_p-x_p\,\mathrm{d}\,y.$$

Now let $H(p,q)=H(1,p)\times H(1,q)$ and let $\Gamma\subset H(p,q)$ be a lattice of matrices with integer entries. Then the manifold $M(p,q)=H(p,q)/\Gamma$ is a compact homogeneous manifold. If $\mathrm{d}\,x_i^1$, $\mathrm{d}\,y^1$, $\mathrm{d}\,z_i^1$, for $i=1,\dots,p$, are invariant forms on $H(1,p)$, and $\mathrm{d}\,x_i^2$, $\mathrm{d}\,y^2$, $\mathrm{d}\,z_i^2$, for $i=1,\dots,q$, are invariant forms on $H(1,q)$ then $\mathrm{d}\,x_i^1\otimes 1$, $\mathrm{d}\,y^1\otimes 1$, $\mathrm{d}\,z_i^1\otimes 1$, $1\otimes\mathrm{d}\,x_i^2$, $1\otimes\mathrm{d}\,y^2$, $1\otimes\mathrm{d}\,z_i^2$ descend to $M(p,q)$ to forms ξ_i^1, η^1, ζ_i^1, ξ_i^2, η^2, ζ_i^2 which are the basis of invariants forms on $M(p,q)$. A 2-form

$$\omega=\xi_1^1\wedge\zeta_1^1+\ldots+\xi_p^1\wedge\zeta_p^1+\xi_1^2\wedge\zeta_1^2+\ldots+\xi_q^2\wedge\zeta_q^2+\eta^1\wedge\eta^2$$

is a symplectic form on $M(p,q)$.

Since the Lie algebra of $H(1,p)$ consists of strictly upper triangular matrices, it is nilpotent, and $M(p,q)$ is a nilmanifold. It means that the cohomology of $M(p,q)$ is encoded in cohomology of invariant forms, which is generated by ξ_i^1, η^1, ζ_i^1, ξ_i^2, η^2, ζ_i^2 with $\mathrm{d}\,\xi_i^1=\mathrm{d}\,\eta^1=\mathrm{d}\,\xi_i^2=\mathrm{d}\,\eta^2=0$, $\mathrm{d}\,\zeta_i^1=-\xi_i^1\wedge\eta^1$, $\mathrm{d}\,\zeta_i^2=-\xi_i^2\wedge\eta^2$. Using this, we could calculate the full cohomology ring of $M(p,q)$ in order to find its Betti numbers and to examine the Hard Lefschetz Property. For example $b^1(M(p,q))=p+q+2$, and so for $p+q$ odd, $M(p,q)$ is not of Kähler type. The calculation of higher Betti numbers is not so short, while still straightforward. In particular, it seems that if b^1 is even then so is b^3. The converse is not true, for example $b^1(M(3,0))=5$ but $b^3(M(3,0))=22$. The author is not aware of any other relation of this kind, but investigating it does not seem viable, for the fact that $M(p,q)$ is not Kähler can be established much easier using the other two criteria.

And so, to check that $M(p,q)$ does not satisfy the Hard Lefschetz Property it is sufficient to consider 1-form ξ_i^1 (or ξ_i^2). Then

$$\begin{aligned}\frac{1}{(p+q)!}\omega^{p+q}\wedge\xi_i^1 &= \pm\xi_1^1\wedge\ldots\wedge\xi_p^1\wedge\eta^1\wedge\zeta_1^1\wedge\ldots\wedge\widehat{\zeta_i^1}\wedge\ldots\wedge\zeta_p^1\\ &\quad\wedge\xi_1^2\wedge\ldots\wedge\xi_q^2\wedge\eta^2\wedge\zeta_1^2\wedge\ldots\wedge\zeta_q^2\\ &= \pm\,\mathrm{d}\Big(\xi_1^1\wedge\ldots\wedge\widehat{\xi_i^1}\wedge\ldots\wedge\xi_p^1\wedge\zeta_1^1\wedge\ldots\wedge\zeta_p^1\\ &\quad\wedge\xi_1^2\wedge\ldots\wedge\xi_q^2\wedge\eta^2\wedge\zeta_1^2\wedge\ldots\wedge\zeta_q^2\Big)\end{aligned}$$

and so $[\xi_i^1][\omega]^{p+q}=0$. To obtain nonformality, we will make use of Proposition 3.8. We compute the triple Massey product $\langle[\eta^1],[\xi_1^1],[\xi_1^1]\rangle$. It is well-defined, since $\eta^1\wedge\xi_1^1=\mathrm{d}\,\zeta_1^1$, and for this particular choice of generators the resulting class is $[\zeta_1^1\wedge\xi_1^1]$, which is clearly nonzero.

Due to Proposition 3.8, we do not need to compute minimal model of $M(p,q)$ to prove, that it is not formal. But actually, in this case, similarly as in Kodaira–Thurston manifold's case, it is quite simple, namely

$$\begin{aligned}\mathbb{R}&\subset\bigwedge\left\{\xi_1^1,\ldots,\xi_p^1,\eta^1,\xi_1^2,\ldots,\xi_q^2,\eta^2\right\}\\ &\subset\bigwedge\left\{\xi_1^1,\ldots,\xi_p^1,\eta^1,\xi_1^2,\ldots,\xi_q^2,\eta^2\right\}\otimes\bigwedge\left\{\zeta_1^1,\ldots,\zeta_p^1,\zeta_1^2,\ldots,\zeta_q^2\right\}.\end{aligned}$$

In fact, all nilmanifolds admit minimal models of this kind. This simplicity of minimal model is sufficient for the following characterization

Proposition 4.4 (cf. [6])**.** *A closed nilmanifold is formal iff it is a torus.*

Proof. Tori are obviously formal, so we need only to prove the other implication. Let M be a closed nilmanifold, $M=G/\Gamma$ for a simply connected nilpotent Lie group G. We will show that if M is formal then $\mathfrak{g}=\mathrm{Lie}(G)$ is Abelian, so G is Abelian. But the unique Abelian simply connected nilpotent Lie group G of dimension n is $\mathbb{R}^n$, which concludes the proof.

Since $\mathfrak{g}$ is nilpotent, its lower central series

$$\mathfrak{D}_1=\mathfrak{g},\qquad\mathfrak{D}_{k+1}=[\mathfrak{g},\mathfrak{D}_k]$$

vanishes, i.e. there exist a minimal $k\geq 1$ such that $\mathfrak{D}_{k+1}=0$. If we take a basis of $\mathfrak{D}_k$, extend it to a basis of $\mathfrak{D}_{k-1}$ and proceed until we obtain a basis for $\mathfrak{g}$. This basis, $\{X_1,\ldots,X_n\}$, will have the property that

$$[X_i,X_j]=\sum_{k<\min\{i,j\}}a_{ij}^kX_k.$$

So we can choose a basis $\{\xi_1,\ldots,\xi_n\}$ of $\mathfrak{g}^*$ (by taking the dual basis of $\{X_1,\ldots,X_n\}$ and reversing the order of indices) with property that

$$\mathrm{d}\,\xi_k=\sum_{k>\max\{i,j\}}b_k^{ij}\xi_i\xi_j.$$

Therefore a minimal model $\mathcal{M}$ of M is just

$$\begin{aligned}\mathbb{R}&\subset\bigwedge\left\{\xi_1\ldots,\xi_{i_1}\right\}\subset\bigwedge\left\{\xi_1\ldots,\xi_{i_1}\right\}\otimes\bigwedge\left\{\xi_{i_1+1}\ldots,\xi_{i_2}\right\}\subset\ldots\\ &\subset\bigwedge\left\{\xi_1\ldots,\xi_{i_1}\right\}\otimes\ldots\otimes\bigwedge\left\{\xi_{i_{s-1}+1}\ldots,\xi_n\right\},\end{aligned}$$

for some sequence $1 = i_0 < i_1 < \ldots < i_{s-1} < i_s = n$. In this notation we have $i_1 = \dim \mathfrak{D}_k = \dim Z(\mathfrak{g})$. From the way the differential is given it is clear that $\mathrm{d}\,\xi_1 \ldots \widehat{\xi_p} \ldots \xi_n = 0$, for $p = 1, \ldots, n$, so the highest cohomology of $\mathfrak{g}$ is $H^n(\mathfrak{g}) = \mathbb{R}\{\xi_1 \ldots \xi_n\} \cong \mathbb{R}$. Let $\rho : \mathcal{M} \to H^*\mathfrak{g}$ be a DGA-morphism inducing isomorphism in cohomology, and let $\rho^1 : \mathfrak{g} \to H^1(\mathfrak{g})$ be its restriction to $\mathcal{M}^1 \cong \mathfrak{g}$. Then $\rho^1(x_i) = [x_i]$ for $i = 1, \ldots, i_1$, and the cohomology classes $[\xi_1], \ldots, [\xi_{i_1}]$ form a basis of $H^1(\mathfrak{g})$. So, if $\{\zeta_{i_1+1}, \ldots, \zeta_n\}$ is a basis for $\ker \rho^1$ then $\{\xi_1, \ldots, \xi_{i_1}, \zeta_{i_1+1}, \ldots, \zeta_n\}$ is a basis of $\mathfrak{g}$. Let A be a transition matrix between this basis and $\{\xi_1, \ldots, \xi_n\}$. Then

$$
\begin{aligned}
0 &= \rho^1(\xi_1) \ldots \rho^1(\xi_{i_1})\rho^1(\zeta_{i_1+1}) \ldots \rho^1(\zeta_n) = \rho(\xi_1 \ldots \xi_{i_1}\zeta_{i_1+1} \ldots \zeta_n) \\
&= (\det A)\rho(\xi_1 \ldots \xi_n) = (\det A)[\xi_1 \ldots \xi_n] \neq 0.
\end{aligned}
$$

This contradiction concludes the proof. □

Further in [12] we can find another two constructions, both based on the complexification of the generalised Heisenberg group, but they are essentially the same. In particular, they have one thing in common, which turns out to be common to all nilmanifolds, namely they are non simply connected. It raises the question how to find analogous examples among simply connected manifolds. Answer to this question is given by introducing the symplectic version of blowing up complex manifolds.

5. Blowing up

In this section we shall consider a symplectic $2n$-dimensional manifold X and its compact symplectic submanifold $M \subset X$ of codimension $2q$, $q \geq 2$. We can endow the normal bundle $TX/TM = E \to M$ with a linear symplectic structure (via $E \cong T^\omega M$), and hence also an almost complex structure. With this structure, we can identify the fibre of E with complex space,

$$\mathbb{C}^q \to E \to M.$$

Let $\widetilde{M}$ denote the bundle over M obtained by fiberwise projectivization of E,

$$\mathbb{C}P^{q-1} \to \widetilde{M} \to M.$$

Over the complex projective space $\mathbb{C}P^{q-1}$, we can construct the *tautological bundle*. Points of $\mathbb{C}P^{q-1}$ are the complex lines $l \subset \mathbb{C}^q$. The tautological bundle $F \to \mathbb{C}P^{q-1}$ can be described as a subset $F \subset \mathbb{C}P^{q-1} \times \mathbb{C}^q$,

$$F = \left\{(l, p) \in \mathbb{C}P^{q-1} \times \mathbb{C}^q : \, p \in l\right\}.$$

It is easy to see that F_0, the space F without the zero section, is diffeomorphic to $\mathbb{C}^p_0 = \mathbb{C}^p \setminus \{0\}$. Now, if we take the tautological bundle over each fiber of $\widetilde{M}$, we get the bundle

$$\mathbb{C} \to \widetilde{E} \to \widetilde{M}.$$

Clearly, $E_0 \cong \widetilde{E}_0$, where E_0 and $\widetilde{E}_0$ are the respective bundles without zero sections and the isomorphism is a restriction of a natural projection $p : \widetilde{E} \to E$. This isomorphism lets us cut out a tubular neighborhood of M in X, which is diffeomorphic to E, and replace it with $\widetilde{E}$. Formally, consider a diffeomorphism $e : E \to e(E) \subset X$, with $e(E)$ being a tubular neighborhood of M and $e(0_E) = M$, and let V be a neighborhood of 0_E (the zero section) in E such that ∂V is a $(2q-1)$-dimensional submanifold of E.

Definition 5.1. The blow-up of a symplectic manifold X along its symplectic submanifold M is a space

$$\widetilde{X} = (X \setminus e(V)) \cup_{\partial V} \left(p^{-1}(\overline{V})\right). \tag{1}$$

Observation.

- While we have made some arbitrary choices in the construction (complex structure on E, tubular neighborhood $e : E \to X$, $V \subset E$), the smooth structure on $\widetilde{X}$ does not depend on these choices (in contrary to, as we will observe, the symplectic structure).
- The projection $p : \widetilde{E} \to E$ induces a map $f : \widetilde{X} \to X$ which is diffeomorphic outside $f^{-1}(M)$.
- We can conduct the same construction for M of codimension 2, but the resulting manifold $\widetilde{X}$ would be just X.
- A neighborhood V in the above construction may be chosen such that the inclusion $p^{-1} : \partial V \to p^{-1}(\overline{V})$ induces an isomorphism in the fundamental groups,
$$p_*^{-1} : \pi_1(\partial V) \cong \pi_1(p^{-1}(\overline{V})).$$

Before constructing the symplectic structure on the blow-up $\widetilde{X}$, we will give its topological description. First, we have the following

Proposition 5.2. *The fundamental group of $\widetilde{X}$ is isomorphic to the fundamental group of X.*

Proof. Applying the Seifert–Van Kampen Theorem to the decomposition (1) we see that $\pi_1(\widetilde{X})$ is the pushout in the diagram

$$\begin{array}{ccc} \pi_1(\partial V) & \xrightarrow{p_*^{-1}} & \pi_1(p^{-1}(\overline{V})) \\ {\scriptstyle e_*}\downarrow & & \downarrow \\ \pi_1(X \setminus e(V)) & \dashrightarrow & \pi_1(\widetilde{X}). \end{array}$$

If we now choose V such that the upper horizontal arrow becomes an isomorphism it is clear, that $\pi_1(\widetilde{X}) \cong \pi_1(X \setminus e(V)) \cong \pi_1(X \setminus M)$, where the second isomorphism comes from the homotopy equivalence of $X \setminus e(V)$ and $X \setminus M$. But the codimension of M in X is greater or equal to 4 and so, by the classical argument in differential topology, $\pi_1(X \setminus M) \cong \pi_1(X)$. $\square$

Now we will describe the cohomology ring of the blow-up in terms of the cohomology rings of M and X. Recall the tautological bundle $\mathbb{C} \to F \to \mathbb{C}P^{q-1}$. The projective space is a Kähler space. Let $\omega_{\mathbb{C}P^{q-1}}$ be a symplectic form on $\mathbb{C}P^{q-1}$ associated with the standard Kähler metric. It has the property that $H^*(\mathbb{C}P^{q-1}) \cong \Lambda \{[\omega_{\mathbb{C}P^{q-1}}]\} / ([\omega_{\mathbb{C}P^{q-1}}]^q)$. The projection $\pi : F \to \mathbb{C}P^{q-1}$ induces isomorphism in cohomology (the base of a vector bundle is a deformation retract of its total space, and the projection is the left inverse of the inclusion). Hence for $a_F = \pi^* [\omega_{\mathbb{C}P^{q-1}}] \in H^2(F)$ we have $H^*(F) \cong \Lambda (a_F) / (a_F^q)$. The following lemma states that this construction can be done simultaneously on each fiber of $\widetilde{E}$.

Lemma 5.3. *There exist a class $a \in H^2(\widetilde{E})$ which restricts to each fiber F_p of the bundle $F \to \widetilde{E} \to M$ as a generator of $H^2(F_p)$ and to $\widetilde{E}_0$ as zero.*

Proof. First, observe that it is sufficient to find a class $w \in \widetilde{M}$ which restriction to each fiber of $\mathbb{C}P^{q-1} \to \widetilde{M} \to M$ generates the second cohomology of this fiber, and then take a equal to the pullback of w. If the bundle $\widetilde{M}$ is trivial, $\widetilde{M} \cong M \times \mathbb{C}P^{q-1}$, we take $[\omega_{\mathbb{C}P^{q-1}}]$. In the general case, we can do this locally and prove that this class is invariant by the action of transition maps on the cohomology of the fiber.

Another way to prove this theorem is to observe that the class $[\omega_{\mathbb{C}P^{q-1}}]$ is just the first Chern class of the dual of the tautological bundle, $[\omega_{\mathbb{C}P^{q-1}}] = c_1(F^*)$. This, together with the naturality of characteristic classes, lets us define $w = c_1(\widetilde{E}^*)$ for $\widetilde{E}^*$, the dual of the vector bundle $\mathbb{C} \to \widetilde{E} \to \widetilde{M}$. □

In order to proceed, we have to notice, that the classes $\{1, a, a^2, \ldots, a^{q-1}\}$ form a basis of $H^*(\widetilde{E})$ as a vector space, and therefore satisfy the assumptions of the Leray-Hirsch Theorem (cf. [23], Theorem 4D.1). So, we obtain

Proposition 5.4. *$H^*(\widetilde{E})$ is a free $H^*(M)$-module with a base $\{1, a, a^2, \ldots, a^{q-1}\}$.*

Knowing the cohomology of $\widetilde{E}$ we can finally calculate the cohomology of $\widetilde{X}$.

Proposition 5.5. *There is a short exact sequence*

$$0 \to H^*(X) \to H^*(\widetilde{X}) \to A^* \to 0,$$

where A^* is a free $H^*(M)$-module with generators $\{a, a^2, \ldots, a^{q-1}\}$.

For the proof we need a "technical" lemma.

Lemma 5.6. *For arbitrary orientable manifolds M, N of dimension n and a map $f : M \to N$ of degree $\deg f \neq 0$, the induced mapping $f^* : H^*(N) \to H^*(M)$ is a monomorphism.*

Proof. For an orientable manifold M the map $H^n \ni [\xi] \mapsto \int_M \xi \in \mathbb{R}$ is a well-defined isomorphism and

$$\int_M f^*\xi = (\deg f) \int_N \xi$$

for every form ξ on N. We can deduce that $f^* : H^n(N) \to H^n(M)$ is an isomorphism. Now, let $0 \neq a \in H^k(N)$. Let $b \in H^{n-k}(M)$ be such, that $a \wedge b \neq 0$. Then, $0 \neq f^*(a \wedge b) = f^*(a) \wedge f^*(b)$ and in particular $f^*(a) \neq 0$. □

Proof of Proposition 5.5. Consider the mapping cylinder $C_f = \widetilde{X} \times [0,1] \sqcup X/\sim$ where the equivalence relation is given by $(x,1) \sim f(x)$. We have the following inclusions

$$i : \widetilde{X} \ni x \mapsto [(x,0)] \in C_f$$
$$j : X \ni x \mapsto [x] \in C_f,$$

and j induces isomorphism in cohomology. The diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\quad f^* \quad} & H^*(\widetilde{X}) \\ {\scriptstyle j^*,\ \cong} \nwarrow & & \nearrow {\scriptstyle i^*} \\ & H^*(C_f) & \end{array}$$

commutes and the horizontal arrow is a monomorphism (Lemma 5.6), so i^* is a monomorphism. Hence the long exact sequence of the pair $(C_f, \widetilde{X})$

$$\ldots \to H^*(C_f, \widetilde{X}) \to H^*(C_f) \xrightarrow{i^*} H^*(\widetilde{X}) \to H^{*+1}(C_f, \widetilde{X}) \to \ldots$$

splits into short exact sequences of the form

$$0 \to H^*(C_f) \xrightarrow{i^*} H^*(\widetilde{X}) \to H^{*+1}(C_f, \widetilde{X}) \to 0.$$

Since $H^*(C_f) \cong H^*(X)$ it remains to see that $H^{*+1}(C_f, \widetilde{X}) = A^*$ is of the form described in the statement of the proposition. Using excision twice, we have

$$H^{*+1}(C_f, \widetilde{X}) \cong H^{*+1}(C_{p|_{p^{-1}(V)}}, p^{-1}(V)) \cong H^{*+1}(C_p, \widetilde{E}).$$

Argument similar to the one given above shows that the long exact sequence of the pair $(C_p, \widetilde{E})$ splits into short exact sequences

$$0 \to H^*(C_p) \to H^*(\widetilde{E}) \to H^{*+1}(C_p, \widetilde{E}) \to 0.$$

Finally, $H^*(C_p) \cong H^*(E) \cong H^*(M)$, $H^*(\widetilde{E}) \cong H^*(M)\left\{1, a, a^2, \ldots, a^{q-1}\right\}$ and $H^{*+1}(C_p, \widetilde{E}) \cong A^*$, which concludes the proof. □

It remains to see that $\widetilde{X}$ is symplectic. While the formal proof of this fact is rather technical (cf. [27]), the idea is quite simple. If ω_X is a symplectic form on X, then $f^*\omega_X$ is a symplectic form outside $f^{-1}(M) \cong 0_{\widetilde{E}} \cong \widetilde{M}$. The pullback of ω_X to $\widetilde{M}$ is closed and nondegenerate in the directions transverse to the fibers of $\mathbb{C}P^{q-1} \to \widetilde{M} \to M$, but it is zero in the fiberwise direction. So we need to add to $f^*\omega_X$ a closed 2-form on $\widetilde{E}$, whose restriction to $\widetilde{M}$ is nondegenerate in the $\mathbb{C}P^{q-1}$ direction. From the description it is clear that we should try to find it among representatives of the class a of Lemma 5.3. In fact, it turns out that such a representative $\alpha \in a$, with the additional property that its support lies in $p^{-1}(V)$, exists. Now, $f^*\omega_X + \alpha$ may not be nondegenerate in general but, due to M being compact, there exists some $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ this sum is nondegenerate, and hence symplectic. Observe that a different choice of ε yields different non-cohomologous symplectic structures.

We can summarize all these results in the following

Theorem 5.7. *For a symplectic manifold X and its symplectic submanifold M of codimension $2q$, there exist a symplectic manifold $\widetilde{X}$, called the blow-up of X along M, and a map $f : \widetilde{X} \to X$ such that*

1. *The dimension of $\widetilde{X}$ coincides with the dimension of X.*
2. *If $q = 1$ then $\widetilde{X} = X$.*
3. *The map $f|_{\widetilde{X}\setminus f^{-1}(M)} : \widetilde{X} \setminus f^{-1}(M) \to X \setminus M$ is a diffeomorphism.*
4. *The map $f|_{f^{-1}(M)} : f^{-1}(M) \to M$ is a bundle with fiber $\mathbb{C}P^{q-1}$.*
5. *$\pi_1(\widetilde{X}) = \pi_1(X)$.*
6. *The cohomology of $\widetilde{X}$ is encoded in the short exact sequence*

$$0 \to H^*(X) \to H^*(\widetilde{X}) \to A^* \to 0,$$

where A^* is a free $H^*(M)$-module with generators $\left\{a, a^2, \ldots, a^{q-1}\right\}$.

We can now construct the McDuff's example.

Example 5.8 (cf. [27])**.** Let M_{KT} be a Thurston nilmanifold (cf. Example 4.1). Tischler proved in [33] that any symplectic manifold with integral symplectic form can be symplectically embedded in $\mathbb{C}P^k$ for k big enough and Gromov in [20] strenghtened this result showing that k can be taken to be $k = 2m+1$, where $2m$ is the dimension of embedded manifold. In particular, M_{KT} can be embedded into $\mathbb{C}P^5$. McDuff's example is a blow-up

$$M_{MD} = \widetilde{\mathbb{C}P^5}^{M_{KT}}$$

of $\mathbb{C}P^5$ along M_{KT}. It is a 10-dimensional, symplectic manifold. It is simple connected, since $\pi_1(M_{MD}) \cong \pi_1(\mathbb{C}P^5) = 0$. Finally, $H^3(M_{MD}) = aH^1(M_{KT})$, so $b^3(M_{MD}) = b^1(M_{KT}) = 3$ and M_{MD} is not of Kähler type.

Observe that the only property of M_{KT} which was used in the above example is the fact that its first Betti number is odd. Therefore we could take instead any other compact symplectic manifold with odd first Betti number, like for example the manifold $M(r,s)$ for $r+s$ odd (cf. Example 4.3). Moreover, if we take any simply connected Kähler manifold X of dimension $2n$ for $n \geq 6$ (such a manifold always exist, for example $\mathbb{C}P^n$) and its blow-up $\widetilde{X}$ along an arbitrary point $p \in X$, the preimage of p by $f : \widetilde{X} \to X$ will be a projective space $\mathbb{C}P^{n-1}$, so now we can embed $M_{KT} \subset \mathbb{C}P^{n-1} \subset \widetilde{X}$ and blow-up $\widetilde{X}$ along M_{KT}, obtaining $\widehat{X}$. Clearly $\pi_1(\widetilde{X}) \cong \pi_1(X) = 0$ and $\pi_1(\widehat{X}) \cong \pi_1\widetilde{X} = 0$ (the codimension of M_{KT} in $\mathbb{C}P^{n-1}$ must be at least 2 for the embedding to exist, and hence codimension of M_{KT} in $\widetilde{X}$ must be at least 4). The cohomology of $\widetilde{X}$ fits into sequence

$$0 \to H^*(X) \to H^*(\widetilde{X}) \to \mathbb{R}\left\{a, a^2, \ldots, a^{n-3}\right\} \to 0,$$

so in particular $H^3(X) \cong H^3(\widetilde{X})$. Now, the short exact sequence for $\widehat{X}$ gives

$$0 \to H^3(\widetilde{X}) \to H^3(\widehat{X}) \to aH^1(M_{KT}) \to 0,$$

and $b^3(\widehat{X}) = b^3(\widetilde{X}) + b^1(M_{KT}) = b^3(X) + b^1(M_{KT})$ is odd, as the sum of an even and odd numbers.

To examine the Hard Lefschetz Property on M_{MD}, we need a slightly better description of the cohomology of a blow-up. Consider again a blow-up of an arbitrary symplectic manifold X of dimension $2n$ along its symplectic submanifold M of cosimension $2q$. We know that we have an isomorphism

$$\Phi : H^*(X) \oplus aH^{*-2}(M) \oplus a^2H^{*-4}(M) \oplus \ldots \oplus a^{q-1}H^{*-2(q-1)}(M) \cong H^*(\widetilde{X}),$$

where $\Phi|_{H^*(X)} = f^*$ and if $j : \widetilde{M} \to \widetilde{X}$ is a natural injection and $p : \widetilde{M} \to M$ a natural projection, then $j^*\Phi(a\xi) = ap^*\xi$. Here we identified a with its image by an isomorphims $H^*(\widetilde{E}) \cong H^*(\widetilde{M})$.

The isomorphism Φ is an isomorphism in the category of vector spaces, not algebras. It means that we still have to find the ring structure on $H^*(\widetilde{X})$. While it is clear that for $v_1, v_2 \in H^*(X)$ and $u_1, u_2 \in H^*(M)$ the following equalities hold

$$\begin{aligned} \Phi(v_1)\Phi(v_2) &= \Phi(v_1v_2), \\ \Phi(v_1)\Phi(a) &= \Phi(a\iota^*v_1), \\ \Phi(au_1)\Phi(au_2) &= \Phi(a^2u_1u_2), \end{aligned}$$

where $\iota : M \to X$, we have to establish to what $\Phi(a)^q$ is equal. From now on, we will omit Φ in notation. From Proposition 5.4, a^q must be of the form

$$a^q = u_q + au_{q-2} + \ldots + a^{q-1}u_1.$$

Observe that if we take any pullback of the bundle $\widetilde{E}$, that is $\varphi^*\widetilde{E} \to M'$ for any smooth $\varphi : M' \to M$, then f^*a will be the counterpart of a in $\varphi^*\widetilde{E} \to M'$ (because a was constructed as a Chern class, which is natural). In particular, $u_1, \ldots, u_q$ are natural, and so they may be obtained from the Chern classes of $E \to M$. The normalization axiom proves that $u_i = -c_i(E)$ for $i = 1, \ldots, q$ and we may write

$$a^q = -c_q(E) - ac_{q-2}(E) - \ldots - a^{q-1}c_1(E).$$

If we now apply these results to McDuff's example, we can show that it does not satisfy the Hard Lefschetz Property. First, $a^3 = -c_2(E)a - c_1(E)a^2$. The additivity of Chern classes gives $c_i(T\mathbb{C}P^5\big|_{M_{KT}}) = c_i(E) + c_i(TM_{KT})$. But M_{KT}, as a homogenous manifold, have a trivial tangent bundle, so all the Chern classes of M_{KT} vanish. Moreover, $T\mathbb{C}P^5\big|_{M_{KT}} = \iota^*T\mathbb{C}P^5$, so the naturality of Chern classes results in the equality $c_i(E) = \iota^*c_i(T\mathbb{C}P^5)$. Cohomology of $\mathbb{C}P^5$ is just

$$H^*(\mathbb{C}P^5) \cong \mathbb{R}\left\{1, \omega_{\mathbb{C}P^5}, \omega^2_{\mathbb{C}P^5}, \omega^3_{\mathbb{C}P^5}, \omega^4_{\mathbb{C}P^5}, \omega^5_{\mathbb{C}P^5}\right\},$$

so there exist constants λ, μ such that

$$a^3 = \lambda a\iota^*[\omega_{\mathbb{C}P^5}]^2 + \mu a^2\iota^*[\omega_{\mathbb{C}P^5}] = \lambda a[\omega_{M_{KT}}]^2 + \mu a^2[\omega_{M_{KT}}].$$

Recall that Kodaira–Thurston example does not satisfy the Hard Lefschetz Property, and $[\xi_4\omega_{M_{KT}}] = 0$. Now we compute

$$\begin{aligned}([\omega_{M_{\mathbb{C}P^5}}] + \varepsilon a)^2 a[\xi_4] &= \left([\omega_{M_{\mathbb{C}P^5}}]^2 + 2\varepsilon a[\omega_{M_{KT}}] + \varepsilon^2 a^2\right) a[\xi_4] \\ &= a[\omega^2_{M_{KT}}\xi_4] + 2\varepsilon a^2[\omega_{M_{KT}}\xi_4] + \varepsilon^2 a^3[\xi_4] = \varepsilon^2 a^3[\xi_4] \\ &= \varepsilon^2\lambda a[\omega^2_{M_{KT}}\xi_4] + \varepsilon^2\mu a^2[\omega_{M_{KT}}\xi_4] = 0.\end{aligned}$$

M_{MD} is also not formal. In fact, we can find a nonvanishing triple Massey product. We know that $[\omega_{\mathbb{C}P^5}]a[\xi_4] = a[\omega_{M_{KT}}\xi_4] = 0$ because $\omega_{M_{KT}}\xi_4 = \mathrm{d}(\xi_1\xi_3)$. Hence $\langle a[\xi_4], [\omega_{\mathbb{C}P^5}], a[\xi_4]\rangle$ is well defined and

$$\langle a[\xi_4], [\omega_{\mathbb{C}P^5}], a[\xi_4]\rangle = 2a[\xi_1][\xi_3][\xi_4] + a[\xi_4]H^4(\widetilde{X}) \not\in a[\xi_4]H^4(\widetilde{X}).$$

Both these results, that is the failure of the Hard Lefschetz Property and nonformality of M_{MD} are just special cases of more general results, whose proofs are analogical to the respective proofs for M_{MD} above.

Proposition 5.9 (cf. [27])**.** *Let M be a symplectic submanifold of a symplectic manifold X such that all Chern classes of M vanish. Then if the Hard Lefschetz Property fails for M it also fails for a symplectic blow up $\widetilde{X}$ of X along M.*

Proposition 5.10 (cf. [5] and also Theorem 5.11 below)**.** *For every $n \geq 5$, the symplectic blow up of $\mathbb{C}P^n$ along a symplectically embedded Kodaira–Thurston manifold is nonformal.*

More careful analysis of the behavior of the Hard Lefschetz Property and Massey products under blowing up can be found in [9]. Among others the author proves the following results

Theorem 5.11. *Let $\widetilde{X}$ be a symplectic blow up of symplectic manifold X of dimension $2n$ along its symplectic submanifold M of codimension $2q$. Then*

(1) *if both X and M satisfy the Hard Lefschetz Property and $2q > n$, then $\widetilde{X}$ also satisfies the Hard Lefschetz Property for properly chosen symplectic form on $\widetilde{X}$,*
(2) *if X has nontrivial triple Massey products, then so does $\widetilde{X}$,*
(3) *if M has nontrivial triple Massey products and $q > 3$, then so does $\widetilde{X}$.*

6. Connected sums

McDuff's example was the first known example of a compact simply-connected symplectic manifold which is not Kähler. The method she introduced, namely the symplectic blowing up, proved to be capable of producing whole families of examples (cf. [5]). But all these examples are of dimension greater or equal to 10, since 4-dimensional symplectic manifold embed in general into $\mathbb{C}P^n$ for $n \geq 5$. Examples of compact simply connected symplectic manifolds in dimensions lower than 10 and not being of Kähler were given for the first time by Gompf in [17]. Among others, Gompf constructed for each finitely presentable group G a compact simply connected symplectic manifold of dimension 4 with G as the fundamental group. To obtain all these results, he translated into the symplectic category the operation of connected sum, known from differential topology. Below we present his construction.

Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds of equal dimension $2n$ and let (N, σ) be a compact symplectic manifold of dimension $2n-2$, embedding symplectically into both M_1 and M_2 with trivial normal bundles E_1 and E_2, respectively. Symplectic neighborhood theorem (cf. [28], Theorem 3.30) states that we can choose a positive constant $\delta > 0$ and tubular neighborhoods of N in M_1 and M_2, denote them by V_1 and V_2, respectively, such that $(V_i, \omega_i|_{V_i})$ is symplectomorphic to $(N \times D(\delta), \sigma + \mathrm{d}\,x \wedge \mathrm{d}\,y)$ for $i = 1, 2$, where $D(\delta) \subset \mathbb{R}^2$ is a 2-dimensional disc of radius δ with the standard symplectic structure $\mathrm{d}\,x \wedge \mathrm{d}\,y$. We can find in dimension 2 an orientation-preserving diffeomorphism $\psi : D(\delta) \setminus \{0\} \to D(\delta) \setminus \{0\}$ which turns the punctured disc inside out. It is sufficient to find any isometric, orientation reversing fiberwise isomorphism of the trivial bundle $N \times \mathbb{R}^2$, restrict it to disc bundle and compose with the turning inside out. We can now use the resulting symplectomorphism $\psi : V_1 \setminus N \to V_2 \setminus N$ to glue $M_1 \setminus N$ with $M_2 \setminus N$, and the resulting manifold

$$M = M_1 \cup_N M_2$$

carries the natural symplectic structure.

This construction might be conducted in more generality, i.e., for nontrivial normal bundles. The only restriction is the existence of orientation-reversing fibrewise isomorphism between them. The existence of such an isomorphism is assured by a condition, that $e(E_1) = -e(E_2)$, so we can speak of connected sum along nontrivially embedded N under this additional assumption.

We will now use the connected sum construction to prove the following

Theorem 6.1 (cf. [17])**.** *For each finitely presentable group G there exist a symplectic 4-dimensional manifold M with $\pi_1(M) = G$ and not of Kähler type.*

Proof. We will only prove the existence of symplectic 4-manifolds and will not address the question of admitting Kähler metric at this point. Then the proof of this statement is in some way similar to the proof of its counterpart in differential topology, namely the theorem that each finitely presentable group is realised as a fundamental group of a smooth 4-manifold. We fix a representation, that is a set of generators $\{g_1, \ldots, g_k\}$ and a set of relations $\{r_1, \ldots, r_l\}$. We take a 4-manifold with a free fundamental group with a sufficient number of generators, and "kill" the relations by an appropriate surgery. Since in the symplectic category we need to refine the classic approach to conserve the symplectic form, we need more space, and hence our starting manifold will have $2k$ generators of its fundamental group. Let F be a surface of genus k. Then $\pi_1(F) = \langle a_1, \ldots, a_k, b_1, \ldots, b_k \rangle$. Choose a family $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ of smoothly immersed circles in F, generating $\pi_1(F)$. We can choose them in such a way that the intersections are $\alpha_i \cdot \beta_j = \delta_{ij}$. Now again, let $\gamma_1, \ldots, \gamma_{k+l}$ be a family of smoothly imbedded circles with $\gamma_i = \beta_i$ for $i = 1, \ldots, k$ and γ_{k+i} generating a word r_i (with a_i substituted for g_i) in $\pi_1(F)$. We can assume that they are in general position. The γ's represent exactly those elements of the fundamental group of F that we need to get rid off.

Consider the manifold $F \times \mathbb{T}^2$ with a symplectic form $\omega = p_1^*\omega_F + p_2^*\omega_{\mathbb{T}^2}$, where p_1, p_2 are natural projections onto the respective factors and ω_F, $\omega_{\mathbb{T}^2}$ are symplectic forms on these factors. For an arbitrary point $x \in \mathbb{S}^1$ let $\alpha = \mathbb{S}^1 \times \{x\}$ and $\beta = \{x\} \times \mathbb{S}^1$ be curves in $\mathbb{T}^2$ and let z be a point in F such that $z \notin \Gamma = \gamma_1 \cup \ldots \cup \gamma_{k+l}$. We would like to conduct a surgery along the tori $\gamma^i \times \alpha$ and $\{z\} \times \mathbb{T}^2$, but only the last one is symplectically embedded, while the others are Lagrangian submanifolds. But, using the following lemma, we can perturb the symplectic form on $F \times \mathbb{T}^2$ in such a way that all these tori will be symplectically embedded.

Lemma 6.2. *For a surface F of genus k and a family of smoothly immersed oriented circles $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ in general position, we can find a surface F' of genus $k' \geq k$ and a family $\Gamma' = \{\gamma'_1, \ldots, \gamma'_{m'}\}$ of smoothly immersed oriented circles in general position such that*

$$\pi_1(F')/\langle [\gamma'_1]_{\pi_1(F')}, \ldots, [\gamma'_{m'}]_{\pi_1(F')} \rangle \cong \pi_1(F)/\langle [\gamma_1]_{\pi_1(F)}, \ldots, [\gamma_m]_{\pi_1(F)} \rangle$$

and there exist a closed 1-form ρ on F' which restricts to volume forms on each oriented circle γ'_i.

The proof of this lemma will be given later.

The fact that we have to enlarge the genus of our surface and the family Γ of circles has no influence on the proof. At the other hand, the existence of ρ makes it possible to perturb the symplectic form in a desired way. In fact, let θ be a 1-form on $\mathbb{T}^2$ obtained as the pullback of the volume form on α by the natural projection $\mathbb{T}^2 \to \alpha$. For t sufficiently small, $\omega + tp_1^*\rho \wedge p_2^*\theta$ is a symplectic form on $F \times \mathbb{T}^2$ and on $\{z\} \times \mathbb{T}^2$ (the nondegeneracy of the form is an open condition). Moreover, its restricion to $\gamma_i \times \alpha$ is clearly symplectic for any i. The last obstruction to performing surgery is the nonempty intersection (and self-intersection) of tori $\mathbb{T}_i^2 = \gamma_i \times \alpha$. But if we consider γ_i as curves in $F \times \beta$, we get an extra dimension, sufficient to perturb γ's to a disjoint family of smoothly embedded curves. Moreover, if we choose the perturbation to be small enough, the tori $\mathbb{T}_i^2$ remain symplectic and disjoint with $\{z\} \times \mathbb{T}^2$. Observe that the normal bundle of $\mathbb{T}_i^2$ in $F \times \mathbb{T}^2$ is isomorphic to the normal bundle of γ_i in $F \times \beta$, and hence is trivial. At this point we need another lemma.

Lemma 6.3. *There exist a simply connected symplectic 4-manifold S with symplectically embedded torus such that the normal bundle of this embedding is trivial and $S \setminus \mathbb{T}^2$ is simply connected..*

In fact there exist a big family of such manifolds. Those are the so called elliptic spaces, which fiber over $\mathbb{C}P^1$ with the torus as a generic fiber. A good description of this family may be found in [18]. The simpliest example is a projective space $\mathbb{C}P^2$ blown up in 9 distinct point, or equivalently $\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ is the projective space with reversed orientation.

Finally, we construct M by cutting out all the tori $\mathbb{T}^2_i$ and $\{z\} \times \mathbb{T}^2$ and gluing in the manifold S. To compute the fundamental group of M let see how it behaves after surgery. For simplicity, we consider gluing in only one copy of S but the general case is clearly analogous. Since $\pi_1(S \setminus \mathbb{T}^2) = 0$, the Seifert–Van Kampen's theorem implies that $\pi_1(M)$ is the pushout in the following diagram

$$\begin{array}{ccc} & \pi_1((F \times \mathbb{T}^2) \setminus \mathbb{T}^2_i) & \\ \nearrow & & \searrow \\ \pi_1((\mathbb{T}^2 \times D(\delta)) \setminus (\mathbb{T}^2 \times \{0\})) & & \pi_1(M) \\ \searrow & & \nearrow \\ & \pi_1(S \setminus \mathbb{T}^2) = 0. & \end{array}$$

Therefore

$$\begin{aligned}\pi_1(M) &\cong \pi_1((F \times \mathbb{T}^2) \setminus \mathbb{T}^2_i)/\operatorname{im}(\pi_1((\mathbb{T}^2 \times D(\delta)) \setminus (\mathbb{T}^2 \times \{0\})) \to \pi_1((F \times \mathbb{T}^2) \setminus \mathbb{T}^2_i)) \\ &\cong \pi_1(F \times \mathbb{T}^2)/\operatorname{im}(\pi_1(\mathbb{T}^2) \to \pi_1(F \times \mathbb{T}^2)).\end{aligned}$$

□

Proof of Lemma 6.2. We have a surface F and a family Γ of smoothly immersed circles in general position. Since the curves in Γ are in general position, we can perceive it as a graph. Consider points $x \neq y \in \mathbb{S}^1$ and curves $\alpha = \mathbb{S}^1 \times \{x\}$, $\beta = \{x\} \times \mathbb{S}^1$, $\gamma = \{y\} \times \mathbb{S}^1$ embedded into some torus $\mathbb{T}^2$ with chosen orientation. Let $D \subset \mathbb{T}^2$ be a disc on the torus centered on some point of γ and disjoint with α and β. Now, for each edge of Γ take a disc D_i centered at some point of this edge and disjoint with all other edges. Let F' be a surface F with copies of $\mathbb{T}^2$ attached to every edge by the diffeomorphism of discs punctured in the center $D^* \cong D^*_i$, which sends $\gamma \cap D^*$ diffeomorphically into $\gamma_i \cap D^*_i$, and preserves orientation. Let Γ' be a set consisting of prolonged curves γ_i together with all copies af α and β. Let ρ_0 be a form on $\mathbb{T}^2$ which vanishes near D and such that $\int_e \rho_0 > 0$ for each edge e of graph $\alpha \cup \beta \cup (\gamma \cap (\mathbb{T}^2 \setminus D))$. We can take a form ρ which is equal to ρ_0 on tori and is zero outside them. It has the property, that for any edge $e \in \Gamma'$ the integral $\int_e \rho > 0$. For each $\gamma \in \Gamma'$ we can choose a volume form θ_γ on γ such that for each edge $e \subset \gamma$ of Γ' the equality $\int_e \theta_\gamma = \int_e \rho$ holds, and in particular $\theta_\gamma - \rho$ is exact, when restricted to γ. We can find a function f_γ with $\mathrm{d}\, f_\gamma = \theta_\gamma - \rho|_\gamma$. Moreover, we may choose f_γ to vanish on the edges so that f on Γ' equal to f_γ on each γ is well defined. Extend f to the whole manifold F' (with the same notation). Then $\rho + \mathrm{d}\, f$ is the desired 1-form. □

We have not proven that these manifolds are not of Kähler type. For many of them it is an immediate consequence of the properties of their fundamental group. We say that the group is Kähler iff it is a fundamental group of some compact Kähler manifold. This condition is in fact quite restrictive, see for example

[1, 2, 3, 34]. The most obvious restriction comes from Hurewicz Theorem which binds first cohomology class of manifold with the non-torsion part of its fundamental group. In particular any group of the form $G = \mathbb{Z}^{2k+1} \oplus G'$, where G' is torsion, is not Kähler. Below are some examples of groups which, for various other reasons, cannot be Kähler.

- Free groups.
- The integral Heisenberg group $H(1,1;\mathbb{Z})$, ie. a subgroup of Heisenberg group consisting of matrices with integral entries.
- The Heisenberg group of Gaussian integers $H(1,1;\mathbb{Z}[i])$.
- $\langle x, y : [[x,y],y]\rangle$.
- $\langle x, y, z : xyxzxzxy\rangle$.
- $\langle x, y, z, w : [x,y][z,w], [[[[y,x],x],x],y]\rangle$.
- $\langle x, y, z, w : [xy, z^2], [xzx, w^3]\rangle$.
- $\langle x, y, z, w : x^3y^{-4}z^2y, y^2z^2\rangle$.
- $\langle x, y, z, w, t : x^2y^2z^2, y^2z^2w^2, z^2w^2t^2\rangle$.
- $\langle x, y, z, w, t : xy^2x, yz^2y, wt^2w\rangle$.

Naturally, Gompf presents a full proof of Theorem 6.1. The methods to show that this construction can be altered to produce manifold not of Kähler type are outside the scope of this note.

7. Other constructions

Let us recapitulate the results above. In section 4 we present a construction of a family of symplectic nilmanifolds $M(p,q)$, where $M(1,0)$ was a Kodaira–Thurston manifold, with representants in every even dimension greater or equal to 4. This family consists of manifolds with both even and odd first Betti numbers, but without Hard Lefschetz Property and without formality for $p+q>0$. In section 5 we were able to present first simply connected examples by using the symplectic blow up, at the price of going up to dimensions greater or equal to 10. Manifolds from this family also do not satisfy Hard Lefschetz Property and have nonvanishing triple Massey products.

Due to the connected sum construction by Gompf, it was possible to obtain simply connected examples in dimension 4. They are obtained as a special case of Theorem 6.1, but Gompf also gave another construction of a symplectic and simply connected 4-manifold without Kähler structure (cf. [17], Example 3.2), which is a connected sum of two Dolgachev surfaces along symplectically embedded tori. Resulting manifold is a closed 4-manifold which is homeomorphic to a complex surface, but is not diffeomorphic to one. Methods used here are 4-dimensional in nature. Surprisingly, the product of such manifold with $\mathbb{S}^2$, while still simply connected and symplectic, is now Kähler.

Therefore to construct simply connected symplectic and non Kähler examples in higher dimensions, we need criteria, which are not specific to dimension 4. We have introduced three such criteria in sections 2 and 3, but in dimension 6 two of them are of no use. By Poincare duality all odd Betti numbers of a closed simply connected 6-dimensional manifold are even. Moreover, Miller proved the following

Theorem 7.1 (cf. [29]). *Every compact $(k-1)$-connected manifold M of dimension* $\dim M \leq 4k-2$ *is formal.*

In particular, every compact simply connected 6-dimensional manifold is formal. Therefore the only useful criterion is the Hard Lefschetz Property. Again, Gompf obtained the following.

Theorem 7.2 (cf. [17], Theorem 7.1)**.** *For any even dimension $n \geq 6$ and finitely presentable group G, there exist a closed symplectic manifold M of dimension n with $\pi_1(M) \cong 6$ and without Hard Lefschetz Property.*

Examples from this family are formal. We have just seen in Theorem 7.1 that in dimensions not greater that 6 all closed simply connected manifolds are formal, and earlier in section 5 that in dimensions greater or equal to 10 we can always construct nonformal examples. Therefore the question about existence of closed symplectic nonformal manifolds is yet to be answered in dimension 8. The answer has been given by Fernandez and Muñoz.

Theorem 7.3 (cf. [16])**.** *There exist an 8-dimensional closed simply connected nonformal symplectic manifold.*

Outline of proof. The starting point is a complex Heisenberg group

$$H_{\mathbb{C}} = H(1,1;\mathbb{C}) = \left\{ A \in GL(3;\mathbb{C}) : \, A = \begin{bmatrix} 1 & u_2 & u_3 \\ & 1 & u_1 \\ & & 1 \end{bmatrix}, u_1, u_2, u_3 \in \mathbb{C} \right\}.$$

Let G be a group $G = H_{\mathbb{C}} \times \mathbb{C}$, where we consider additive group structure on the second factor and let Λ be a lattice $\Lambda = \mathbb{Z} + \mathbb{Z}e^{2i\pi/3} \subset \mathbb{C}$. Then manifold

$$M = G/\Lambda = \{(A, u_4) \in H_{\mathbb{C}} \times \mathbb{C} : U_1, u_2, u_3, u_4 \in \Lambda\}$$

is a compact 8-dimensional nilmanifold. We know (by Proposition 4.4) that it is not formal. But it is not simply connected. To fix that, introduce an action of $\mathbb{Z}_3$ on G generated by

$$\rho : \, (u_1, u_2, u_3, u_4) \to (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4),$$

where $\zeta = e^{2i\pi/3}$. Since $\rho(\Lambda) \subset \Lambda$, this action descends to M.

Basis of complex invariant forms on G is given by 1-forms $\mu = \mathrm{d}\, u_1$, $\nu = \mathrm{d}\, u_2$, $\theta = \mathrm{d}\, u_3 - u_2 \,\mathrm{d}\, u_1$ and $\eta = \mathrm{d}\, u_4$. The 2-form

$$\omega = i\mu \wedge \overline{\mu} + \nu \wedge \theta + \overline{\nu} \wedge \overline{\theta} + i\eta \wedge \overline{\eta}$$

is a real symplectic form, invariant by ρ. So, the quotient $\widehat{M} = M/\mathbb{Z}_3$ is a symplectic orbifold. This orbifold has all the properties required in the statement of the theorem, except for not being a smooth manifold. Indeed, we have

Lemma 7.4. *Orbifold $\widehat{M}$ is simply connected and all its cohomology groups in odd dimensions vanish. Moreover it is not formal.*

Now it remains to desingularize the orbifold $\widehat{M}$. It is done by blowing up manifold M in fixed points of the action of $\mathbb{Z}_3$ and extending the action to blow up so that it does not have any fixed point. The quotient of the resulting manifold by the extended action can be shown to exhibit all desired properties. □

Outline of proof of Lemma 7.4. The first part of the statement follows from the observation that the mapping $\Lambda \cong \pi_1(M) \to \pi_1(\widehat{M})$ induced by the obvious projection is epimorphism, since ρ has fixed points. Now it suffices to consider image of generators of Λ, which is trivial. The second part of the statement follows from direct computation of $H^3(M)$ (based on Theorem 4.2), checking that its $\mathbb{Z}_3$-invariant part is trivial and utilizing Poincare duality.

For the proof of the last part, consider the following forms on M

$$\alpha = \mu \wedge \overline{\mu}, \qquad \beta_1 = \nu \wedge \overline{\nu}, \qquad \beta_2 = \nu \wedge \overline{\eta}, \qquad \beta_3 = \overline{\nu} \wedge \eta,$$
$$\gamma_1 = -\theta \wedge \overline{\mu} \wedge \overline{\nu}, \quad \gamma_2 = -\theta \wedge \overline{\mu} \wedge \overline{\eta}, \quad \gamma_3 = \overline{\theta} \wedge \mu \wedge \eta.$$

They are all $\mathbb{Z}_3$-invariant, hence they all descend to $\widehat{M}$. The forms α and β_i are closed and $\mathrm{d}\,\gamma_i = \alpha \wedge \beta_i$. Direct computiations proves that

$$[\gamma_1 \wedge \gamma_2 \wedge \beta_3 + \gamma_2 \wedge \gamma_3 \wedge \beta_1 + \gamma_3 \wedge \gamma_1 \wedge \beta_2] = 2[\mu \wedge \overline{\mu} \wedge \nu \wedge \overline{\nu} \wedge \theta \wedge \overline{\theta} \wedge \eta \wedge \overline{\eta}] \neq 0,$$

while direct computations in minimal model, with assumption of formality, lead to conclusion that this class should vanish. □

The manifold constructed above does not satisfy Hard Lefschetz Property. Question of existence of a closed simply connected nonformal symplectic manifold satisfying this property would be interesting, because it would lighten possible relationships between these notions. Proof that there is no implication from formality to Hard Lefschetz Property is given by Theorem 7.2 for $n = 6$ and $G = 0$. Proof that the reversed implication does not hold also has recently been found (cf. [10]). It is in fact slightly modified example from Theorem 7.3 above. The only change is that the orbifold $\widehat{M}$ is blown up along some tori disjoint from singular set to obtain Hard Lefschetz Property, and then desingularized as before.

There are many constructions which has not been mentioned above. One of them is the generalization of symplectic nilmanifolds, where instead of nilpotent Lie group we consider solvable one. A compact quotient of such group is known as solvmanifold. Every nilmanifold is a solvmanifold, but the converse is not true. While nilpotent and solvable Lie groups present many similarities, there are also striking differences, mainly because we do not have a counterpart of Theorem 4.2 in general (there exist analogous result for some subclass of solvmanifolds, cf. [24]). In particular, the proof of the Theorem 4.4 cannot be adapted to this setting. After many attempts and partial results (cf. [4, 7, 21, 31]) it has been established

Theorem 7.5 (cf. [22])**.** *A compact solvmanifold admits a Kahler structure iff it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.*

As a result, theory of solvmanifolds is quite different. A good introduction is given in [31]. While the class of solvmanifolds is harder to investigate than class of nilmanifolds, it repays the study with many interesting results. For example, regarding formality and Hard Lefschetz Property, we can find compact symplectic solvmanifolds satisfying neither of those (cf. [14]) in dimension 6, only one of them (cf. [7]) in dimension 8 or finally both of them, while still not being Kähler (cf. [15]), in dimension 4.

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