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A note on circle actions on sympletic manifolds

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A NOTE ON CIRCLE ACTIONS ON SYMPLECTIC MANIFOLDS

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Abstract. We construct a six dimensional symplectic manifold which admits a smooth circle action, but not a symplectic one, at least with respect to any symplectic form deformation equivalent to the standard form. We discuss the possible approach to prove that in fact this example does not admit any symplectic circle action.

1. Introduction

The fundamental theorem of symplectic topology is that any two symplectic manifolds of the same dimension are locally symplectomorphic. It follows that there are no local invariants for symplectic manifolds. As a consequence of that, the group of symplectomorphisms of a symplectic manifold is large, in fact infinite dimensional, and usually hard to describe. For example consider a symplectic manifold $M = \mathbb{S}^2 \times \mathbb{S}^2$ with a symplectic structure $\omega = \mu\omega_0 + \omega_0$, where μ is a real constant $\mu \geq 1$ and ω_0 is the standard volume form on $\mathbb{S}^2$. For $\mu = 1$ Gromov showed [8] that $\mathrm{Symp}(\mathbb{S}^2 \times \mathbb{S}^2, \omega_0 + \omega_0)$ is homotopy equivalent to $\mathrm{SO}(3) \times \mathrm{SO}(3)$, but for arbitrary $\mu > 1$ only partial results on the topology of symplectomorphisms group are obtained, using nonelementary methods (cf. [1, 2]).

So, as we have seen above, the study of the symplectomorphism group as a whole can be difficult even for the simplest of symplectic manifolds. But hopefully there are some related notions, which might prove to be more accessible and at the same time still give some insight into symplectic symmetries of the manifold. One of these notions is the notion of symplectic degree of symmetry. Degree of symmetry of a closed manifold, denote it by $\mathrm{Sym}(M)$, is a maximum over dimensions of compact Lie groups that can act smoothly and effectively on this manifold. Since for every compact Lie group there exists invariant metric, clearly

$$\mathrm{Sym}(M) \leq \max_g \dim \mathrm{Isom}(M, g) \leq \frac{n(n+1)}{2},$$

where the maximum is taken over metrics on M and n is a dimension of M. Now, symplectic degree of symmetry, $\mathrm{SSym}(M)$, will be the maximum over dimensions of compact Lie groups acting smoothly, effectively and symplectically on M, with respect to any symplectic structure on M. Equivalently $\mathrm{SSym}(M)$ is the dimension of the maximal compact Lie group contained as a subgroup in group of symplectomorphisms.

There is an obvious upper bound for degree of symplectic symmetry of $2n$-dimensional symplectic manifold, namely $\mathrm{SSym}(M) \leq \mathrm{Sym}(M) \leq n(2n + 1)$. In fact, for dimensions greater than two, the last inequality can be strenghtened to $\mathrm{SSym}(M) < n(2n+1)$, since the only manifolds with maximal degree of symmetry are spheres and real projective planes, which are not symplectic in those dimensions.

The product of spheres considered in the first paragraph is an example when this two degrees coincide,

$$\mathrm{SSym}(\mathbb{S}^2 \times \mathbb{S}^2) = \mathrm{Sym}(\mathbb{S}^2 \times \mathbb{S}^2) = 6.$$

Similarly, $\mathrm{SSym}(\mathbb{C}P^2) = \mathrm{Sym}(\mathbb{C}P^2) = 8$. These manifolds provide examples of what we could call highly symmetric symplectic manifolds. At the other hand there are symplectic manifolds like $K3$-surface, with $\mathrm{Sym}(K3) = 0$ and hence in particular $\mathrm{SSym}(K3) = 0$. We could say that $K3$-surface is symplectically unsymmetric.

For the symplectic degree of symmetry to be a meaningful notation, it needs to be essentially different from the metric one. That is why it is an interesting question, whether there exist an example of symplectic manifold with $\mathrm{Sym}(M) > 0$ and $\mathrm{SSym}(M) = 0$. It seems that such an example is yet unknown.

What we know, due to the result of Baldridge [5], is that existence of any smooth circle action with fixed points on a 4-manifold (and so for example the existence of any compact Lie group action on a manifold with nonzero Euler characteristic) implies the existence of a symplectic circle action. While the general case, without the assumption on the existence of fixed points, is still open, it is commonly believed to be true. Therefore it seems reasonable to start looking for our example in dimension 6. While we do not succeed in finding one, we prove the following theorem.

Theorem 1.1. *There exists a symplectic* 6*-manifold* (M, ω) *which admits a smooth circle action, but does not admit any circle action compatible with any symplectic form deformation equivalent to* ω*.*

The problem with obtaining $\mathrm{SSym}(M) = 0$ lies, not surprisingly, in the potential variety of non equivalent symplectic structures on 6-manifolds and lack of tools to classify them. Still, symplectic structures on 6-manifolds appear to exhibit interesting behaviour, pointing toward the conjecture, that this variety can, in some sense, be controlled. This rather vague statement is made substansialised a little in the last section of this note.

The outline of the paper is as follows. In Section 2 we recall the standard technique from topology, which we call "stabilization", and which can be used to produce examples of manifolds in dimensions greater than with some interesting properties. In Section 3 we recall definitions and recall some results concerning symplectic and Hamiltonian circle actions and in Section 4 we do the same for the Todd genus. Section 5 is devoted to the construction of an example constituting the proof of the Theorem 1.1. Finally, in Section 6 we discuss perspectives on improving this result, concluding with a statement of aforementioned conjecture concerning symplectic structures on 6-manifolds.

Remark. All manifolds are considered to be smooth and all cohomology rings are considered to be with real coeffitients unless stated otherwise.

2. Stabilization

Proposition 2.1. *Let* M *and* N *be simply connected* 4*-manifolds. If* M *and* N *are homotopy equivalent, then for each manifold* F *of positive dimension,* $M \times F$ *and* $N \times F$ *are diffeomorphic.*

Proof. It is well known, that two simply connected 4-manifolds M and N are homotopy equivalent iff they intersection forms agree. Now, by [21] the latter condition is equivalent to M and N being h-cobordant. Let W be an h-cobordism between M and N. Then $W \times F$ is an h-cobordism between $M \times F$ and $N \times F$. Since the smooth h-cobordism theorem works in dimensions greater than four, if F is simply connected than our proof is finished.

For general F we have to use s-cobordism theorem. In our case it says that W is cobordant to a cylinder modulo boundary (and so, in particular $M \times F$ and $N \times F$ are diffeomorphic) iff Whitehead torsion $\tau(\iota \times \mathrm{id_F}) \in \mathrm{Wh}(\pi_1(M \times F))$ of the inclusion $\iota \times \mathrm{id_F} : M \times F \to W \times F$ vanishes. Using product formula for Whitehead torsion (cf. [13], Corollary 1.3), we obtain

$$\tau(\iota \times \mathrm{id_F}) = \chi(F) j_* \tau(\iota) + \chi(W) j'_* \tau(\mathrm{id_F})$$

for some homomorphisms j_* and j'_* from Whitehaed groups of factors into Whitehead group of product. But $\mathrm{Wh}(\pi_1(M)) = 0$ since M is simply connected and clearly $\tau(\mathrm{id_F}) = 0$. □

This "stabilization" procedure is often used to produce examples of various phenomena in higher dimensions. Below we give two of these examples from the literature.

Proposition 2.2 ([19])**.** *There exists a 6-manifold with two symplectic structures which are not deformation equivalent.*

Proof. Let M be a Barlow surface and $N := \mathbb{C}P^2 \sharp 8 \overline{\mathbb{C}P^2}$. They are homotopy equivalent, hence $M \times \mathbb{S}^2 = N \times \mathbb{S}^2$ as smooth manifolds. Product symplectic structures induced from M and N, even though they induce the same homotopy class of almost complex structures, are not deformation equivalent. They can be distinguished by invariants of Gromov type. □

Proposition 2.3 ([9])**.** *There exist symplectic manifolds with circle action which is not compatible with any symplectic structure.*

Proof. Let M and N be homotopy eqiuvalent manifolds such that M is symplectic and N is not. Then $M \times \prod_{i=1}^k \mathbb{S}^2 = N \times \prod_{i=1}^k \mathbb{S}^2$ as smooth manifolds. Product manifold is symplectic, and we can equip it with smooth circle action given as a diagonal action on $N \times \prod_{i=1}^k \mathbb{S}^2$, with action on N being trivial and action on product of spheres having exactly l fixed points. Then the diagonal action has non-symplectic fixed point set $\coprod_{i=1}^l N$, hence it cannot be symplectic for any symplectic structure (cf. Proposition 3.6 below). □

3. Symplectic circle actions

Consider a smooth circle action on a manifold M, ie. a smooth homomorphism $\mathbb{S}^1 \to \mathrm{Diff}(M)$. For each element $\lambda \in \mathbb{S}^1$ we denote the corresponding diffeomorphism by the same letter. If M is symplectic and the image of the homomorphism is contained in the subgroup of symplectomorphisms, we call such an action symplectic. More precisely

Definition 3.1. A circle action on M compatible with a symplectic structure ω is a homomorphism $\mathbb{S}^1 \to \mathrm{Symp}(M, \omega)$. Equivalently, a circle action on M is compatible with ω iff for any $\lambda \in \mathbb{S}^1$ we have $\lambda^* \omega = \omega$.

Any circle action induces a nonvanishing vector field ξ on M, known as the fundamental vector field. Since symplectic form ω is nondegenerate, the 1-form $\iota_\xi \omega$ is also nonvanishing.

Remark 3.2. Circle action on M is compatible with symplectic structure ω iff its fundamental field satisfies $\mathcal{L}_\xi \omega = 0$ or equivalently, by Cartan's formula, iff $\iota_\xi \omega$ is a closed 1-form.

We distinguish a special class of symplectic circle action, which are usually much easier to describe, namely the Hamiltonian actions.

Definition 3.3. A circle action on a symplectic manifold M compatible with a symplectic structure ω is called Hamiltonian, if the 1-form $\iota_\xi \omega$ is exact. Any smooth function $\Phi : M \to \mathbb{R}$ such, that $\mathrm{d}\,\Phi = \iota_\xi \omega$ is called a Hamiltonian for the action.

One of the important question of the theory of symplectic actions, in particular symplectic circle actions, is which symplectic circle actions are Hamiltonian. In dimension 4 the full answer is known.

Proposition 3.4 ([15])**.** *A symplectic circle action on a symplectic* 4*-manifold is Hamiltonian iff it has at least one fixed point.*

It is clear, that nontrivial Hamiltonian action must have at least two connected components of fixed point set, corresponding to the minimum and maximum of the Hamiltonian, but the proof of the converse is nontrivial. Further, in dimension 4 full classification of Hamiltonian circle action is known (cf. [12]).

In higher dimensions the analog of Proposition 3.4 no longer holds. In fact, in the same paper that contained this proposition, McDuff gave an example of symplectic 6-manifold with symplectic circle action which is not Hamiltonian. In this example, connected components of fixed point set are 2-dimensional manifolds. It is still unknown, whether there can exist a symplectic circle action with isolated fixed points which is not Hamiltonian. Still, there are many other criterions for a symplectic circle action to be Hamiltonian, including the one given below.

Proposition 3.5 ([18], see also [16], Theorem 5.5)**.** *Let M be a symplectic $2n$-manifold with symplectic structure ω. If ω satisfies weak Lefschetz property, ie.*

$$\cup[\omega]^{n-1} : H^1(M) \to H^{2n-1}(M)$$

is an isomorphism, then every circle action on M compatible with ω is Hamiltonian.

One of the methods of analyzing symplectic circle actions is analyzing its fixed point set. There are many results of this kind, often obtained by the equivariant version of Atiyah-Singer Index Theorem (various "localization" formulas, cf. Proposition 4.8 and Proposition 3.8 below). The following result, on the other hand, is quite elementary.

Proposition 3.6. *Every component of a fixed point set of a circle action on manifold M compatible with some symplectic structure ω is a symplectic submanifold of (M, ω).*

In the case of Hamiltonian actions, fixed points of action correspond to critical points of the Hamiltonian. It can be shown that level sets of Hamiltonians are always connected, so we have two important components of fixed point set, Φ_{min} being the set of points where Hamiltonian reaches its minimum and Φ_{max} where it reaches its maximum. To analyze those components, we can utilize the fact, that Hamiltonian is a Morses-Bott function, and hence the machinery of Morse theory applies. In this way it can be shown, that:

Proposition 3.7 ([14])**.** *Let M be a symplectic manifold with symplectic structure ω and Hamiltonian, with respect to this structure, circle action assosiated with Hamiltonian Φ. Then*

$$\pi_1(\Phi_{min}) = \pi_1(\Phi_{max}) = \pi_1(M).$$

At the other hand, the "localization" techniques mentioned earlier give the following result.

Proposition 3.8. *Under assumptions as in Proposition 3.7 above,*

$$\mathrm{td}(\Phi_{min}, \omega_{\Phi_{min}}) = \mathrm{td}(M, \omega).$$

The proof of this proposition will be given in Section 4.

4. Todd genus

The common reference for all definitions and results in this section is [6]. See also [7]

Definition 4.1. Let Ω be a cobordism ring and let R be integral domain. A genus is a homomorphism $\Omega \to R$.

If $\Omega = \Omega_*^{SO}$ is a ring of oriented cobordisms, we say that the genus is real and if $\Omega = \Omega_*^{U}$ is a ring of stably almost complex cobordisms, we say that the genus is complex.

Let $Q(x) = 1 + a_2x^2 + a_4x^4 + \ldots \in R[[x]]$ be an even power series over R with leading term equal to 1. The series $Q(x_1)\ldots Q(x_n)$ is symmetric in indeterminates x_i, so it can be exspressed by elementary symmetric polynomials. Let x_i's be of weight 2. Then

$$Q(x_1)\ldots Q(x_n) = 1 + K_1(p_1) + \ldots + K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0) + \ldots,$$

where $K_r(p_1, \ldots, p_r)$ is the term of weight $2r$, K_r is a polynomial over R in r variables and p_i's are elementary symmetric polynomials of x_i^2's. Observe that the terms K_r's are stable, ie. K_r does not depent on n for $n \geq r$.

Proposition 4.2. *The map which assigns to any $4n$-dimensional manifold M an element $\langle K_n(p_1, \ldots, p_n), [M]\rangle \in R$, where $p_1, \ldots, p_n$ are Pontrjagin classes of M, induces a well-defined real genus*

$$\varphi_Q : \Omega_*^{SO} \to R$$

associated with an even series Q.

If we take an arbitrary series $Q(x) = 1 + a_1x + a_2x^2 + \ldots$ with leading term equal to 1, we obtain

$$Q(x_1)\ldots Q(x_n) = 1 + K_1(c_1) + \ldots + K_n(c_1, \ldots, c_n) + K_{n+1}(c_1, \ldots, c_n, 0) + \ldots,$$

where c_i's are elementary symmetric polynomials of x_i's, and the similar result holds.

Proposition 4.3. *The map which assigns to any stably almost complex $2n$-dimensional manifold M an element $\langle K_n(c_1,\dots,c_n),[M]\rangle \in R$, where $c_1,\dots,c_n$ are Chern classes of M, induces a well-defined complex genus*

$$\varphi_Q : \Omega_*^U \to R$$

associated with a series Q.

Moreover, we have the following

Proposition 4.4. *Any real or complex genus is associated with some power series.*

Example 4.5.

(1) The L-genus is the real genus associated with series

$$Q(x) = \frac{x}{\tanh(x)} \in \mathbb{Q}[[x]]$$

The famous Hirzebruch's signature theorem [10] states that $L(M) = \sigma(m)$.

(2) The $\hat{A}$-genus is the real genus associated with series

$$Q(x) = \frac{x/2}{\sinh(x/2)} \in \mathbb{Q}[[x]]$$

For spin manifolds this genus is an integer, equal to the index of the Dirac operator.

(3) The Todd genus is the complex genus associated with series

$$Q(x) = \frac{x}{1-e^{-x}} \in \mathbb{Q}[[x]].$$

The Todd genus can be characterized as the unique complex genus which evaluates to 1 on each complex projective space.

(4) The χ_y-genus is the complex genus associated with series

$$Q(x) = \frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}} \in (\mathbb{Q}[y])[[x]].$$

Lemma 4.6. *Composed with the coeffitient homomorphism*

$$\mathrm{ev}_{y_0} : \mathbb{Q}[y] \ni P(y) \mapsto P(y_0) \in \mathbb{Q},$$

the χ_y-genus is equal to

(1) *the signature, for $y_0 = 1$,*
(2) *the Todd genus, for $y_0 = 0$,*
(3) *the Euler characteristic, for $y_0 = -1$.*

Proof. Take $Q(x) = \frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}}$.

(1)

$$Q(x)|_{y=1} = \frac{x(1+e^{-2x})}{1-e^{-2x}} = \frac{x}{\tanh(x)},$$

so $\chi_1(M) = L(M)$ and, by Hirzebruch's signature theorem, this equals the signature of M.

(2)

$$Q(x)|_{y=0} = \frac{x}{1-e^{-x}}.$$

(3) Clearly, the exspression for $Q(x)$ given above is not defined for $y = -1$. But

$$\lim_{y_0 \to -1} Q(x)|_{y=y_0} = 1 + x$$

and so for $2n$-dimensional almost complex manifold M with almost complex structure J we get

$$\chi_y(M, J)|_{y=-1} = \langle c_n(M, J), [M] \rangle = \chi(M).$$

□

Observe that a priori complex genus of a manifold is undefined - it may, and in fact it does, depend on the choice of stably almost complex structure. But complex genera are invariants of a homotopy class of this structures. In particular, for symplectic manifold M with a symplectic structure, comples genus of a pair (M, ω) is well-defined. In particular we can define $\chi_y(M, \omega)$ and $\mathrm{td}(M, \omega)$. Yet, there are cases, when complex genera are invariants of smooth (even homotopy) type.

Proposition 4.7. *If M is a symplectic manifold of dimension 2 or 4 then its Todd genus is independent of the choice of symplectic structure and equals*

(1) $\mathrm{td}(M) = \frac{\chi(M)}{2}$ *if* $\dim(M) = 2$ *and*
(2) $\mathrm{td}(M) = \frac{\sigma(M)+\chi(M)}{4}$ *if* $\dim(M) = 4$ *and*

Proof. Direct computation yields $\mathrm{td}(M, \omega) = \frac{1}{2}\langle c_1(M, \omega), [M] \rangle = \frac{1}{2}\chi(M)$ in the 2-dimensional case and

$$\begin{aligned} \mathrm{td}(M, \omega) &= \frac{1}{12}\langle c_1^2(M, \omega) + c_2(M, \omega), [M] \rangle = \frac{1}{12}\langle p_1(M) + 3c_2(M, \omega), [M] \rangle \\ &= \frac{1}{12}(3L(M) + 3\chi(M)) = \frac{\sigma(M) + \chi(M)}{4} \end{aligned}$$

in the 4-dimensional case. □

The important property of χ_y-genus, which can be derived from the equivariant version of Atiyah-Singer Index Formula is the following

Proposition 4.8. *For a circle action on (M, ω) compatible with ω, the following localization formula holds*

$$\chi_y(M, \omega) = \sum_{\nu} (-y)^{d_\nu} \chi_y(M_\nu, \omega_\nu),$$

where ν indexes the connected components M_ν of fixed points set, d_ν is the dimension of the subbundle of the normal bundle to M_ν on which circle acts with negative weights and ω_ν is a symplectic structure on M_ν restricted from M.

Proof of Proposition 3.8. Φ_{min} is the unique connected component of the fixed point set which has only nonnegative weights of induced action on normal bundle. In language of the Proposition 4.8 it means that Φ_{min} is the only component with $d = 0$. Hence

$$\chi_y(M, \omega) = \chi_y(\Phi_{min}, \omega_{\Phi_{min}}) - y \sum_{M_\nu \neq \Phi_{min}} (-y)^{d_\nu - 1} \chi_y(M_\nu, \omega_\nu).$$

In particular

$$\mathrm{td}(M, \omega) = \chi_y(M, \omega)|_{y=0} = \chi_y(\Phi_{min}, \omega_{\Phi_{min}})|_{y=0} = \mathrm{td}(\Phi_{min}, \omega_{\Phi_{min}}).$$

□

5. Proof of the Theorem 1.1

The proof follows from a series of lemmas.

Lemma 5.1. *For every nonnegative integer n there exist a symplectic manifold homotopy equivalent to $3\mathbb{C}P^2\sharp(4+n)\overline{\mathbb{C}P^2}$.*

Proof. Clearly, it is sufficient to find a symplectic 4-manifold $K \simeq 3\mathbb{C}P^2\sharp 4\overline{\mathbb{C}P^2}$, ie. to solve the $n = 0$ case. Then, for positive n we just blow up K in n distinct points. Such an example is constructed in [4]. But for many cases there exist "smaller" examples, in particular there exist irreducible examples for $n = 1$ and $3 \leq n \leq 15$ (see [3] and references therein). □

Manifold $3\mathbb{C}P^2\sharp(19+n)\overline{\mathbb{C}P^2}$, which we will further denote as M_n, admits a circle action or even a torus action. Indeed, we can use equivariant connected sum to construct this action from suitable linear actions on projective spaces. We don't know, whether this action is compatible with almost complex structure on M_n, because the almost complex structure might not survive under connected sum. In fact $\mathbb{C}P^2\sharp\mathbb{C}P^2$ does not admit any almost complex structure, since in its integral cohomology ring $H^*(\mathbb{C}P^2\sharp\mathbb{C}P^2;\mathbb{Z})$ there is no candidate for the first Chern class.

From the lemma above and from the stabilization (Proposition 2.1) we conclude that for any 2-dimensional oriented surface F_g of genus g, manifold $M_n \times F_g$ is symplectic and supports a smooth circle (torus) action. We denote this manifold by $M(n, g)$.

Assume now, that $M(n, g)$ admits a symplectic circle action for some symplectic structure ω. We have the following

Lemma 5.2. *Any symplectic circle action on $M(n, g)$ is Hamiltonian.*

Proof. We will show that $M(n, g)$ is weakly Hamiltonian. From Künneth's formula we have $H^2(M(n, g)) \cong H^2(M_n) \oplus H^2(F_g)$, hence $[\omega]$ admits a decomposition $[\omega] = a + b$ for some $a \in H^2(M_n)$ and $b \in H^2(F_g)$. Since the symplectic form ω is nondegenerate, we have $0 \neq [\omega]^3 = 3a^2b$ and in particular $a, a^2, b \neq 0$. Using Künneth's formula again, we get $H^1(M(n, g)) \cong H^0(M_n) \otimes H^1(F_g)$ and $H^5(M(n, g)) \cong H^4(M_n) \otimes H^1(F_g)$. Under this identification $\cup[\omega]^2$ becomes

$$(\cup a^2) \otimes \mathrm{id}_{\mathrm{H^1(F_g)}} : H^0(M_n) \otimes H^1(F_g) \to H^4(M_n) \otimes H^1(F_g),$$

which clearly is an isomorphism. Now, the lemma follows from Proposition 3.5. □

We recall that for a Hamiltonian circle action with Hamiltonia Φ there exist unique connected component Φ_{min} of fixed point set where Hamiltonian attains its minimum. By Proposition 3.7 we see, that Φ is either an oriented surface of genus g or a symplectic 4-manifold with $b_1 = 2g$. Hence

Lemma 5.3. *For any Hamiltonian circle action on $M(n, g)$ with Hamiltonian Φ the inequality $\mathrm{td}(\Phi_{min}) \geq 1 - g$ is satisfied.*

Proof. If $\Phi_{min} = F_g$ then, by Proposition 4.7,

$$\mathrm{td}(\Phi_{min}) = \frac{2 - 2g}{2} = 1 - g.$$

If Φ_{min} is 4-dimensional with $b_1(\Phi_{min}) = 2g$, then, by Proposition 4.7 again,

$$\begin{aligned} \operatorname{td}(\Phi_{min}) &= \frac{b_2^+(\Phi_{min}) - b_2^-(\Phi_{min}) + 2 - 2b_1(\Phi_{min}) + b_2^+(\Phi_{min}) + b_2^-(\Phi_{min})}{4} \\ &= \frac{2 + 2b_2^+(\Phi_{min})}{4} - g \geq 1 - g, \end{aligned}$$

since for symplectic manifold b_2^+ is always positive. □

At the other hand, we have

Lemma 5.4. *For any symplectic structure ω on $M(n, g)$ deformation equivalent to a product structure we have* $\operatorname{td}(M(n, g), \omega) = 2(1 - g)$.

Proof. If ω is deformation equivalent to a product form, then we can compute Todd genus of $M(n, g)$ with respect to ω as

$$\operatorname{td}(M(n, g), \omega) = \operatorname{td}(M_n) \operatorname{td}(F_g).$$

Now, by Proposition 4.7

$$\begin{aligned} \operatorname{td}(M_n) &= \frac{\sigma(M_n) + \chi(M_n)}{4} = \frac{-16 - n + 24 + n}{4} = 2, \\ \operatorname{td}(F_g) &= \frac{\chi(F_g)}{2} = \frac{2 - 2g}{2} = 1 - g. \end{aligned}$$

□

Lemmas 5.2, 5.3 and 5.4 together imply the following

Corollary 5.5. *For $g \geq 2$ manifold $M(n, g)$ admits no circle action compatible with any symplectic structure deformation equivalent to product structure.*

Hence the family of manifolds $M(n, g)$ with $g \geq 2$ is an infinite family of examples of manifolds satisfying conditions of Theorem 1.1. Observe, that for $g = 0$ or $g = 1$ the manifold $M(n, g)$ supports a symplectic circle action, induced from a symplectic action on the factor F_g.

6. Discussion

As mentioned in the introduction, the result of Theorem 1.1 is highly unsatisfactory. It is rather a partial result on a way to answer the following conjecture in a positive way.

Conjecture 6.1. *There exists a symplectic manifold which supports a smooth circle action but does not support an action compatible with any symplectic structure.*

The only stage of the construction, where we used the additional assumption about symplectic structure being deformation equivalent to a product one, is the proof of Lemma 5.4. We cannot in general compute the Todd genus of a product manifold $M = N \times F$ as a product of Todd genera of N and F, unless the product is in Ω_*^U, ie. almost complex structure on M is homotopy equivalent to a product of some almost complex structures on N and F. By Wall's classification of almost complex structures on 6-manifolds, for any such structure J on M to satisfy this condition is equivalent to

$$\begin{aligned} \langle c_1(M, \omega), [F] \rangle &= 2 - 2g, \\ \langle c_1^2(M, \omega), [N] \rangle &= 3\sigma(N) + 2\chi(N) = 0, \end{aligned}$$

but the same classification tells us that most of these structures do not satisfy them. Still, there are some clues pointing toward the following statement.

Conjecture 6.2. *Distinct symplectic structures on* 6*-manifolds cannot be distinguished by the associated homotopy classes of almost complex structures.*

To the best of author's knowledge, all known examples of exotic (ie. nonisotropic) symplectic structures on symplectic 6-manifolds are either deformation equivalent (cf. [17]) or at least they induce the same homotopy class of almost complex structures (cf. [11, 19, 20]) and has to be distinguished by invariants of Gromov type. In particular none of them provides the counterexample for the conjecture above.

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