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Sympletic Lefschetz pencils

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SYMPLECTIC LEFSCHETZ PENCILS

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ABSTRACT. We review the theory of Lefschetz pencils in symplectic context. In particular, we present the Donaldson's result about the existence of pencils on symplectic manifolds and the converse result of Gompf. Then we proceed to present higher-order generalisations.

1. INTRODUCTION

The first to introduce Lefschetz pencils into the context of symplectic manifolds was Donaldson. In [5] he proved that every symplectic manifold, up to a deformation of symplectic form, admits a Lefschetz pencil compatible with a symplectic structure. His proof is an extension of his earlier proof [4] of existence of symplectic submanifolds of codimension 2.

The notion of Lefschetz pencil has been adapted from the complex geometry, where it has been introduced by Lefschetz himself. A pencil is a division of a manifold into smoothly parametrized family of fibres of codimension 2. It is not surprising then that pencils carry many similarities to fibre bundles. Yet there are two differences. First, the minor one, is that all the fibres intersect in some codimension-4 submanifold. Second is that we allow a finite number of singular fibres in a pencil, and this one proves to be the major one. In fact, as we will see, topology of a total space of a pencil is governed mostly by its singular fibres. Finding a way to describe this topology turns out to be quite important since, by the aforementioned result of Donaldson and the result of Gompf about existence of symplectic form on the total space of a Lefschetz pencil under some reasonable conditions, questions about the topology of symplectic manifolds can be translated to questions about the topology of Lefschetz pencils.

While the topology of Lefschetz pencils is well-understood and can be described in an almost purely combinatorial way, the results of this approach to symplectic topology are not as numerous as one could hope. The main problem lies in the "translation" of symplectic problems to this combinatorial language. Still, many interesting results have been obtained both of the constructive and classifying type. An example of the result of the first type is a construction of an infinite family of symplectic 4-manifolds not admitting complex structure in [12]. A nice example of a result of the second type is a stable classification [3], up to blow-up and connected sum with some universal element, of integral symplectic 4-manifolds by their signature, Euler characteristic, symplectic volume and the evaluation of a class $c_1[\omega]$ on the fundamental class.

This note is organized as follows. In Section 2 we shortly review the theory of symplectic fibre bundles, which is a natural background for further sections. In Section 3 we introduce Lefschetz fibrations and show how their topology is determined by their monodromy and in Section 4 we show how does the considerations about the topology of Lefschetz fibrations fit into symplectic context. In Section 5 we finally introduce Lefschetz pencils and present the results of Donaldson and Gompf. Section 6 is devoted to providing a sketch of Donaldson's proof. Finally, in Section 7 some generalisations of the notation of Lefschetz pencil are presented.

2. Warm up: symplectic fibrations

Definition 2.1. [1] A fiber bundle $p : M \to B$ with fiber F is called a *symplectic fiber bundle* if F is a symplectic manifold and the structure group $\mathrm{Diff}(F)$ of p reduces to the symplectomorphisms group $\mathrm{Symp}(F)$.

Given such a definition, it is natural to ask whether a symplectic manifold M admits a structure of symplectic fiber bundle, somehow compatible with a symplectic structure on M, and conversely, whether a total space of a symplectic fiber bundle is always a symplectic manifold, again with symplectic structure somehow induced from the bundle structure.

As it turns out, the answer to the first question is positive if we can fiber the manifold in question by its symplectic submanifolds.

Proposition 2.2. [11, Lemma 6.2] *Let (M, ω_M) be a symplectic manifold and let $p : M \to B$ be a fiber bundle such that for any point $b \in B$ the fiber $p^{-1}(b)$ is a symplectic submanifold of M. Then p is a symplectic fiber bundle.*

The second question have been addressed for the first time by Thurston, at least for fibers of dimension 2, when he used such fibrations to construct the first known example of symplectic and non-Kähler manifold.

Proposition 2.3. [13] *Let $p : M \to B$ be a symplectic fiber bundle with symplectic base space and 2-dimensional fiber F. If $[F] \neq 0 \in H_2(M)$, then M admits a symplectic structure with respect to which each fiber is symplectic.*

This propositions generalises naturally to fiber bundles of arbitrary fiber dimensions, as in Proposition 2.4 below. The only condition we have to impose is the existence of a candidate for the cohomology class of a symplectic form. While this conditions is usually satisfied for 2-dimensional fiber (for example if fiber is not a torus it is always satisfied), in general it can be fairly restrictive.

Proposition 2.4. [11, Theorem 6.3] *Let $p : M \to B$ be a symplectic fiber bundle with symplectic base space and a cohomology class $a \in H^2(M)$ which restricts to a fiber as a cohomology class of a symplectic form, $\iota_F^* a = [\omega_F] \in H^2(F)$. Then M admits a symplectic structure with respect to which each fiber is symplectic.*

Gompf gave a further generalization of Proposition 2.4, in which the map p is no longer a fibre bundle, but an arbitrary holomorphic map, phenomenon which we will see more of in further sections. To state his result we need a preparatory definition.

Definition 2.5. Let M be a manifold with almost-complex structure J. We say that an arbitrary 2-form α

(1) tames J if for any vector $v \neq 0 \in TM$ we have $\alpha(v, Jv) > 0$;
(2) is J-compatible if $\alpha = \alpha \circ J \times J$.

[1]In this section all manifolds are considered to be closed and connected.

Proposition 2.6. [6] *Let M and N be almost-complex manifolds with almost complex structures J_M and J_N, let $p : M \to N$ be a J-holomorphic map and let ω_N be a symplectic form on N taming J. If for each point $y \in N$ there exists a neighbourhood W_y of $p^{-1}(y)$ and a closed 2-form η_y on W_y such that $[\eta_y] = \iota^*_{W_y \hookrightarrow M} c$ for some fixed class $c \in H^2(M; \mathbb{R})$ and such that η_y tames $J_M|\ker \mathrm{d}_x p$ for each point $x \in W_y$, then there exist a symplectic form ω_M on M.*

3. Lefschetz fibrations

If we loosen up the conditions on symplectic fibrations in a proper way, we obtain a structure which, while still being relatively easy to study by topological methods, can be imposed on much wider class of symplectic manifolds. Recall that in the category of closed and connected manifolds, a map $p : M \to B$ is a fiber bundle with total space M and base B if and only if it is a submersion. The proper way to weaken this condition turns out to be to admit a finite number of singular points, where the differential $\mathrm{d}p$ is not surjective, while retaining control over the behaviour of p around this points. What we obtain is called a Lefschetz fibration.

Definition 3.1. A Lefschetz fibration on an oriented manifold M of dimension $2n$ is a map $p : M \to \mathbb{C}P^1$ together with a finite set $C = \{c_1, \ldots, c_k\} \subset M$ of critical points, such that[2]

(1) the restriction of p to $M \setminus C$ is a submersion,
(2) $p(c_i) \neq p(c_j)$ for $c_i \neq c_j$,
(3) for each critical point $c_i \in C$ there exists a complex coordinate chart $(z_1, \ldots, z_n)$ around c_i, compatible with orientation, in which c_i corresponds to 0 and p has the form $(z_1, \ldots, z_k) \mapsto z_1^2 + \ldots + z_k^2$.

Topology of Lefschetz fibrations. Below we give a brief discussion of topology of Lefschetz fibrations which can be found, usually in more depth, in numerous sources, e.g. [2, 6, 7, 10]. Many of these sources deal exclusively with fibrations on manifolds of dimension 4, but the theory is similar. No further references for the material in this section will be made.

If we restrict p to

$$p|_{p^{-1}(\mathbb{C}P^1 \setminus p(C))} : p^{-1}(\mathbb{C}P^1 \setminus p(C)) \to \mathbb{C}P^1 \setminus p(C)$$

it becomes an ordinary fiber bundle. In particular there exists a manifold F of dimension $2n - 2$ which is diffeomorphic to each fiber of this restriction. We call F a regular fiber of p, while every fiber $F_i = p^{-1}(p(c_i))$ is called a singular fiber. Notice that while every regular fiber of F is diffeomorphic to F, this diffeomorphism is not canonical.

Lemma 3.2. *For each critical fiber F_i with unique critical point c_i there exist a corresponding embedded sphere $\mathbb{S}_i^{n-1} \subset F$, called a vanishing cycle, and a map $f_i : F \to F_i$ which maps the sphere $\mathbb{S}_i^{n-1}$ to a critical point but otherwise is a diffeomorphism, i.e. the restriction $f|_{F \setminus \mathbb{S}_i^{n-1}} : F \setminus \mathbb{S}_i^{n-1} \to F_i \setminus \{c_i\}$ is a diffeomorphism.*

Argument. Fix a critical point $c_i \in C$ and coordinates $(z_1, \ldots, z_n)$ as in point (3) of Definition 3.1. The critical fiber F_i containing c_i corresponds to the solution space of $z_1^2 + \ldots + z_n^2 = 0$, with critical point in $z_1 = \ldots = z_n = 0$. For each nearby

[2]Some authors consider more general structures, with $p : M \to \Sigma$ where Σ is any oriented surface, and also call them Lefschetz fibrations.

regular fiber we can find real parameters $\lambda, \theta \in \mathbb{R}$ such that this fiber is locally given by $z_1^2 + \ldots + z_n^2 = e^{i\theta}\lambda$. If we now restrict this equation to the subspace $\{(e^{i\theta/2}x_1, \ldots, e^{i\theta/2}x_n), x_1, \ldots, x_n \in \mathbb{R}\} \subset \mathbb{C}^n$, the solution space is just a sphere $x_1^2 + \ldots + x_n^2 = \lambda$. As we approach the critical fiber, the radius of the sphere decreases and eventually the sphere collapses to the critical point. □

Remark 3.3. In the preceding lemma neither the sphere $\mathbb{S}_i^{n-1}$ (the vanishing cycle) nor the map f_i is canonical.

Fix a regular value $y \in \mathbb{C}P^1$ and an identification of the fibre F_y with the model regular fibre F. Let γ be any loop in $\Omega(\mathbb{C}P^1, y)$, i.e. a smooth map $\gamma : [0,1] =: I \to M$ with $\gamma(0) = \gamma(1) = y$. If γ does not intersect any critical value, we can pull back the fibre bundle $p|_{p^{-1}(\gamma(I))}$ to obtain a fibre bundle over I, which is necessarily trivial. The choice of smooth trivialisation of this bundle corresponds to the choice of self-diffeomorphism of F_y, so in turn to the choice of element in $\mathrm{Diff}(F)$, the self-diffeomorphisms group of F. Different trivialisations will induce different elements in $\mathrm{Diff}(F)$, but all these elements will be pairwise isotopic. Hence we have well-defined map $\Omega(\mathbb{C}P^1 \setminus p(C), y) \to \mathrm{Map}(F)$, where $\mathrm{Map}(F)$ is the mapping class group of F, i.e. the group $\mathrm{Diff}(F)$ modulo the isotopies. This map reduces to a monodromy map

$$m_p : \pi_1(\mathbb{C}P^1 \setminus p(C)) \to \mathrm{Map}(F),$$

because homotopic loops induce the same element in $\mathrm{Map}(F)$.

As it turns out, the local model of p near critical points is enough to compute the monodromy map.

Lemma 3.4. *Let γ_i be a loop encircling exactly one critical value $p(c_i)$ in $\mathbb{C}P^1$. Then the image $m_p([\gamma_i]) \in \mathrm{Map}(F)$ is a class of the generalized Dehn twist around the corresponding vanishing cycle.*

Recall that a generalised Dehn twist around the sphere $\mathbb{S}^{n-1}$ is a smooth map $\rho : T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$ with support in a small neighbourhood of the zero section and restricting to $-\mathrm{id}_{\mathbb{S}^n}$ on this section.[3]

We will now see to what degree does the monodromy map determinate the topology of the total space of the fibration.

Proposition 3.5. *Let $y' \neq y$ be another regular value of p and let D' denote a small open disc around y', whose closure contains no critical values of p. Let $D = \mathbb{C}P^1 \setminus \{y'\}$, this time containing all critical values of p. The monodromy map $m_p : D \setminus p(C) \to \mathrm{Map}(F)$ completely determines the smooth structure on the total space of the restricted Lefschetz fibration $p|_{p^{-1}(D)} : p^{-1}(D) \to D$.*

Proof. Let $D_0 \subset D$ be a small closed disc around y containing no critical points. Clearly, p restricts to a trivial fibre bundle over D_0, so we can choose a smooth trivialisation $p^{-1}(D_0) \xrightarrow{\cong} D_0 \times F$. Let now $\gamma_1', \ldots, \gamma_k'$ be smooth paths in D connecting y to critical values $p(c_1), \ldots, p(c_k)$ respectively, pairwise non-intersecting except for their common vertex y and let $U_1, \ldots, U_k$ be their small regular neighbourhoods. Notice that the element $m_p([\gamma_i]) \in \mathrm{Map}(F)$, where γ_i's are as before, determines the vanishing cycle up to smooth isotopy. So if we choose to attach an n-handle $D^n \times D^n$ to a boundary fibre of $p^{-1}(D_0)$ by a map embedding $\mathbb{S}^{n-1} \times \{0\}$ into a vanishing cycle with a framing of the normal bundle given by the monodromy, the

[3]For a nice visualization of a Dehn twist on a surface reader is referred to [9].

smooth structure on the resulting manifold will be well-defined. But this manifold is diffeomorphic to $p^{-1}(D_0 \cup U_1)$.

Now we repeat the procedure for all the other critical points, each time attaching an n-handle to a corresponding vanishing cycle with appropriate framing. This way we obtain handle decomposition completely determining the smooth structure on $p^{-1}(D_0 \cup \bigcup_{i=1}^{k} U_i)$, Finally, it remains to observe that $D_0 \cup \bigcup_{i=1}^{k} U_i$ is a deformation retract of D and thus there exist unique way, up to smooth isotopy, of extending the bundle p on D. □

Since all points of $D \cap D' \simeq \mathbb{S}^1$ are regular values of p, the restriction of p to $D \cap D'$ is a fibre bundle. Since p extends to D', this bundle must be trivial. But the trivialisations of $p|_{p^{-1}(D\cap D')}$ coming from D and D' need not coincide. This, together with the Proposition 3.5 above leads to the following

Corollary 3.6. *The smooth structure on the total space of a Lefschetz fibration (p, C) with regular fibre F is completely determined by the associated monodromy map $m_p : \pi_1(\mathbb{C}P^1 \setminus p(C)) \to \mathrm{Map}(F)$ and an element in $\pi_1(\mathrm{Diff}(F), \mathrm{id}_F)$.*[4]

There are still two observations to be made. Firstly, Lefschetz fibration on a disc extends to a Lefschetz fibration over the sphere $\mathbb{C}P^1$ exactly when the restriction of the fibration to the boundary of the disc is trivial, so the monodromy maps $m_p : \pi_1(D \setminus p(C)) \to \mathrm{Map}(F)$ corresponding to Lefschetz fibration over spheres are exactly those with $m_p([\gamma_k]) \circ \ldots \circ m_p([\gamma_1]) = 1 \in \mathrm{Map}(F)$, where γ_i's are as before. Secondly, changing the base point y or the identification $F \cong F_y$ over the base point or the cyclic orientation of curves changes the monodromy map by what is called an elementary move, and all these monodromies induce the same Lefschetz fibration. All this concludes in the following theorem

Theorem 3.7.

(1) *Let D be a disc. For any choice of*
  - *a regular fibre F,*
  - *a non-negative integer k and*
  - *an equivalence class of maps*

$$m : \pi_1(D \setminus \{y_1, \ldots, y_k\}) \to \mathrm{Map}(F),$$

  where $y_1, \ldots, y_k$ are arbitrary but fixed, different points, and m maps each of the generators $[\gamma_1], \ldots, [\gamma_k]$, such that γ_i is a curve encircling the corresponding point y_i once and encircling no other point of $\{y_1, \ldots, y_k\}$, to a class of some generalised Dehn twists in $\mathrm{Map}(F)$, up to equivalence given by elementary moves,

  there exists a smooth compact manifold with boundary M, unique up to diffeomorphism, and a Lefschetz fibration $p : M \to D$ corresponding to these choices.

(2) *For any choice of*
  - *F and k as above,*
  - *m as above with an additional condition that*

$$m([\gamma_k]) \circ \ldots \circ m([\gamma_1]) = 1 \in \mathrm{Map}(F)$$

  and

[4]Specifying the base point is necessary in this case since $\mathrm{Diff}(F)$ is usually not path-connected.

- *an element* $\rho \in \pi_1\left(\mathrm{Diff}(F), \mathrm{id}_F\right)$,

there exists a smooth closed manifold M*, unique up to diffeomorphism, and a Lefschetz fibration* $p : M \to \mathbb{C}P^1$ *corresponding to these choices.*

4. Lefschetz fibrations on symplectic manifolds.

Definition 4.1. Let (p, C) be a Lefschetz fibration on a symplectic manifold (M, ω) of dimension $2n$. We say that this fibration is compatible with ω if each fibre of p is a symplectic submanifold of M, away from possible critical points, and coordinates of point (3) of Definition 3.1 can be chosen in such a way that ω is a $(1,1)$-form with respect to them.

Again we will try to encode the information about a total space of this fibration in some algebraical way, i.e. we will aim for the symplectic version of Theorem 3.7. The methods will be analogous to those utilised above and the same references are still adequate, so again we will omit them in the discussion.

The difference between the current situation is the presence of a symplectic form ω_F on a regular fibre F. This form is not canonical in a same way as F, i.e. it is given by a choice of a regular fibre of p and a diffeomorphism of this fibre with F.

Let y, y', D and D' be as before, i.e. y is a regular value of the fibration $p : \widetilde{M}^A \to \mathbb{C}P^1$, D is a disc around y containing all critical values of p, y' is a point in $\mathbb{C}P^1 \setminus D$ and D' is a small disc around y' containing no critical values of p and such that $D \cup D' = \mathbb{C}P^1$. The compatibility condition of Definition 4.1 results in two following observations.

Fact 4.2. Vanishing cycles are represented by Lagrangian spheres. In particular their neighbourhood are symplectomorphic with the neighbourhood of the zero section in $T^*\mathbb{S}_i^{n-1}$ with canonical symplectic structure on cotangent bundle.

Fact 4.3. Monodromy map on a disc D containing, as before, all critical points reduces to $m_p : \pi_1(D \setminus p(C)) \to \mathrm{Map}(F, \omega_F)$, where $\mathrm{Map}(F, \omega_F)$ is the space of symplectomorphisms of (F, ω_F) modulo symplectic isotopies.

The procedure of reconstructing the total space of p over the disc D by attaching handles can now, by the two observations above, by conducted so as to preserve the symplectic structure. So we can recover the symplectic structure on a total space of a restricted bundle $p|_{p^{-1}(D)} : p^{-1}(D) \to D$. Just as before, a fibration over D extends to $\mathbb{C}P^1$ if and only if for the generators $[\gamma_1], \ldots, [\gamma_k] \in \pi_1\left(D \setminus p(C)\right)$ chosen in a usual way, the equality $m_p([\gamma_k]) \circ \ldots \circ m_p([\gamma_1]) = 1$ on monodromy holds, only this time in the space $\mathrm{Map}(F, \omega_F)$, and the extensions correspond to the elements of $\pi_1\left(\mathrm{Symp}(F, \omega_F)\right)$.

This considerations lead us to a conclusion, that an aforementioned symplectic version of Theorem 3.7 is the following.

Theorem 4.4.

(1) *Let* D *be a disc. For any choice of*
- *a symplectic regular fibre* (F, ω_F),
- *a non-negative integer* k *and*
- *an equivalence class of maps*

$$m : \pi_1\left(D \setminus \{y_1, \ldots, y_k\}\right) \to \mathrm{Map}(F, \omega_F),$$

where $y_1, \ldots, y_k$ *are arbitrary but fixed, different points, and* m *maps each of the generators* $[\gamma_1], \ldots, [\gamma_k]$*, such that* γ_i *is a curve encircling the corresponding point* y_i *once and encircling no other point of* $\{y_1, \ldots, y_k\}$*, to a class of some generalised symplectic Dehn twists in* $\mathrm{Map}(F, \omega_F)$*, up to equivalence given by elementary moves,*

there exists a smooth compact symplectic manifold with boundary (M, ω)*, unique up to symplectic isotopy, and a Lefschetz fibration* $p : M \to D$*, compatible with* ω*, corresponding to these choices.*

(2) *For any choice of*
- (F, ω_F) *and* k *as above,*
- m *as above with an additional condition that*

$$m([\gamma_k]) \circ \ldots \circ m([\gamma_1]) = 1 \in \mathrm{Map}(F, \omega_F)$$

and
- *an element* $\rho \in \pi_1(\mathrm{Symp}(F, \omega_F), \mathrm{id}_F)$*,*

there exists a smooth closed symplectic manifold (M, ω)*, unique up to symplectic isotopy, and a Lefschetz fibration* $p : M \to \mathbb{C}P^1$*, compatible with* ω*, corresponding to these choices.*

5. Lefschetz pencils

Lefschetz pencil is a further generalisation of a Lefschetz fibration, justified by the celebrated result of Donaldson (Theorem 5.5 below).

Definition 5.1. [5] A topological Lefschetz pencil on an oriented manifold M of dimension $2n$ is a triple (p, A, C), where
1. $A \subset M$ is a submanifold of codimension 4,
2. (p, C) is a Lefschetz fibration on $M \setminus A$,
3. for each point $a \in A$ there is a complex coordinate chart $(z_1, \ldots, z_n)$ around A, compatible with orientation, in which A coincides with the subspace $z_1 = z_2 = 0$ and p has the form $(z_1, \ldots, z_k) \mapsto [z_1 : z_2]$.

Let us take a closer look at the map p in the coordinate chart of point (3) above. While p itself clearly cannot be extended to A, its fibres can. For any point $y \in \mathbb{C}P^1$, the closure $\overline{p^{-1}(y)}$ is just $p^{-1}(y) \cup A$, and the behaviour of p near A assures us, that smooth structure on $p^{-1}(y)$, away from a possible critical point, extends to this closure. So we can, and will, call $\overline{p^{-1}(y)}$ a fibre of the pencil. These fibres are quite unusual, because they intersect in a common submanifold A.

Example 5.2. Consider an algebraic surface

$$K := \{[z_0 : z_1 : z_2 : z_3] | z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{C}P^3.$$

This is of course an example of a $K3$-surface. An intersection A of the linear subspace $A_0 = \{[0 : 0 : z_2 : z_3]\} \subset \mathbb{C}P^3$ and K consists of 4 points,

$$A = \{[e^{i\pi/4} : 1], [e^{i3\pi/4} : 1], [e^{i5\pi/4} : 1], [e^{i7\pi/4} : 1]\}.$$

All linear subspaces of $\mathbb{C}P^3$ of complex codimension 1 and containing A_0 are parametrized by points of $\mathbb{C}P^1$ in a following manner

$$A_{[w_0:w_1]} = \{[w_0 : w_1 : z_2 : z_3]\}.$$

All these subspaces add up to the whole space $\mathbb{C}P^3$ and for any pair of points $[w_0 : w_1] \neq [w_0', w_1'] \in \mathbb{C}P^1$ the corresponding subspaces intersect exactly in A. So we can define a map $p : K \setminus A \to \mathbb{C}P^1$ which assigns to any point $x \in K$ the unique point $[w_0 : w_1] \in \mathbb{C}P^1$ such that $x \in A_{[w_0:w_1]}$. It is not hard to check that this map is in fact a Lefschetz pencil on K.

Before stating the results concerning Lefschetz pencils on symplectic manifolds it is natural to demand some compatibility between the pencil structure and the symplectic form.

Definition 5.3. A topological Lefschetz pencil on a symplectic manifold (M, ω) of dimension $2n$ is said to be a symplectic Lefschetz pencil on M if

(1) coordinate charts described in point (3) of Definition 3.1 and point (3) of Definition 5.1 can be chosen such that the symplectic form ω is a $(1, 1)$-form with respect to them,
(2) A is a symplectic submanifold of M,
(3) any fibre of the pencil is a symplectic submanifold of M away from critical points,
(4) the homology class of any regular fibre of the pencil is Poincaré dual to the cohomology class of the symplectic form.

Remark 5.4. Since A is a symplectic submanifold of M we can take a symplectic blow-up $\widetilde{M}^A$ of M along A. It is an easy exercise to check that, by point (1) above, the map p extends to a compatible Lefschetz fibration on a symplectic manifold $(\widetilde{M}^A, \widetilde{\omega})$.

We are finally ready to state the aforementioned theorem of Donaldson.

Theorem 5.5. [5] *Let (M, ω) be a symplectic manifold such that ω is integral, i.e. $[\omega] \in \operatorname{im}(H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R}))$. Then for integral k large enough, $(M, k\omega)$ admits a symplectic Lefschetz pencil.*

This theorem and the Remark 5.4 lead to the following

Corollary 5.6. *Every symplectic manifold with integral symplectic structure admits, after symplectic blow-up of a submanifold of codimension* 4*, a compatible structure of a Lefschetz fibration.*

Remark 5.7. The restriction that a symplectic form is integral is in fact a minor one. Every symplectic form ω can be perturbed, by an arbitrary small perturbation in an arbitrary small open set, to a rational symplectic form ω', i.e. such that $[\omega'] \in \operatorname{im}\left(H^2(M; \mathbb{Q}) \to H^2(M; \mathbb{R})\right)$. Then it is just a matter of multiplying by an appropriate positive integer to obtain an integral one.

Corollary 5.6 together with Remark 5.7 implies in particular that symplectic manifolds, up to a blow-up, can be described topologically as in Theorem 4.4.

As it turns out, even the inconvenience of blowing-up can be omitted, namely we can find an analogue of Theorem 4.4 for Lefschetz pencils. To do this we must repeat the discussion before this theorem but with some additional care. Let (p, C) be a Lefschetz fibration obtained by blowing-up of a symplectic Lefschetz pencil (p, A, C). Consider the monodromy map of this fibration restricted to a disc D as before,

$$m_p : \pi_1 (D \setminus \{y_1, \dots, y_k\}) \to \operatorname{Map}(F, \omega_F).$$

If we are not careful with our choices of representatives of classes in $\operatorname{Map}(F, \omega)$, it may happen that, while the reconstructed symplectic structure extends to the total space of a fibration over a sphere, it extends in such a way, that the blow-up of a neighbourhood of A cannot be blown-down. To avert that, we must retain control of how A and its neighbourhood behave under monodromy.

Since, for $[\gamma_1], \ldots, [\gamma_k]$ as usual, $m_p([\gamma_1]), \ldots, m_p([\gamma_k])$ have representatives with supports disjoint with some small neighbourhood of A, we can reduce the monodromy map to

$$m_p : \pi_1 (D \setminus \{y_1, \ldots, y_k\}) \to \mathrm{Map}(F, \omega_F; A),$$

where $\mathrm{Map}(F, \omega_F; A)$ denotes the space of symplectomorphisms of (F, ω_F) leaving A and $TF|_A$ fixed, modulo symplectic isotopies with the same property. But the equality $m([\gamma_k]) \circ \ldots \circ m([\gamma_1]) = 1$ no longer holds in $\mathrm{Map}(F, \omega_F; A)$. On the contrary, there exist a non-trivial element $\varsigma_A \in \mathrm{Map}(F, \omega_F; A)$, depending only on an embedding $A \subset F$, such that $m([\gamma_k]) \circ \ldots \circ m([\gamma_1]) = \varsigma_A \in \mathrm{Map}(F, \omega_F; A)$. So clearly, we must at least impose this equality as a condition on a monodromy map if we hope to obtain a Lefschetz fibration corresponding to a Lefschetz pencil. In fact, as proven by Gompf, this is almost a sufficient condition. Namely, we have the following

Theorem 5.8. [7] *Let D be a disc. For any choice of*

- *a symplectic regular fibre (F, ω_F),*
- *a symplectic submanifold $A \subset F$ of codimension 2 satisfying $[A] = \mathrm{PD}(\omega_F)$,*
- *a non-negative integer k and*
- *an equivalence class of maps*

$$m : \pi_1 (D \setminus \{y_1, \ldots, y_k\}) \to \mathrm{Map}(F, \omega_F; A),$$

  as in part (1) of Theorem 4.4, satisfying

$$m([\gamma_k]) \circ \ldots \circ m([\gamma_1]) = \varsigma_A \in \mathrm{Map}(F, \omega_F),$$

such that either $\dim(F) > 2$ or $\dim(F) = 2$ and each vanishing cycle is non-separating[5]*, there exists a smooth closed symplectic manifold M, unique up to symplectic isotopy, and a Lefschetz pencil $p : M \setminus A \to \mathbb{C}P^1$ corresponding to these choices.*

6. Outline of the proof of the Theorem 5.5

We present here a very cursory review of Donaldson's proof. Common references for the material below are two papers of Donaldson from Journal of Differential Geometry [4, 5]

Let (M, ω) be a symplectic manifold of dimension $2n$ with an integral symplectic form. We identify $H^*(M; \mathbb{Z})$ with its image in $H^*(M; \mathbb{R})$ by a coordinate change map, so in particular we write $[\omega] \in H^2(M; \mathbb{Z})$. It is a classical result that complex line bundles over a smooth manifold are classified by the second integral cohomology of this manifold, with the correspondence set by the first Chern class. So in particular we can choose a complex line bundle $\mathbb{C} \to L \to M$ with $c_1(L) = [\omega]$. By the Chern-Weil theory, we can impose a connection on L with curvature $-i\omega$.

The choice of a connection induces a covariant derivative ∇, so for any section $s \in \Gamma L$ we have a map $\nabla s : \Gamma TM \to \Gamma L$. Covariant derivative is $\mathcal{C}^\infty(M)$-linear in its vector field coordinate, so, for a fixed section s, it induces a map $(\nabla s)_x : T_x M \to L_x$

[5] If $\dim(F) = 2$, i.e. F is a surface, then vanishing cycles are just smooth loops on F. Therefore they are of codimension 1 and the notion of separating or non-separating vanishing cycles makes sense.

over each point $x \in M$. Fix an ω-compatible almost-complex structure J on M. It gives an identification of T_xM with $\mathbb{C}^n$, in which ω corresponds to a standard complex structure. Similarly, complex structure on L identifies each fibre L_x with $\mathbb{C}$. So we can consider $(\nabla s)_x$ as an $\mathbb{R}$-linear map $\mathbb{C}^n \to \mathbb{C}$. We only have linearity over $\mathbb{R}$, since no compatibility of ∇ with almost-complex structures of TM and L is given.

Consider an arbitrary nondegenerate $\mathbb{R}$-linear map $a : \mathbb{C}^n \to \mathbb{C}$ and let $\omega_{\mathbb{C}^n}$ denote the standard symplectic structure on $\mathbb{C}^n$. Let $a = a' + a''$ be a unique splitting of a into $\mathbb{C}$-linear and anti-$\mathbb{C}$-linear part respectively. Donaldson provided a very geometric proof of the following lemma.

Lemma 6.1. *If $\|a''\| < \|a'\|$, where the norm $\|\cdot\|$ is induced by the standard metrics on $\mathbb{C}^n$ and $\mathbb{C}$, then* $\ker a$ *is a symplectic subspace of $\mathbb{C}P^n$.*

Proof. Choose an arbitrary oriented $\mathbb{R}$-linear subspace $V \subset \mathbb{C}^n$ of dimension $2k$. The space $\bigwedge^{2k} V^*$ is 1-dimensional, hence there exist a real parameter $\eta_V \in \mathbb{R}$ such that $\frac{1}{k!}\omega_{\mathbb{C}^n}^k|_V = \eta_V \Omega_V$, where Ω_V is an orientation form induced by the orientation of V and the standard metric on $\mathbb{C}^n$ restricted to V. It is easy to check, that $\eta_V \in [-1, 1]$ and $\eta_V = 1$ exactly when V is a $\mathbb{C}$-linear subspace. Clearly, V is a symplectic subspace if and only if $\eta_V > 0$.

We can choose unique real number $\theta_V \in [0, \pi]$, called the Kähler angle, such that $\cos\theta_V = \eta_V$. It is a matter of computation to check that for a as above

$$\tan\theta_{\ker a} = \frac{2\sqrt{\|a'\|^2\|a''\|^2 - \|\langle a', \overline{a''}\rangle\|^2}}{\|a'\|^2 - \|a''\|^2}.$$

□

The standard notation for the $\mathbb{C}$-linear and anti-$\mathbb{C}$-linear part of ∇s is ∂s and $\overline{\partial} s$ respectively. Assume that we can choose s to be transversal to the zero section of L and to satisfy $\|\overline{\partial} s\| < \|\partial s\|$ on the zero set $A := \{s = 0\} \subset M$, where $\|\cdot\|$ is determined by a metric on M associated to J and ω, and a Hermitian metric on L. Then A is a codimension-2 and, by the lemma above, symplectic submanifold. Clearly, $\mathrm{PD}(A) = c_1(L) = [\omega]$. While in general L does not support such a section, Donaldson proved that, for k large enough, $L^{\otimes k}$ does. This leads to his famous submanifold theorem.

Theorem 6.2. [4] *Let (M, ω) be a symplectic manifold of dimension $2n$. For integer k large enough there exists a symplectic submanifold A of (M, ω) representing homology class $k\,\mathrm{PD}(\omega) \in H_{2n-2}(M; \mathbb{Z})$.*

The guiding idea of the Donaldson's proof of Theorem 5.5 is to extend this argument to two sections s_0, s_1 of $L^{\otimes k}$ with additional transversality conditions imposed. This transversality conditions are expressed in the language of ε-transversality.

Remark 6.3. Auroux took one step further and proved the existence of three such sections, cf. Remark 7.10. The problem of existence of even greater number of such sections is still open.

For any linear map T we define $\nu(T)$ as the square root of the least positive eigenvalue of TT^*. This notion translates to the setting of sections of bundles, so we can formulate the following definition

Definition 6.4. [5] Fix $\varepsilon > 0$. Section s of some complex bundle is ε-transverse to 0 if there is no point where $\|s\| < \varepsilon$ and $\nu(\nabla s) < \varepsilon$ at the same time, with respect to norms defined by the choice of metrics on the base and on the bundle.

Definition 6.5. [5] Fix $\varepsilon > 0$. A pair of sections $s_0, s_1 \in \Gamma L^{\otimes k}$ is said to satisfy the ε-transversality condition if

(1) the section s_0 is ε-transverse to 0,
(2) the section $(s_0, s_1) \in \Gamma\left(L^{\otimes k} \oplus L^{\otimes k}\right)$ is ε-transverse to 0 and
(3) for $W_\infty := \{s_0 = 0\}$ and for $F := s_1/s_0 : M \setminus W_\infty \to \mathbb{C}$ the derivative ∂F is ε-transverse to 0.

The norm used in the definition above is defined with respect to the metric g_k on M, associated to J and $k\omega$, and a Hermitian metric on $L^{\otimes k}$. We can use this metrics and Levi-Civita connection of g_k to define, in a standard way, norms $\|\cdot\|_{\mathcal{C}^r}$.

Definition 6.6. [5] For $C > 0$ we say that the section $s \in \Gamma L^{\otimes k}$ is C-bounded if $\|s\|_{\mathcal{C}^3} < C$ and $\|\overline{\partial} s\|_{\mathcal{C}^2} < C/\sqrt{k}$.

Donaldson's proof splits into two parts. The first part is the proof of the following proposition.

Proposition 6.7. *For k large enough, if there exist constants $C, \varepsilon > 0$ and sections $s_0, s_1 \in \Gamma L^{\otimes k}$ which are C-bounded and satisfy the ε-transversality condition, then there exist a symplectic Lefschetz pencil on M.*

The second part, for which we refer the reader to Sections 3,4,5,6 and 7 of [5], is the construction of a pair of sections $s_0, s_1 \in \Gamma L^{\otimes k}$ satisfying, for k large enough, the hypotheses of Proposition 6.7.

Sketch of proof of the Proposition 6.7. The common zero-set of sections s_0 and s_1,

$$A := \{s_0 = s_1 = 0\},$$

is, by a point (2) of Definition 6.5, a smooth submanifold of codimension 4 in M. On the complement of A we have a map

$$F : M \setminus A \ni x \to [(s_0)_x : (s_1)_x] \in \mathbb{C}P^1.$$

It is clear that this map is well-defined and restricts to a map $F : M \setminus W_\infty \to \mathbb{C}$ of point (3) of Definition 6.5 (hence the collision in notation). Let $\Gamma \subset M \setminus A$ consist of points where $\|\overline{\partial} F\| \geq \|\partial F\|$. By the discussion analogous to this preceding the formulation of the Theorem 6.2, restriction of F to $M \setminus A \setminus \Gamma$ is a submersion with symplectic fibres.

It can be shown by a simple analysis that Γ is compact. Points where $\partial F = 0$ all lie in Γ and, by point (3) of Definition 6.5, are isolated, so $\Delta := \{\partial F = 0\}$ is finite. Some more analysis based on Inverse Function Theorem is needed to show that for k sufficiently large we can find small balls around points of Δ which are disjoint and cover Γ. Finally, even more analysis allows to modify F in this balls so that points of Δ become critical points with local model as in definition of Lefschetz pencil. Now it suffices to further modify F in a tubular neighbourhood of A to make it coincide with the appropriate local model. □

7. Linear systems

The notion of Lefschetz pencil is very susceptible to generalisations, for which the dimension of a basis is higher than 2. A common setting of these generalisations is a notion of a linear k-system.

Definition 7.1. [7] Let M be a manifold of dimension $2n$ and let k be a positive integer, $k \leq n-1$. A linear k-system (p, A, J) consists of an almost-complex structure J on M, a submanifold $A \subset M$ of codimension $2(k+1)$ and a smooth map $p : M \setminus A \to \mathbb{C}P^k$, such that

1. (1) for a standard symplectic structure $\omega_{\mathbb{C}P^k}$ on $\mathbb{C}P^k$, the 2-form $p^*\omega_{\mathbb{C}P^k}$ tames J and
2. (2) the normal bundle νA of $A \subset M$ admits a complex structure, such that for some identification of νA with a tubular neighbourhood of A, the map p is just a projectivisation over each fibre.

Notice that a Lefschetz pencil is an example of a linear 1-system, but with strong control over the critical points of p absent from the general definition.

Remark 7.2. Notice that, just as in the case of Lefschetz pencils, we can prolong each fibre $p^{-1}(y)$ in a smooth way to a fibre $F_y = \overline{p^{-1}(y)} = p^{-1}(y) \cup A$.

Remark 7.3. The inclusion $\iota : M \setminus A \to M$ induces an isomorphism in cohomology

$$\iota^* : H^{\leq 2k}(M; \mathbb{Z}) \xrightarrow{\cong} H^{\leq 2k}(M \setminus A; \mathbb{Z}).$$

In particular for a unique generator α of $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$ with $\langle \alpha^{n-1}, [\mathbb{C}P^{n-1}] \rangle > 0$ we have a distinguished class $c_p = (\iota^*)^{-1} p^* \alpha \in H^2(M; \mathbb{Z})$.

Just as in Section 2, two questions naturally arise: under what conditions does a symplectic manifold admit a k-linear system, compatible in some sense, and under what conditions does a total space of a linear k-system admit a symplectic form, again compatible in some sense.

The answer to the latter question has been provided by Gompf in two stages, generalising the Propositions 2.3 and 2.6 respectively. First, in [8], he distinguished a class of $2n$-dimensional $(n-1)$-linear systems, where the fibre is 2-dimensional, which he called hyperpencils, and proved that they admit a symplectic structures. Loosely speaking, a hyperpencil is a linear system with some mild control over the critical points. To quote Gompf's definition we need to introduce a, somewhat technical, notion of a wrapped critical point.

Definition 7.4. [8] Let f be a smooth map between smooth manifolds M and N and denote by $\mathrm{Crit}(M)$ the set of critical points of f. A point $x \in M$ is *wrapped if the linear subspace*

$$\mathrm{span}(\overline{\bigcup_{y \in M \setminus \mathrm{Crit}(M)} \ker \mathrm{d}_y f} \cap T_x M) \subset T_x M$$

is of codimension at most 2.

Definition 7.5. [8] A hyperpencil on an oriented manifold M of dimension $2n$ consists of a finite set $A \subset M$ and a map $p : M \setminus A \to \mathbb{C}P^{n-1}$ such that

1. (1) around each point of A there exist complex coordinates $(z_1, \ldots, z_n)$ in which p is given by the projectivisation,

(2) each critical point of p is wrapped,
(3) around each critical point of p there exists locally defined $p^*\omega_{\mathbb{C}P^{n-1}}$-compatible almost-complex structure J,
(4) for a point $y \in \mathbb{C}P^{n-1}$, each connected component of the set of regular points of p in the fibre $\overline{p^{-1}(y)}$ intersects A.

Theorem 7.6. [8] *For any hyperpencil (p, A) on an oriented manifold M of dimension $2n$ there exists a symplectic structure ω in a cohomology class c_p of Remark 7.3, that is compatible with a complex structure near points of A given by coordinate charts of point (1) of Definition 7.5.*

Later, in [7], Gompf gave an answer for any dimension of a fibre. His main result, [7, Theorem 2.3], is somewhat complicated, so we split it into Lemma 7.7, Theorem 7.8 and Lemma 7.9 below.

Lemma 7.7. *For any linear k-system (p, A, J), the submanifold A is J-holomorphic and the almost complex structure J_ν on the normal bundle νA mentioned in Definition 7.1 can be chosen such that $J = J|_{TA} \oplus J_\nu$ on $TM|_A$. For this choice of J_ν there exist a Hermitian metric on νA reducing the structure group of this bundle to $U(1) \to U(k+1)$ acting diagonally.*

Theorem 7.8. *Let (p, A, J) be a linear k-system on M. If there exists a closed 2-form ζ on M such that*

- *ζ restricts to a symplectic form on A taming $J|_A$,*
- *$[\zeta] = c_p \in H^2(M)$,*
- *$\zeta|_{TF|_A} = \zeta|_{TA} \oplus \zeta|_\nu$ and*
- *on each J-complex line subbundle of ν, ζ agrees with a Hermitian metric of Lemma 7.7.*

and for each point $y \in \mathbb{C}P^k$ the fibre F_y has a neighbourhood W_y with a closed 2-form η_y

- *taming $J|_{\ker d_x p}$ for all $x \in W_y \setminus A$,*
- *agreeing with ζ on each $TF_z|_A$ for z near y in $\mathbb{C}P^k$ and*
- *satisfying $[\eta_y - \zeta] = 0 \in H^2(W_y, A)$.*

Then M admits a symplectic form in cohomology class c_p.

Lemma 7.9. *Let (p, A, J) be a linear k system on M. If ω_A is a symplectic form on A taming $J|_A$ and with $[\omega_A] = c_p|_A \in H^2(A; \mathbb{Z})$, then it extends to a closed 2-form ζ on M as in Theorem 7.8.*

Remark 7.10. Answer to the existence question is only known for $k = 2$. Auroux first showed in [1] that every 4-dimensional symplectic map admits a branched cover, which is a linear 2-system with controlled behaviour near critical submanifolds, and then extended this result in [2], where he showed that every symplectic manifold admits a quasiholomorphic map to $\mathbb{C}P^2$.

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