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On some smooth circle actions on symplectic manifolds

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1 Introduction

This thesis is devoted to the study of circle actions on symplectic manifolds. We are mostly interested in answering the following question.

Question. Assume that a closed symplectic manifold (M, ω) admits a smooth circle action. Does that imply that for some other symplectic form ω' on M, a symplectic manifold (M, ω') admits a symplectic circle action?

In dimension 4 the answer has been partially provided in the form of the following theorem of Baldridge [3].

Theorem. *If a closed symplectic 4-manifold admits a non-trivial smooth circle action with fixed points, then it is diffeomorphic to either rational or ruled surface. In particular, it admits a symplectic circle action, possibly with respect to another symplectic form.*

In the same paper Baldridge conjectures that the assumption about the existence of fixed points can be dropped. Indeed, it seems that this conjecture has recently been proven by Bowden [5].

Despite the success in dimension 4, it seems highly unlikely that the answer to this question is positive in general. On the other hand, no counterexample is currently known, at least up to author's knowledge.

If we ask a weaker question, the answer is usually easier to find. And so, if we fix a smooth circle action on a symplectic manifold it is easy to find symplectic forms which are not invariant by this action. In fact every invariant form can be made into a non-invariant one by an arbitrary small deformation. Hajduk, Pawałowski and Tralle [13] (and more recently Kaluba and Politarczyk [17]) constructed examples of smooth circle actions on symplectic manifolds with non-symplectic fixed point sets, showing that they cannot be symplectic with respect to any symplectic form. Yet their manifolds may admit other symplectic circle actions. Allday [1] on the other hand found examples of cohomologically symplectic manifolds with smooth and not symplectic circle action, but it is not clear whether this manifolds are actually symplectic.

In this thesis we find a similar result, but in a slightly different direction. We construct a smooth circle action on a symplectic 6-manifold and then we prove that this manifold does not admit a symplectic circle action for a whole equivalence class of symplectic structures. The main result is the following.

Theorem 1.1. *There exists a closed symplectic 6-manifold (M, ω) such that M admits a non-trivial smooth circle action and for any symplectic form ω' on M equivalent to ω, a symplectic manifold (M, ω') does not admit any non-trivial symplectic circle actions.*

To prove this, we had to find some criterium for the existence of a symplectic circle action depending only on the equivalence class of a symplectic form. This criterium is given in Lemma 7.4. While this criterium is stated in the language of an underlying almost-complex structure, it is symplectic in nature. Its proof is based on properties of a moment map of the action as a Morse–Bott function.

Of course, to confirm a negative answer to the question asked at the beginning of this introduction, we would have to extend the conclusion of the Theorem 1.1 from any symplectic form equivalent to a given one to any symplectic form at all. Our approach fails for symplectic forms for which the Todd genus of a compatible almost-complex structure satisfies some inequality. The Wall's classification of almost-complex structures on 6-manifolds [34] proves, that there are many almost-complex structures on our manifold which satisfy this inequality, but we do not know, if any of them is compatible with some symplectic form.

Section 2 contains a brief discussion of the current state of knowledge about the Chern numbers of non-equivalent symplectic structures. It may be summarized by saying that we do not know, whether non-equivalent symplectic structures can ever be distinguished by their Chern numbers. This is different to the complex case, where examples of complex structures on a given 6-manifold with different Chern numbers are known [20]. If it turned out that the Todd genus of a symplectic manifold, or at least a symplectic 6-manifold, depends only on the underlying smooth manifold, our method would provide an answer to the main question.

Sections 2 to 6 of this thesis contain short introductions to the various topics related with the proof of the Theorem 1.1 and introduce some results necessary in this proof. And so, Section 2 contains a brief recollection on symplectic manifolds. Of particular interest is Example 2.9, a possible building block for a manifold constructed in the proof of the main theorem, the definition of equivalence of symplectic forms, and Observation 2.15. Section 3 concerns smooth circle actions. Some basic notions and results are presented, to be used later. This section concludes with Example 3.23, another building block for the main theorem. Section 4 discusses symplectic and Hamiltonian circle actions. Some basic properties of a moment map are recalled, as well as the the important criterium for a symplectic circle action to be Hamiltonian (Proposition 4.17). Section 5 quotes the theorem of Li [21] about fundamental groups of Hamiltonian $\mathbb{S}^1$-manifolds. The original proof from Li's paper is modified, to better suit our needs. Another important tool, the Todd genus, is the topic of Section 6. The genus is introduced in the spirit of Hirzebruch's books [15, 16]. Essential results of this chapter are computations of the genus for low-dimensional manifolds (Lemma 6.9) and Corollary 6.11. Finally, all of this tools, along with the "stabilization procedure" described in Lemma 7.3, are used to give a proof of the main theorem in Section 7.

Unless explicitely stated otherwise, the term "manifold" will always refer to a smooth ($\mathcal{C}^\infty$) connected manifold without boundary, not necessarily compact. Compact manifolds will be described using the term "closed manifold". Similarly, unless explicitely stated otherwise, the term "map" will always refer to a smooth ($\mathcal{C}^\infty$) map.

2 Symplectic manifolds

A fundamental notion for this thesis is that of a symplectic manifold.

Definition 2.1. A differential 2-form ω on a manifold M is *symplectic* if ω is closed and non-degenerate. The pair (M, ω) is known as a *symplectic manifold.*

By saying that a 2-form ω on M is non-degenerate, we mean that for each point $p \in M$ and for each non-zero vector $v_p \in \mathrm{T}_p M$, the 1-form $\mathrm{i}(v_p)\omega_p$ is non-zero. Equivalently, the dimension of M is even, $\dim M = 2n$, and ω^n is a volume form on M. As as immediate consequence we can say that every symplectic manifold is canonically oriented.

Definition 2.2. A map $f : M_1 \to M_2$ between symplectic manifolds (M_1, ω_1) and (M_2, ω_2) such that $f^*\omega_2 = \omega_1$ is called a *symplectic map* and denoted $f : (M_1, \omega_1) \to (M_2, \omega_2)$.

The standard example of a symplectic manifold in dimension $2n$ is the following.

Example 2.3. Let $p_1, \ldots, p_n, q_1, \ldots, q_n$ denote the standard coordinates on the Euclidean space $\mathbb{R}^{2n}$. A 2-form ω_{std} given by $\omega_{std} = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \ldots + \mathrm{d}p_n + \mathrm{d}q_n$ is a symplectic form on $\mathbb{R}^{2n}$ known as the *standard symplectic form.*

This example is, in some sense, the most important one, due to the following theorem, known as the Darboux theorem (see for example McDuff and Salamon's book [25, Theorem 3.15]).

Theorem 2.4. *Let (M, ω) be a symplectic manifold of dimension $2n$. Every point $p \in M$ has a neighbourhood symplectomorphic to an open subset of $(\mathbb{R}^{2n}, \omega_{std})$.*

Another important example is that of a symplectic form on the contangent bundle of an arbitrary manifold. Again, its importance stems from a neighbourhood theorem, this one associated with Weinstein (see Theorem 2.6 below).

Example 2.5. Let L be an arbitrary n-dimensional manifold. Canonical 1-form θ on the contangent bundle T^*L is the unique 1-form characterised by the condition that for any 1-form σ on L the equality $\sigma^*\theta = \sigma$ holds. The 2-form $\mathrm{d}\theta$ is the standard symplectic form on T^*L.

Theorem 2.6. *Let (M, ω) be a symplectic $2n$-manifold and let $L \subset M$ be an n-dimensional submanifold such that $\omega|_L \equiv 0$. Such L is known as a* Lagrange *submanifold. The inclusion of L into T^*L as a zero-section extends to a symplectomorphism from a neighbourhood of L in (M, ω) to a neighbourhood of the zero-section in $(\mathrm{T}^*L, \mathrm{d}\theta)$.*

Both examples given above are examples of open symplectic manifolds. Historically, the first closed manifolds known to admit a symplectic structure were Kähler manifolds, such as the complex projective space.

Example 2.7. Complex projective space $\mathbb{CP}^n$ admits a symplectic form ω_{FS}, the Fubini–Study form. This is the unique form with the property that for each parametrisation $\varphi_i : \mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto [z_0 : \ldots : z_{i-1} : 1 : z_i : \ldots : z_n] \in \mathbb{CP}^n$, where $i \in \{0, \ldots, n\}$, the pullback $\varphi_i^*\omega_{FS}$ is equal to

$$\varphi_i^*\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(|z_1|^2 + \ldots + |z_n|^2 + 1).$$

One way of obtaining new examples of closed symplectic manifolds is by taking a symplectic submanifolds of known closed symplectic manifold. If (M, ω) is a symplectic manifold, then a submanifold $K \subset M$ is *symplectic* if $\omega|_K$ is a symplectic form on K. Finding symplectic submanifolds in complex projective spaces is particularly easy thanks to the following observation.

Observation 2.8. *Every algebraic submanifold of* $\mathbb{CP}^n$ *is a symplectic manifold.*

Example 2.9. A quadric in $\mathbb{CP}^3$ described by the equation

$$K = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

is a 4-dimensional closed symplectic manifold with induced symplectic form $\omega_{FS}|_K$. The Chern classes of K are easily computed and equal to

$$c_1(K) = 0 \text{ and } \langle c_2(K), [K]\rangle = 24.$$

Hence $\chi(K) = 24$ and $\sigma(K) = -16$. At the other hand, using the Lefschetz hyperplane theorem we obtain $\pi_1(K) = 1$. Putting all of this together, we see that $b^2_+(K) = 3$ and $b^2_-(K) = 19$. The symplectic manifold $(K, \omega_{FS}|_K)$ is known as the *symplectic K3 surface*.

One obstruction to the existence of a symplectic form on a given oriented manifold M is as follows: a cohomology class of a symplectic form ω on M satisfies $\langle[\omega]^n, [M]\rangle > 0$, where $2n$ is the dimension of M. Another obstruction is the existence of a compatible almost-complex structure on M.

Definition 2.10. An almost-complex structure J on a symplectic manifold (M, ω) is *compatible* if for each pair of vector fields X and Y on M the following conditions hold:

$$\omega(X, JX) > 0 \qquad \text{and} \qquad \omega(JX, JY) = \omega(X, Y).$$

Equivalently, J is a compatible almost-complex structure if $(X, Y) \mapsto \omega(X, JY)$ is a Riemannian metric on M. Such metric is also said to be compatible.

Proposition 2.11. *For a symplectic manifold* (M, ω) *the underlying manifold* M *admits almost-complex structures compatible with* ω*. The family of all such structures is a non-empty contractible subset of the space of all almost-complex structures on* M*.*

For a symplectic manifold (M, ω) and a compatible almost-complex structure J, we can compute the Chern numbers $c_I(M, J)$. Since all such almost-complex structures are homotopic, those numbers do not depend on the choice of J. We can write $c_I(M, \omega)$.

Above, we have seen two obstructions for the existence of a symplectic form: the existence of an appropriate cohomology class and the existence of an almost-complex structure. Surprisingly, especially in higher dimensions, not many other obstructions are known.

If a manifold M admit a symplectic form, it automatically admits many other. In fact, if (M, ω) is a symplectic manifold and α is an arbitrary 1-form on M with compact support, then $\omega + \varepsilon\, d\alpha$ is also a symplectic manifold for $\varepsilon > 0$ small enough. Similarly, for any positive constant $c \in \mathbb{R}$, the form $c\omega$ is also symplectic. Finally, for any self-diffeomorphism $f : M \to M$, the form $f^*\omega$ is yet another symplectic form on M. Of course, all of these forms are "similar", and if we tried to classify symplectic forms on a given manifold, we would not consider these forms as different. This similarity is encapsulated in the notion of *equivalence* of symplectic forms (Definition 2.14 below).

Definition 2.12. Two symplectic forms ω and ω' on a manifold M are *symplectomorphic*, if the symplectic manifolds (M, ω) and (M, ω') are symplectomorphic by a symplectomorphism preserving the orientation.

Since every symplectic form on a manifold determines an orientation, we can treat admitting symplectic forms as a property of oriented manifolds. This is expressed in the definition above. In particular, $\mathbb{CP}^2$ with standard orientation admits a symplectic form, while $\mathbb{CP}^2$ with non-standard orientation, denoted as $\overline{\mathbb{CP}^2}$, does not, as seen for example using the cohomological criterium given earlier.

Definition 2.13. Two symplectic forms ω and ω' on a manifold M are *deformation equivalent*, if there exists a smooth path $\{\omega_t\}_{t=0}^1$ of symplectic forms on M with $\omega = \omega_0$ and $\omega' = \omega_1$.

Definition 2.14. Two symplectic forms ω and ω' on a manifold M are *equivalent*, if there exists a sequence $\omega_0, \dots, \omega_k$ with the property that $\omega = \omega_0$, $\omega' = \omega_k$ and for each $i = 1, \dots, k$, the form ω_i is either symplectomorphic or deformation equivalent to ω_{i-1}.

Notice, that as integer classes, the Chern classes of a symplectic manifold (M, ω) do not change when we deform the symplectic form ω. For a self-diffeomorphism f of M, the Chern classes of (M, ω) and $(M, f^*\omega)$ will, in general, differ. But an orientation-preserving self-diffeomorphism of M acts trivially on the highest homology and cohomology of M, so the Chern numbers of (M, ω) and $(M, f^*\omega)$ will coincide. This leads us to the following conclusion.

Observation 2.15. *If ω_1 and ω_2 are two equivalent symplectic forms on a manifold M, then all Chern numbers of (M, ω_1) and (M, ω_2) coincide.*

It might seem that based on this observation it should be relatively easy to find examples of non-equivalent symplectic forms on a given symplectic manifold. But the reality is quite different. There are known examples of inequivalent symplectic forms in dimension 4 [26, 31, 32], but they have to be distinguished by their Chern classes, not Chern numbers, since the latter are topological invariants. Ruan [30] constructs examples of non-equivalent symplectic forms on 6-dimensional manifolds, but this forms turn out to induce the same Chern numbers. They are distinguished by their Gromov invariants, which will not be discussed here.

3 Smooth circle actions

In this section we recall some standard results concerning compact Lie group actions, stated for the case when this Lie group is the circle. This allowed us to simplify the presentation. Moreover, we have chosen to present only a small part of this rich theory, focusing on what will be needed in further sections. Much broader introduction to the general theory of compact Lie group actions can be found for example in the classical book of Bredon [6].

Definition 3.1. A map $\lambda : \mathbb{S}^1 \times M \to M$ is a *smooth circle action* on a manifold M if it satisfies the following conditions:

1. $\lambda(1, p) = p$ for every $p \in M$,
2. $\lambda(e^{it}, \lambda(e^{is}, p)) = \lambda(e^{i(t+s)}, p)$ for every $e^{it}, e^{is} \in \mathbb{S}^1$ and every $p \in M$.

Example 3.2. Any manifold admits at least one smooth circle action, the *trivial action*. It is given by the projection $\lambda : \mathbb{S}^1 \times M \ni (e^{it}, p) \mapsto p \in M$.

Any smooth circle action λ gives rise to a family of maps

$$\lambda_{e^{it}} : M \ni p \mapsto \lambda(e^{it}, p) \in M$$

parametrized by elements $e^{it} \in \mathbb{S}^1$. The map $\lambda_{e^{it}}$ is known as the *translation by* e^{it} or the *action of* e^{it}. Conditions 1. and 2. of Definition 3.1 above imply, that the map $\mathbb{S}^1 \ni e^{it} \mapsto \lambda_{e^{it}} \in \mathrm{Diff}\,(M)$ is a group homomorphism. Conversely, every continuous homomorphism $\mathbb{S}^1 \to \mathrm{Diff}\,(M)$ is induced by a smooth circle action [7, Theorem 5]. Sometimes, when there is no risk of confusion, we will write $e^{it} \cdot p$ instead of $\lambda_{e^{it}}(p)$.

A map $f : M_1 \to M_2$ between two manifolds with circle actions λ_1 and λ_2 respectively is called *equivariant* if $(\lambda_2)_{e^{it}} f = f(\lambda_1)_{e^{it}}$ for every $e^{it} \in \mathbb{S}^1$. A category of *smooth* $\mathbb{S}^1$*-manifolds* is a category whose objects are pairs (M, λ), where M is a manifold and λ is a smooth circle action on M, and whose morphisms are equivariant maps.

Remark. As mentioned before, we assume all maps to be smooth. If instead we only require λ in Definition 3.1 to be continuous, we obtain the notion of a *continuous circle action* and a category of *continuous* $\mathbb{S}^1$*-manifolds* with morphisms being continuous equivariant maps. We could also consider general *circle actions*, with λ being a map in the set-theoretic sense. Many of the results we mention below still hold in those broader categories, possibly in some weaker form or after imposing some additional conditions. We will not elaborate on that. Moreover, since we work almost exclusively in smooth category, we usually omit "smooth" in "smooth circle action", assuming every action to be smooth unless explicitely stated otherwise.

Let now λ be some circle action on a manifold M and let $p \in M$ be a point in M. Elements $e^{it} \in \mathbb{S}^1$ which fix p form a closed subgroup of $\mathbb{S}^1$. This subgroup is known as the *isotropy subgroup of* p and denoted $\mathrm{Iso}\,(p, \lambda)$ or $\mathbb{S}^1_p$, when the action is clear from the context. As a closed subgroup of the circle, $\mathbb{S}^1_p$ is either finite or equal to the whole group $\mathbb{S}^1$. In the latter case, we say that the point p is *fixed* by the action. The set of all fixed points of the action is denoted by

Fix (λ) or simply $M^{\mathbb{S}^1}$. If all isotropy subgroups Iso (p, λ) for $p \in M$ are trivial, then the action is *free*.

Another notion, in some sense complementary to the notion of isotropy subgroup, is that of an *orbit*. The orbit of p, denoted by Orb (p, λ) or $\mathbb{S}^1 \cdot p$, is a subset of M consisting of all points of M which can be obtained from p by a translation by some element of $\mathbb{S}^1$. So, $\mathbb{S}^1 \cdot p = \{e^{it} \cdot p : e^{it} \in \mathbb{S}^1\}$. As we will see below, $\mathbb{S}^1 \cdot p$ is always an embedded submanifold of M. In the special case when $\mathbb{S}^1 \cdot p$ is the whole manifold M we say that the action is *transitive*.

Lemma 3.3. *If (M, λ) is an $\mathbb{S}^1$-manifold and p is a point in M then*

$$\mathcal{O}_p : \mathbb{S}^1/\mathbb{S}^1_p \ni [e^{it}] \mapsto e^{it} \cdot p \in M$$

s a well-defined embedding onto $\mathbb{S}^1 \cdot p$. We will call this map the orbit map of p.

Before proceeding to a proof, let us first prove another lemma.

Lemma 3.4. *If (M_1, λ_1) and (M_2, λ_2) are $\mathbb{S}^1$-manifolds and $f : M_1 \to M_2$ is equivariant, then the rank of f is constant on every orbit of λ_1.*

Proof. Let p and q be two points of M_1 which lie in the same orbit of λ_1. Then there exists $e^{it} \in \mathbb{S}^1$ such that $(\lambda_1)_{e^{it}}(p) = q$. By differentiating the equivariance condition $(\lambda_2)_{e^{it}} f = f(\lambda_1)_{e^{it}}$ at p, we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{T}_p M_1 & \xrightarrow{\mathrm{d}_p f} & \mathrm{T}_{f(p)} M_2 \\ {\scriptstyle \mathrm{d}_p(\lambda_1)_{e^{it}}}\Big\downarrow\cong & & \cong\Big\downarrow{\scriptstyle \mathrm{d}_{f(p)}(\lambda_2)_{e^{it}}} \\ \mathrm{T}_q M_1 & \xrightarrow[\mathrm{d}_q f]{} & \mathrm{T}_{f(q)} M_2, \end{array}$$

in which the vertical arrows are isomorphisms. Hence rank of f at p and q is the same. ☐

Proof of Lemma 3.3. The only non-trivial part is that $\mathcal{O}_p$ is an embedding. It is clearly a continuous injection. Since $\mathbb{S}^1/\mathbb{S}^1_p$ is compact, it is in fact a homeomorphism onto its image. If p is a fixed point, then $\mathbb{S}^1/\mathbb{S}^1_p$ is a one-point space and $\mathcal{O}_p$ is trivially an embedding. Assume then, that p is not a fixed point. We can draw a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{e^{it} \mapsto e^{it} \cdot p} & M \\ {\scriptstyle e^{it} \mapsto [e^{it}]}\Big\downarrow & \nearrow{\scriptstyle \mathcal{O}_p} & \\ \mathbb{S}^1/\mathbb{S}^1_p & & \end{array}$$

with horizontal and vertical maps which are clearly smooth. Moreover, since $\mathbb{S}^1_p$ is a finite subgroup of $\mathbb{S}^1$, the vertical arrow is a finite covering. This proves that $\mathcal{O}_p$ is smooth. The map $\mathcal{O}_p$ is clearly equivariant with respect to the natural transitive action of $\mathbb{S}^1$ on $\mathbb{S}^1/\mathbb{S}^1_p$ given by $e^{it} \cdot [e^{is}] = [e^{i(t+s)}]$. Hence, by Lemma 3.4, it is of constant rank on $\mathbb{S}^1/\mathbb{S}^1_p$. This rank may be equal to 0 or 1, but since p is not fixed, the former is not possible. ☐

We can use a circle action λ on a manifold M to define an equivalence relation $\sim_\lambda$ on M as follows: points p and q of M are in relation, $p \sim_\lambda q$, if and only if there exists $e^{it} \in \mathbb{S}^1$ such that $\lambda_{e^{it}}(p) = q$. In other words, two points are related if they belong to the same orbit of the action. The quotient space $M/{\sim_\lambda}$ is called the *orbit space* associated with λ and is sometimes denoted by $M/\mathbb{S}^1$. With a natural quotient topology $M/\mathbb{S}^1$ becomes a Hausdorff space, compact if M is compact. More can be said about the topology of the orbit space when the action is free.

Proposition 3.5. *If λ is a free circle action on a manifold M, then the orbit space $M/\mathbb{S}^1$ of this action admits a smooth structure such that the projection $M \to M/\mathbb{S}^1$ is an $\mathbb{S}^1$-bundle.*

Many results in the theory of group actions apply only to *effective* group actions. A group action is effective if every non-trivial element of the group acts in a non-trivial way. As we will see in Remark 3.6 below, the effectiveness of the action is rarely the issue in case of circle actions.

Remark 3.6. If a manifold M admits a non-trivial circle action λ, it admits an effective circle action λ_e induced from λ in a canonical way.

Proof. First, observe that a circle action λ on a manifold M is effective if and only if $\mathbb{S}^1_\lambda := \bigcap_{p\in M} \operatorname{Iso}(p,\lambda) = \{1\}$. If we assume that λ is non-trivial, then the subgroup $\mathbb{S}^1_\lambda < \mathbb{S}^1$ is finite. The quotient group $\mathbb{S}^1/\mathbb{S}^1_\lambda$ is isomorphic to $\mathbb{S}^1$, and we can define $\mathbb{S}^1/\mathbb{S}^1_\lambda$-action λ_e on M by $\lambda_e : \mathbb{S}^1/\mathbb{S}^1_\lambda \times M \ni ([e^{it}], p) \mapsto \lambda(e^{it}, p) \in M$. It is an easy exercise to check that this is a well-defined map and an effective circle action on M. Of course, if we start with effective λ, then $\lambda_e = \lambda$. □

Every circle action λ on a manifold M induces a vector field ξ_λ defined by $(\xi_\lambda)_p = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \lambda_{e^{it}}(p)$. This vector field is called the *fundamental vector field of* λ.

Remark 3.7. If (M, λ) is an $\mathbb{S}^1$-manifold and p is a point of M, then p is a fixed point of λ if and only if the fundamental vector field ξ_λ of λ vanishes at p.

Proof. This is an immediate consequence of Lemma 3.4 and the fact that the map $e^{it} \mapsto \lambda_{e^{it}}(p)$ is equivariant with respect to the obvious transitive action of $\mathbb{S}^1$ on $\mathbb{S}^1$. □

Corollary 3.8. *If a closed manifold M admits a circle action with no fixed points, then its Euler characteristic $\chi(M) = 0$.*

Proof. By Poincare-Hopf index theorem (see for example Milnor [28, p. 35]), the Euler characteristic of a closed manifold is computed by zeroes of a generic vector field. If M admits a circle action λ with no fixed points, then we can compute $\chi(M)$ using ξ_λ. But this vector field has no zeroes, by Remark 3.7 above, hence $\chi(M) = 0$. □

Example 3.9. Any continuous homomorphism $\phi : \mathbb{S}^1 \to \mathrm{SO}(n)$ determines a circle action on $\mathbb{R}^n$ such that each element $e^{it} \in \mathbb{S}^1$ acts in a linear way. Such action is called an *n-dimensional real representation* of $\mathbb{S}^1$. Similarly, we define *n-dimensional complex representations*, induced by continuous homomorphisms $\phi : \mathbb{S}^1 \to \mathrm{U}(n)$. Natural inclusion $\mathrm{U}(n) \to \mathrm{SO}(2n)$ makes any n-dimensional complex representation a $2n$-dimensional real representation.

Two real (complex) n-dimensional representations ϕ_1 and ϕ_2 are *isomorphic* if there exists an $\mathbb{R}$-linear automorphism of $\mathbb{R}^n$ ($\mathbb{C}$-linear automorphism of $\mathbb{C}^n$) satisfying $\rho\phi_1(e^{it}) = \phi_2(e^{it})\rho$ for any $e^{it} \in \mathbb{S}^1$.

Example 3.10. We can easily classify all 1-dimensional representation, both real and complex. SO(1) is a trivial group, so the only 1-dimensional real representation is the trivial action on $\mathbb{R}$. We will denote this representation by $\mathbb{R}_0$. For the complex case, observe that every continuous homomorphism $\mathbb{S}^1 \to \mathrm{U}(1)$ is of the form $\mathbb{S}^1 \ni e^{it} \mapsto \left[e^{ikt}\right] \in \mathrm{U}(1)$ for some integer k. The corresponding action on $\mathbb{C}$ is given by $e^{it} \cdot z = e^{ikt}z$ for every $z \in \mathbb{C}$. We will denote this representations by $\mathbb{C}_k$.

Remark 3.11. Two representations $\mathbb{C}_k$ and $\mathbb{C}_l$ are isomorphic as complex representations if and only if $k = l$, and as real representations if and only if $k = \pm l$.

Example 3.12. We can combine two representations into a new one as follows. Let λ_i be an n_i dimensional real representation for $i = 1, 2$. Denote the corresponding homomorphisms $\mathbb{S}^1 \to \mathrm{SO}(n_i)$ with λ_i as well. There is a natural, diagonal inclusion $\iota : \mathrm{SO}(n_1) \oplus \mathrm{SO}(n_2) \to \mathrm{SO}(n_1 + n_2)$ which we can use to define a new homomorphism $\lambda = \iota(\lambda_1, \lambda_2) : \mathbb{S}^1 \to \mathrm{SO}(n_1 + n_2)$. The $(n_1 + n_2)$-dimensional representation obtained this way is called the *direct sum* of representations λ_1 and λ_2. We use the notation $\lambda = \lambda_1 \oplus \lambda_2$. There is an obvious way to extend this to complex representations and to any finite number of summands.

Using this construction and the 1-dimensional "building blocks" of Example 3.9 above, we can define for any n-tuple of integers $(k_1, \ldots, k_n)$ a n-dimensional complex representation $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$, or a $(2n + m)$-dimensional real representation $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n} \oplus m\mathbb{R}_0$. The former is a circle action on $\mathbb{C}^n$ given by

$$e^{it} \cdot (z_1, \ldots, z_n) = (e^{ik_1 t}z_1, \ldots, e^{ik_n t}z_n),$$

and the latter is a circle action on $\mathbb{R}^{2n+m}$ given by

$$\begin{aligned} &e^{it} \cdot (x_1, y_1, \ldots, x_n, y_n, t_1, \ldots, t_m) \\ &\quad = (\cos(k_1 t)x_1 - \sin(k_1 t)y_1, \sin(k_1 t)x_1 + \cos(k_1 t)y_1, \ldots, \\ &\qquad \cos(k_n t)x_n - \sin(k_n t)y_n, \sin(k_n t)x_n + \cos(k_n t)y_n, t_1, \ldots, t_m). \end{aligned}$$

Remark 3.13. Two representations $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ and $\mathbb{C}_{k'_1} \oplus \ldots \oplus \mathbb{C}_{k'_n}$ are isomorphic as complex representations if and only if the n-tuples $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ coincide up to some permutation of indices.

A real representation $\mathbb{C}_{k_1} \oplus \ldots \mathbb{C}_{k_n} \oplus 2m\mathbb{R}_0$ of even dimension $2n + 2m$ is isomorphic as a real representation to $\mathbb{C}_{k_1} \oplus \ldots \mathbb{C}_{k_n} \oplus m\mathbb{C}_0$. It means that with every $2n$-dimensional real representation of Example 3.12 above we can also associate an n-tuple of integers $(k_1, \ldots, k_n)$. Two such n-tuples $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ determine isomorphic real representations if and only if they coincide up to some permutation of indices and some change of signs (compare with Remark 3.11).

We can introduce an intermediate notion of an *oriented real representation*. An n-dimensional oriented real orientation is just an n-dimensional real representation together with a choice of orientation on $\mathbb{R}^n$. Two oriented representations are considered to be isomorphic if they are isomorphic as real representations, and the isomorphism can be chosen so that it preserves the orientation.

If we endow the representations of Example 3.12 above with standard orientation, then n-tuples $(k_1, \dots, k_n)$ and $(k'_1, \dots, k'_n)$ determine isomorphic oriented real representations if and only if they coincide up to some permutation of indices and some "even" change of signs, that is if there exists a permutation $\sigma \in \Sigma_n$ and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ such that $k'_i = \epsilon_{\sigma(i)} k_{\sigma(i)}$ for all $i = 1, \dots, n$ and $\epsilon_1 \cdots \epsilon_n = 1$.

The importance of the Example 3.12 above stems from the following, well-known result from representation theory (see for example Hall [14, Proposition 4.36]).

Theorem 3.14. *Every real or complex representation is isomorphic to one of the representations in Example 3.12.*

Example 3.15. Let λ be a circle action on a n-dimensional manifold M with non-empty fixed point set. For a fixed point p, the homomorphism

$$\mathbb{S}^1 \ni e^{it} \mapsto \mathrm{d}_p \lambda_{e^{it}} \in \mathrm{GL}(\mathrm{T}_p\, M)$$

defines a circle action on $\mathrm{T}_p\, M$ called the *representation of λ at p*.

To any representation at a fixed point we can assign an n-dimensional real representation in the previous sense. We can find a Riemannian metric g on M such that all translations $\lambda_{e^{it}}$ are g-isometries [11, Corollary B.12]. We say that such a metric is *invariant*. The homomorphism $\mathbb{S}^1 \to \mathrm{GL}(\mathrm{T}_p\, M)$ above reduces to the homomorphism $\mathbb{S}^1 \to \mathrm{SO}(\mathrm{T}_p\, M, g)$. By choosing a g_p-orthonormal basis in $\mathrm{T}_p\, M$ we can identify $\mathrm{SO}(\mathrm{T}_p\, M, g)$ with $\mathrm{SO}(n)$, obtaining an n-dimensional real representation $\mathbb{S}^1 \to \mathrm{SO}(n)$. It is easy to check that different choices of invariant metrics or orthonormal bases lead to isomorphic representations. In particular, every such representation is, by Theorem 3.14, isomorphic to one of the representations from Example 3.12.

As we will see in Lemma 3.16 below, the action in a neighbourhood of a fixed point is fully determined by its representation at this fixed point. This will let us perform some constructions on manifolds with circle actions, which we will describe later.

Lemma 3.16. *Let (M, λ) be an $\mathbb{S}^1$-manifold and let $p \in M$ be a fixed point of λ. Then there exists a neighbourhood of p in M equivariantly diffeomorphic to the representation of λ at p.*

Proof. Let g be an invariant Riemannian metric on M. For any $\varepsilon > 0$ define $D_\varepsilon = \{v \in \mathrm{T}_p\, M : g_p(v, v) < \varepsilon^2\}$. The exponential map $\exp_g : \mathrm{T}_p\, M \to M$ is equivariant[1] and its derivative at p is an isomorphism, hence for $\varepsilon > 0$ small enough, the restriction $\exp_g|_{D_\varepsilon}$ is an equivariant diffeomorphism onto some open neighbourhood of p. It suffices to compose $\exp_g|_{D_\varepsilon}$ with an equivariant diffeomorphism $\mathrm{T}_p\, M \to D_\varepsilon$. □

Corollary 3.17. *Let (M, λ) be an $\mathbb{S}^1$-manifold. Every connected component of the fixed point set of λ is a submanifold of M.*

[1] This is an immediate consequence of uniqueness of geodesics (see Kobayashi and Nomizu's book [18, Chapter III, Theorem 6.4]). The translation $\lambda_{e^{is}}$ is an isometry for every $e^{is} \in \mathbb{S}^1$, so the curve $\gamma : t \mapsto \lambda_{e^{is}} \exp_g(tv)$ is a geodesic with initial condition $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \gamma(t) = \mathrm{d}_p \lambda_{e^{is}}(v)$. It means that γ has to coincide with $t \mapsto \exp_g(t\, \mathrm{d}_p \lambda_{e^{is}}(v))$. Setting $t = 1$ gives us equiariance of the exponens.

Going in another direction, we can also use representations to define some concrete actions, as seen in Examples 3.18 and 3.19.

Example 3.18. Any n-dimensional complex representation restricts to a circle action on the sphere $\mathbb{S}^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_n|^2 = 1\}$. These are called *linear actions* on the sphere. If the weights of the representation are non-zero, the action has no fixed points. If all of the weights are equal to 1 or -1, the action is also free. If all the weights are equal to 1, the fibre bundle $\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}/\mathbb{S}^1$ over $\mathbb{S}^{2n-1}/\mathbb{S}^1 \cong \mathbb{CP}^{n-1}$ is known as the *Hopf bundle*.

In a similar way, we can use real representations to produce linear circle actions on even-dimensional spheres. Those actions always have non-empty fixed point sets. This is clear, both from the definition and from Corollary 3.8.

Example 3.19. Let $\lambda = \mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ and $\lambda' = n\mathbb{C}_1$ be two n-dimensional complex representations. For any $e^{it}, e^{it'} \in \mathbb{S}^1$ the translations $\lambda_{e^{it}}$ and $\lambda'_{e^{it'}}$ commute, so the representation λ descends to a circle action $\tilde{\lambda}$ on the orbit space of λ', which, as we have seen in Example 3.18, is diffeomorphic to $\mathbb{CP}^{n-1}$. In other words, there exists unique circle action $\tilde{\lambda}$ on $\mathbb{CP}^{n-1}$ such that for each $e^{it} \in \mathbb{S}^1$, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{S}^{2n-1} & \xrightarrow{\lambda_{e^{it}}} & \mathbb{S}^{2n-1} \\ \downarrow & & \downarrow \\ \mathbb{CP}^{n-1} & \xrightarrow[\tilde{\lambda}_{e^{it}}]{} & \mathbb{CP}^{n-1}. \end{array}$$

In homogeneous coordinates on $\mathbb{CP}^{n-1}$ the action $\tilde{\lambda}$ is given by

$$e^{it} \cdot [z_1 : \ldots : z_n] = [e^{ik_1 t} z_1 : \ldots : e^{ik_n t} z_n].$$

To understand the fixed point set $(\mathbb{CP}^{n-1})^{\mathbb{S}^1}$, assume that the weights are arranged in an ascending order, so that

$$k_{i_0+1} = \ldots = k_{i_1} < k_{i_1+1} = \ldots = k_{i_2} < \ldots < k_{i_{p-1}+1} = \ldots = k_{i_p}$$

for some integers $0 = i_0 < i_1 < \ldots < i_{p-1} < i_p = n$. In that case, the fixed point set has p connected components, each of which is an embedded projective space. The j-th component is a $\mathbb{CP}^{i_j - i_{j-1} - 1}$ and in homogeneous coordinates the embedding is given by

$$[w_1 : \ldots : w_{i_j - i_{j-1}}] \mapsto [0 : \ldots : 0 : \underbrace{w_1}_{i_{j-1}+1} : \ldots : \underbrace{w_{i_j - i_{j-1}}}_{i_j} : 0 : \ldots : 0].$$

This picture simplifies when all weights are pairwise different. Then the fixed point set consists of exactly n distinct points, $(\mathbb{CP}^{n-1})^{\mathbb{S}^1} = \{p_1, \ldots, p_n\}$, where $p_1 = [1 : 0 : \ldots : 0]$, $p_2 = [0 : 1 : 0 : \ldots : 0]$ and so on. We can compute the weights of this action at each of these points. This weights at p_i are

$$k_1 - k_i, \ldots, k_{i-1} - k_i, k_{i+1} - k_i, \ldots, k_n - k_i.$$

Example 3.20. The direct sum of representations (Example 3.12) is a special case of a more general construction. If M_1 and M_2 are two $\mathbb{S}^1$-manifolds, then we can define an $\mathbb{S}^1$-action on $M_1 \times M_2$ by $e^{it} \cdot (p_1, p_2) = (e^{it} \cdot p_1, e^{it} \cdot p_2)$ for $p_1 \in M_1$ and $p_2 \in M_2$. This action is called the *product action* on $M_1 \times M_2$. Clearly, the isotropy subgroup $\mathbb{S}^1_{(p_1,p_2)}$ of a point $(p_1, p_2) \in M_1 \times M_2$ is just $\mathbb{S}^1_{(p_1,p_2)} = \mathbb{S}^1_{p_1} \cap \mathbb{S}^1_{p_2}$, so if the action on M_1 or M_2 is non-trivial/free, then the product action is non-trivial/free as well.

As mentioned before, the knowledge about the local behaviour of circle actions near fixed points can be used to perform various kinds of surgeries in the category of $\mathbb{S}^1$-manifolds. For example, we can define the equivariant connected sum. As we recall, the connected sum of two oriented n-dimensional manifolds is obtained by removing a point from each of these manifolds and gluing a neighbourhood of the removed point from the first manifold with a neighbourhood of the removed point from the second manifold by an orientation-reversing diffeomorphism. This translates to the equivariant setting, as seen in the following definition.

Definition 3.21. Let M_1 and M_2 be two oriented n-dimensional $\mathbb{S}^1$-manifolds. Let $p_1 \in M_1^{\mathbb{S}^1}$ and $p_2 \in M_2^{\mathbb{S}^1}$ be some fixed points. If there exists a circle action on $\mathbb{R}^n$ fixing 0, an equivariant, orientation-preserving embedding ι_1 of $\mathbb{R}^n$ into M_1, sending 0 to p_1, and an equivariant, orientation-reversing embedding ι_2 of $\mathbb{R}^n$ into M_2 sending 0 to p_2, then the connected sum

$$M_1 \sharp M_2 = M_1 \setminus \{p_1\} \cup_{\iota_2(\iota_1)^{-1}} M_2 \setminus \{p_2\}$$

admits unique circle action which restricts to given circle actions on $M_1 \setminus \{p_1\}$ and $M_2 \setminus \{p_2\}$. The resulting $\mathbb{S}^1$ manifold $M_1 \sharp M_2$ is called the *equivariant connected sum* of M_1 and M_2.

Example 3.22. We can prove that if a manifold M of even dimension $2n$ admits a non-trivial circle action with non-empty fixed point set, then so do $M \sharp \mathbb{CP}^n$ and $M \sharp \overline{\mathbb{CP}^n}$. Indeed, let $p \in M^{\mathbb{S}^1}$ be a fixed point. Choose integers $k_1, \ldots, k_n$ such that there exists an oriented equivariant linear isomorphism of the representation of the action at p and $\mathbb{C}_{k_1} \oplus \cdots \oplus \mathbb{C}_{k_n}$. Endow $\mathbb{CP}^n$ with a linear action induced from the representation $\mathbb{C}_{10} \oplus \mathbb{C}_{k_1} \oplus \cdots \oplus \mathbb{C}_{k_{n-1}} \oplus \mathbb{C}_{-k_n}$. By Lemma 3.16, the neighbourhood of p in M and that of $[1 : 0 : \ldots : 0]$ in $\mathbb{CP}^n$ are equivariantly diffeomorphic by orientation-reversing diffeomorphism. Hence, we can form equivariant connected sum $M \sharp \mathbb{CP}^n$. Similarly, if we endow $\mathbb{CP}^n$ with a linear action induced from the representation $\mathbb{C}_0 \oplus \mathbb{C}_{k_1} \oplus \cdots \oplus \mathbb{C}_{k_{n-1}} \oplus \mathbb{C}_{k_n}$, we can form equivariant sum $M \sharp \overline{\mathbb{CP}^n}$. These new $\mathbb{S}^1$-manifolds have non- empty fixed point set, because there are natural inclusions $M^{\mathbb{S}^1} \setminus \{p\} \to (M \sharp \mathbb{CP}^n)^{\mathbb{S}^1}$ and $(\mathbb{CP}^n)^{\mathbb{S}^1} \setminus \{[1 : 0 : \ldots : 0]\} \to (M \sharp \mathbb{CP}^n)^{\mathbb{S}^1}$, and $(\mathbb{CP}^n)^{\mathbb{S}^1} \setminus \{[1 : 0 : \ldots : 0]\}$ consists of at least n points

$$[0 : 1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0, 1].$$

Example 3.23. As an immediate consequence of the previous example, we have that $k\mathbb{CP}^n \sharp l\overline{\mathbb{CP}^n}$ admits a non-trivial circle action for any pair of positive integers k, l.

4 Symplectic circle actions

Definition 4.1. A *symplectic circle action* λ on a symplectic manifold (M, ω) is a circle action such that $\lambda^*_{e^{it}}\omega = \omega$ for each $e^{it} \in \mathbb{S}^1$. The triple (M, ω, λ) is called a symplectic $\mathbb{S}^1$-manifold.

Remark 4.2. A circle action λ on a symplectic manifold (M, ω) is symplectic if and only if $\mathrm{i}(\xi_\lambda)\omega$ is a closed 1-form.

Proof.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda^*_{e^{it}}\omega = \mathcal{L}_{\xi_\lambda}\omega = \mathrm{d}\mathrm{i}(\xi_\lambda)\omega + \mathrm{i}(\xi_\lambda)\,\mathrm{d}\omega = \mathrm{d}\mathrm{i}(\xi_\lambda)\omega.$$

□

We have seen that with any symplectic form ω one can associate a compatible metric g and almost complex structure J. It turns out that if ω is invariant with respect to some circle action, then g and J can be chosen to be invariant as well.

Remark 4.3. If (M, ω, λ) is a symplectic $\mathbb{S}^1$-manifold then there exists a metric g and an almost complex structure J on M which are compatible with ω and λ-invariant.

Proof. We can start with any ω-compatible metric and average it by the action. The obtained invariant metric will still be ω-compatible. An invariant, ω-compatible almost complex structure can then be produced using this metric. □

We have already seen that for an $\mathbb{S}^1$-manifold (M, λ) every fixed point p of λ has a neighbourhood equivariantly diffeomorphic with the representation of λ at p (Lemma 3.16). In a presence of a symplectic form, we can refine this result to the following.

Lemma 4.4. *Let (M, ω, λ) be a symplectic $\mathbb{S}^1$-manifold and let $p \in M$ be a fixed point of λ. Let $k_1, \ldots, k_n$ be the weights of λ at p. There exists a neighbourhood of p in M equivariantly symplectomorphic to a disc in $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ with the standard symplectic structure.*

Proof. Let ω_0 denote the standard symplectic structure on $\mathbb{C}^n$, the underlying manifold of $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$. A choice of λ-invariant metric gives an equivariant diffeomorphism of $\mathrm{T}_p M$ with a neighbourhood of p in M, as in the proof of Lemma 3.16. If we choose the metric to be also compatible with ω and choose a compatible λ-invariant almost-complex structure, then we obtain an equivariant diffeomorphism of $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ to a neighbourhood of p in M such that the pullback ω_1 of ω by this diffeomorphism is an equivariant symplectic form on $\mathbb{C}^n$ which agrees with ω_0 at 0. Now, the key to the proof is the so-called equivariant Darboux theorem, quoted in Proposition 4.5. Using this theorem for $(\mathbb{C}^n, \omega_0)$ and $(\mathbb{C}^n, \omega_1)$ concludes the proof. □

Proposition 4.5. *Let (M, λ) be an $\mathbb{S}^1$-manifold with fixed point p and let ω_0, ω_1 be two λ-invariant symplectic forms on M which coincide in p. There exist invariant neighbourhoods U_0 and U_1 of p and an equivariant symplectomorphism $(U_0, \omega_0|_{U_0}) \to (U_1, \omega_1|_{U_1})$.*

The proof of a slightly more general version can be found for example in an appendix to a paper of Bates and Lerman [4].

Observe that in the representation of λ at p the fixed point set corresponds to the direct summand $m\mathbb{C}_0 \subset m\mathbb{C}_0 \oplus \mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_{n-m}}$. This leads to the following corollary.

Corollary 4.6. *If (M, ω, λ) is a symplectic $\mathbb{S}^1$-manifold then every connected component of the fixed point set of λ is a symplectic submanifold of (M, ω).*

Another corollary of the discussion above is that the notion of weights of λ at a fixed point p is well-defined (see Remark 3.13). Even more, weights are locally constant on fixed point set, so we have a well-defined notion of weights of λ at a connected component F of a fixed point set of λ. The point-wise decompositions $\mathrm{T}_p M \cong m_1\mathbb{C}_{k_1} \oplus \ldots \oplus m_{k_p}\mathbb{C}_{k_p}$ for $k_1 < \ldots < k_p$ glue to the bundle decomposition $TM|_F = N_{k_1} \oplus \ldots \oplus N_{k_p}$, where $TF = N_0$.

Definition 4.7. The number $\sum_{i:k_i<0} m_i$, which computes the number of negative weights with multiplicities, is called the index of F and denoted as $d(F)$.

From this moment we will focus on a subclass of symplectic $\mathbb{S}^1$-manifolds, the Hamiltonian $\mathbb{S}^1$-manifolds.

Definition 4.8. A symplectic circle action λ on a symplectic manifold (M, ω) is called *Hamiltonian* if the 1-form $\mathrm{i}(\xi_\lambda)\omega$ is exact. Any function $\phi : M \to \mathbb{R}$ such that $\mathrm{i}(\xi_\lambda)\omega = -\,\mathrm{d}\phi$ is called a *moment map* of the action. The quadruple $(M, \omega, \lambda, \phi)$ is called a *Hamiltonian $\mathbb{S}^1$-manifold.*

Example 4.9. $\mathbb{C}_k$ can be considered as a symplectic $\mathbb{S}^1$-manifold, with standard symplectic structure on $\mathbb{C}$. The map $\phi : \mathbb{C} \ni z \mapsto \frac{k}{2} z\overline{z} \in \mathbb{R}$ is a moment map for $\mathbb{C}_k$, making it a Hamiltonian $\mathbb{S}^1$-manifold.

Similarly, $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ with the standard symplectic form is a Hamiltonian $\mathbb{S}^1$-manifold with moment map $\phi : \mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto \sum_{i=1}^n \frac{k_i}{2} z_i\overline{z_i}$.

Hamiltonian $\mathbb{S}^1$-manifolds are often studied through their associated moment maps, which have some particularly nice properties. We will discuss these properties later. Let us first make some easy observations about Hamiltonian circle actions.

Remark 4.10. Let (M, ω, λ) be a symplectic $\mathbb{S}^1$-manifold.

1. If $\phi_1, \phi_2 : M \to \mathbb{R}$ are moment maps of λ, then there exists a constant $c \in \mathbb{R}$ such that $\phi_1 \equiv \phi_2 + c$.

2. If ϕ is a moment map of λ, then $\mathrm{Fix}\,(M, \lambda) = \mathrm{Crit}(\phi)$.

Proof. Point 1. follows from $\mathrm{d}(\phi_1 - \phi_2) = 0$. Point 2. is a consequence of Remark 3.7 and nondegeneracy of ω. □

Of course, all results about symplectic $\mathbb{S}^1$-manifolds are valid in the context of Hamiltonian $\mathbb{S}^1$-manifolds. Lemma 4.4 can be strenghtened to incorporate the moment maps.

Lemma 4.11. *Consider $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ as a Hamiltonian $\mathbb{S}^1$-manifold with standard symplectic form and with moment map described in Example 4.9. Let $(M, \omega, \lambda, \phi)$ be a Hamiltonian $\mathbb{S}^1$-manifold and let $p \in M$ be a fixed point of λ.*

Let $k_1, \ldots, k_n$ be weights of λ at p. There exists a neighbourhood of p in M equivariantly symplectomorphic to a disc in $\mathbb{C}_{k_1} \oplus \ldots \oplus \mathbb{C}_{k_n}$ by a symplectomorphism which commutes the moment maps.

The importance of Lemma 4.11 lies in the fact that it describes a behaviour of a moment map near its critical points. In particular, in the coordinates given by this Lemma, the Hessian of ϕ at a critical point p is gicen by a diagonal matrix

$$\operatorname{Hess}_p \phi = \begin{bmatrix} k_1 & & & & \\ & k_1 & & & \\ & & \ddots & & \\ & & & k_n & \\ & & & & k_n \end{bmatrix}.$$

This means that in the normal direction, $\operatorname{Hess}_p \phi$ is nondegenerate. This makes ϕ a Morse–Bott function. Morse–Bott theory is essentially similar to Morse theory. Good reference to the latter is Milnor's book [27]. The former, in the context of moment maps, is presented in Audin's book [2]. Of particular use to us is Audin's Theorem 3.1.1 from Chapter III. Notice, that the index $\operatorname{ind}_F \phi$ of a Morse–Bott function ϕ at a critical submanifold F is not identical to the index $d(F)$ defined in Definition 4.7. On the contrary, $\operatorname{ind}_F \phi = 2d(F)$. Having that in mind, we see that the straightforward consequence of the aforementioned theorem [2, Theorem III.3.1.1] is the following.

Proposition 4.12. *If $(M, \omega, \lambda, \phi)$ is a closed Hamiltonian $\mathbb{S}^1$-manifold, then all level sets of ϕ are connected.*

In particular it leads to the following.

Corollary 4.13. *The global minimum ϕ_0 of a moment map ϕ is the unique connected component of the fixed point set with index equal to zero.*

To use the results above, we will need some way to confirm that a given symplectic $\mathbb{S}^1$-manifold is in fact Hamiltonian. There are many interesting results concerning the question when is a symplectic circle action Hamiltonian. The obvious sufficient condition is the following.

Remark 4.14. If the manifold M is simply-connected then every symplectic circle action on (M, ω) is Hamiltonian.

At the other hand, there is the following simple necessary condition.

Remark 4.15. If $(M, \omega, \lambda, \phi)$ is a closed Hamiltonian $\mathbb{S}^1$-manifold, then its fixed point set is non-empty.

Proof. If ϕ is a moment map of the action, then it reaches both a minimum and a maximum. In particular $\operatorname{Crit}(\phi) \neq \emptyset$. Now we use point 2. of Remark 4.10. □

For symplectic 4-manifolds this is a sufficient condition as well, as shown by McDuff [24]. However in general the situation is not that simple, as illustrated by an example of a symplectic 6-manifold with a symplectic circle action which is not Hamiltonian, found in the same paper. One possible extension of McDuff's criterium to higher dimensions is given by Ono [29] (see also McDuff and Salamon [25, Theorem 5.5]). To state it we need to introduce another definition.

Definition 4.16. A closed symplectic manifold (M, ω) of dimension $2n$ is said to have the weak Lefschetz property if and only if the map

$$\wedge[\omega]^{n-1} : H^1(M) \to H^{2n-1}(M)$$

is an isomorphism.

Proposition 4.17. *If a closed symplectic manifold (M, ω) has the weak Lefschetz property then any symplectic circle action on (M, ω) is Hamiltonian if and only if it has fixed points.*

Proof. One implication is just Remark 4.15 above. To prove the other, choose a symplectic circle action λ with fixed points and denote by A the homology class $(\mathcal{O}_p)_*[\mathbb{S}^1] = [\mathbb{S}^1 \cdot p] \in H_1(M; \mathbb{R})$ of some non-trivial orbit $\mathbb{S}^1 \cdot p$ of λ. Let $\gamma : [0,1] \to M$ be a curve connecting p to some fixed point of the action. The map

$$\mathbb{S}^1 \times [0,1] \xrightarrow{\mathrm{id}_{\mathbb{S}^1} \times \gamma} \mathbb{S}^1 \times M \xrightarrow{\lambda} M$$

is a homotopy between $e^{it} \mapsto e^{it} \cdot p$ and the constant map, so in homology we have

$$A = (\mathcal{O}_p)_*[\mathbb{S}^1] = [\mathbb{S}^1 \cdot p] = \frac{1}{|\mathbb{S}^1_p|} \lambda_*[\mathbb{S}^1 \times \{p\}] = 0 \in H_1(M; \mathbb{R}).$$

Now, take any 1-form α on M. By averaging over the action, we can find an invariant 1-form $\widetilde{\alpha}$ such that $[\widetilde{\alpha}] = [\alpha]$. In particular $\mathrm{i}(\xi_\lambda)\widetilde{\alpha}$ is constant on $\mathbb{S}^1_p$, $\mathrm{i}(\xi_\lambda)\widetilde{\alpha} \equiv c \in \mathbb{R}$. If c would be non-zero, then $\widetilde{\alpha}$ would integrate to a non-zero number on $\mathbb{S}^1 \cdot p$, which is impossible. Hence $\mathrm{i}(\xi_\lambda)\widetilde{\alpha} \equiv 0$. As a result,

$$\begin{aligned} \langle [\alpha] \wedge [\mathrm{i}(\xi_\lambda)\omega^n], [M] \rangle = \langle [\widetilde{\alpha}] \wedge [\mathrm{i}(\xi_\lambda)\omega^n], [M] \rangle &= \int_M \widetilde{\alpha} \wedge \mathrm{i}(\xi_\lambda)\omega^n \\ &= \int_M \mathrm{i}(\xi_\lambda)\widetilde{\alpha} \wedge \omega^n = 0. \end{aligned}$$

By Poincare duality, $[\mathrm{i}(\xi_\lambda)\omega] \wedge [\omega]^{n-1} = [\mathrm{i}(\xi_\lambda)\omega^n] = 0$, and so, by weak Lefschetz property, $[\mathrm{i}(\xi_\lambda)\omega] = 0$. ☐

5 The theorem of Li

Using the Morse–Bott theory of moment maps, Li [21] has shown that for any closed Hamiltonian $\mathbb{S}^1$-space, the fundamental group of the minimum of the moment map, the fundamental group of the reduced space at any value of the moment map and the fundamental group of the whole manifold are isomorphic. Here we present this theorem together with Li's proof, with some modifications. The fundamental group of the reduced space, which has not been and will not be introduced in this thesis, has been omitted in the statement of the theorem. Also, in the proof, Lemma 0.7 of [21], which relied on reduction, has been replaced by Lemma 5.1 below. For a Hamiltonian $\mathbb{S}^1$-manifold $(M, \omega, \lambda, \phi)$ by M^c we denote the set $\{p \in M : \phi(p) \leq c\}$.

Lemma 5.1. *If $(M, \omega, \lambda, \phi)$ is a closed Hamiltonian $\mathbb{S}^1$-manifold, then for every $p \in M$ the orbit map $\mathcal{O}_p$ is homotopic to a constant map inside $M^{\phi(p)}$.*

Proof. Consider a curve γ in $M^{\phi(p)}$ with $\gamma(0) = p$ and $\gamma(1)$ a fixed point of the action. Let h_γ be a homotopy $h_\gamma : \mathbb{S}^1 \times [0,1] \ni (e^{it}, s) \mapsto e^{it} \cdot \gamma(s) \in M^{\phi(p)}$. If we can choose γ in such a way, that $\mathbb{S}^1_p \subset \mathbb{S}^1_{\gamma(s)}$ for each $s \in [0,1]$, then h_γ descends to a map $\mathbb{S}^1/\mathbb{S}^1_p \times [0,1] \to M^{\phi(p)}$ which is a homotopy from $\mathcal{O}_p$ to a constant map.

To see that this can be done choose a λ-invariant almost complex structure J compatible with ω. Consider a vector field $-J\xi_\lambda$ where ξ_λ is the fundamental vector field of λ. The flow φ_τ of $-J\xi_\lambda$ commutes with the action and the value of ϕ decreases along this flow. The curve $\varphi_\tau(p)$ can be reparametrized so that $\varphi_{\tau(s)}(p)$ is a fixed point for $s = 1$. Take $\gamma(s) = \varphi_{\tau(s)}(p)$. For $e^{it_0} \in \mathbb{S}^1_p$:

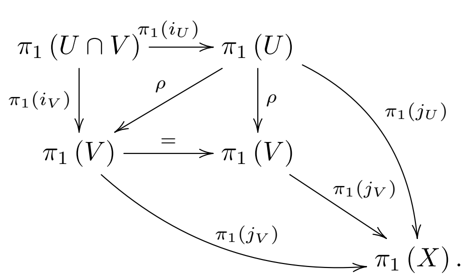
$$e^{it_0} \cdot \gamma(s) = e^{it_0} \cdot \varphi_{\tau(s)}(p) = \varphi_{\tau(s)}(e^{it_0} \cdot p) = \varphi_{\tau(s)}(p) = \gamma(s).$$

As a result, $\mathbb{S}^1_p \subset \mathbb{S}^1_{\gamma(s)}$. ☐

Lemma 5.2. *Let X be a topological space and let $U, V \subset X$ be its open, path connected subsets such that $U \cup V = X$ and $U \cap V$ is path connected and let $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to X$ and $j_V : V \to X$ denote the inclusions. If the map $\pi_1(i_U) : \pi_1(U \cap V) \to \pi_1(U)$ is an epimorphism and if $\ker(\pi_1(i_U)) < \ker(\pi_1(i_V))$, then $\pi_1(j_V) : \pi_1(V) \to \pi_1(X)$ is an isomorphism.*

Remark 5.3. As usual, we can take U and V to be arbitrary subsets of X as long as $U \cup V = X$ and there exists a neighbourhood U' of U and a neighbourhood V' of V such that U' is homotopy equivalent to U, V' is homotopy equivalent to V and $U' \cap V'$ is homotopy equivalent to $U \cap V$.

Proof of Lemma 5.2. The condition $\ker(\pi_1(i_U)) < \ker(\pi_1(i_V))$ means that there exists a map $\rho : \pi_1(U) \to \pi_1(V)$ such that $\rho\, \pi_1(i_U) = \pi_1(i_V)$. We can build a diagram:



The commutativity of this diagram is clear, except for the rightmost triangle. But $\pi_1(j_U)\,\pi_1(i_U) = \pi_1(j_V)\,\pi_1(i_V) = \pi_1(j_V)\,\rho\,\pi_1(i_U)$ and $\pi_1(i_U)$ is assumed to be an epimorphism, so $\pi_1(j_U) = \pi_1(j_V)\,\rho$. Now, by Seifert–van Kampen's theorem, $\pi_1(j_V)$ has to be an isomorphism. □

Lemma 5.4. *Let $\phi : M \to \mathbb{R}$ be a Morse–Bott function on M such that the image of ϕ is $\phi(M) = [a, b]$. Let $c \in (a, b]$ be a critical value of ϕ such that the corresponding critical submanifold $F \subset M$ is connected and $\operatorname{ind}_F \phi \geq 3$. Then for $\varepsilon > 0$ sufficiently small the inclusion $M^{c-\varepsilon} \to M^{c+\varepsilon}$ induces an isomorphism $\pi_1(M^{c-\varepsilon}) \to \pi_1(M^{c+\varepsilon})$.*

Proof. For $\varepsilon > 0$ sufficiently small we have a commutative diagram

$$\begin{array}{ccc} M^{c+\varepsilon} & \xrightarrow{\ \simeq\ } & M^{c-\varepsilon} \cup_{\mathbb{S}(D^-)} D^- \\ \uparrow & \nearrow & \\ M^{c-\varepsilon} & & \end{array},$$

where D^- is a negative disc bundle over F, $\mathbb{S}(D^-)$ is the corresponding sphere bundle with fiber $\mathbb{S}^{\operatorname{ind}_F \phi - 1}$ and the horizontal arrow is a homotopy equivalence. This is a fundamental result of the theory of Morse–Bott. Its proof is analogous to that of a corresponding result in Morse theory. For proof of the latter, see for example the classical book of Milnor [27].

To prove the lemma using this diagram, it is enough to prove that the inclusion $M^{c-\varepsilon} \to M^{c-\varepsilon} \cup_{\mathbb{S}(D^-)} D^-$ induces an isomorphism on the level of fundamental groups. The projection in the disc bundle $D^- \to F$ is a homotopy equivalence, so we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathbb{S}(D^-)) & \longrightarrow & \pi_1(D^-) \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle\cong} \\ & & \pi_1(F) \end{array},$$

where the diagonal arrow is an isomorphism from the homotopy exact sequence of the fiber bundle $\mathbb{S}(D^-) \to F$ with simply-connected fiber $\mathbb{S}^{\operatorname{ind}_F \phi - 1}$. This means that $\pi_1(\mathbb{S}(D^-) \to D^-)$ is an isomorphism and for $X = M^{c-\varepsilon} \cup_{\mathbb{S}(D^-)} D^-$, $U = D^-$, and $V = M^{c-\varepsilon}$ the hypothesis of Lemma 5.2 is satisfied (up to Remark 5.3). □

Lemma 5.5. *Let $(M, \omega, \lambda, \phi)$ be a Hamiltonian $\mathbb{S}^1$-manifold such that the image of ϕ is $\phi(M) = [a, b]$. Let $c \in (a, b)$ be a critical value of ϕ such that the corresponding critical submanifold $F \subset M$ is connected and $\operatorname{ind}_F \phi = 2$. Then for $\varepsilon > 0$ sufficiently small the inclusion $M^{c-\varepsilon} \to M^{c+\varepsilon}$ induces an isomorphism $\pi_1(M^{c-\varepsilon}) \to \pi_1(M^{c+\varepsilon})$.*

Proof. We want to proceed as before, that is to use Lemma 5.2 for the space $M^{c-\varepsilon} \cup_{\mathbb{S}(D^-)} D^-$. But this time the map $\pi_1(\mathbb{S}(D^-) \to D^-)$ is no longer an isomorphism. The homotopy exact sequence for the fiber bundle $\mathbb{S}(D^-) \to F$

gives us

$$\begin{array}{ccccccc} & & & \nearrow & \pi_1\left(D^-\right) & & \\ & & & & \downarrow \cong & & \\ \pi_1\left(\mathbb{S}^1\right) & \longrightarrow & \pi_1\left(\mathbb{S}(D^-)\right) & \longrightarrow & \pi_1\left(F\right) & \longrightarrow & 1, \end{array}$$

so the map $\pi_1\left(\mathbb{S}(D^-) \to D^-\right)$ is an epimorphism with kernel equal to the image of the map $\pi_1\left(\mathbb{S}^1 \to \mathbb{S}(D^-)\right)$. But, using Lemma 4.11 we see that after possibly shrinking ε this image is generated by the loop $\mathcal{O}_p : \mathbb{S}^1 \to M^{c-\varepsilon}$ for some point $p \in M^{c-\varepsilon}$. Now, using Lemma 5.1, we see that $[\mathcal{O}_p] = 1 \in \pi_1\left(M^{c-\varepsilon}\right)$. As a result, $\ker \pi_1\left(\mathbb{S}(D^-) \to D^-\right) < \ker \pi_1\left(\mathbb{S}(D^-) \to M^{c-\varepsilon}\right)$ and the hypothesis of Lemma 5.2 is again satisfied. □

Remark. In Lemmas 5.4 and 5.5 we assumed that the level set $\phi^{-1}(c)$ contained only one connected component of the fixed point set. This assumption was made only for simplicity, and it can easily be dropped. Indeed, if $\phi^{-1}(c)$ contains many components of the fixed point set, we can glue them in one by one, each time reasoning as in the proof of Lemma 5.4 or Lemma 5.5.

Theorem 5.6. *Let $(M, \omega, \lambda, \phi)$ be a closed Hamiltonian $\mathbb{S}^1$-manifold. Let ϕ_0 denote the connected submanifold of M consisting of all points of M on which ϕ reaches its minimum. For simplicity, we will refer to this submanifolds as the minimum of ϕ. The induced map $\pi_1\left(\phi_0\right) \to \pi_1\left(M\right)$ is an isomorphism.*

Proof. Let $a_0 < a_1 < \ldots < a_k$ be all critical values of ϕ. Since M is closed, ϕ_0 is compact. In that case, Lemma 4.11 can be used to show, that for ε_0 small enough, ϕ_0 is a deformation retract of $M^{a_0+\varepsilon_0}$. Choose $\varepsilon_1, \ldots, \varepsilon_k$ as in Lemmas 5.4 and 5.5. Then $\pi_1\left(\phi_{min}\right) \to \pi_1\left(M^{a_0+\varepsilon_0}\right)$ is an isomorphism by previous considerations, $\pi_1\left(M^{a_i-\varepsilon_i}\right) \to \pi_1\left(M^{a_i+\varepsilon_i}\right)$ are isomorphisms for $i = 1, \ldots, k$ by Lemmas 5.4 and 5.5, and $\pi_1\left(M^{a_{i-1}+\varepsilon_{i-1}}\right) \to \pi_1\left(M^{a_i-\varepsilon_i}\right)$ are isomorphisms for $i = 1, \ldots, k$ by some classical result in Morse–Bott theory (see [27, Theorem 3.1] for Morse-theoretic version). Gathering all of the above we obtain:

$$\pi_1\left(\phi_{min}\right) \xrightarrow{\cong} \pi_1\left(M^{a_k+\varepsilon_k}\right) = \pi_1\left(M\right).$$

□

6 Todd genus

In this section, we give a short introduction to the notion of complex genera and quote the localization theorem for χ_y-genus on closed symplectic $\mathbb{S}^1$-manifolds. Our presentation is based on books of Hirzebruch [15] and Hirzebruch, Berger and Jung [16].

Let R be any integral domain with distinguished unitary monomorphism $\mathbb{Q} \to R$.

Definition 6.1. A *complex genus* with values in R is a unitary ring homomorphism $\varphi : \Omega^{\mathbb{C}} \otimes \mathbb{Q} \to R$, where $\Omega^{\mathbb{C}} \otimes \mathbb{Q}$ is a rational stably almost complex cobordism ring.

Let $R[Y_1, \ldots]$ denote the direct limit of the polynomial rings

$$R[Y_1, \ldots] = \varinjlim_i R[Y_1, \ldots, Y_i].$$

Introduce a gradation $R[Y_1, \ldots] = \bigoplus_i R[Y_1, \ldots]_i$ determined by the property, that the polynomial $Y_{i_1} \cdot \ldots \cdot Y_{i_k}$ has weight $i_1 + \ldots + i_k$. Choose a family $\mathcal{K} = \{K_i\}_{i=1,\ldots}$, where $K_i \in R[Y_1, \ldots]_i$. We call such a family a *multiplicative sequence* if for any positive integers $m, n > 0$ and for any elements $y_1, \ldots, y_m, y'_1, \ldots, y'_n \in R$ the following identity holds:

$$K_{m+n}\Big(\sum_{\substack{p+q=1\\ p=0,\ldots,m\\ q=0,\ldots,n}} y_p y'_q, \ldots, \sum_{\substack{p+q=m+n\\ p=0,\ldots,m\\ q=0,\ldots,n}} y_p y'_q\Big) = K_m(y_1, \ldots, y_m) K_n(y'_1, \ldots, y'_n), \tag{6.1}$$

where $y_0 = y'_0 = 1 \in R$.

We can use the family $\mathcal{K}$ and a monomorphism $\mathbb{Q} \to R$, to assign to any $2m$-dimensional almost complex manifold (M, J) an element

$$\langle K_m(c_1(M, J), \ldots, c_m(M, J)), [M]\rangle \in R.$$

This is a linear combination of Chern numbers of (M, J) with coeffitients in R and Chern numbers are invariants of stably almost complex cobordism, so this assignment induces a map $\varphi_{\mathcal{K}} : \Omega^{\mathbb{C}} \otimes \mathbb{Q} \to R$.

Proposition 6.2. *The map $\varphi_{\mathcal{K}}$ is a genus.*

Proof. Additivity of $\varphi_{\mathcal{K}}$ is straightforward, while multiplicativity is an easy consequence of (6.1). □

There is a simple way of producing multiplicative sequences over R from formal power series in one variable over R with free term equal to 1. Denote by $\sigma_k(X_1, \ldots, X_n)$ the k-th elementary symmetric polynomial in variables $X_1, \ldots, X_n$.

Lemma 6.3. *For any formal power series $Q = 1 + a_1 X + a_2 X^2 + \ldots \in R[[X]]$, there exists a unique multiplicative sequence $\mathcal{K}^Q = \{K_i^Q\}_{i=1,\ldots}$ with the property that $a_i = K_i^Q(1, 0, \ldots, 0)$ for $i = 1, \ldots$.*

Proof. The product $Q(X_1)\cdot\ldots\cdot Q(X_n)$ is symmetric in variables $X_1,\ldots,X_n$, so it can be represented as a formal power series in elementary symmetric polynomials of $X_1,\ldots,X_n$. In other words, there exist polynomials $K_i^n \in R[Y_1,\ldots,Y_i]$ such that

$$(Q(X_1)\cdot\ldots\cdot Q(X_n))_i = K_i^n(\sigma_1(X_1,\ldots,X_n),\ldots,\sigma_i(X_1,\ldots,X_n)). \tag{6.2}$$

Here by $(Q(X_1)\cdot\ldots\cdot Q(X_n))_i$ we denote the homogeneous part of the series $Q(X_1)\cdot\ldots\cdot Q(X_n)$ of degree i. As en element in $R[Y_1,\ldots]$, K_i^n is clearly of weight i. Moreover, for any $n,n'\geq i$, the polynomials K_i^n and $K_i^{n'}$ are equal in $R[Y_1,\ldots]$. Thus, they define an element $K_i^Q \in R[Y_1,\ldots]_i$ which depends only on Q. Multiplicativeness of $\{K_i^Q\}_i$ follows from (6.2) and the observation that

$$\sigma_{m+n}(X_1,\ldots,X_{n+m}) = \sum_{\substack{p+q=1\\ p=0,\ldots,m\\ q=0,\ldots,n}} \sigma_p(X_1,\ldots,X_n)\sigma_q(X_{n+1},\ldots,X_{n+m}).$$

To obtain the condition $a_i = K_i^Q(1,0,\ldots,0)$ we can just substitute $X_1=1$ and $X_2=\ldots=X_n=0$ to (6.2). □

Clearly, the correspondence

$$\left\{\begin{matrix}\text{formal power series over } R\\ \text{starting with 1}\end{matrix}\right\} \ni Q \mapsto \mathcal{K}^Q \in \left\{\text{multipl. sequences over } R\right\}$$

is one-to-one.

We can choose any formal power series Q as above, find an associated multiplicative sequence $\mathcal{K}^Q$ and then construct a genus $\varphi_{\mathcal{K}^Q}$. This genus will be denoted by φ_Q. This way a series Q determines a genus φ_Q. The converse also holds.

Proposition 6.4. *If φ is a complex genus with values in R, then there exists exactly one formal power series in $R[[X]]$ starting with 1 such that $\varphi=\varphi_Q$.*

Below we present a rough sketch of a proof. Complete and far more elegant proof can be found in [16, pp. 14–15].

Sketch of proof of Proposition 6.4. The idea is to construct a sequence $a_1,\ldots$ of elements in R such that if $Q_i = 1+a_1X+\ldots+a_iX^i \in R[X]\subset R[[X]]$, then φ_{Q_i} coincides with φ on complex projective planes $\mathbb{CP}^1,\ldots,\mathbb{CP}^i$. For this sequence, the genus φ_Q induced by the power series $Q=1+a_1X+a_2X^2+\ldots\in R[[X]]$ would coincide with φ on all complex projective spaces, which generate the whole ring $\Omega^{\mathbb{C}}\otimes\mathbb{Q}$. The sequence $a_1,\ldots$ can be computed algorithmically, as shown below.

If $Q_1=1+a_1X$, then $\varphi_{Q_1}(\mathbb{CP}^1)=2a_1$, so a_1 has to be equal to $\varphi(\mathbb{CP}^1)/2$. So, Q_2 has to be of the form $Q_2 = 1+(\varphi(\mathbb{CP}^1)/2)X+a_2X^2$. As epected, $\varphi_{Q_2}(\mathbb{CP}^1)=\varphi(\mathbb{CP}^1)$. Moreover, $\varphi_{Q_2}(\mathbb{CP}^2)=3a_2+3\varphi^2(\mathbb{CP}^1)/4$. Therefore, we have to choose $a_2=(\varphi(\mathbb{CP}^2)/3)-\varphi^2(\mathbb{CP}^1)/4$. Similarly, if we have already chosen $a_1,\ldots,a_n$, then by evaluating $\varphi_{Q_{n+1}}$ on $\mathbb{CP}^{n+1}$ we will obtain a linear equation on a_{n+1}. □

We can summarize this considerations as follows. The diagram of sets and maps between sets:

$$\begin{array}{ccc} \left\{\begin{array}{l}\text{formal power series over } R\\ \text{starting with } 1\end{array}\right\} & \longrightarrow & \left\{\text{multiplicative sequences over } R\right\} \\ \uparrow & \swarrow & \\ \left\{\text{complex genera with values in } R\right\} & & \end{array}$$

is commutative. In particular, all the maps are bijective.

Example 6.5. Take $R = \mathbb{Q}[y]$, a polynomial ring over $\mathbb{Q}$ and let Q be the formal power series $Q(x) = \frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}} = 1 - \frac{1}{2}(y-1)x + \frac{1}{12}(y+1)^2x^2 + \ldots$. The corresponding genus is known as the χ_y-genus.

If we take a homomorphism $f : R \to R'$ in the category of integral domains with distinguished unitary monomorphism from $\mathbb{Q}$ and a genus $\varphi : \Omega^{\mathbb{C}} \otimes \mathbb{Q} \to R$, we can produce a new genus $f\varphi : \Omega^{\mathbb{C}} \otimes \mathbb{Q} \to R'$. For any rational number $c \in \mathbb{Q}$ we have a "substitution homomorphism" $f_c : \mathbb{Q}[y] \to \mathbb{Q}$ taking y to c. The genus $f_c\chi_y$ will be denotes as χ_c.

Example 6.6. χ_1, a genus induced by $Q(x) = \frac{x}{\tanh x}$, is known as the L-genus. The famous signature theorem by Hirzebruch [15] states that for any closed almost-complex manifold (M, J), its genus $\chi_1(M, J)$ is equal to the signature of M. In particular it depends only on the topology of the underlying manifold, and not on the almost-complex structure.

Example 6.7. Genus χ_{-1} corresponds to the formal power series

$$Q(x) = \lim_{y \to -1} \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}} = 1 + x.$$

It is easy to compute $\chi_{-1}(M, J)$ for a $2n$-dimensional closed almost-complex manifold (M, J). The result is $\chi_{-1}(M, J) = \langle c_n(M, J), [M]\rangle$, which is known to be equal to the Euler characteristic $\chi(M)$ of M. Again, this depends only on the topology of M.

Example 6.8. Genus χ_0 is known as the Todd genus. We use the notation $\chi_0(M, J) =: \mathrm{td}(M, J)$. The corresponding formal power series is

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots.$$

Below we present the computations of the Todd genus for closed manifolds of low dimensions. This results will be useful to us later.

Lemma 6.9. *Let (M, J) be a closed almost-complex manifold of dimension $2n$ for $n = 1, 2, 3$. The Todd genus $\mathrm{td}(M, J)$ is equal to*

$$\mathrm{td}(M, J) = \begin{cases} \frac{1}{2}\chi(M), & \textit{for } n = 1, \\ \frac{1}{4}(\sigma(M) + \chi(M)), & \textit{for } n = 2, \\ \frac{1}{24}\langle c_1(M, J)c_2(M, J), [M]\rangle, & \textit{for } n = 3. \end{cases}$$

Remark. Notice, that for closed almost-complex manifolds of dimensions 2 and 4 the Todd genus depends only on the topology of the underlying manifold. In these cases we will write $\mathrm{td}(M)$ instead of $\mathrm{td}(M, J)$.

Proof of Lemma 6.9. Straightforward calculation yield:

$$\mathrm{td}(M,J) = \begin{cases} \frac{1}{2}\langle c_1(M,J),[M]\rangle, & \text{for } n=1,\\ \frac{1}{12}\langle c_1^2(M,J)+c_2(M,J),[M]\rangle, & \text{for } n=2,\\ \frac{1}{24}\langle c_1(M,J)c_2(M,J),[M]\rangle, & \text{for } n=3.\end{cases}$$

Cases $n = 1$ and $n = 3$ are now clear. For $n = 2$ observe, that

$$\begin{aligned}\langle c_1^2(M,J)+c_2(M,J),[M]\rangle &= \langle (c_1^2(M,J)-2c_2(M,J))+3c_2(M,J),[M]\rangle\\ &= 3\chi_1(M,J)+3\chi(M)\end{aligned}$$

and the proof follows from the aforementioned signature theorem. □

As mentioned earlier (Proposition 2.11), each closed symplectic manifold (M, ω) admits a family of ω-compatible almost-complex structures. All these structures are homotopic, hence the almost-complex manifolds are cobordant and the genus $\chi_y(M, \omega)$ is well-defined. The genus $\chi_y(M, \omega)$ behaves especially well in the presence of a symplectic circle action (or in fact an action preserving an almost-complex structure). Let (M, ω, λ) be a closed symplectic $\mathbb{S}^1$-manifold. Let $\{M_\nu\}_\nu$ denote the family of connected components of the fixed point set of λ. Recall, that each M_ν is a closed symplectic submanifold of (M, ω) and has well-defined index $d_\nu = d(M_\nu)$. The χ_y-genus of (M, ω) can be computed from this data, as shown in the following theorem [16, Section 5.7].

Theorem 6.10. *For a symplectic $\mathbb{S}^1$-manifold (M, ω, λ) the genus $\chi_y(M, \omega)$ can be computed as*

$$\chi_y(M,\omega) = \sum_\nu (-y)^{d_\nu} \chi_y(M_\nu, \omega|_{M_\nu}). \tag{6.3}$$

As observed for example by Fel'dman [9], this theorem leads easily to the following corollary.

Corollary 6.11. *Let $(M, \omega, \lambda, \phi)$ be a Hamiltonian $\mathbb{S}^1$-manifold. The Todd genus $\mathrm{td}(M, \omega)$ of this manifold is localized at the minimum:*

$$\mathrm{td}(M,\omega) = \mathrm{td}(\phi_0, \omega|_{\phi_0})\,.$$

Proof. When $y = 0$, in the formula (6.3) the contributions from all submanifolds M_ν with positive index $d_\nu > 0$ vanish, so $\mathrm{td}(M, \omega) = \sum_{\nu:d_\nu=0} \mathrm{td}(M_\nu, \omega|_{M_\nu})$. But there is only one such submanifold, and it is exactly ϕ_0 (Corollary 4.13). □

7 Proof of Theorem 1.1

We start with any closed simply-connected symplectic 4-manifold (K, ω_K) such that the underlying smooth manifold K satisfies $b_+^2(K) > 1$. We have seen one such manifold in Section 2 (Example 2.9), but there are many others. Some of them can be found in Gompf's paper [12]. K does not admit any circle actions. To see this, observe that $\chi(K) > 0$ so any circle action on K have non-empty fixed point set (by Corollary 3.8). By the aforementioned result of Baldridge [3], K would have to be either a blow-up[2] of $\mathbb{CP}^2$ or a blow-up of a sphere bundle over a surface, and these possibilities conflict with $b_+^2(K) > 1$ and $\pi_1(K) = 1$. Despite that many of these manifolds admit topological circle actions (all of them do after blowing up at a point), as explained in the following two lemmas.

Lemma 7.1. *If the intersection form of K is indefinite and odd, then K is homeomorphic to an $\mathbb{S}^1$-manifold.*

Proof. This is just a simple application of the well-known topological classification of 4-manifolds. By Serre's result on the classification of symmetric unimodular quadratic forms, the intersection form of K is isomorphic to that of $k\mathbb{CP}^2 \sharp l\overline{\mathbb{CP}^2}$, where $k = b_+^2(K) > 0$ and $l = b_-^2(K) > 0$. Hence, by Freedman's classification [10], K is homeomorphic to $k\mathbb{CP}^2 \sharp l\overline{\mathbb{CP}^2}$ which can be endowed with many non-trivial circle actions, as described in Example 3.23. □

Lemma 7.2. *If $(\widetilde{K}, \widetilde{\omega_K})$ is a one point blow-up of (K, ω_K), then the intersection form of $\widetilde{K}$ is indefinite and odd.*

Proof. Let μ_K be the intersection form of K. The intersection form $\mu_{\widetilde{K}}$ of $\widetilde{K}$ is equal to $\mu_K \oplus \left[-1\right]$, so it is clearly odd and not positively definite. But it is not negatively definite, since $[\widetilde{\omega_K}]^2 > 0$. □

From this moment we assume that K is homeomorphic to some $\mathbb{S}^1$-manifold N. Our next step is to modify K so that it becomes an $\mathbb{S}^1$-manifold and retains its symplectic structure. We do it using a "stabilisation" procedure, described in the Lemma 7.3 below.

Lemma 7.3. *If K and N are homeomorphic closed simply-connected 4-manifolds and F is any closed manifold of positive dimension, then $K \times F$ is diffeomorphic to $N \times F$.*

Proof. According to a theorem of Wall [33], K and N are h-cobordant. Let W be this h-cobordism and let $\iota : K \to W$ denote the inclusion. $W \times F$ is an h-cobordism between $K \times F$ and $N \times F$. Since $\dim(K \times F) = \dim(N \times F) > 4$, we can hope to use the s-cobordism theorem of Barden, Mazur and Stallings. A nice introduction to this topic can be found for example in Lück's book [22]. The obstruction for $W \times F$ to be an s-cobordism is a Whitehead torsion $\mathrm{Wh}\,(\iota \times \mathrm{id}_F)$ of a map $\iota \times \mathrm{id}_F : K \times F \to W \times F$. It is an element in the Whitehead group $\mathrm{Wh}\,(\pi)$ of the fundamental group of $W \times F$, $\pi := \pi_1(W \times F) \cong \pi_1(F)$. To

[2] Here and later, by "blow-up" we understand the so-called symplectic blow-up. This procedure, analogous to the complex blow-up, has been proposed by Gromov and described in the paper of McDuff [23]. See also chapter 7 of McDuff and Salamon's book [25].

compute this element, we can use the product formula of Kwun and Szczarba [19]. We obtain

$$\mathrm{Wh}\,(\iota \times \mathrm{id}_F) = \chi\,(F)\, j_* \,\mathrm{Wh}\,(\iota) + \chi\,(W)\, j'_* \,\mathrm{Wh}\,(\mathrm{id}_F)$$

for some homomorphisms j_* and j'_*. The torsion $\mathrm{Wh}\,(\iota)$ is an element of the trivial group $\mathrm{Wh}\,(\pi_1\,(W))$ (since W is simply-connected), and so $\mathrm{Wh}\,(\iota) = 0$. On the other hand, $\mathrm{Wh}\,(\mathrm{id}_F) = 0$. Thus we can conclude that $\mathrm{Wh}\,(\iota \times \mathrm{id}_F) = 0$ as well. We now use the s-cobordism theorem to finish the proof. □

We will stabilize K using a surface Σ of genus $g > 1$. We can easily find a symplectic form ω_Σ on Σ. The manifold $(K \times \Sigma, \omega_K \times \omega_\Sigma)$ is then a symplectic manifold. If we chose $g = 0$ or 1, then the product $(K \times \Sigma, \omega_K \times \omega_\Sigma)$ would inherit a non-trivial symplectic circle action from (Σ, ω_Σ), but since for $g > 1$ the surface Σ does not admit any non-trivial smooth circle action, we may hope that in this case no non-trivial symplectic circle actions will appear on $(K \times \Sigma, \omega_K \times \omega_\Sigma)$. At the other hand, $K \times \Sigma \cong N \times \Sigma$ by Lemma 7.3 above[3], so $K \times \Sigma$ has a non-trivial product circle action.

To prove that $(K \times \Sigma, \omega_K \times \omega_\Sigma)$ does indeed provide an example for Theorem 1.1, we have to formulate some condition for this manifold to admit a symplectic circle action depending only on the equivalence class of the symplectic form. This condition is given by the following lemma.

Lemma 7.4. *Let K, g, Σ be as above and let ω be any symplectic form on $K \times \Sigma$. If $(K \times \Sigma, \omega)$ admits a symplectic circle action then* $\mathrm{td}(K \times \Sigma, \omega) \geq (1 - g)$.

Proof. First, notice that $\chi\,(K \times \Sigma) = 2\chi\,(K)\,(1 - g) < 0$, so every circle action on $K \times \Sigma$ has fixed points (Corollary 3.8). Now, we use Proposition 4.17 to show that any symplectic circle action on $K \times \Sigma$ is Hamiltonian. To do that we have to show that $(K \times \Sigma, \omega)$ has the weak Lefschetz property.

By Künneth formula, the cohomology vector space of $K \times \Sigma$ is given by

$$H^*(K \times \Sigma; \mathbb{R}) \cong H^*(K; \mathbb{R}) \otimes H^*(\Sigma; \mathbb{R}),$$

with the product taken in the category of graded algebras. In particular, there exist classes $a \in H^2(K; \mathbb{R})$ and $b \in H^2(\Sigma; \mathbb{R})$ such that $[\omega] = a \otimes 1 + 1 \otimes b$, and the following diagram commutes:

$$\begin{array}{ccc} H^1(K \times \Sigma; \mathbb{R}) & \xrightarrow{\wedge[\omega]^2} & H^5(K \times \Sigma; \mathbb{R}) \\ \cong \uparrow & & \uparrow \cong \\ H^0(K; \mathbb{R}) \otimes H^1(\Sigma; \mathbb{R}) & \xrightarrow{\wedge a^2 \otimes 1} & H^4(K; \mathbb{R}) \otimes H^1(\Sigma; \mathbb{R}). \end{array}$$

The lower horizontal arrow is an isomorphism, because $0 \neq [\omega]^3 = 3a^2 \otimes b$ implies $a^2 \neq 0$. Hence $(K \times \Sigma, \omega)$ has the weak Lefschetz property, as expected.

Fix a symplectic circle action on $(K \times \Sigma, \omega)$. By the discussion above, it is Hamiltonian with some moment map ϕ. Let ϕ_0 denote, as before, the minimum of ϕ. By Proposition 4.12 ϕ_0 is connected, and by Corollary 4.6 it is a symplectic submanifold of $(K \times \Sigma, \omega)$. In particular, $(\phi_0, \omega|_{\phi_0})$ is a symplectic manifold of dimension 0, 2, or 4. We compute $\mathrm{td}(\phi_0)$ in each of these cases.

[3]Alternatively, we could argue as follows. $K \times \Sigma$ is diffeomorphic to $N \times \Sigma$ because they are h-cobordant and the group $\mathrm{Wh}\,(\pi_1\,(K \times \Sigma)) \cong \mathrm{Wh}\,(\pi_1\,(\Sigma))$ is trivial, by the theorem of Farrell and Jones [8].

ϕ_0 is 0-dimensional Then ϕ_0 is a point and $\mathrm{td}(\phi_0) = 1$.

ϕ_0 is 2-dimensional Then ϕ_0 is a surface and, since by Theorem 5.6 we have $\pi_1(\phi_0) \cong \pi_1(K \times F) \cong \pi_1(F)$, it is a surface of genus g. In particular, by Lemma 6.9, $\mathrm{td}(\phi_0) = 1 - g$.

ϕ_0 is 4-dimensional As before, Theorem 5.6 gives us $\pi_1(\phi_0) \cong \pi_1(F)$. So, $b^1(\phi_0) = 2g$. Simultaneously, as a manifold admitting a symplectic form ϕ_0 satisfies $b^2_+(\phi_0) \geq 1$. Now, using Lemma 6.9 we obtain $\mathrm{td}(\phi_0) \geq 1 - g$.

In each of these cases, we obtain $\mathrm{td}(\phi_0) \geq 1 - g$. The lemma is a consequence of this and Corollary 6.11. □

This, together with Observation 2.15, implies that if ω is a symplectic form on $K \times \Sigma$ equivalent to the product form $\omega_K \times \omega_\Sigma$ and $(K \times \Sigma, \omega)$ is a symplectic $\mathbb{S}^1$-manifold, then $\mathrm{td}(K \times \Sigma, \omega_K \times \omega_\Sigma) \geq 1 - g$. But clearly:

$$\mathrm{td}(K \times \Sigma, \omega_K \times \omega_\Sigma) = \mathrm{td}(K, \omega_K)\, \mathrm{td}(\Sigma, \omega_\Sigma) = \frac{1}{2}(1 + b^2_+(K))(1 - g) < 1 - g.$$

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