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Some properties of conjugacy classes in Coxeter groups

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Abstract. We review some combinatorial properties of conjugacy classes in Coxeter groups, discuss their applications in the theory of Hecke algebras, and prove a new result about conjugacy classes in right-angled Coxeter groups, yielding a description of the space of central functionals on its Hecke algebra analogous to the interpretation of central functionals on a group algebra as functions on conjugacy classes.

1. Introduction

In this paper we review some theorems about conjugacy classes in Coxeter groups and extend their scope to some additional cases, on which we would like to work further, in connection with our long-term research project of investigating the Hecke-von Neumann algebras.

A Hecke algebra $\mathbb{C}_{\mathbf{q}}[W]$ is a certain deformation of the group algebra of a Coxeter system (W, S). It carries a natural structure of a Hilbert algebra in the sense sense of Chapter 5 of [3] (see [4]). Thus it can be completed to a von Neumann algebra $\mathcal{N}_{\mathbf{q}}(W)$ acting on the Hilbert completion $L^2_{\mathbf{q}}(W)$ of $\mathbb{C}_{\mathbf{q}}[W]$ (actually, as there are two actions of $\mathbb{C}_{\mathbf{q}}[W]$ on $L^2_{\mathbf{q}}(W)$, from the left and from the right, there exist two natural completions of the Hecke algebra). We are primarily interested in describing the centers of these von Neumann algebras. To a central element $A \in \mathcal{N}_{\mathbf{q}}(W)$ there corresponds, as in the case of the classical group von Neumann algebra, its *symbol* $AT_e \in L^2_{\mathbf{q}}(W)$, which in turn defines a functional $|AT_e\rangle$ on $L^2_{\mathbf{q}}(W)$, and in particular on $\mathbb{C}_{\mathbf{q}}[W]$. This functional satisfies, for any $X, Y \in \mathbb{C}_{\mathbf{q}}[W]$ the identity $\langle XY|AT_e\rangle = \langle YX|AT_e\rangle$. We call such functionals *central functionals*. We wish to better understand the purely algebraic space of central functionals, hoping that this will help us describe the centers of Hecke-von Neumann algebras.

It turns out that if the group W satisfies some combinatorial conditions, which we call the *descent property* and the *connected base property*, the situation is similar to the case of ordinary group algebra, where the central functionals correspond to class functions. Basically, the descent property says that in the graph obtained from W by adding edges between elements which are conjugate by a generator, any element can be connected to an element of minimal length in its conjugacy class by an edge-path, along which the word length in non-increasing. The connected base property on the other hand says that any two elements of minimal word length in the same conjugacy class can be obtained one from another by a sequence of conjugations satisfying some additional properties involving the word length. Currently it is known that finite Weyl groups satisfy the descent property and the connected base property—which is uninteresting from our point of view, as the resulting algebras are finite-dimensional. In this note we show that right-angled Coxeter groups also have these properties.

The paper is organized as follows. Section 2 is devoted to presenting the main definitions and facts in the theory of Coxeter groups. It recalls some elementary combinatorial theory of these groups. In Section 3 we formulate the descent property and the connected base property in more detail. We also recall some other combinatorial properties of conjugacy classes in Coxeter groups, which are related to the contents of this paper.

In Section 4 we first define the Hecke algebra of a Coxeter group. Then we analyze its space of central functionals. A functional on the Hecke algebra of W can be interpreted as a complex function on W. We describe the relations this function must satisfy in order to represent a central functional. We show that if the descent property and the connected base property are satisfied, then such a function is uniquely defined by its values on elements of minimal length in their conjugacy classes. These results were pointed out in [6].

Finally, in Section 5 we define right-angled Coxeter groups and prove that they satisfy the descent property and connected base property. The proof is based on a strengthening of the exchange condition, which holds in the right-angled case.

2. Coxeter groups

A pair (W, S) where W is a group and $S \subseteq W$ is a finite generating set of involutions, i.e. elements of order 2, is called a *pre-Coxeter system.* A pre-Coxeter system with presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle_{s,t \in S}, \tag{1}$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$ for $s \neq t$, is called a *Coxeter system.* The group W itself is then a *Coxeter group.* It is *irreducible* if S cannot be decomposed into a sum of two mutually commuting subsets.

As for any group with a distinguished set of generators, we may define a length function on W by

$$\ell(w) = \min\{n : (\exists s_1, \dots, s_n \in S) w = s_1 s_2 \cdots s_n\}. \tag{2}$$

If $w = s_1 \cdots s_n$, with $s_i \in S$ and $n = \ell(w)$, we say that $s_1 \cdots s_n$ (treated as a word in the alphabet S, not an actual element of W) is a reduced expression for w. An important property of length in a Coxeter group is that $\ell(sw) = \ell(w) \pm 1$ for all $w \in W$ and $s \in S$. To see this, just notice that the map $w \mapsto (-1)^{\ell(w)}$ is a group homomorphism onto $\{-1, 1\}$, hence $\ell(w) \neq \ell(sw)$, and the difference is at most 1, by the triangle inequality.

A fundamental result characterizing the Coxeter groups among all groups with a distinguished set of generating involutions is the following.

Theorem 2.1 (Exchange condition)**.** *Suppose (W, S) is a Coxeter system. Let $s_1 \cdots s_n$ be a reduced expression for an element $w \in W$, and let $t \in S$ be such that $\ell(tw) < \ell(w)$. Then, for some $1 \leq i \leq n$, we have*

$$tw = s_1 \cdots \hat{s_i} \cdots s_n. \tag{3}$$

The same holds if we replace tw with wt.

For the proof we refer to [1, Theorem 3.3.4]. As a corollary, we obtain the equivalent Folding condition.

Corollary 2.2 (Folding condition)**.** *Suppose (W, S) is a Coxeter system. Let $w \in W$, and $s, t \in S$. If $\ell(sw) = \ell(wt) = \ell(w) + 1$, then either $\ell(swt) = \ell(w) + 2$ or $swt = w$.*

Proof. Let $s_1 \cdots s_n$ be a reduced word representing w. By our assumption $ss_1 \cdots s_n$ and $s_1 \cdots s_n t$ are also reduced. If $\ell(swt) < \ell(w) + 2$, then $\ell(swt) = \ell(w)$ and we may apply the Exchange condition to the word wt and generator s and conclude that $swt = w$, since if a cancellation occurred at any s_i, then the same cancellation could be used to obtain a shorter word for sw—a contradiction. □

2.1. **Finite Weyl groups.** Finite Weyl groups are a subclass of finite Coxeter groups. Since one of the theorems we quote later refers to Weyl groups, we will enumerate them here.

A presentation of a Coxeter system (W, S) can be encoded in a labeled graph Γ, called the *Coxeter diagram*, in the following way. As the set of vertices of Γ take S. Vertices s and t are joined by an edge with label m_{st} whenever $m_{st} > 2$. Thus, in particular, a Coxeter system is irreducible if and only if the corresponding graph is connected. When drawing the graph Γ, we omit labels equal to 3. Table 2.1 lists all finite Weyl groups. The families A_n, $B_n = C_n$, and D_n are parametrized by the number of generators n.

Type
A_n
$B_n = C_n$
D_n
$I_2(6)$
F_4
E_6
E_7
E_8

Coxeter diagram

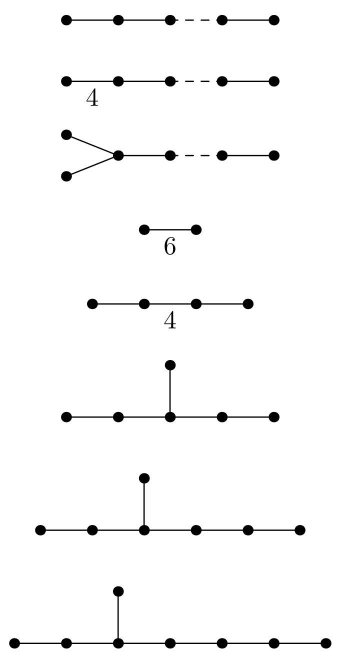

Table 1. Classification of finite Weyl groups

3. Properties of conjugacy classes

3.1. **The conjugacy graph.** To better visualize the notions we will consider, we introduce a graph representing the structure of conjugacy classes. Let G be a group, and let $X \subseteq G$ be symmetric, i.e. if $x \in X$, then also $x^{-1} \in X$. The *conjugacy graph* of G with respect to X is an undirected graph $\Gamma(G, X)$ defined as follows:

- the set of vertices of $\Gamma(G, X)$ is G,
- there is an edge between (not necessarily distinct) vertices g and h if and only if $h = x^{-1}gx$ for some $x \in X$.

By a *path* in $\Gamma(G, X)$ we understand a sequence $p = (p_0, \dots, p_n)$ of elements of G such that p_i and p_{i+1} are joined by an edge in $\Gamma(G, X)$ for all $i < n$.

If we fix X, we may speak about local properties of a function $f : G \to \mathbb{R}$. We say that f has a *local minimum* (resp. *local maximum*, and *is locally constant*) at $g \in G$, if $f(g) \leq f(h)$ (resp. $f(g) \geq f(h)$, and $f(g) = f(h)$) for all neighbors h of g in $\Gamma(G, X)$.

Now let us return to the case of a Coxeter system (W, S). Let $C \subseteq W$ be a conjugacy class. Denote by

$$C_{\min} = \{w \in W : \ell(w) = \min_{u \in C} \ell(u)\} \tag{4}$$

the set of elements of minimal length in C. We will call it the *base* of C. Similarly we define $C_{\max}$. A conjugacy class C is said to be *flat*, if it is equal to its base, i.e. ℓ is constant on C.

3.2. **Bad and evil elements.** Let $R = \{w^{-1}sw : w \in W, s \in S\}$ stand for the set of reflections in (W, S). Perkins and Rowley defined in [7] the following special types of elements of a conjugacy class C:

- $w \in C \setminus C_{\min}$ is *bad downward* if it is a local minimum of ℓ in $\Gamma(W, R)$
- $w \in C \setminus C_{\max}$ is *bad upward* if it is a local maximum of ℓ in $\Gamma(W, R)$
- $w \in C$ is *bad* if it is either bad upward or bad downward
- $w \in C$ is *evil* if C is not flat, but ℓ is locally constant at w in $\Gamma(W, R)$.

The main result of [8] is the following.

Theorem 3.1. *Let W be a Coxeter group without finite irreducible direct factors. Then W contains no bad upward elements.*

Also, in [7] the following was shown.

Theorem 3.2. *A Coxeter group contains no evil elements.*

3.3. **Connectivity properties.** Geck and Pfeiffer introduced in [6] two properties of a Coxeter group, which we call *the descent property* and *the connected base property*:

the descent property: any $w \in W$ can be connected with the base of its conjugacy class by a path p in $\Gamma(W, S)$, such that ℓ is non-increasing along p,

the connected base property: for any u, v in the base of the same conjugacy class C there exists a sequence

$$u = w_0, w_1, \dots, w_n = v$$

of elements of $C_{\min}$ such that $w_{i+1} = x_i^{-1} w_i x_i$ with $x_i \in W$ satisfying either $\ell(x_i^{-1} w_i) = \ell(x_i) + \ell(w_i)$, or $\ell(w_i x_i) = \ell(w_i) + \ell(x_i)$.

In the same paper they have shown the following.

Theorem 3.3. *A finite Weyl group has the descent property and the connected base property.*

4. Applications to Hecke algebras

4.1. **Hecke algebras.** Hecke algebras are deformations of group algebras of Coxeter groups. Although they can be defined over arbitrary rings, here we will consider only Hecke algebras over the field of complex numbers, as this is the case we hold primary interest in, because of its connections with weighted L^2-cohomology of Coxeter groups ([2, 4]).

Throughout this section we will consider an arbitrary Coxeter system (W, S). Let $\mathbf{q} = (q_s)_{s\in S}$ be a system of positive real numbers, such that $q_s = q_t$ whenever s and t are conjugate in W. There is a simple criterion allowing to determine when two generators are conjugate by just looking at the Coxeter diagram of (W, S).

Lemma 4.1. *Two generators $s, t \in S$ are conjugate in W if and only if there exists a path in the Coxeter diagram of (W, S), joining s with t, such that all labels along this path are odd integers.*

Proof. If s and t are joined by an edge with odd label m, then the relation $(st)^m = 1$ can be rewritten as

$$(st)^{(m-1)/2} s (ts)^{(m-1)/2} = t.$$

By transitivity of the conjugacy relation, if two generators are joined in the Coxeter diagram Γ by a path with odd labels, they are conjugate.

Now denote by T the set of all generators $t' \in S$ which can be connected with t by a path in Γ with odd labels, and suppose that $s \notin T$. Consider a Coxeter system $(W', S \setminus T)$ defined by the Coxeter diagram obtained from Γ by removing all the vertices in T. We claim that the map $\phi\colon S \to W'$ given by

$$\begin{aligned} \phi(s') &= s' \qquad \text{for } s' \notin T \\ \phi(t') &= 1 \qquad \text{for } t' \in T \end{aligned}$$

extends to a group homomorphism $\phi\colon W \to W'$. Indeed, the relations between two generators both belonging either to $S \setminus T$ or T are satisfied in W', and if $s' \in S \setminus T$ and $t' \in T$, then $m_{s't'}$ is even or infinite by definition of T, hence in W' we have $\phi(s')^{m_{s't'}} = 1$.

The elements $\phi(s)$ and $\phi(t)$ are non-conjugate in W', hence s and t are non-conjugate in W. □

A *Hecke algebra* of (W, S) with multiparameter $\mathbf{q}$ is an associative $\mathbb{C}$-algebra $\mathbb{C}_{\mathbf{q}}[W]$ with basis $\{T_w : w \in W\}$, satisfying the relations

$$T_u T_w = T_{uw} \qquad \text{if } \ell(uw) = \ell(u) + \ell(w), \tag{5}$$

$$T_s T_w = (q_s - 1)T_w + q_s T_{sw} \qquad \text{if } s \in S \text{ and } \ell(sw) < \ell(w). \tag{6}$$

It follows directly from these relations that the right-sided analogue of (6) also holds, and that the T_s, and hence all T_w are invertible. The inverse of T_s is $T_s^{-1} = q_s^{-1}(T_s + 1 - q_s)$.

It is not readily visible why such an algebra should exist. It is a standard theorem that it indeed does and is unique.

Theorem 4.2. *For every Coxeter system (W, S) and every multiparameter $\mathbf{q}$ such that $q_s = q_t$ whenever s and t are conjugate, there exists a unique associated Hecke algebra.*

For a proof consult [5].

4.2. **Central functionals.** Consider a Hecke algebra $\mathbb{C}_{\mathbf{q}}[W]$. A functional $f \in \mathbb{C}_{\mathbf{q}}[W]^*$ is *central* if

$$f(XY) = f(YX) \tag{7}$$

for any $X, Y \in \mathbb{C}_{\mathbf{q}}[W]$. The space of central functionals will be denoted by $\mathbb{C}_{\mathbf{q}}[W]^*_{\mathrm{ctr}}$. There is a canonical isomorphism between $\mathbb{C}_{\mathbf{q}}[W]^*$ and the space of complex functions on W. Thus, for a functional $f \in \mathbb{C}_{\mathbf{q}}[W]$ we will often write $f(w)$ instead of $f(T_w)$.

In [6] some connections between the descent and connected base properties, and central functionals on Hecke algebras were pointed out. We will present them in this subsection, with somewhat more detailed proofs.

Lemma 4.3. *Suppose that the Coxeter system (W, S) satisfies the connected base property, and let $f \in \mathbb{C}_{\mathbf{q}}[W]^*_{\mathrm{ctr}}$ be a central functional on its Hecke algebra with multiparameter $\mathbf{q}$. If $u, v \in W$ both belong to the base of the same conjugacy class C, then $f(u) = f(v)$.*

Proof. It is sufficient to consider the case when $v = x^{-1}ux$ with either $\ell(x^{-1}u) = \ell(x) + \ell(u)$ or $\ell(ux) = \ell(u) + \ell(x)$. In the first case, notice that since $vx^{-1} = x^{-1}u$, we have

$$\ell(vx^{-1}) = \ell(x^{-1}u) = \ell(x) + \ell(u) = \ell(x) + \ell(v),$$

and therefore

$$T_u = T_{x^{-1}}^{-1} T_{x^{-1}} T_u = T_{x^{-1}}^{-1} T_{vx^{-1}} = T_{x^{-1}}^{-1} T_v T_{x^{-1}}.$$

Thus T_u and T_v are conjugate in $\mathbb{C}_{\mathbf{q}}[W]$, and $f(u) = f(v)$. In the second case we proceed analogously, obtaining

$$T_u = T_u T_x T_x^{-1} = T_{xv} T_x^{-1} = T_x T_v T_x^{-1}.$$

□

Lemma 4.4. *Let $f \in \mathbb{C}_{\mathbf{q}}[W]^*$. The functional f is central if and only if for every $w \in W$ and $s \in S$ such that $\ell(w) < \ell(sw) = \ell(ws) < \ell(sws)$, the following relations are satisfied*

$$f(sw) = f(ws) \tag{8}$$

$$f(sws) = (q_s - 1)f(sw) + q_s f(w) \tag{9}$$

Proof. First suppose that $f \in \mathbb{C}_{\mathbf{q}}[W]^*$ is central. Take s and w such that $\ell(w) < \ell(sw) = \ell(ws) < \ell(sws)$. Then

$$f(sw) = f(T_s T_w) = f(T_w T_s) = f(ws),$$

and

$$\begin{aligned} f(sws) &= f(T_w T_s^2) = f(T_w((q_s - 1)T_s + q_s)) \\ &= f((q_s - 1)T_{ws} + q_s T_w) = (q_s - 1)f(sw) + q_s f(w). \end{aligned}$$

Now let $f \in \mathbb{C}_{\mathbf{q}}[W]^*$ be any functional satisfying conditions of the Lemma. In order to prove it is central, it suffices to show that for any $u \in W$ and $s \in S$ we have

$$f(T_u T_s) = f(T_s T_u) \tag{10}$$

If the double coset $\{u, su, us, sus\}$ has four elements, then, by the Folding condition (Corollary 2.2), it contains a unique element w of minimal length. There are thus four cases to consider.

If $u = w$, then (10) follows directly from (8). If $u = sw$, then we compute

$$f(T_uT_s) = f(sws) = (q_s - 1)f(sw) + q_s f(w) = \\ = (q_s - 1)f(u) + q_s f(w) = f(T_sT_u).$$

Similarly, if $u = ws$, we have

$$f(T_sT_u) = f(sws) = (q_s - 1)f(sw) + q_s f(w) = \\ = (q_s - 1)f(u) + q_s f(w) = f(T_uT_s).$$

Finally, if $u = sws$, we get

$$f(T_sT_u) = (q_s - 1)f(sws) + q_s f(ws) = \\ = (q_s - 1)f(sws) + q_s f(sw) = f(T_uT_s).$$

If the double coset $\{u, su, us, sus\}$ has only two elements we have two cases. If $w = u = sus$, then

$$f(T_sT_u) = f(T_{su}) = f(T_{us}) = f(T_uT_s),$$

and if $w = su = us$, then

$$f(T_sT_u) = (q_s - 1)f(T_u) + q_s f(T_w) = f(T_uT_s).$$

In each case (10) is satisfied. □

Lemma 4.5. *Suppose (W, S) satisfies the descent property. Let $\{X_s\}_{s\in S}$ be a system of indeterminates, such that $X_s = X_t$ whenever s and t are conjugate in W. Then for every $w \in W$ and $v \in C_{\min}$ for some conjugacy class $C \subseteq W$ there exists a polynomial $Q_{w,v} \in \mathbb{Z}[X_s]_{s\in S}$ such that*

1. *$Q_{w,v} = 0$ for $\ell(v) > \ell(w)$,*
2. *for every multiparameter $\mathbf{q}$ and every central functional $f \in \mathbb{C}_{\mathbf{q}}[W]^*_{\mathrm{ctr}}$ we have*

$$f(w) = \sum_v Q_{w,v}(\mathbf{q}) f(v). \tag{11}$$

Proof. We construct $Q_{w,v}$ recursively. If $w \in C_{\min}$ for some conjugacy class C, we put $Q_{w,w} = 1$ and $Q_{w,v} = 0$ for $v \neq w$. Now denote by U the set of all $w \in W$ for which all the polynomials $Q_{w,v}$ were already defined. Let w be an element of $W \setminus U$ of minimal length. By the descent property, and possibly changing w, we may assume that some conjugate sws with $\ell(sws) \leq \ell(w)$ is in U. If $\ell(sws) = \ell(w)$, we put $Q_{w,v} = Q_{sws,v}$. If $\ell(sws) < \ell(w)$, then $sw, sws \in U$ by minimality of $\ell(w)$, and we can define

$$Q_{w,v} = (X_s - 1)Q_{sw,v} + X_s Q_{sws,v}.$$

Since there are finitely many elements of W of any given length, this process will define the polynomials $Q_{w,v}$ for all w and v. □

Corollary 4.6. *Suppose (W, S) satisfies the descent property and the connected base property. Let X_s be as in the previous lemma. Then there exists a family of polynomials $Q_{w,C} \in \mathbb{Z}[X_s]_{s\in S}$ indexed by elements $w \in W$ and conjugacy classes $C \subseteq W$, such that*

(1) *$Q_{w,C} = 0$ for all but finitely many C,*
(2) *for every multiparameter $\mathbf{q}$ and every central functional $f \in \mathbb{C}_{\mathbf{q}}[W]^*$ we have*

$$f(w) = \sum_C Q_{w,C}(\mathbf{q}) f(C_{\min}), \tag{12}$$

where by $f(C_{\min})$ we understand the value of f on any $v \in C_{\min}$.

Proof. We can just take

$$Q_{w,C} = \sum_{v \in C_{\min}} Q_{w,v},$$

where $Q_{w,v}$ are given by Lemma 4.5. By the connected base property and Lemma 4.3 $f(v)$ is independent of the choice of $v \in C_{\min}$. □

5. Right-angled Coxeter groups and their Hecke algebras

A Coxeter group is *right-angled* if $m_{st} \in \{2, \infty\}$ for $s \neq t$. A strengthened version of the Exchange condition holds for such groups.

Proposition 5.1. *Suppose (W, S) is a right-angled Coxeter system. Let $s_1 \cdots s_n$ be a reduced expression for an element $w \in W$, and let $t \in S$ be such that $\ell(tw) < \ell(w)$. Then, for some $1 \leq i \leq n$, we have*

(1) *$tw = s_1 \cdots \hat{s}_i \cdots s_n$,*
(2) *$s_i = t$,*
(3) *t commutes with all s_j, where $j < i$.*

The same holds if we replace tw with wt.

Proof. Part (1) follows from the Exchange condition (Theorem 2.1). To show part (2), notice that we have a homomorphism $\phi\colon W \to \mathbb{Z}_2$, killing all generators of W except t. Since $\phi(tw) = \phi(s_1 \cdots \hat{s}_i \cdots s_n)$, we must have $\phi(ts_i) = 0$, hence $t = s_i$.

Finally, to prove (3) we proceed by induction on n. For $n = 1$ there is nothing to prove, so let $n > 1$. We may assume that i was chosen to be minimal possible. If $i = 1$, then (3) is trivially satisfied, so let $i > 1$.

We thus have

$$ts_1 \cdots s_{i-1} = s_1 \cdots s_{i-1} t,$$

and

$$\ell(ts_1 \cdots s_{i-1}) = i,$$

for otherwise we could apply the exchange condition to t and w with some $j \leq i-1$, contradicting the minimality of i. In particular, it follows that $s_1 \neq t$. Therefore

$$\ell(s_1 t s_1 \cdots s_{i-1}) = \ell(s_2 \cdots s_{i-1} t) = i - 1 < \ell(ts_1 \cdots s_{i-1}),$$

and by the induction hypothesis

$$s_1 t s_1 \cdots s_{i-1} = ts_1 \cdots \hat{s}_j \cdots s_{i-1},$$

where $s_j = s_1 \neq t$, and s_1 commutes with t. We obtain

$$tw = s_1 t s_2 \cdots s_n = s_1 \cdots \hat{s}_i \cdots s_n,$$

and by canceling out the initial s_1, we may apply the induction assumption to t and $w' = s_2 \cdots s_n$. □

Proposition 5.2. *Let (W, S) be a right-angled Coxeter system, and let $C \subseteq W$ be a conjugacy class. If $u \in C$ and $v \in C_{\min}$, then there exists a path p in $\Gamma(W, S)$, starting at u and ending at v, such that ℓ is weakly decreasing along p.*

Proof. For a path $p = (p_0, \dots, p_n)$ in $\Gamma(W, S)$, define

$$A(p) = \sum_{i=0}^{n} \ell(p_i).$$

Let $q = (q_0, q_1, \dots, q_n)$ be a path in $\Gamma(W, S)$ such that $q_0 = u$ and $q_n = v$. Suppose that ℓ does not weakly decrease along q. Since $v \in C_{\min}$, it means that there exists i such that

$$\ell(q_{i-1}) < \ell(q_i) \qquad \text{and} \qquad \ell(q_i) \geq \ell(q_{i+1}).$$

Denote $w = q_{i-1}$, $sws = q_i$, and $tswst = q_{i+1}$. If

$$\ell(tsws) = \ell(swst) > \ell(sws),$$

then by the Folding condition (Corollary 2.2), $sws = tswst$, and by removing q_i from the path q we obtain a shorter path q'. Otherwise, without loss of generality we may assume that

$$\ell(tsws) < \ell(sws).$$

Now we have two cases. If $s = t$, then $tswst = w$ and by removing q_i and q_{i+1} from q we obtain a shorter path q'. If $s \neq t$, then by the Exchange condition (Theorem 2.1) we obtain $tsws = ss_1 \cdots \hat{s}_i \cdots s_n s$, where $s_1 \cdots s_n$ is a reduced word representing w, and t commutes with $s, s_1, \dots, s_{i-1}$. Thus, if we replace $q_i = sws$ by twt, we still get a path q' in $\Gamma(W, S)$. Furthermore,

$$\ell(tw) = \ell(ts_1 \cdots s_i \cdots s_n) = \ell(s_1 \cdots \hat{s}_i \cdots s_n) < \ell(w),$$

and therefore $\ell(twt) \leq \ell(w)$, so $A(q') < A(q)$. In all cases passing from q to q' decreases the value of $A(q)$, which is a non-negative integer. This process has to terminate yielding the desired path p. □

Corollary 5.3. *A right-angled Coxeter system (W, S) satisfies the descent property and the connected base property.*

Proof. The descent property is a direct consequence of Proposition 5.2. To see that the connected base property also holds, consider $u, v \in C_{\min}$ for some conjugacy class C. They are connected by a path p in $\Gamma(W, S)$, on which ℓ weakly decreases, which in this case implies it is constant. So $u = p_0, p_1, \dots, p_n = v \in C_{\min}$, and $p_{i+1} = s_i p_i s_i$ for some $s_i \in S$. We may assume that $p_i \neq p_{i+1}$ for $i < n$. If both $\ell(s_i p_i) = \ell(p_i) - 1$ and $\ell(p_i s_i) = \ell(p_i) - 1$ hold, then by the Folding condition (Corollary 2.2) applied to $w = s_i p_i$ and $s = t = s_i$ we obtain a contradiction $p_{i+1} = p_i$. Therefore either $\ell(s_i p_i) = \ell(p_i) + 1$ or $\ell(p_i s_i) = \ell(p_i) + 1$, and the connected base property is satisfied. □

Denote by $\sim$ the relation of conjugation on W, and by $\mathbb{C}[[W/{\sim}]]$ the $\mathbb{C}$-linear space of complex functions on the set of conjugacy classes on W.

Corollary 5.4. *For a right-angled Coxeter system (W, S) the map $j \colon \mathbb{C}_{\mathbf{q}}[W]^*_{\mathrm{ctr}} \to \mathbb{C}[[W/{\sim}]]$, defined by $j(f)(C) = f(C_{\min})$ is injective.*

Proof. This is a direct consequence of Corollaries 5.3 and 4.6, which show that any central functional on $\mathbb{C}_{\mathbf{q}}[W]$ is uniquely determined by its values on bases of conjugacy classes. □

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