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Analogs of principal series representations for Thompson's
groups F and T

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ANALOGS OF PRINCIPAL SERIES REPRESENTATIONS FOR THOMPSON'S GROUPS F AND T

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ABSTRACT. We define series of representations of the Thompson's groups F and T , which are analogs of principal series representations of $SL(2, \mathbb{R})$. We show that they are irreducible and classify them up to unitary equivalence. We also prove that they are different from representations induced from finite-dimensional representations of stabilizers of points under natural actions of F and T on the unit interval and the unit circle, respectively.

1. INTRODUCTION

Suppose that a group G acts on a measure space (X, μ) , leaving μ quasi-invariant. Such an action gives rise to a series of unitary representations π_s on the complex Hilbert space $L^2(X, \mu)$, given by

$$(1) \quad \pi_s(\gamma)f = \left(\frac{d\gamma_*\mu}{d\mu} \right)^{1/2+is} f \circ \gamma^{-1},$$

where $s \in \mathbb{R}$. If G is a semisimple Lie group, a construction of this type can lead to a part of the principal series of G , and therefore to a large family of pairwise inequivalent irreducible representations (see e.g. [1], Appendix E).

If G is a "large" group, such as the group of diffeomorphisms of a smooth manifold, the representations of the form (1) are irreducible ([6, 3]). In this paper we study the irreducibility of representations of Thompson's groups F and T acting on the unit interval I and unit circle S^1 . Common wisdom says that they are "large" subgroups of the groups of piecewise linear homeomorphisms of I and S^1 . One might therefore expect that the corresponding representations are irreducible. In Section 3 we show that this is the case. Moreover, we show that the only equivalences between them arise from trivial reasons.

The main example we had in mind when considering the representations (1) was the principal series of $SL(2, \mathbb{R})$. It consists of two parts, one of which is obtained from the action of $SL(2, \mathbb{R})$ on the one-point compactification of \mathbb{R} by Möbius transformations. The same representations can be obtained through induction from a unitary character of the subgroup $P \leq SL(2, \mathbb{R})$ consisting of upper-triangular matrices. This subgroup happens to be the stabilizer of the point ∞ , hence it can be recovered from the action of $SL(2, \mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$.

It turns out that the analogy between the representations π_s of Thompson's groups, and the principal series representations of $SL(2, \mathbb{R})$ breaks here. In Section 4 we will show that there is a large class of pairwise inequivalent irreducible representations of F , obtained through induction from finite-dimensional unitary representations of stabilizers of points. This class, however, does not contain any of the representations π_s .

It is easy to see that if the restrictions of representations π_s to a subgroup of G are irreducible or inequivalent, then so are the original representations. It follows

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that the analogs of the principal series representations of many naturally occurring groups which contain F , such as the group of piecewise linear or piecewise projective homeomorphisms of the unit interval, are irreducible.

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2. PRELIMINARIES

2.1. Thompson's groups F and T . The Thompson's group F consists of all orientation-preserving piecewise linear homeomorphisms ϕ of the unit interval $[0, 1]$ such that

- (1) ϕ preserves the set $\mathbb{Z}[1/2] \cap [0, 1]$ of dyadic rational numbers in $[0, 1]$,
- (2) ϕ has finitely many nondifferentiability points, which all lie in $\mathbb{Z}[1/2]$,
- (3) the slopes of ϕ are of the form 2^k for $k \in \mathbb{Z}$.

The surjection $\eta: [0, 1] \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$ given by $\eta(t) = e^{2\pi it}$ induces a faithful action of F on S^1 . Denote by R the group of rotations of S^1 with angles in $\pi\mathbb{Z}[1/2]$. The Thompson's group T is the group of homeomorphisms of S^1 generated by F , viewed as a group of homeomorphisms of S^1 , and R . We will identify F with the subgroup of T fixing the point $1 \in S^1$.

2.2. Technical lemmas.

Lemma 1. *Let $I \subseteq [0, 1]$ be a nondegenerate closed interval of length less than 1. If for a function $f \in L^1([0, 1])$ there exists a dense subset D of $S = \{s \in \mathbb{R} : s + I \subseteq [0, 1]\}$ such that for every $s \in D$ we have $f(x) = f(x + s)$ almost everywhere on I , then f is constant.*

Proof. For any continuous function $\phi \in C(I)$ we may define $f_\phi: S \rightarrow \mathbb{C}$ by

$$(2) \quad f_\phi(s) = \int_I f(x+s)\phi(x) dx = \int_{I+s} f(x)\phi(x-s) dx.$$

Then f_ϕ is continuous, and constant on D , hence constant on S . For any subinterval $J \subseteq I$ we may take a uniformly bounded sequence $\phi_n \in C(I)$ converging pointwise to the characteristic function χ_J , obtaining, by the dominated convergence theorem, that the value of the integral $\int_{J+s} f(x) dx$ is independent of $s \in S$. It is now easy to check that

$$(3) \quad \int_a^b f(x) dx = \frac{1}{b-a} \int_0^1 f(x) dx$$

for any $0 \leq a < b \leq 1$, hence f is constant. \square

Lemma 2 ([2], Lemma 4.2). *If $0 = x_0 < x_1 < x_2 < \dots < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \dots < y_n = 1$ are partitions of the interval $[0, 1]$ consisting of dyadic rational numbers, then there exists $\gamma \in F$ such that $\gamma(x_i) = y_i$ for all $i = 0, 1, \dots, n$. Furthermore, if $x_{i-1} = y_{i-1}$ and $x_i = y_i$ for some i , then γ can be chosen so that $\gamma(x) = x$ for $x \in [x_{i-1}, x_i]$.*

Proof. This follows from the observation that any two dyadic rational numbers can be written in the form $2^{\alpha_1} + 2^{\alpha_2} + \dots + 2^{\alpha_k}$, where $\alpha_i \in \mathbb{Z}$, with the same number of summands. Such decompositions of $x_i - x_{i-1}$ and $y_i - y_{i-1}$ allow us to define γ on $[x_{i-1}, x_i]$. \square

In particular, Lemma 2 implies that for any closed interval $I \subseteq (0, 1)$, and $h \in \mathbb{Z}[1/2]$ such that $I + h \subseteq (0, 1)$, there exists $\gamma_{I,h} \in F$ satisfying

$$(4) \quad \gamma_{I,h}(x) = x + h$$

for $x \in I$.

3. A SERIES OF REPRESENTATIONS OF THE GROUP F

The Lebesgue measure on $[0, 1]$ is quasi-invariant under the action of F , which allows us to consider a series of representations π_s of F , given by formula (1). In this section we will show that they are irreducible and pairwise inequivalent, except for trivial cases.

3.1. Restrictions on intertwining operators. For $0 \leq a < b \leq 1$ we define $F_{a,b}$ as the subgroup of F consisting of homeomorphisms which are identity outside $[a, b]$, and $P_{a,b}$ as the orthogonal projection onto the space $L^2([a, b])$, treated as a subspace of $L^2([0, 1])$. Finally, for $\phi \in L^\infty([0, 1])$, the corresponding multiplication operator on $L^2([0, 1])$ is denoted by M_ϕ , and the algebra of all such operators by \mathcal{M} . It is a maximal commutative operator algebra on $L^2([0, 1])$ (e.g. [5], Proposition 4.7.6), hence $\mathcal{M} = \mathcal{M}'$. Moreover, \mathcal{M} is equal to the von Neumann algebra generated by the operators $P_{a,b}$.

Lemma 3. Fix $s \in \mathbb{R}$. If $f \in L^2([0, 1])$ satisfies $\pi_s(\gamma)f = f$ for all $\gamma \in F_{a,b}$, then $f|_{[a,b]} = 0$.

Proof. If $f|_{[a,b]} \neq 0$, then there exist dyadic rational numbers $c < d$ such that $[c, d] \subset (a, b)$ and $f|_{[c,d]} \neq 0$. By Lemma 2, for a positive integer n there exists $\gamma_n \in F_{a,b}$ such that $\gamma_n(c) = c$ and $\gamma_n|_{[c,d]}$ is linear with slope 2^{-n} .

We have

$$(5) \quad \int_c^d \left| \pi_s(\gamma_n^{-1})f \right|^2 = \int_c^{c+2^{-n}(d-c)} |f|^2 \xrightarrow{n \rightarrow \infty} 0,$$

and therefore $f \neq \pi_s(\gamma_n^{-1})f$ for some n . \square

Lemma 4. For any $s, t \in \mathbb{R}$ the space $\text{Int}(\pi_s, \pi_t)$ of intertwining operators is contained in \mathcal{M} .

Proof. From Lemma 3 it follows that any $T \in \text{Int}(\pi_s, \pi_t)$ commutes with the projections $P_{a,b}$. We therefore have

$$(6) \quad \mathcal{M} = \{P_{a,b} : 0 \leq a < b \leq 1\}'' \subseteq \text{Int}(\pi_s, \pi_t)',$$

so $\text{Int}(\pi_s, \pi_t) \subseteq \text{Int}(\pi_s, \pi_t)'' \subseteq \mathcal{M}' = \mathcal{M}$. \square

3.2. Irreducibility and non-equivalence.

Proposition 5. The representations π_s are irreducible for all $s \in \mathbb{R}$.

Proof. Let $T \in \text{Int}(\pi_s, \pi_s)$. It follows from Lemma 4 that there exists $\phi \in L^\infty([0, 1])$ such that $T = M_\phi$. The condition $T\pi_s(\gamma)1 = \pi_s(\gamma)T1$ implies

$$(7) \quad \phi = \phi \circ \gamma^{-1}$$

for every $\gamma \in F$. This is satisfied in particular by the elements $\gamma_{1,h}$, so by Lemma 1 the function ϕ is constant and T is a scalar operator. \square

Proposition 6. The representations π_s and π_t are unitarily equivalent if and only if $s - t = 2k\pi / \log 2$ for some $k \in \mathbb{Z}$.

Proof. If $s - t = 2k\pi / \log 2$ then $(d\gamma_*\mu/d\mu)^{1/2+is} = (d\gamma_*\mu/d\mu)^{1/2+it}$ and the representations are equivalent.

Now, suppose that there exists a unitary intertwining operator $U \in \text{Int}(\pi_s, \pi_t)$. Then $U = M_\phi$, where $\phi \in L^\infty([0, 1])$, and

$$(8) \quad \frac{\phi \circ \gamma^{-1}}{\phi} = \left(\frac{d\gamma_*\mu}{d\mu} \right)^{i(s-t)}$$

for every $\gamma \in F$. Putting $\gamma_{I,h}$ into (8) we get $\phi(x) = \phi(x+h)$ almost everywhere on I . Therefore, by Lemma 1 the function ϕ is constant. From (8) it now follows that $(d\gamma_*\mu/d\mu)^{i(s-t)}$ is constant for all $\gamma \in F$, which is possible only if $s-t = 2k\pi/\log 2$ for some $k \in \mathbb{Z}$. \square

4. REPRESENTATIONS INDUCED FROM STABILIZERS OF POINTS

4.1. Irreducibility and nonequivalence criterion. Let $H_1, H_2 \leq G$ be discrete groups, and let σ_i be an irreducible finite-dimensional unitary representation of H_i . Denote $\rho_i = \text{Ind}_{H_i}^G \sigma_i$. The following theorem is a slightly weaker version of Theorems 1–3 in [4]:

Theorem 7.

- (1) If $[H_1 : H_1 \cap gH_1g^{-1}] = \infty$ for every $g \in G - H_1$, then ρ_1 is irreducible.
- (2) If $H_1 = H_2$, and $[H_1 : H_1 \cap gH_1g^{-1}] = \infty$ for every $g \in G - H_1$, then ρ_1 and ρ_2 are inequivalent whenever σ_1 is not equivalent to σ_2 .
- (3) If $[H_1 : H_1 \cap gH_2g^{-1}] = \infty$ for every $g \in G$, then ρ_1 and ρ_2 are not equivalent.

4.2. Representations induced from stabilizers of points. For $p \in (0,1)$ let F_p denote the stabilizer of the point p under the action of F . Notice that if $p \notin \mathbb{Z}[1/2]$, the map

$$(9) \quad \gamma \mapsto (\log_2 \gamma'_+(0), \log_2 \gamma'(p), \log_2 \gamma'_-(1))$$

is a homomorphism from F_p onto \mathbb{Z}^3 (for $p \in \mathbb{Z}[1/2]$ we just take two one-sided derivatives at p , obtaining a homomorphism onto \mathbb{Z}^4). This homomorphism allows us to construct a large family of characters of F_p .

Lemma 8. Let $p, q \in (0,1)$.

- (1) If p and q belong to different orbits of F , then $[F_p : F_p \cap \gamma F_q \gamma^{-1}] = \infty$ for every $\gamma \in F$.
- (2) $[F_p : F_p \cap \gamma F_p \gamma^{-1}] = \infty$ for every $\gamma \in F - F_p$.

Proof. The group $F_p \cap F_q$ is the stabilizer of q under the action of F_p , hence the index $[F_p : F_p \cap F_q]$ equals the cardinality of the orbit $F_p q$. It follows from Lemma 2 that $|F_p q| = \infty$ for any $q \notin \{0, p, 1\}$.

If p and q belong to different orbits of F , then $p \neq \gamma(q)$ for any $\gamma \in F$, and if $\gamma \in F - F_p$, then $p \neq \gamma(p)$, hence both conclusions follow. \square

If H is a subgroup of a discrete group G , and σ is a unitary representation of gHg^{-1} , where $g \in G$, we define a unitary representation σ^g of H by $\sigma^g(x) = \sigma(gxg^{-1})$. It is irreducible if and only if σ is irreducible. Moreover, it is not hard to check that the induced representations $\text{Ind}_{gHg^{-1}}^G \sigma$ and $\text{Ind}_H^G \sigma^g$ are equivalent.

Proposition 9. Let $p, q \in (0,1)$, and let σ_p and σ_q be irreducible finite dimensional unitary representations of F_p and F_q , respectively. Then the representations $\text{Ind}_{F_p}^F \sigma_p$ and $\text{Ind}_{F_q}^F \sigma_q$ are irreducible. Moreover, they are equivalent if and only if there exists $\gamma \in F$ such that $q = \gamma(p)$ and σ_p is equivalent to σ_q^γ .

Proof. Irreducibility and nonequivalence in case when p and q are in different orbits of F follow directly from Theorem 7 and Lemma 8. Now suppose that $q = \gamma(p)$ for some $\gamma \in F$. Then $\text{Ind}_{F_q}^F \sigma_q$ is equivalent to $\text{Ind}_{F_p}^F \sigma_q^\gamma$. But $\text{Ind}_{F_p}^F \sigma_q^\gamma$ is equivalent to $\text{Ind}_{F_p}^F \sigma_p$ if and only if σ_q^γ and σ_p are equivalent. \square

4.3. The representations π_s are not induced from finite-dimensional representations of stabilizers of points.

Proposition 10. *Let $p \in (0, 1)$ and let σ be a finite-dimensional unitary representation of F_p . Then for all $s \in \mathbb{R}$ the representations $\text{Ind}_{F_p}^F \sigma$ and π_s are inequivalent.*

Proof. The restriction $\text{Res}_{F_p}^F \text{Ind}_{F_p}^F \sigma$ contains σ . Therefore it suffices to prove that the restriction $\text{Res}_{F_p}^F \pi_s$ does not contain a finite-dimensional subrepresentation.

The spaces $L^2([0, p])$ and $L^2([p, 1])$ are $\text{Res}_{F_p}^F \pi_s$ -invariant. We will show that $L^2([0, 1]) = L^2([0, p]) \oplus L^2([p, 1])$ is a decomposition of $\text{Res}_{F_p}^F \pi_s$ into irreducible representations. Let $\rho = (\text{Res}_{F_p}^F \pi_s)|_{L^2([0, p])}$, and take $T \in \text{Int}(\rho, \rho)$. If $[a, b] \subseteq [0, p]$, then $F_{a,b} \leq F_p$, and by Lemma 3 the operator T commutes with the projection $P_{a,b}|_{L^2([0, p])}$. Hence, by the same argument as in the proof of Lemma 4, T is a multiplication operator given by a function $\phi \in L^\infty([0, p])$. Proceeding as in the proof of Proposition 5 we show that ϕ is constant. In the same way we show that $(\text{Res}_{F_p}^F \pi_s)|_{L^2([p, 1])}$ is irreducible. Both these representations are infinite-dimensional, so their sum can't contain a finite-dimensional representation. \square

5. REPRESENTATIONS OF THE GROUP T

The pushforward of the Lebesgue measure on $[0, 1]$ through the map η from Subsection 2.1 is quasi-invariant under the action of T , hence we can define representations ρ_s of T through formula (1). The map η induces a unitary isomorphism of $L^2([0, 1])$ and $L^2(S^1)$, which establishes equivalence of the representations π_s and $\text{Res}_F^T \rho_s$. We immediately obtain the following proposition:

Proposition 11. *The representations ρ_s of T are irreducible. The representations ρ_s and ρ_t are unitarily equivalent if and only if $s - t = 2k\pi / \log 2$ for some $k \in \mathbb{Z}$.*

For $p \in S^1$ denote by T_p the stabilizer of p under the action of T . By repeating the arguments of Section 4, one can show the following analog of Proposition 9 for the group T :

Proposition 12. *Let $p, q \in S^1$, and let σ_p and σ_q be irreducible finite dimensional unitary representations of T_p and T_q , respectively. Then the representations $\text{Ind}_{T_p}^T \sigma_p$ and $\text{Ind}_{T_q}^T \sigma_q$ are irreducible. Moreover, they are equivalent if and only if there exists $\gamma \in T$ such that $q = \gamma(p)$ and σ_p is equivalent to σ_q^γ .*

Besides the stabilizers of points of S^1 , there exists one more natural subgroup of T , namely the group of rotations R with angles in $\pi\mathbb{Z}[1/2]$.

Lemma 13.

- (1) $[R : R \cap \gamma T_p \gamma^{-1}] = \infty$ for every $\gamma \in T$ and $p \in S^1$.
- (2) $[R : R \cap \gamma R \gamma^{-1}] = \infty$ for every $\gamma \in T - R$.

Proof. Since $\gamma T_p \gamma^{-1} = T_{\gamma(p)}$, and R acts freely on S^1 , we obtain (1). For the proof of (2) consider the action of T on $L^2(S^1)$ through the representation ρ_0 . Then R is the stabilizer T_1 of the function $1 \in L^2(S^1)$, and $\gamma R \gamma^{-1}$ equals the stabilizer T_f of the function $f = \rho_0(\gamma)1 = (d\gamma_*\mu/d\mu)^{1/2}$. For $\gamma \in T - R$ it is a non-constant simple function on S^1 , hence its orbit under the action of R is infinite, and so is the index $[R : R \cap \gamma R \gamma^{-1}]$. \square

Proposition 14. *Take $p \in S^1$ and let ρ and σ be finite-dimensional unitary representations of R and T_p , respectively. Then*

- (1) the representation $\text{Ind}_R^T \rho$ is irreducible,
- (2) the representations $\text{Ind}_R^T \rho$ and $\text{Int}_{T_p}^T \sigma$ are inequivalent,
- (3) for any $s \in \mathbb{R}$ the representations ρ_s and $\text{Ind}_R^T \rho$ are inequivalent.

Proof. The proofs of (1) and (2) follow from Theorem 7 and Lemma 13. To prove (3) it suffices to show that $\text{Res}_{T_p}^T \rho_s$ does not contain a finite-dimensional representation. But, treating F as a subgroup of T we have

$$(10) \quad \text{Res}_{F_p}^{T_p} \text{Res}_{T_p}^T \rho_s = \text{Res}_{F_p}^F \pi_s,$$

and we already know that this restriction, and therefore also $\text{Res}_{T_p}^T \rho_s$, do not contain a finite-dimensional representation. \square

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