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Boundary representations of hyperbolic groups

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I dedicate this thesis to the memory of my mother Eugenia

ABSTRACT

To any Gromov hyperbolic group Γ one can associate its boundary—a topological space $\partial\Gamma$ with an action of Γ by homeomorphisms. The boundary of Γ can be endowed with an additional structure of a measure metric space, depending on the choice of a left-invariant hyperbolic metric d on Γ, quasi-isometric to the word metric. The measure μ_d arising from this construction is called the Patterson-Sullivan measure. It is quasi-invariant under the action of Γ, which allows one to define a unitary representation π_d of the group Γ on the space $L^2(\partial\Gamma, \mu_d)$ by the formula

$$[\pi_d(g)f](\xi) = \left[\frac{dg_*\mu_d}{d\mu_d}(\xi)\right]^{1/2} f(g^{-1}\xi).$$

We call it the *boundary representation associated to d*.

The main theorem of this thesis states that the boundary representations are irreducible. This gives an explicit faithful irreducible unitary representation (and in fact, many such representations) of an arbitrary hyperbolic group. It can also be seen as a generalization of the standard fact that the Patterson-Sullivan measures are ergodic. Such result was previously known, due to Bader and Muchnik, in the case where Γ is the fundamental group of a closed negatively curved manifold, endowed with a metric induced by an orbit map of the action of Γ on the universal cover.

The second result of the thesis deals with classification of the boundary representations up to unitary equivalence. The considered family of metrics on Γ can be equipped with a natural equivalence relation of *rough similarity*—two metrics are roughly similar if one of them is similar to a bounded perturbation of the other. The Patterson-Sullivan measures coming from roughly similar metrics are equivalent, and this translates into unitary equivalence of the corresponding boundary representations. We manage to turn this implication into an equivalence, under an additional assumption of double ergodicity of the involved Patterson-Sullivan measures. This assumption is likely to be automatically satisfied—Uri Bader and Alex Furman intend to publish the full proof in their forthcoming paper.

Streszczenie

Każdej grupie Γ hiperbolicznej w sensie Gromowa można przypisać przestrzeń topologiczną $\partial\Gamma$, na której Γ działa przez homeomorfizmy, zwaną brzegiem Γ. Brzeg można wyposażyć w metrykę oraz miarę, które zależą od wyboru niezmienniczej, hiperbolicznej metryki d na grupie Γ, quasi-isometrycznej z metryką słów. Powstała w tej konstrukcji miara μ_d nazywana jest miarą Pattersona-Sullivana. Jest ona quasi-niezmiennicza ze względu na działanie Γ, co pozwala zdefiniować reprezentację unitarną π_d grupy Γ na przestrzeni Hilberta $L^2(\partial\Gamma, \mu_d)$ wzorem

$$[\pi_d(g)f](\xi) = \left[\frac{dg_*\mu_d}{d\mu_d}(\xi)\right]^{1/2} f(g^{-1}\xi).$$

Reprezentacje te nazywamy *reprezentacjami brzegowymi*.

Główne twierdzenie rozprawy głosi, iż reprezentacje brzegowe są nieprzywiedlne. Daje to jawnie opisane, wierne reprezentacje nieprzywiedlne dowolnej grupy hiperbolicznej. Wynik ten można również traktować jako uogólnienie standardowego faktu ergodyczności działania grupy hiperbolicznej na jej brzegu z miarą Pattersona-Sullivana. Rezultat tego typu był dowiedziony wcześniej przez Badera i Muchnika dla grup podstawowych zamkniętych rozmaitości ujemnie zakrzywionych, z metrykami indukowanymi przez bijekcje z orbitami w nakryciach uniwersalnych.

Drugi z wyników to klasyfikacja reprezentacji brzegowych z dokładnością do unitarnej równoważności. Rodzina rozważanych przez nas metryk na Γ jest wyposażona w naturalną relację równoważności, generowaną przez podobieństwa i równości z dokładnością do ograniczonego zaburzenia (tj. quasi-izometrie ze stałą multyplikatywną równą 1). Miary Pattersona-Sullivana pochodzące od równoważnych metryk są równoważne, co tłumaczy się na unitarną równoważność reprezentacji brzegowych. Zamieniamy tę implikację w równoważność przy dodatkowym założeniu podwójnej ergodyczności odpowiednich miar Pattersona-Sullivana. Założenie to jest prawdopodobnie automatycznie spełnione; Uri Bader i Alex Furman zamierzają opublikować dowód w nadchodzącej pracy.

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Contents

Chapter 1

Introduction

Any action of a group G on a measure space (X, μ) preserving the class of μ induces an action on the space of measurable functions on X. It can be normalized to obtain a unitary representation of G on $L^2(X, \mu)$. This construction generalizes the notion of a quasi-regular representation, which we obtain when X is a homogeneous space for G; we will still refer to these generalized representations as quasi-regular.

Irreducibility of such quasi-regular representations is a mixing condition stronger than ergodicity. Indeed, for non-ergodic actions the space $L^2(X, \mu)$ decomposes into spaces of functions supported on the nontrivial invariant sets, and any ergodic action of an abelian group, such as the action of $\mathbb{Z}$ on the circle by powers of an irrational rotation, gives a reducible representation. There are many natural examples of actions yielding irreducible quasi-regular representations:

- the natural action of the group of diffeomorphisms of a manifold M, or some of its subgroups preserving additional structure on M [30, 22],
- the action of the Thompson's groups F and T on the unit interval and the unit circle [21],
- the action of a lattice in a Lie group on its Furstenberg boundary [4, 15],
- the action of the automorphism group of a regular tree on its boundary [17],
- the action of a free group on its boundary [18, 19],

- the action of the fundamental group of a compact strictly negatively curved Riemannian manifold M on the boundary of the universal cover of M [3].

The exact relationship between irreducibility of the quasi-regular representation and the dynamical properties of the action of G on X is fully understood only in the case of discrete groups acting on discrete spaces [5, 11, 14]. The genuine quasi-regular representations are also better understood, via the notion of imprimitivity system [27]. For general locally compact groups, irreducibility of the quasi-regular representations was conjectured in [3] for another broad class of actions.

Conjecture. *For a locally compact group G and a spread-out probability measure μ on G, the quasi-regular representation associated to the action of G on the μ-boundary of G is irreducible.*

In this work we study the representations of hyperbolic groups associated with actions on their Gromov boundaries endowed with the Patterson-Sullivan measures. Following [3], we call them *boundary representations*. Our main result states that they are always irreducible. Moreover, when the metric on the group is quasi-isometrically perturbed, the class of the Patterson-Sullivan measure varies, thus leading to a potentially vast supply of non-equivalent irreducible representations. Indeed, under an additional assumption of double ergodicity, we show that the only unitary equivalences between the boundary representations arise from rough similarities of the corresponding metrics. Our results thus generalize the work of Bader and Muchnik [3].

The assumption of double ergodicity is most likely automatically satisfied. Unfortunately, no written proof exists in the generality of our paper. Uri Bader and Alex Furman are planning to include the proof of a stronger result of double metric ergodicity of the Patterson-Sullivan measures in their forthcoming article [2]. They explained the ideas of the proof to us. Unfortunately, we were not able to fill in all the technical details and produce a complete proof before submitting the thesis. We want to emphasize that our main result—irreducibility of the boundary representations—does not rely on double ergodicity, and is proven in full detail.

The irreducibility of the quasi-regular representations can also be seen in a slightly different light. As far as we know, this is the first general construction of a family of faithful irreducible unitary representations of an arbitrary hyperbolic group. In general, providing such constructions for large classes of groups, for which there is no structural description allowing to reduce the problem to some better understood cases, seems to be a difficult task.

The line of our proof can be said to lie within bounded distance from the arguments of Bader and Muchnik, which we generalize to the setting of arbitrary hyperbolic groups, circumventing some of the difficulties they had to deal with. Basically, we construct a family of operators in the von Neumann algebra of the representation, analogous to the operators used in their approach. However, since they try to obtain them as weak operator limits of some arithmetic averages, in order to prove convergence they need to resort to a result of Margulis, describing the asymptotic behavior of the number of certain geodesic segments in a manifold. By using weighted averages and choosing suitable weights, we omit the necessity of knowing such asymptotics, and obtain a more self-contained and simpler proof, applicable in a wider context.

Recently, Uri Bader has informed us about an unpublished work of Roman Muchnik, establishing irreducibility of quasi-regular representations of hyperbolic groups associated with their actions on Poisson boundaries of finitely supported symmetric random walks. This is also a special case of our result, which we explain in Section 3.7.2.

The core of the thesis is Chapter 3, which can be read as a self-contained article, independent from the remainder of the thesis. As such, it assumes familiarity with hyperbolic spaces and groups, their boundaries, visual metrics, and Patterson-Sullivan measures, at least in the geodesic context. It is supplemented by Chapter 2, containing an informal introduction to these subjects. We refrained ourselves from writing proofs of all the standard results, as they already can be found in many places. Instead, we tried to provide all the necessary intuitions, accompanied by references to complete proofs.

The text of Chapter 3 is organized as follows. In Sections 3.1 and 3.2 we introduce some notational conventions and definitions from geometric group theory, discuss some basic results concerning hyperbolic groups and their boundaries, and finally define the class of representations we are going to consider. All the geometry is contained in Section 3.3, where we explore some subsets of the group, estimate their growth and show that they are nicely distributed. Section 3.4 uses these estimates to construct certain operators in the von Neumann algebras of the boundary representations. In Section 3.5 we gather all the previous results into the proof of irreducibility of the boundary representations. We also explain why they are weakly contained in the regular representation. Section 3.6 contains the classification of the boundary representations with respect to unitary equivalence, using the assumption of double ergodicity of the Patterson-

Sullivan measures. Finally, in Section 3.7 we discuss two examples with more explicitly defined groups and metrics. We have a closer look at the case of fundamental groups of negatively curved manifolds, previously studied by Bader and Muchnik, and we also explain how the conjecture mentioned in the Introduction follows for a certain class of random walks on a hyperbolic group.

Chapter 2

Prelude: hyperbolic groups

Let X be a metric space. A geodesic segment (ray, line) in X is an isometric embedding of a segment (resp. ray, line) into X. By abuse of terminology one usually does not distinguish between such a map and its image. The space X is geodesic if any two points in X can be joined by a geodesic segment. Any connected Riemannian manifold is an example of a geodesic space.

The notion of Gromov hyperbolicity is an attempt to generalize the idea of negative curvature (more specifically, pinched sectional negative curvature) of Riemannian manifolds to the setting of geodesic metric spaces (and even further). In order to achieve this goal one must find properties of negatively curved manifolds, which are possible to formulate in the language of geodesic metric spaces. One of such properties is the *slim triangle condition*. A geodesic triangle, i.e. a triple of points together with a choice of geodesic segments joining them, is said to be *δ-slim* for some $\delta \geq 0$ if each of its sides lies in the δ-neighborhood of the union of the remaining two sides. This condition turns out to be enough to define the notion of *δ-hyperbolicity*: a metric space X is δ-hyperbolic if any geodesic triangle in X is δ-slim. From now on, we will assume that X is δ-hyperbolic.

There are other definitions of δ-hyperbolicity, and we will explore one of them further in order to drop the assumption of geodesicity. They are all equivalent, but with different values of δ. Hence, one should not grow too attached to the precise value of δ. It is the existence of δ that matters, except one particular case: if $\delta = 0$, then X is a generalized tree. Thus, one usually drops δ from the notation and speaks just about hyperbolic spaces.

From the formulation of the slim triangle condition it immediately follows that it is invariant under bounded perturbations, at the expense of changing δ. The class of geodesic hyperbolic metric spaces is in fact invariant under mappings called *quasi-isometries* [10, Theorem III.H.1.9], which are bounded perturbations of bi-Lipschitz maps. More precisely, a map $\phi\colon X \to Y$ is a quasi-isometry, if it satisfies

$$\frac{1}{L} d_X(p,q) - C \leq d_Y(\phi(p), \phi(q)) \leq L d_X(p,q) + C \tag{2.1}$$

for some $L \geq 1$ and $C \geq 0$, and there exists a map $\psi\colon Y \to X$, playing the role of an "approximate inverse" of ϕ, satisfying a similar estimate, such that $d_X(p, \psi\phi(p))$ and $d_Y(q, \phi\psi(q))$ are uniformly bounded functions. It is worth noting that this invariance implies that the notion of hyperbolicity is insensitive to local structure of the space.

An important feature of hyperbolic spaces is the behavior of geodesic rays. Two geodesic rays γ_1 and γ_2 starting at the same point are either asymptotic—the distance $d(\gamma_1(t), \gamma_2(t))$ remains bounded by 2δ—or after an initial period of traveling along each other they spread apart and travel along a third geodesic in opposite directions. Before explaining this in more detail it is convenient to discuss the notion of the Gromov boundary, which offers a better view on the situation.

Let us first recall a variant of the Arzela-Ascoli Theorem, namely if X and Y are metric spaces with X separable and Y compact, then every sequence of equicontinuous mappings $f_n\colon X \to Y$ has a subsequence that converges uniformly on compact subsets to a continuous map $f\colon X \to Y$. We will use this fact in the following context. Let $X = [0, \infty)$, and let Y be a locally compact (not necessarily compact!) metric space. A *generalized geodesic ray* $\gamma\colon [0, \infty) \to Y$ is either a geodesic ray or a geodesic segment extended by a constant map after its endpoint, i.e. a map such that $\gamma|_{[0,t_0]}$ is a geodesic segment and $\gamma(t) = \gamma(t_0)$ when $t > t_0$ for some t_0. If f_n is a sequence of generalized geodesic rays, then it is equicontinuous, as it consists of 1-Lipschitz functions. Moreover, if the initial points $f_n(0)$ are contained in a bounded area, then the images of restrictions $f_n|_{[0,t_0]}$ lie in a bounded, and hence compact, ball. Therefore it is possible to apply the Arzela-Ascoli theorem piecewise and conclude that the sequence f_n has a convergent subsequence.

Now we will define the boundary of a locally compact geodesic hyperbolic space X. First, choose a basepoint $o \in X$. The set Ω_o of all geodesic rays in X emanating from o can be given the compact-open topology. We define two geodesic rays $\gamma_1, \gamma_2 \in \Omega_o$ to be asymptotic if the function $d(\gamma_1(t), \gamma_2(t))$ is bounded. A simple application of the slim triangle condition shows that in this case the bound can always be taken to be 2δ. The boundary $\partial_o X$ of X is defined as the quotient space of Ω_o by the relation of asymptoticity. It can be seen as the space of directions visible from the point o, or as a way to compactify X by completing every geodesic ray starting at o with an endpoint in a natural way.

The boundary can also be defined without using a basepoint. Simply instead of considering the space Ω_o we take the whole space Ω of geodesic rays in X with the relation of asymptoticity, and define the boundary ∂X. The inclusion $\Omega_o \subseteq \Omega$ induces a continuous injection $\partial_o X \to \partial X$ of the corresponding boundaries. It is actually a bijection: if γ is a geodesic ray representing the element $[\gamma] \in \Omega$, then we may consider a sequence of generalized geodesic rays γ_n, joining o to $\gamma(n)$. By the slim triangle condition, after some initial period the rays γ_n travel along γ. Hence, the limit of a convergent subsequence of γ_n is a geodesic ray starting at o and asymptotic to γ.

It remains to observe that $\partial_o X$ is compact, and hence the natural bijections between different boundaries are homeomorphisms. Since sequential compactness follows from the Arzela-Ascoli Theorem, it is enough to show that $\partial_o X$ is first countable. This can be done by observing that for $[\gamma] \in \partial_o X$ the sets

$$V_n = \{[\gamma'] \in \partial_o X : d(\gamma(t), \gamma'(t)) \leq 2\delta \text{ for } t < n\} \tag{2.2}$$

form a basis of (not necessarily open) neighborhoods of $[\gamma]$.

An important point in the above discussion is the following visibility property of the boundary ∂X. For any $x \in X$ and $\xi \in \partial X$ there exists a geodesic ray starting at x and ending at ξ. Every point of the boundary is "visible" from every point of X. This is just a reformulation of the earlier statement that the spaces ∂X and $\partial_x X$ are homeomorphic. Even more is true. Every point of the boundary is visible from any other point in ∂X, i.e. for any $\xi, \eta \in \partial X$ there exists a geodesic γ in X with endpoints ξ and η. It is obtained as a limit point of the sequence of geodesics joining $\gamma_1(n)$ with $\gamma_2(n)$, where γ_1 and γ_2 are geodesic rays with a common initial point, representing ξ and η.

We may now return to the remark about behavior of geodesic rays. If γ_1 and γ_2 are geodesic rays starting at o, with endpoints ξ_1 and ξ_2, there exists a geodesic γ joining ξ_1 with ξ_2. After an initial period of going together, the rays γ_i start traveling along γ in opposite directions.

Hyperbolicity has another definition, which does not depend on the notion of a geodesic. First, in any metric space one can define the *Gromov product* of two points x, y with respect to a basepoint o as

$$(x, y)_o = \frac{1}{2}(d(o, x) + d(o, y) - d(x, y)). \tag{2.3}$$

Then, the space X is hyperbolic if and only if its Gromov product satisfies the inequality

$$(x, y)_o \geq \min\{(x, z)_o, (z, y)_o\} - \delta \tag{2.4}$$

for all $x, y, z, o \in X$ and some $\delta \geq 0$, not necessarily the same as in the slim triangle condition. The proof of equivalence passes through some more conditions for hyperbolicity, and can be found in [10, Chapter III.H, Propositions 1.17 and 1.22].

Better understanding of the Gromov product can shed some more light on this definition. Suppose that γ and γ' are geodesic segments starting at $o \in X$ with endpoints x, y. If $t < (x, y)_o$, double application of inequality (2.4) yields

$$(\gamma(t), \gamma'(t)) \geq \min\{(\gamma(t), x)_o, (x, y)_o, (y, \gamma'(t))_o\} - 2\delta = t - 2\delta, \tag{2.5}$$

which after expanding the definition of the Gromov product on the left transforms into

$$d(\gamma(t), \gamma'(t)) \leq 4\delta. \tag{2.6}$$

In other words, the Gromov product of x and y with respect to o measures how long the geodesic segments from o to x and y travel near each other.

The condition (2.4) can now be interpreted intuitively. It means that two geodesic segments emanating from the same point and traveling near each other behave approximately as if they were parallel: if they both travel near a third geodesic segment for some time, they are near each other for at least the same time (approximately, up to δ). Extreme case of this behavior can be observed for a tree, where geodesic segments starting from the same point and traveling sufficiently near each other are actually identical. This can be contrasted with the case of Euclidean plane, where any two rays starting at the same point travel within some fixed distance from their bisector for much longer than they are within the same distance from each other.

Although the hyperbolicity condition based on the Gromov product allows us to consider non-geodesic spaces, in this informal introduction we will stick with the geodesic case. Apart form enabling a new definition of hyperbolicity, the Gromov product can be used to endow the boundary ∂X with extra structure. Before we do this, we need to extend the Gromov product to $X \cup \partial X$. This can be done, although there is no canonical extension, and a continuous extension might not exist. Basically, to define $(\xi, \eta)_o$ for $\xi, \eta \in \partial X$, we consider the lower limit $\liminf_{t\to\infty}(\gamma(t), \gamma'(t))_o$, where γ and γ' are the corresponding representatives, and take supremum over all choices of γ and γ'. We could replace suprema by infima or even use an arbitrary choice of representatives, obtaining different extensions, which nonetheless can be shown to differ by at most a uniform constant, depending only on δ [10, Remark III.H.3.17(5)]. The extended Gromov product of a point in X and a point in ∂X can be defined similarly. Any such extension satisfies condition (2.4) with adjusted value of δ, and we may just choose and fix one of them.

Recall that neighborhoods of a point $\xi = [\gamma] \in \partial X$ can be defined as sets consisting of the endpoints η of geodesic rays traveling near γ for sufficiently long. This amounts to the Gromov product (ξ, η) (we will omit the basepoint from notation) being sufficiently large. Since we may now quantify for how long two geodesic rays travel along each other, we should be able to define a metric on the boundary.

If X is a tree with a basepoint o, we may identify its boundary with the set of all geodesic rays starting at o. The Gromov product of two geodesic rays is just the length of their common initial segment. Then for any $\epsilon > 0$ the function

$$\rho_\epsilon(\xi, \eta) = e^{-\epsilon(\xi,\eta)} \tag{2.7}$$

is a metric. For a general hyperbolic space we could try to use the same formula. Unfortunately, there is no reason for ρ_ϵ to satisfy the triangle condition. Instead, the following weak variant of the ultrametric inequality is satisfied.

$$\rho_\epsilon(\xi, \eta) \leq e^{\epsilon\delta} \max\{\rho_\epsilon(\xi, \zeta), \rho_\epsilon(\zeta, \eta)\}. \tag{2.8}$$

There is however a general procedure to turn such a function into a metric. Namely, we can use ρ_ϵ to measure lengths of "polygonal curves" in ∂X. Such a curve is just a finite sequence of points, and its ρ_ϵ-length is the sum of ρ_ϵ-distances between consecutive pairs of points. We may define $d_\epsilon(\xi, \eta)$ as the infimum of ρ_ϵ-lengths of all polygonal curves joining ξ to η. This always gives a pseudo-metric satisfying $d_\epsilon(\xi, \eta) \leq \rho_\epsilon(\xi, \eta)$. It can be proved that if ϵ is sufficiently small, there exists a constant $C > 0$ such that

$$C\rho_\epsilon(\xi, \eta) \leq d_\epsilon(\xi, \eta) \tag{2.9}$$

for all ξ and η [10, Proposition III.H.3.21], and in particular d_ϵ is a metric compatible with the topology of ∂X. Such metrics are called *visual metrics*.

We have thus turned the boundary of a hyperbolic space into a metric space. There is now a canonical way to construct a measure, which at some level resembles the process of turning ρ_ϵ into a metric. We begin by fixing $\alpha \geq 0$, the "dimension" of our measure. Now, we can think of balls of radius r as having "volume" r^α. This by no means extends to a measure on the boundary, but we may counter this as follows. Let $E \subseteq \partial\Gamma$ be a Borel subset. For any $\theta > 0$ we may cover it by countably many balls $B(\xi_i, r_i)$ with radii $r_i < \theta$. We may therefore consider the infimum of the sum of the "volumes" of the balls,

$$\inf \Big\{ \sum_i r_i^\alpha : E \text{ is covered by balls } B(\xi_i, r_i) \text{ with } r_i < \theta \Big\}. \tag{2.10}$$

When we decrease θ, we restrict the family of covers, and thus the infimum increases. It turns out that its limit with $\theta \to 0$ defines a measure $\mathcal{H}^\alpha$ on $\partial\Gamma$, called the α-dimensional Hausdorff measure [29]. It is an easy observation that if for some α we have $\mathcal{H}^\alpha(E) < \infty$ then $\mathcal{H}^\beta(E) = 0$ for $\beta > \alpha$, and similarly, if $\mathcal{H}^\alpha(E) > 0$ then $\mathcal{H}^\beta(E) = \infty$ for $\beta < \alpha$. The natural measure to consider on ∂X is the Hausdorff measure corresponding to the critical dimension D such that $\mathcal{H}^\beta(\partial X) = \infty$ for $\beta < D$ and $\mathcal{H}^\beta(\partial X) = 0$ for $\beta > D$—if such D, called the Hausdorff dimension of ∂X, exists, and the corresponding measure is well-behaved.

Let us finish the discussion of Hausdorff measures in general, by returning to our standard example—an infinite tree X with base vertex o. If we consider X to be rooted at o, we may use the term *children* for the set of neighbors of a given vertex v, which are separated from the root by v. We will also denote by $|v|$ the distance $d(o, v)$. For the sake of simplicity, we will assume X to be regular of degree $q + 1$.

The boundary ∂X (which we will again identify with the set of geodesic rays starting at o) endowed with the visual metric d_ϵ is an ultrametric space. We will explain in a more detailed way that its Hausdorff dimension is $D = \log q/\epsilon$. Observe that any ball $B(\xi, r)$ consists of rays sharing with ξ an initial segment of length $-\log r/\epsilon$. It is thus equal to ∂X_v, the boundary of the subtree of X rooted at the vertex $v = \xi(\lceil -\log r/\epsilon \rceil)$. Its minimal radius (in this setting the radius and center of a ball are not unique) is $e^{-\epsilon|v|}$, and we have $(e^{-\epsilon|v|})^D = q^{-|v|}$. Any ball ∂X_v can be decomposed into q disjoint balls ∂X_w, corresponding to children of v, with $|w| = |v| + 1$. Moreover, ∂X_v is compact, and hence from any cover with balls we may choose a finite subcover and then refine it to get a disjoint one. This means that for $E = \partial X_v$, the infimum in (2.10), and hence its D-dimensional Hausdorff measure, are equal to $q^{-|v|}$. This fully identifies the measure.

Until now, we were speaking of general hyperbolic spaces, without any mention of groups. The more general approach we could now develop would be to consider groups acting by isometries on hyperbolic spaces. However, in this informal introduction we will stick to a more specialized case. Let Γ be a finitely generated group with a finite symmetric generating set S. We may define the *Cayley graph* of Γ with respect to S, denoted by

$\mathrm{Cay}(\Gamma, S)$ as the graph with vertex set Γ, and undirected edges of the form $\{g, gs\}$ where $s \in S$. It is connected, and hence endowed with the natural path metric d, whose restriction to Γ is referred to as the *word metric*. The Cayley graph has a natural basepoint, the vertex 1, and we denote $|g| = d(1, g)$. The group Γ acts isometrically on $\mathrm{Cay}(\Gamma, S)$ by left multiplication.

Changing the generating set S leads to a quasi-isometric Cayley graph, and thus we may define Γ to be *hyperbolic* if any of its Cayley graphs is. Equivalently, Γ is hyperbolic if it is a hyperbolic space when considered with any of its word metrics. The Švarc-Milnor Lemma says that if Γ acts by isometries, properly, and cocompactly on a geodesic metric space X, then the orbit map $\Gamma \to X$ sending g to gx for some fixed $x \in X$ is a quasi-isometry between Γ (with any of the word metrics) and X. Thus, if X is hyperbolic, so is Γ. It follows that for instance the fundamental group of a negatively curved compact manifold M is hyperbolic, as it acts by deck-transformations on the universal cover of M, which is in particular a hyperbolic geodesic metric space.

The boundary $\partial\Gamma$ of Γ is the boundary of any of the Cayley graphs of Γ. This is a well defined topological space, since quasi-isometries can be shown to induce homeomorphisms of boundaries [10, Theorem III.H.3.9]. By choosing a generating set S for Γ and sufficiently small $\epsilon > 0$, we fix a visual metric d_ϵ on the boundary $\partial\Gamma$. As exhibited in Theorem 3.6.5, different choices of S can lead to nonequivalent measures on the boundary. Since Γ acts on $\mathrm{Cay}(G, S)$ by isometries, its action induces an action on $\partial\Gamma$ by homeomorphisms. This allows to define a class of measures on $\partial\Gamma$, called quasi-conformal measures, which turn out to be equivalent to the Hausdorff measure.

For $\xi \in \partial\Gamma$ and $g \in \Gamma$ define the *Busemann function*

$$\beta_\xi(g) = |g| - 2(g, \xi). \tag{2.11}$$

This can be defined for arbitrary hyperbolic spaces, and looking at the example of a tree shows that β_ξ can be interpreted as "distance from ξ", renormalized in such a way that $\beta_\xi(o) = 0$ for the basepoint o. Indeed, for two vertices v and w of a tree, the equality $\beta_\xi(v) = \beta_\xi(w)$ holds if and only if the meeting point of geodesic rays starting from v and w, and hitting the boundary at ξ, is equidistant from v and w.

Now, a regular Borel measure μ on $\partial\Gamma$ is said to be *quasi-conformal of dimension D* if it is quasi-invariant under the action of Γ and the Radon-Nikodym derivatives of its push-forwards satisfy the estimate

$$\frac{1}{C} e^{-\epsilon D \beta_\xi(g)} \leq \frac{d g_* \mu}{d\mu}(\xi) \leq C e^{-\epsilon D \beta_\xi(g)} \tag{2.12}$$

for all $g \in \Gamma$ and $\xi \in \partial\Gamma$, with a uniform constant C. It can be shown [13, Proposition 7.4] that such a measure satisfies the property called *Ahlfors regularity*, namely for $r \leq \operatorname{diam}(\partial\Gamma)$ the measure of any ball of radius r satisfies the estimate

$$\frac{1}{C'} r^D \leq \mu(B(\xi, r)) \leq C' r^D \tag{2.13}$$

uniformly in ξ and r. From this, it is easy to deduce that such a measure is equivalent to the D-dimensional Hausdorff measure $\mathcal{H}^D$ in a strong sense: there exists a constant $C'' > 0$ such that for any Borel set $E \subseteq \partial\Gamma$

$$\frac{1}{C''} \mu(E) \leq \mathcal{H}^D(E) \leq C'' \mu(E). \tag{2.14}$$

In particular, a finite non-zero quasi-conformal measure on $\partial\Gamma$ may exist only for D equal to the Hausdorff dimension of $\partial\Gamma$. One can exhibit such a measure as follows. Define a measure ν_s supported on Γ by

$$\nu_s = \sum_{g \in \Gamma} e^{-s|g|} \delta_g, \tag{2.15}$$

where δ_g is the Dirac mass at g. Cardinalities of balls in Γ grow at most exponentially fast, and so there exists a critical exponent λ such that $\nu_s(\Gamma) < \infty$ for $s > \lambda$. It can be shown that $\nu_s(\Gamma) = \infty$ for $s \leq \lambda$ [12, Lemma 2.5.1]. We can thus normalize ν_s to a probability measure on the compact space $\Gamma \cup \partial\Gamma$. By weak-* compactness, the normalized measures $\nu_s / \nu_s(\Gamma)$ have a limit point ν as s goes down to λ, supported on $\partial\Gamma$, which turns out to be quasi-conformal of dimension $D = \lambda / \epsilon$ [13, Theorem 5.4].

Chapter 3

Boundary representations

3.1 Preliminaries

In this section we introduce the basic notions associated with hyperbolic spaces and groups. We start by fixing some notational conventions for various kinds of estimates, which we will use throughout the paper in order to avoid the aggregation of non-essential constants and hopefully making the presentation more lucid. Then we introduce the basic terminology related to quasi-isometries, define hyperbolic spaces and groups, and finally, discuss the notion of the Gromov boundary. For details on these subjects see [10, Chapters III.H.1 and III.H.3].

3.1.1 Estimates

In the paper we will work with additive and multiplicative estimates. In order to avoid the escalation of constants coming from such estimates, we will suppress them using the following notation. Let f, g be functions on a set X. If there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all x, we write $f \prec g$. If both $f \prec g$ and $g \prec f$ hold, we write $f \asymp g$. Analogously, for additive estimates, $f \lesssim g$ if there exists c such that $f \leq g + c$, and $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$ hold. The variables in which the estimates are assumed to be uniform will be either clear from context or explicitly mentioned. Sometimes, to indicate that we do not care whether the estimate is uniform in some of the variables (which does not mean that we claim it is not), we write them as subscripts to the symbol of the corresponding estimate, e.g. $f(x,y) \lesssim_x g(x,y)$ need not be uniform in x.

3.1.2 Quasi-isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. Take $L \geq 1$ and $C \geq 0$. A map $\phi\colon X \to Y$ satisfying the condition

$$\frac{1}{L} d_X(p,q) - C \leq d_Y(\phi(p), \phi(q)) \leq L d_X(p,q) + C \tag{3.1.1}$$

for all $p, q \in X$ is called an *(L,C)-quasi-isometric embedding*. If the image of ϕ is a C-net in Y, i.e. its C-neighborhood covers Y, or equivalently, if there exists a quasi-isometric embedding $\psi\colon Y \to X$, called the *quasi-inverse of* ϕ, such that $d_X(x, \psi\phi(x))$ and $d_Y(y, \phi\psi(y))$ are uniformly bounded functions on X and Y respectively, then ϕ is an *(L,C)-quasi-isometry*. A $(1, C)$-quasi-isometry is called a *C-rough isometry*. A quasi-isometry ϕ satisfying $d_Y(\phi(x), \phi(y)) \approx L d_X(x,y)$ with additive constant C is an *(L,C)-rough similarity*.

An (L, C)-quasi-isometric embedding $\gamma\colon \mathbb{R} \to X$ is called an *(L,C)-quasi-geodesic* in X. Similarly one defines quasi-geodesic rays and segments, and their roughly geodesic variants. We say that X is an (L, C)-*quasi-geodesic space*, if any two points in X can be joined by an (L, C)-quasi-geodesic segment. A *C-roughly geodesic space* is defined in the same manner. We will later fix the constants L and C and suppress them from notation.

3.1.3 Hyperbolic spaces and groups

Let (X, d) be a metric space. For any basepoint $o \in X$ one defines the Gromov product $(\cdot, \cdot)_o\colon X \times X \to [0, \infty)$ with respect to o as

$$(x,y)_o = \frac{1}{2}(d(x,o) + d(y,o) - d(x,y)). \tag{3.1.2}$$

A different choice of the basepoint leads to another Gromov product, satisfying

$$\left|(x,y)_o - (x,y)_p\right| \leq d(o,p). \tag{3.1.3}$$

If the Gromov product on X satisfies the estimate

$$(x,y)_o \gtrsim \min\{(x,z)_o, (y,z)_o\} \tag{3.1.4}$$

for some (equivalently, for every—but with a different constant) basepoint $o \in X$, the space X is said to be *hyperbolic*. We may iterate (3.1.4), to obtain

$$(x_1, x_n)_o \gtrsim \min\{(x_1,x_2)_o, (x_2,x_3)_o, \ldots, (x_{n-1}, x_n)_o\}, \tag{3.1.5}$$

with constants depending only on n. The property of being hyperbolic is preserved by quasi-isometries within the class of geodesic spaces. In case of general metric spaces, it is possible to quasi-isometrically perturb a hyperbolic metric and obtain a non-hyperbolic one (see [8, Proposition A.11]).

A finitely generated group Γ is hyperbolic if its Cayley graph with respect to some finite set of generators is hyperbolic. As Cayley graphs of a given group are geodesic and quasi-isometric to each other, this notion does not depend on the generating set. The quasi-isometric metrics induced on the group by the path metrics on its Cayley graphs are called the *word metrics*. We will denote by $\mathcal{D}(\Gamma)$ the class of all hyperbolic left-invariant metrics on Γ (not necessarily coming from an action on a geodesic space), quasi-isometric to a word metric (i.e. the identity map is a quasi-isometry). Finally, a hyperbolic group is *non-elementary* if it does not contain a cyclic subgroup of finite index.

3.1.4 The Gromov boundary

Now, assume that X is hyperbolic and has a fixed basepoint $o \in X$, which we will omit in the notation for the Gromov product. We will also denote $|x| = d(x, o)$. A sequence $(x_n) \subset X$ *tends to* ∞ if

$$\lim_{i,j\to\infty} (x_i, x_j) = \infty. \tag{3.1.6}$$

Two such sequences (x_n) and (y_n) are *equivalent* if $\lim_{n\to\infty}(x_n, y_n) = \infty$. By (3.1.3) these notions are independent of the basepoint. The boundary of X, denoted ∂X, is the set of equivalence classes of sequences tending to infinity. The space $\overline{X} = X \cup \partial X$ can be given a natural topology making it a compactification of X, on which the isometry group $\mathrm{Isom}(X)$ acts by homeomorphisms.

The Gromov product can be extended (in a not necessarily continuous way) to $\overline{X}$ in such a way that the estimate (3.1.4) is still satisfied (with different constants). One simply represents elements of X as constant sequences, and for $x, y \in \overline{X}$ defines

$$(x, y) = \sup \liminf_{i,j\to\infty} (x_i, y_j), \tag{3.1.7}$$

where the supremum is taken over all representatives (x_i) and (y_i) of x and y. A sequence $(x_i) \subset X$ converges to $\xi \in \partial X$ if and only if $(x_i, \xi) \to \infty$, so in particular, representatives of ξ are exactly the sequences in X converging to ξ. By [10, Remark 3.17], we have

$$\liminf_{i,j\to\infty}(x_i, y_j) \approx (x, y) \tag{3.1.8}$$

whenever $x_i \to x$ and $y_i \to y$.

The topology of ∂X is metrizable. For sufficiently small $\epsilon > 0$ there exists a metric d_ϵ on ∂X, compatible with its topology, satisfying

$$d_\epsilon(\xi, \eta) \asymp_\epsilon e^{-\epsilon(\xi,\eta)}. \tag{3.1.9}$$

Such a metric is called a *visual metric*.

We will later use the fact that for a hyperbolic group Γ the only Γ-equivariant homeomorphism ϕ of $\partial\Gamma$ is the identity map. It follows from the fact that any element of Γ of infinite order has exactly one attracting point in $\partial\Gamma$, which is therefore fixed by ϕ, and the attracting points of all such elements form a dense subset [25, Proposition 4.2 and Theorem 4.3].

3.2 The geometric setting

In this section we describe some ways to deal with non-geodesic hyperbolic metrics. As the representations we will consider depend on the metric on the group, this will allow to investigate a class of representations much wider than those obtained from the word metrics. Everything we need in this regard is contained in the papers [8, 9].

In [8] the notions of a quasi-ruler and quasi-ruled space are introduced, and the fundamental properties of the Patterson-Sullivan measures for quasi-ruled hyperbolic spaces, generalizing the results of [13], which apply only to metrics coming from proper actions on geodesic spaces, are developed. The article [9] studies boundaries of almost geodesic hyperbolic spaces, and is a useful reference for some basic lemmas.

It turns out that the classes of hyperbolic quasi-ruled spaces and hyperbolic almost geodesic spaces are the same and equal to the class of roughly geodesic hyperbolic spaces. We discuss the notion of a quasi-ruled space only in order to formulate Theorem 3.2.1. Afterwards, all the arguments will be based on the notion of a rough geodesic.

3.2.1 Roughly geodesic hyperbolic spaces

For $\tau \geq 0$ a *τ-quasi-ruler* is a quasi-geodesic $\gamma\colon \mathbb{R} \to X$ satisfying for all $s < t < u$ the condition

$$(\gamma(s), \gamma(u))_{\gamma(t)} \leq \tau. \tag{3.2.1}$$

The space X is said to be (L, C, τ)*-quasi-ruled* if it is a (L, C)-quasi-geodesic space, and every (L, C)-quasi-geodesic is a τ-quasi-ruler. By [8, Theorem A.1], if $\phi\colon X \to Y$ is a quasi-isometry with X hyperbolic and geodesic, then Y is hyperbolic if and only if it is quasi-ruled. It follows that for a hyperbolic group Γ, all the metrics in the class $\mathcal{D}(\Gamma)$ are quasi-ruled.

By [8, Lemma A.2], for every L, C, and τ there exists $K > 0$ such that every (L, C, τ)-quasi-ruled space is K-roughly geodesic. On the other hand, it is clear that a K-roughly geodesic space is $(1, K, 3K/2)$-quasi-ruled. By [9, Proposition 5.2(1)], the hyperbolic spaces studied therein are also exactly the roughly geodesic hyperbolic spaces.

Now suppose that X is a roughly geodesic hyperbolic space. Every roughly geodesic ray γ in X converges to an endpoint $\gamma(\infty)$ in the boundary. The converse statement is also true, i.e. every point in ∂X is the endpoint of some K-roughly geodesic ray, where K depends only on X [9, Proposition 5.2(2)]. In a similar fashion, every pair of distinct points of ∂X can be joined by a K-roughly geodesic line [9, Proposition 5.2(3)]. From now on, when we use the terms *roughly geodesic segment/ray/line* without specifying the constant, we always think of the universal constants from the definition of a roughly geodesic space and the remark above.

By [9, Proposition 5.5], if the map $\phi\colon X \to Y$ is a (L, C)-quasi-isometry of roughly geodesic hyperbolic spaces, their Gromov products satisfy estimates

$$\frac{1}{L}(x, y)_z \lesssim (\phi(x), \phi(y))_{\phi(z)} \lesssim L(x, y)_z \tag{3.2.2}$$

uniformly for all $x, y, z \in X$. As a consequence, for a hyperbolic group Γ all the metrics in $\mathcal{D}(\Gamma)$ give rise to exactly the same boundary.

3.2.2 Quasi-conformal measures

Consider a roughly geodesic hyperbolic space X with a basepoint $o \in X$, and a non-elementary hyperbolic group $\Gamma \subseteq \mathrm{Isom}(X)$ which acts on X properly and cocompactly. A measure μ on $(\partial X, d_\epsilon)$ is said to be Γ*-quasi-conformal of dimension D* if it is quasi-invariant under the action of Γ, and the corresponding Radon-Nikodym derivatives satisfy the estimate

$$\frac{dg_*\mu}{d\mu}(\xi) \asymp e^{\epsilon D(2(go,\xi)-d(o,go))} \tag{3.2.3}$$

uniformly in ξ and g. Since $d(o, go) \approx_{o,p} d(p, gp)$, this notion is independent of the choice of o. Moreover, being Γ-quasi-conformal does not depend on ϵ, as for different values of ϵ only the dimension D changes. Finally, μ is *Ahlfors regular of dimension D* if it satisfies the estimate

$$\mu(B_{\partial X}(\xi, \rho)) \asymp \rho^D \tag{3.2.4}$$

uniformly in ξ and $\rho \leq \operatorname{diam} \partial X$. In particular, since ∂X is compact, any Ahlfors regular measure on ∂X is finite.

Recall that the Hausdorff measure of a metric space Y is defined as follows. First, for $\alpha \geq 0$ one defines the α-dimensional Hausdorff measure $\mathcal{H}_\alpha$ as

$$\mathcal{H}_\alpha(E) = \lim_{\theta \to 0^+} \inf \left\{ \sum_i (\operatorname{diam} U_i)^\alpha : E \subseteq \bigcup_i U_i \text{ and } \operatorname{diam} U_i \leq \theta \right\} \tag{3.2.5}$$

for every Borel set $E \subseteq Y$. Then, the Hausdorff dimension of Y is the number

$$\dim_H Y = \inf\{\alpha : \mathcal{H}_\alpha(Y) = 0\} = \sup\{\alpha : \mathcal{H}_\alpha(Y) = \infty\}. \tag{3.2.6}$$

The Hausdorff measure on Y is the $(\dim_H Y)$-dimensional Hausdorff measure.

Now, take $x \in X$ and denote

$$D = \limsup_{R \to \infty} \frac{1}{\epsilon R} \log |B_X(x, R) \cap \Gamma x|, \tag{3.2.7}$$

and $\omega = e^{D\epsilon}$. We then have the following.

Theorem 3.2.1 ([8, Theorem 2.3])**.** *Suppose that X is a proper roughly geodesic hyperbolic space, and $\Gamma \subseteq \operatorname{Isom}(X)$ is a non-elementary hyperbolic group, acting properly and cocompactly. Then the Hausdorff dimension of $(\partial X, d_\epsilon)$ is equal to D, defined in* (3.2.7)*, and the corresponding Hausdorff measure μ is Γ-quasi-conformal of dimension D and Ahlfors regular of dimension D. Furthermore, any Γ-quasi-conformal measure μ' on ∂X is equivalent to μ with Radon-Nikodym derivative $d\mu'/d\mu \asymp 1$ a.e., and $|B_X(x, R) \cap \Gamma x| \asymp \omega^R$.*

In particular, this theorem implies that the quasi-conformal measures associated to different choices of ϵ are equivalent, so the above considerations lead to a unique measure class on ∂X (in fact a class of the finer relation of equivalence with Radon-Nikodym derivatives bounded away from 0 and ∞), depending only on the metric d, called the *Patterson-Sullivan class*. Also, by Ahlfors regularity, the boundary has no isolated points.

We say that a measure class preserving action of a group G on a measure space (X, ν) is *doubly ergodic*, if the induced diagonal action of G on (X^2, ν^2) is ergodic. In the classification of the boundary representations, double ergodicity of Patterson-Sullivan measures will be crucial. This result is known to experts, but apparently the proof has never been written down. It was communicated to us by Uri Bader that the full proof of a stronger property called *double metric ergodicity* will appear in a forthcoming joint paper with Alex Furman [2].

3.2.3 Boundary representations

We will now fix some notation for the rest of the paper. Let Γ be a non-elementary hyperbolic group. Fix a metric $d \in \mathcal{D}(\Gamma)$, and choose $1 \in \Gamma$ as the basepoint. Since Γ acts on itself by isometries freely and cocompactly, we are in the setting of Section 3.2.2. Pick a sufficiently small $\epsilon > 0$, and let D be the Hausdorff dimension of $(\partial\Gamma, d_\epsilon)$. Denote by μ the corresponding Hausdorff measure. We may normalize μ and d_ϵ in such a way that $\mu(\partial\Gamma) = 1$ and $\operatorname{diam} \partial\Gamma = 1$. Since D is inversely proportional to ϵ, by choosing sufficiently small ϵ, we may also assume that $D > 1$. Now, denote

$$P_g(\xi) = \frac{dg_*\mu}{d\mu}(\xi) \asymp \omega^{2(g,\xi)-|g|}. \tag{3.2.8}$$

The *boundary representation* π of Γ associated to μ is the unitary representation of Γ on the Hilbert space $L^2(\partial\Gamma, \mu)$ given by

$$[\pi(g)\phi](\xi) = P_g^{1/2}(\xi)\phi(g^{-1}\xi) \tag{3.2.9}$$

for $\phi \in L^2(\partial\Gamma, \mu)$ and $g \in \Gamma$. If we take a measure ν equivalent to μ, then the unitary isomorphism $T_{\mu\nu} \colon L^2(\partial\Gamma, \mu) \to L^2(\partial\Gamma, \nu)$ defined by

$$T_{\mu\nu}\phi = \left(\frac{d\mu}{d\nu}\right)^{1/2} \phi \tag{3.2.10}$$

intertwines the corresponding boundary representations. We therefore obtain a unique (up to unitary equivalence) representation of Γ associated to the class of Γ-quasi-conformal measures on $\partial\Gamma$ with respect to d.

3.3 Shadows and cones

In this section we will work with Γ in order to estimate the cardinalities of some of its subsets. First, we introduce the classical notion of the shadow cast by an element of the group onto its boundary. Then, for a ball B in the boundary, we define the cone over B as the set of all elements Γ whose shadows intersect B. It turns out that the growth of such a cone behaves as one could expect, i.e. its intersection with a large ball in Γ comprises approximately $\mu(B)$ of the ball's volume.

We then move on to define double shadows in $\partial\Gamma^2$. A double shadow of g is the product of the shadows of g and g^{-1}. We show that they form a nice cover of $\partial\Gamma^2$, just as in the case of ordinary shadows in $\partial\Gamma$.

3.3.1 Shadows

We begin by observing that for any element g of Γ there exists a roughly geodesic ray emanating from 1 and passing within a uniform distance from g. In terms of the Gromov product this can be stated as follows.

Lemma 3.3.1. *The estimate*

$$\sup_{\xi \in \partial\Gamma} (g, \xi) \approx |g| \tag{3.3.1}$$

holds uniformly for $g \in \Gamma$.

Proof. We get the upper estimate $\sup(g, \xi) \leq |g|$ from the triangle inequality.

The Gromov product on Γ satisfies the identity $(g, h) + (g^{-1}, g^{-1}h) = |g|$, which, after extension to $\overline{\Gamma}$, takes the form

$$(g^{-1}, g^{-1}\xi) \approx |g| - (g, \xi). \tag{3.3.2}$$

If we fix two distinct points $\xi_1, \xi_2 \in \partial\Gamma$, then

$$\max_i (g^{-1}, g^{-1}\xi_i) \approx |g| - \min_i (g, \xi_i) \gtrsim |g| - (\xi_1, \xi_2), \tag{3.3.3}$$

which gives the estimate from below for g^{-1}. □

Using Lemma 3.3.1, for every $g \in \Gamma$ we may fix $\hat{g} \in \partial\Gamma$ such that $(g, \hat{g}) \approx |g|$. We will also denote by $\check{g}$ the point in the boundary corresponding to g^{-1}. The point $\hat{g}$ plays the same role as the endpoint of the geodesic ray starting at the basepoint and passing through g in a CAT(0) space. In particular, we have

$$(\xi, g) \approx \min\{|g|, (\hat{g}, \xi)\} \tag{3.3.4}$$

for all $\xi \in \Gamma$. This estimate will be usually used in the form

$$\omega^{(\xi,g)} \asymp \min\{\omega^{|g|}, d_\epsilon(\hat{g}, \xi)^{-D}\}. \tag{3.3.5}$$

By Theorem 3.2.1, the growth of Γ satisfies $|B_\Gamma(1, R)| \asymp \omega^R$. Fix $r > 0$ and for $R > 0$ define the annulus

$$A_R = \{g \in \Gamma : R - r \leq |g| \leq R + r\}, \tag{3.3.6}$$

We will assume that r is sufficiently large for the following three conditions to hold:

(1) the lower bound for the cardinality of the ball $B_\Gamma(1, R + r)$ exceeds the upper bound for the cardinality of $B_\Gamma(1, R - r)$, so that the annuli A_R satisfy the growth estimate $|A_R| \asymp \omega^R$,

(2) for any roughly geodesic ray γ from 1 to $\xi \in \partial\Gamma$ we have $\gamma(R) \in A_R$,

(3) r satisfies the bound obtained in the proof of Proposition 3.3.5[1], ensuring that the elements of Γ of length approximately R constructed therein are in A_R.

For $\sigma > 0$ define the *shadow* $\Sigma(g, \sigma)$ of g as the closed ball

$$\Sigma(g, \sigma) = B_{\partial\Gamma}(\hat{g}, e^{-\epsilon(|g| - \sigma)}). \tag{3.3.7}$$

The following fundamental property of shadows is classical, and we include its very short proof. This standard lemma has also a second part, saying that the multiplicity of the cover of $\partial\Gamma$ by shadows is uniformly bounded in R, but we will not need that statement.

Lemma 3.3.2. *For sufficiently large σ, the family of shadows $\{\Sigma(g, \sigma) : g \in A_R\}$ is a cover of $\partial\Gamma$ for any $R \geq 0$.*

Proof. For $\xi \in \partial\Gamma$ take a roughly geodesic ray γ from 1 to ξ. Then $g = \gamma(R) \in A_R$ and we have $(\hat{g}, \xi) \gtrsim \min\{(\hat{g}, g), (g, \xi)\} \approx R$, so for sufficiently large σ we get $\xi \in \Sigma(g, \sigma)$. □

We may now define $\Sigma(g) = \Sigma(g, \sigma)$ with σ sufficiently large to satisfy the conclusion of Lemma 3.3.2.

[1] To precisely formulate condition (3) we need Lemma 3.3.4, which asserts that there exists some universal constant τ related to cancellations of elements in the group. Until Proposition 3.3.5, only conditions (1) and (2) will be used, so the reader need not be afraid of a circular definition.

3.3.2 Cones over balls in the boundary

For $\xi \in \partial\Gamma$ and $\theta > 0$ define the *cone over* $B_{\partial\Gamma}(\xi, e^{-\epsilon\theta})$ as

$$C(\xi,\theta) = \{g \in \Gamma : \Sigma(g) \cap B_{\partial\Gamma}(\xi, e^{-\epsilon\theta}) \neq \emptyset\}, \tag{3.3.8}$$

and denote

$$C_R(\xi,\theta) = A_R \cap C(\xi,\theta). \tag{3.3.9}$$

Lemma 3.3.3. *The growth of the cone $C(\xi,\theta)$ satisfies the estimates*

$$\omega^{R-\theta} \prec |C_R(\xi,\theta)| \prec \omega^R \tag{3.3.10}$$

uniformly in R, θ, ξ. When $R \geq \theta$, the tight estimate

$$|C_R(\xi,\theta)| \asymp \omega^{R-\theta} \tag{3.3.11}$$

holds.

Proof. The upper bound $|C_R(\xi,\theta)| \prec \omega^R$ follows from the estimate on $|A_R|$. For the lower bound, observe that by Lemma 3.3.2 the shadows $\Sigma(g)$ of $g \in C_R(\xi,\theta)$ cover the ball $B(\xi, e^{-\epsilon\theta})$. Hence,

$$\omega^{-\theta} \asymp \mu(B_{\partial\Gamma}(\xi, e^{-\epsilon\theta})) \leq \sum_{g \in C_R(\xi,\theta)} \mu(\Sigma(g)) \asymp |C_R(\xi,\theta)|\, \omega^{-R}, \tag{3.3.12}$$

so $|C_R(\xi,\theta)| \succ \omega^{R-\theta}$.

Now, assume that $R \geq \theta$. Let γ be a roughly geodesic ray from 1 to ξ. If $g \in C_R(\xi,\theta)$, we may pick some $\eta \in \Sigma(g) \cap B_{\partial\Gamma}(\xi, e^{-\epsilon\theta})$. We then have

$$(g, \gamma(\theta)) \gtrsim \min\{(g,\hat{g}), (\hat{g},\eta), (\eta,\xi), (\xi,\gamma(\theta))\} \gtrsim \theta, \tag{3.3.13}$$

and in consequence

$$d(g,\gamma(\theta)) \lesssim R - \theta. \tag{3.3.14}$$

Hence, $C_R(\xi,\theta) \subseteq B_\Gamma(\gamma(\theta), R-\theta+C)$ for some constant C, and the last estimate follows from the bound on the growth of Γ. □

3.3.3 Shadows in the square of the boundary

The next lemma will allow us to understand the distribution of the points $(\hat{g}, \check{g})$ in $\partial\Gamma^2$. It generalizes the observation that if we take two elements g, h of a nonabelian free group expressed in the standard generators, then after possibly changing the last letter of g, there is no cancellation in the product gh. We thought that such a natural result should be well-known, but to our surprise we did not find it in any of the standard references for hyperbolic groups. Therefore we present it together with its full proof.

Lemma 3.3.4. *Let Γ be a non-elementary hyperbolic group, endowed with a metric $d \in \mathcal{D}(\Gamma)$. There exists $\tau > 0$ such that for any $g_0, h \in \Gamma$ one can find $g \in B_\Gamma(g_0, \tau)$ such that $|gh| \geq |g| + |h| - 2\tau$.*

Proof. For every $g \in \Gamma$ fix a roughly geodesic segment $\gamma_g \colon [0, |g|] \to \Gamma$ joining 1 to g, and its reverse $\overline{\gamma}_g(t) = \gamma_g(|g| - t)$. Now, take any $\tau > 0$ and suppose that for all $g \in B_\Gamma(g_0, \tau)$ we have the opposite inequality $|gh| < |g| + |h| - 2\tau$, or equivalently $(1, gh)_g > \tau$ (see Fig. 3.1a). In particular,

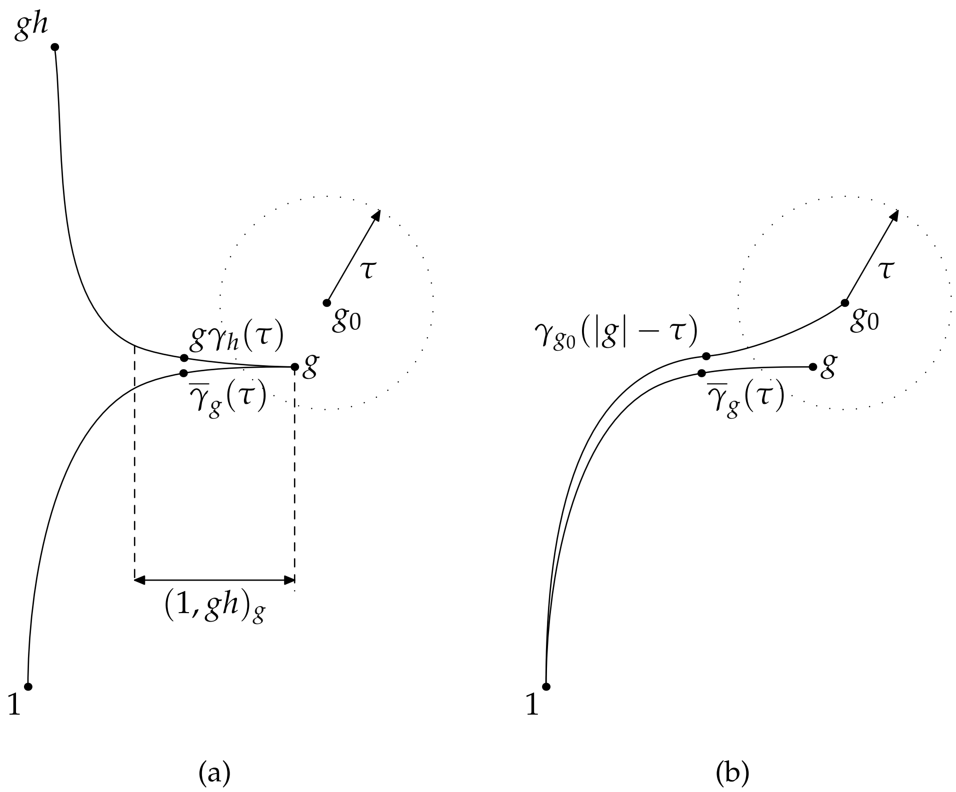

Figure 3.1

this implies that $|g|, |h| > \tau$, and we may compute, with estimates being uniform in τ:

$$(\overline{\gamma}_g(\tau), g\gamma_h(\tau))_g \gtrsim \\ \gtrsim \min\{(\overline{\gamma}_g(\tau), 1)_g, (1, gh)_g, (gh, g\gamma_h(\tau))_g\} \gtrsim \tau, \tag{3.3.15}$$

and in consequence, $d(\overline{\gamma}_g(\tau), g\gamma_h(\tau)) \approx 0$. On the other hand (see Fig. 3.1b),

$(g, g_0) \geq |g| - \tau$, so

$$(\overline{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \gtrsim \\ \gtrsim \min\{(\overline{\gamma}_g(\tau), g), (g, g_0), (g_0, \gamma_{g_0}(|g| - \tau))\} \approx |g| - \tau, \tag{3.3.16}$$

and thus $d(\overline{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \approx 0$. Finally, we obtain

$$d(g\gamma_h(\tau), \gamma_{g_0}(|g| - \tau)) \leq \\ \leq d(g\gamma_h(\tau), \overline{\gamma}_g(\tau)) + d(\overline{\gamma}_g(\tau), \gamma_{g_0}(|g| - \tau)) \approx 0. \tag{3.3.17}$$

It follows that the injective map $g \mapsto g\gamma_h(\tau)$ sends the ball $B_\Gamma(g_0, \tau)$ into a fixed radius neighborhood of the interval $\overline{\gamma}_{g_0}([0, 2\tau])$. Since Γ is non-elementary, the volume of the ball grows exponentially with τ, while the neighborhood of the interval $\gamma_{g_0}([0, 2\tau])$ has linear growth, hence for sufficiently large τ, independent of g_0 and h, we obtain a contradiction. □

For $\sigma > 0$ we now define the *double shadow* of $g \in \Gamma$ as

$$\Sigma_2(g, \sigma) = B_{\partial\Gamma}(\hat{g}, e^{-\epsilon(|g|/2 - \sigma)}) \times B_{\partial\Gamma}(\check{g}, e^{-\epsilon(|g|/2 - \sigma)}) \subseteq \partial\Gamma^2. \tag{3.3.18}$$

Thanks to the factor of $1/2$ in the exponent, the measure of a double shadow of $g \in A_R$ is approximately proportional to $1/|A_R|$. Just as ordinary shadows, the double shadows of elements of A_R form a cover.

Proposition 3.3.5. *For sufficiently large $\sigma > 0$ the family $\{\Sigma_2(g, \sigma) : g \in A_R\}$ of double shadows is a cover of $\partial\Gamma^2$ for all $R > 0$.*

Proof. Take $(\xi_1, \xi_2) \in \partial\Gamma^2$, and for $i = 1, 2$ let γ_i be a roughly geodesic ray from 1 to ξ_i. Put $g_i = \gamma_i(R/2)$. By Lemma 3.3.4 there exist a universal constant τ and $g \in B_\Gamma(g_1, \tau)$ such that $|gg_2^{-1}| \approx R$, and, as it was mentioned in the definition of the annulus A_R, we may assume that its thickness is sufficiently large for it to contain the element gg_2^{-1}. We have $(gg_2^{-1}, g) \approx R/2$ and $(g_2 g^{-1}, g_2) \approx R/2$, and in consequence

$$(gg_2^{-1}, \xi_1) \gtrsim \min\{(gg_2^{-1}, g), (g, g_1), (g_1, \xi_1)\} \gtrsim R/2 \tag{3.3.19}$$

and

$$(g_2 g^{-1}, \xi_2) \gtrsim \min\{(g_2 g^{-1}, g_2), (g_2, \xi_2)\} \gtrsim R/2. \tag{3.3.20}$$

Hence, for sufficiently large σ, the double shadow $\Sigma_2(gg_2^{-1}, \sigma)$ contains the pair (ξ_1, ξ_2). □

Similarly as in the case of shadows, we will denote $\Sigma_2(g) = \Sigma_2(g, \sigma)$ for some fixed σ sufficiently large for Proposition 3.3.5 to hold.

3.4 Operators in the positive cone

By the *positive cone of the representation* π we will understand the weak operator closure of the cone spanned by $\pi(\Gamma)$ in $\mathcal{B}(L^2(\partial\Gamma, \mu))$. The purpose of this section is to prove Proposition 3.4.4, which states that operators arising from positive kernels in $L^\infty(\partial\Gamma^2)$ are contained in the positive cone of π. The operators in question will be constructed as weak operator limits of sequences of weighted averages of normalized operators $\pi(g)$ with $g \in A_R$. Convergence will be first tested on Lipschitz functions, and then established using density of Lipschitz functions in L^2 and uniform boundedness of the averages.

3.4.1 Uniform boundedness of averages of $P_g^{1/2}$

Recall that $P_g = dg_*\mu/d\mu$. Let us define

$$\widetilde{P}_g = \frac{P_g^{1/2}}{\|P_g^{1/2}\|_1}. \tag{3.4.1}$$

We begin by finding an estimate for the norm $\|P_g^{1/2}\|_1$, and using it to get a more manageable approximation of the function $\widetilde{P}_g$.

Lemma 3.4.1. *The L^1-norms of $P_g^{1/2}$ satisfy the estimate*

$$\|P_g^{1/2}\|_1 \asymp \omega^{-|g|/2}(1 + |g|) \tag{3.4.2}$$

uniformly in g. Moreover,

$$\widetilde{P}_g(\xi) \asymp \frac{\omega^{(g,\xi)}}{1 + |g|} \prec \frac{d_\epsilon(\hat{g}, \xi)^{-D}}{1 + |g|}. \tag{3.4.3}$$

Proof. By estimate (3.3.4), we have

$$P_g^{1/2}(\xi) \asymp \omega^{(g,\xi)-|g|/2} \asymp \omega^{-|g|/2} \min\{\omega^{|g|}, d_\epsilon(\hat{g}, \xi)^{-D}\}. \tag{3.4.4}$$

Using Ahlfors regularity and the Fubini's theorem, we calculate

$$\begin{aligned} \omega^{|g|/2}\|P_g^{1/2}\|_1 &\asymp \int_{\partial\Gamma} \min\{\omega^{|g|}, d_\epsilon(\hat{g},\xi)^{-D}\}\, d\mu(\xi) = \\ &= \int_0^{\omega^{|g|}} \mu\{\xi : d_\epsilon(\hat{g},\xi)^{-D} > t\}\, dt = \\ &= 1 + \int_1^{\omega^{|g|}} \mu\{\xi : d_\epsilon(\hat{g},\xi)^{-D} > t\}\, dt = \\ &= 1 + \int_1^{\omega^{|g|}} \mu(B_{\partial\Gamma}(\hat{g}, t^{-1/D}))\, dt \asymp \\ &\asymp 1 + \int_1^{\omega^{|g|}} t^{-1}\, dt \asymp 1 + |g|\,. \end{aligned} \tag{3.4.5}$$

The second part follows by combining this estimate with (3.2.8) and (3.3.5). □

Now we prove the crucial result, stating that the averages of the functions $\widetilde{P}_g$ over A_R are uniformly bounded in the L^∞ norm. Later on, the problem of uniform boundedness of weighted averages of suitably normalized operators $\pi(g)$ will be reduced to this estimate.

Proposition 3.4.2. *The estimate*

$$\sum_{g\in A_R} \widetilde{P}_g(\eta) \prec \omega^R \tag{3.4.6}$$

holds uniformly in R *and* η.

Proof. By Lemma 3.4.1 and (3.3.5) we have

$$\sum_{g\in A_R} \widetilde{P}_g(\eta) \asymp \frac{1}{(1+R)} \sum_{g\in A_R} \min\{\omega^R, d_\epsilon(\hat{g},\eta)^{-D}\}. \tag{3.4.7}$$

The sum on the right can be estimated in a similar fashion as in the proof of Lemma 3.4.1, yielding

$$\begin{aligned} &\sum_{g\in A_R} \min\{\omega^R, d_\epsilon(\hat{g},\eta)^{-D}\} \prec \\ &\qquad \prec \omega^R + \int_1^{\omega^R} \Big|\{g \in A_R : d_\epsilon(\hat{g},\eta)^{-D} > t\}\Big|\, dt \le \\ &\qquad \le \omega^R + \int_1^{\omega^R} |C_R(\eta, \log t/D\epsilon)|\, dt \end{aligned} \tag{3.4.8}$$

But for $1 < t < \omega^R$ we have $0 < \log t/D\epsilon < R$, so we may apply Lemma 3.3.3 to obtain

$$\int_1^{\omega^R} |C_R(\eta, \log t/D\epsilon)|\, dt \asymp \int_1^{\omega^R} \omega^R t^{-1}\, dt = \omega^R R \log \omega, \tag{3.4.9}$$

which ends the proof. □

3.4.2 Approximation on the space of Lipschitz functions

Denote by $\mathrm{Lip}(\partial\Gamma)$ the vector space of Lipschitz functions on $(\partial\Gamma, d_\epsilon)$. Let $\lambda(\phi)$ be the Lipschitz constant of $\phi \in \mathrm{Lip}(\partial\Gamma)$. By the Lebesgue differentiation theorem [23, Theorem 1.8], which is valid for any Ahlfors regular metric measure space, the characteristic functions of balls span a dense subspace of $L^2(\partial\Gamma, \mu)$, and since they can be approximated by Lipschitz functions, it follows that $\mathrm{Lip}(\partial\Gamma)$ is a dense subspace of $L^2(\partial\Gamma, \mu)$.

Define the normalized operator $\widetilde{\pi}(g) = \pi(g)/\|P_g^{1/2}\|_1$. Since

$$\|P_g^{1/2}\|_1 = \langle \pi(g)\mathbf{1}, \mathbf{1}\rangle = \langle \mathbf{1}, \pi(g^{-1})\mathbf{1}\rangle = \|P_{g^{-1}}^{1/2}\|_1, \tag{3.4.10}$$

where $\mathbf{1}$ is the constant function with value 1, the operators $\widetilde{\pi}(g)$ satisfy $\widetilde{\pi}(g)^* = \widetilde{\pi}(g^{-1})$. Moreover, as the next lemma shows, it turns out that on Lipschitz functions $\widetilde{\pi}(g)$ can be approximated in a particularly nice way.

Lemma 3.4.3. *For $\phi, \psi \in \mathrm{Lip}(\partial\Gamma)$ we have*

$$\left|\langle \widetilde{\pi}(g)\phi, \psi\rangle - \phi(\check{g})\overline{\psi(\hat{g})}\right| \prec \frac{\lambda(\phi)\,\|\psi\|_\infty + \lambda(\psi)\,\|\phi\|_\infty}{(1+|g|)^{1/D}}, \tag{3.4.11}$$

uniformly in g, ϕ, and ψ.

Proof. We have

$$\begin{aligned} &\left|\langle \widetilde{\pi}(g)\phi, \psi\rangle - \phi(\check{g})\overline{\psi(\hat{g})}\right| \le \\ &\qquad \le |\langle \widetilde{\pi}(g)\phi, \psi - \psi(\hat{g})\mathbf{1}\rangle| + \left|\overline{\psi(\hat{g})}\Big(\langle \phi, \widetilde{\pi}(g^{-1})\mathbf{1}\rangle - \phi(\check{g})\Big)\right| \le \\ &\qquad \le \|\phi\|_\infty \int_{\partial\Gamma} \widetilde{P}_g(\xi)\left|\psi(\xi) - \psi(\hat{g})\right| d\mu(\xi) + \\ &\qquad + \|\psi\|_\infty \int_{\partial\Gamma} \widetilde{P}_{g^{-1}}(\xi)\left|\phi(\xi) - \phi(\check{g})\right| d\mu(\xi). \end{aligned} \tag{3.4.12}$$

Both terms are similar, so we will estimate only the first one. Since ψ is Lipschitz, we get

$$\int_{\partial\Gamma} \widetilde{P}_g(\xi) \Big| \psi(\xi) - \psi(\hat{g}) \Big| \, d\mu(\xi) \leq \lambda(\psi) \int_{\partial\Gamma} \widetilde{P}_g(\xi) d_\epsilon(\hat{g}, \xi) \, d\mu(\xi), \quad (3.4.13)$$

and by integrating separately on some ball $B = B_{\partial\Gamma}(\hat{g}, \rho)$ and its complement, we obtain

$$\int_B \widetilde{P}_g(\xi) d_\epsilon(\hat{g}, \xi) \, d\mu(\xi) \leq \rho \|\widetilde{P}_g\|_1 = \rho \quad (3.4.14)$$

and, using Lemma 3.4.1,

$$\int_{\partial\Gamma \setminus B} \widetilde{P}_g(\xi) d_\epsilon(\hat{g}, \xi) \, d\mu(\xi) \prec \int_{\partial\Gamma \setminus B} \frac{d_\epsilon(\hat{g}, \xi)^{1-D}}{1 + |g|} \, d\mu(\xi) \prec \frac{\rho^{1-D}}{1 + |g|}, \quad (3.4.15)$$

since $D > 1$. We finish by taking $\rho = (1 + |g|)^{-1/D}$. □

3.4.3 Constructing operators in the positive cone

For a function $K \in L^\infty(\partial\Gamma^2, \mu^2)$, define the operator $T_K \in \mathcal{B}(L^2(\partial\Gamma, \mu))$ with kernel K by

$$\langle T_K \phi, \psi \rangle = \int_{\partial\Gamma^2} \phi(\xi) \overline{\psi(\eta)} K(\xi, \eta) \, d\mu^2(\xi, \eta). \quad (3.4.16)$$

Proposition 3.4.4. *The operator T_K with kernel $K \geq 0$ is in the positive cone of π.*

Proof. We will construct a one-parameter family of operators S_R, with $R \geq 0$, in the positive cone of π, converging to T_K in the weak operator topology. We start by fixing R and taking a measurable partition $\mathcal{V} = \{V_g : g \in A_R\}$ of $\partial\Gamma^2$ such that $V_g \subseteq \Sigma_2(g)$, which can be obtained for instance by putting a linear order on the finite set A_R and taking

$$V_g = \Sigma_2(g) \setminus \bigcup_{h < g} \Sigma_2(h). \quad (3.4.17)$$

By Proposition 3.3.5, $\mathcal{V}$ indeed covers $\partial\Gamma^2$, and is disjoint by definition. Now, put

$$w_g = \int_{V_g} K \, d\mu^2 \quad (3.4.18)$$

and define

$$S_R = \sum_{g \in A_R} w_g \widetilde{\pi}(g). \tag{3.4.19}$$

For any $\phi, \psi \in \mathrm{Lip}(\partial\Gamma)$ we get, using Lemma 3.4.3 and the fact that $V_g \subseteq \Sigma_2(g)$

$$\begin{aligned}
&\Big|\langle S_R\phi, \psi\rangle - \langle T_K\phi, \psi\rangle\Big| \leq \\
&\quad \leq \sum_{g \in A_R} \int_{V_g} K\, d\mu^2 \Big|\langle \widetilde{\pi}(g)\phi, \psi\rangle - \phi(\check{g})\overline{\psi(\hat{g})}\Big| + \\
&\quad + \sum_{g \in A_R} \int_{V_g} \Big|\phi(\check{g})\overline{\psi(\hat{g})} - \phi(\xi)\overline{\psi(\eta)}\Big| K(\xi,\eta)\, d\mu^2(\xi,\eta) \prec \\
&\quad \prec \|K\|_1 \frac{\lambda(\phi)\, \|\psi\|_\infty + \lambda(\psi)\, \|\phi\|_\infty}{(1+R)^{1/D}} + \\
&\quad + \int_{\partial\Gamma^2} e^{-\epsilon R/2} \Big(\lambda(\phi)\, \|\psi\|_\infty + \lambda(\psi)\, \|\phi\|_\infty\Big) K(\xi,\eta)\, d\mu^2(\xi,\eta),
\end{aligned} \tag{3.4.20}$$

so $\langle S_R\phi, \psi\rangle \xrightarrow[R\to\infty]{} \langle T_K\phi, \psi\rangle$.

By density of $\mathrm{Lip}(\partial\Gamma)$ it now remains to show that the operators S_R are uniformly bounded. First, observe that $w_g \leq \|K\|_\infty\, \mu(\Sigma_2(g)) \prec \omega^{-|g|}$ uniformly in g, and thus by Proposition 3.4.2

$$\|S_R\mathbf{1}\|_\infty = \sup_{\xi \in \partial\Gamma} \sum_{g \in A_R} w_g \widetilde{P}_g(\xi) \prec 1 \tag{3.4.21}$$

uniformly in R. Moreover, since A_R is symmetric, the same estimate holds for the adjoint operator S_R^*. Now, let $\phi, \psi \in L^4(\partial\Gamma, \mu) \subseteq L^2(\partial\Gamma, \mu)$. The system of weights $\{w_g\}$ can be treated as a measure on A_R; using the Schwarz inequality in the space $L^2(\partial\Gamma \times A_R, \mu \otimes w)$, we obtain

$$\begin{aligned}
|\langle S_R\phi, \psi\rangle|^2 &= \left| \int\limits_{\partial\Gamma \times A_R} \widetilde{P}_g(\xi)\phi(g^{-1}\xi)\psi(\xi)\, d\xi dg \right|^2 \leq \\
&\leq \int\limits_{\partial\Gamma \times A_R} \widetilde{P}_g(\xi) \left|\phi(g^{-1}\xi)\right|^2 d\xi dg \int\limits_{\partial\Gamma \times A_R} \widetilde{P}_g(\xi)\, |\psi(\xi)|^2\, d\xi dg = \\
&= \langle S_R\, |\phi|^2, \mathbf{1}\rangle \langle S_R\mathbf{1}, |\psi|^2\rangle \leq \|\phi^2\|_1\, \|S_R^*\mathbf{1}\|_\infty\, \|S_R\mathbf{1}\|_\infty\, \|\psi^2\|_1 \prec \|\phi\|_2^2\, \|\psi\|_2^2
\end{aligned} \tag{3.4.22}$$

uniformly in R, and thus S_R are uniformly bounded and converge to T_K in the weak operator topology as desired. □

3.5 The boundary representations

In this section we show that the boundary representations are irreducible and weakly contained in the regular representation.

3.5.1 Irreducibility

We will prove irreducibility by using the following standard observation.

Lemma 3.5.1. *Let σ be a unitary representation of a group G on a Hilbert space $\mathcal{H}$. If there exists a cyclic vector $\phi \in \mathcal{H}$ such that the orthogonal projection P_ϕ onto the subspace $\mathbb{C}\phi$ is contained in the von Neumann algebra generated by $\sigma(G) \subseteq \mathcal{B}(\mathcal{H})$, then the representation σ is irreducible.*

Proof. Let $\mathcal{H}_0 \leq \mathcal{H}$ be a closed nonzero invariant subspace. We may take $\psi \in \mathcal{H}_0$ with $\langle \phi, \psi \rangle \neq 0$, for otherwise

$$\langle \sigma(g)\phi, \psi \rangle = \langle \phi, \sigma(g^{-1})\psi \rangle = 0 \tag{3.5.1}$$

for all $\psi \in \mathcal{H}_0$ and $g \in G$, which by cyclicity of ϕ yields $\mathcal{H}_0 = 0$. Since P_ϕ is in the von Neumann algebra generated by $\sigma(G)$, the nonzero vector $P_\phi \psi = \lambda \phi$ belongs to $\mathcal{H}_0$. Hence, $\mathcal{H}_0$ contains a cyclic vector of σ and equals $\mathcal{H}$. □

Proving the irreducibility of the boundary representations is now a mere formality.

Theorem 3.5.2. *For any metric $d \in \mathcal{D}(\Gamma)$ the associated boundary representation is irreducible.*

Proof. For a positive function $\phi \in L^\infty(\partial\Gamma)$ the kernel $(\xi, \eta) \mapsto \phi(\eta)$ yields a one-dimensional operator T given by $T\psi = \langle \psi, \mathbf{1} \rangle \phi$. By Proposition 3.4.4 it is contained in the von Neumann algebra of π, and $T\mathbf{1} = \phi$ is contained in the weakly closed span of $\pi(\Gamma)\mathbf{1}$. But $L^\infty(\partial\Gamma)$ is dense in $L^2(\partial\Gamma)$, and weakly closed subspaces are closed, so $\mathbf{1}$ is a cyclic vector of π. For $\phi = \mathbf{1}$ the operator T is the orthogonal projection onto $\mathbb{C}\mathbf{1}$, so by Lemma 3.5.1 we are done. □

3.5.2 Weak containment in the regular representation

An important property of any unitary representation is its weak containment in the regular representation. In case of quasi-regular representations we may apply the criterion of [26], which states that if the action of a discrete group G on a standard Borel space (X, ν) is amenable, then all representations obtained by twisting the quasi-regular representation (3.2.9) of G on $L^2(X, \nu)$ with a cocycle—in our case trivial—are weakly contained in the regular representation.

Amenability of the action on the Gromov boundary was established in [1, Theorem 5.1]. Namely, for any finite Borel measure μ on $\partial\Gamma$, quasi-invariant under the action of Γ, the action of Γ on $(\partial\Gamma, \mu)$ is amenable. We thus obtain the following.

Proposition 3.5.3. *The boundary representations of* Γ *are weakly contained in the regular representation.*

3.6 Classification

Here, we investigate unitary equivalence between representations arising from different metrics on Γ. We show that the corresponding boundary representations are equivalent if and only if the metrics are roughly similar, provided that the corresponding Patterson-Sullivan measures are doubly ergodic.

For the entire section, let $d' \in \mathcal{D}(\Gamma)$ be another metric with associated boundary representation π'. Prime will be used to indicate objects associated to d', analogous to the ones defined for d.

3.6.1 Preparatory lemmas

We begin by extending the scope of Proposition 3.4.4 to orthogonal projections onto subspaces $L^2(E, \mu)$. This will be used to show that any unitary isomorphism of boundary representations preserves this family of projections and thus has to be induced by a map of boundaries.

Lemma 3.6.1. *For any measurable set* $E \subseteq \partial\Gamma$ *the orthogonal projection* P_E *onto* $L^2(E, \mu) \subseteq L^2(\partial\Gamma, \mu)$ *is contained in the positive cone of* π.

Proof. Fix $E \subseteq \partial\Gamma$. For $\rho > 0$ define $K_\rho \in L^\infty(\partial\Gamma^2, \mu^2)$ as

$$K_\rho(\xi, \eta) = \frac{1}{\mu(B_{\partial\Gamma}(\xi, \rho))} \chi_E(\xi) \chi_{B_{\partial\Gamma}(\xi,\rho)}(\eta) \tag{3.6.1}$$

and let T_ρ be the operator with kernel K_ρ. By Proposition 3.4.4 it is contained in the weak operator closure of the cone spanned by $\pi(\Gamma)$. For

$\phi, \psi \in \mathrm{Lip}(\partial\Gamma)$ we have

$$\begin{aligned}
&\left|\langle T_\rho\phi, \psi\rangle - \langle P_E\phi, \psi\rangle\right| \leq \\
&\quad \leq \left| \int_E \frac{\phi(\xi)}{\mu(B_{\partial\Gamma}(\xi,\rho))} \int_{B_{\partial\Gamma}(\xi,\rho)} \left(\overline{\psi(\eta)} - \overline{\psi(\xi)}\right) d\mu(\eta) d\mu(\xi) \right| \leq \\
&\quad \leq \mu(E) \, \|\phi\|_\infty \, \lambda(\psi)\rho \xrightarrow[\rho\to 0]{} 0,
\end{aligned} \tag{3.6.2}$$

so to finish the proof it only remains to show that the family of operators T_ρ is uniformly bounded. We may estimate their operator norms using the Schwarz inequality, obtaining

$$\|T_\rho\| \leq \left\| \int_{\partial\Gamma} K_\rho(\xi,\eta) \, d\mu(\xi) \right\|_\infty^{1/2} \left\| \int_{\partial\Gamma} K_\rho(\xi,\eta) \, d\mu(\eta) \right\|_\infty^{1/2}. \tag{3.6.3}$$

By Ahlfors regularity we get

$$\begin{aligned}
\int_{\partial\Gamma} K_\rho(\xi,\eta) \, d\mu(\xi) &= \int_E \frac{\chi_{B_{\partial\Gamma}(\xi,\rho)}(\eta)}{\mu(B_{\partial\Gamma}(\xi,\rho))} \, d\mu(\xi) \asymp \\
&\asymp \int_E \frac{\chi_{B_{\partial\Gamma}(\eta,\rho)}(\xi)}{\mu(B_{\partial\Gamma}(\eta,\rho))} \, d\mu(\xi) \leq 1,
\end{aligned} \tag{3.6.4}$$

while the second integral is simply equal to $\chi_E(\xi) \leq 1$. □

Lemma 3.6.2. *The measure class of μ^2 contains a Γ-invariant measure ν on $\partial\Gamma^2$. It satisfies*

$$\nu(E) \asymp \int_E d_\epsilon(\xi,\eta)^{-2D} \, d\mu^2(\xi,\eta). \tag{3.6.5}$$

If the action of Γ on $(\partial\Gamma, \mu)$ is doubly ergodic, then ν is unique up to scaling.

Proof. Let $\tilde{\nu} = d_\epsilon(\xi,\eta)^{-2D} d\mu^2(\xi,\eta)$ be the measure on the right-hand side of (3.6.5). We have μ^2-a.e.

$$\begin{aligned}
&\log_\omega\left(\frac{dg_*\tilde{\nu}}{d\tilde{\nu}}(\xi,\eta)\right) = \\
&\quad = \log_\omega\left(\left(\frac{d_\epsilon(g^{-1}\xi, g^{-1}\eta)}{d_\epsilon(\xi,\eta)}\right)^{-2D} P_g(\xi)P_g(\eta)\right) \approx \\
&\quad \approx 2(g^{-1}\xi, g^{-1}\eta) - 2(\xi,\eta) + 2(g,\xi) + 2(g,\eta) - 2\,|g| \approx 0,
\end{aligned} \tag{3.6.6}$$

where the last estimate is obtained by expanding the definition of the Gromov product, after replacing ξ and η by sequences $x_n \to \xi$ and $y_n \to \eta$.

It follows that the Radon-Nikodym derivatives $dg_*\tilde{\nu}/d\tilde{\nu}$ are uniformly bounded. It is a classical result that in such a situation $\tilde{\nu}$ can be replaced by an equivalent Γ-invariant measure $\nu = \rho\tilde{\nu}$. One just has to solve the cohomological equation

$$\rho(\xi, \eta) = \frac{dg_*\tilde{\nu}}{d\tilde{\nu}}(\xi, \eta)\rho(g^{-1}\xi, g^{-1}\eta). \tag{3.6.7}$$

The function

$$\rho = \sup_{g \in \Gamma} \frac{dg_*\tilde{\nu}}{d\tilde{\nu}}(\xi, \eta) \tag{3.6.8}$$

is bounded away from 0 and ∞, and can be seen to satisfy equation (3.6.7) by applying supremum over h to the cocycle identity

$$\frac{d(gh)_*\tilde{\nu}}{d\tilde{\nu}}(\xi, \eta) = \frac{dg_*\tilde{\nu}}{d\tilde{\nu}}(\xi, \eta)\frac{dh_*\tilde{\nu}}{d\tilde{\nu}}(g^{-1}\xi, g^{-1}\eta). \tag{3.6.9}$$

We thus obtain an invariant measure satisfying (3.6.5). Its uniqueness follows from the double ergodicity of μ—if ν' is another invariant measure, then $d\nu'/d\nu$ is an invariant function, and must be constant. □

3.6.2 Equivalence in terms of measurable structures

By an *isomorphism* of measure spaces (X, μ_X) and (Y, μ_Y) we will understand a Lebesgue isomorphism, i.e. a Borel map $F\colon X \to Y$, for which there exist two subsets $N \subseteq X$ and $N' \subseteq Y$ of measure 0, such that the restriction $F\colon X \setminus N \to Y \setminus N'$ is a Borel isomorphism, and the pushforward $F_*\mu_X$ is equivalent to μ_Y. Such an isomorphism will be called equivariant if the corresponding equivariance condition is satisfied almost everywhere.

Lemma 3.6.3. *Suppose that the representations π and π' are unitarily equivalent. Then there exists a Γ-equivariant isomorphism $F\colon (\partial\Gamma, \mu) \to (\partial\Gamma, \mu')$.*

Proof. Suppose $T\colon L^2(\partial\Gamma, \mu) \to L^2(\partial\Gamma, \mu')$ is a unitary intertwining operator. It induces a Γ-equivariant isomorphism

$$\hat{T}\colon \mathcal{B}(L^2(\partial\Gamma, \mu')) \to \mathcal{B}(L^2(\partial\Gamma, \mu)), \tag{3.6.10}$$

of von Neumann algebras endowed with the conjugation actions of the corresponding representations, given by $\hat{T}(S) = T^*ST$. The isomorphism $\hat{T}$ maps the positive cone of π' onto the positive cone of π, and preserves orthogonal projections.

Now, observe that the subalgebra $L^\infty(\partial\Gamma, \mu') \leq \mathcal{B}(L^2(\partial\Gamma, \mu'))$ of multiplication operators is generated by the orthogonal projections P_E onto $L^2(E, \mu')$. They can be characterized as the orthogonal projections P such that both P and $I - P$ are in the positive cone of π'. Indeed, one inclusion is a consequence Lemma 3.6.1. For the other one observe that if both P and $I - P$ are in the positive cone, then they preserve the natural partial order on functions in $L^2(\partial\Gamma, \mu')$. Since the only possible decompositions $\mathbf{1} = P\mathbf{1} + (I - P)\mathbf{1}$ into a sum of two orthogonal positive functions are of the form $\mathbf{1} = \chi_E + \chi_{E^c}$, for bounded positive ϕ we get

$$P\phi \leq P(\|\phi\|_\infty \mathbf{1}) \leq \|\phi\|_\infty \chi_E \tag{3.6.11}$$

for some fixed $E \subseteq \partial\Gamma$, so P sends $L^2(\partial\Gamma, \mu')$ into $L^2(E, \mu')$. Similarly, the image of $I - P$ is contained in $L^2(E^c, \mu')$, so $P = P_E$.

It follows that $\hat{T}$ restricts to a Γ-equivariant isomorphism between the spaces $L^\infty(\partial\Gamma, \mu')$ and $L^\infty(\partial\Gamma, \mu)$. By [16, Theorem 4, p. 238], any such isomorphism is induced by an isomorphism $F\colon (\partial\Gamma, \mu) \to (\partial\Gamma, \mu')$, which is uniquely determined up to perturbations on sets of measure 0. Thus, Γ-equivariance of $\hat{T}$ implies that F is also Γ-equivariant. □

3.6.3 Equivalence in terms of metric structures

Recall that in a metric space (X, d) the cross-ratio of a quadruple of distinct points $x, y, z, w \in X$ is defined as

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}. \tag{3.6.12}$$

The next lemma is an adaptation of a classical argument from ergodic theory (see e.g. the proof of [20, Theorem 6.2]); we include the detailed proof for the sake of self-containment.

Lemma 3.6.4. *If π and π' are unitarily equivalent, and the action of Γ on $(\partial\Gamma, \mu')$ is doubly ergodic, then we have*

$$d'_\epsilon(\xi, \eta) \asymp d_\epsilon(\xi, \eta)^{D/D'}. \tag{3.6.13}$$

Proof. It follows from Lemma 3.6.3 that there exists a Γ-equivariant isomorphism $F\colon (\partial\Gamma, \mu) \to (\partial\Gamma, \mu')$. Let ν and ν' be the invariant measures on $\partial\Gamma^2$, corresponding to the metrics d and d', constructed in Lemma 3.6.2. Then the pushforward $F^2_*\nu$ is an invariant measure on $\partial\Gamma^2$, equivalent to μ'^2, and thus equal to ν' by double ergodicity. It follows that a.e.

$$d'_\epsilon(\xi, \eta)^{-2D'} \asymp d_\epsilon(F^{-1}(\xi), F^{-1}(\eta))^{-2D} \frac{dF_*\mu}{d\mu'}(\xi) \frac{dF_*\mu}{d\mu'}(\eta) \tag{3.6.14}$$

After raising this to the power $-1/2D'$ and plugging into the definition of the cross-ratio, the Radon-Nikodym derivatives of μ cancel out, and we obtain that the estimate

$$[F(\xi_1), F(\xi_2), F(\eta_1), F(\eta_2)]' \asymp [\xi_1, \xi_2, \eta_1, \eta_2]^{D/D'} \tag{3.6.15}$$

holds on a subset $E \subseteq \partial\Gamma^4$ of measure 1 (with respect to μ^4).

If we take $\rho > 0$ and (ξ_2, η_2) such that the corresponding section

$$\{(\xi_1, \eta_1) : (\xi_1, \xi_2, \eta_1, \eta_2) \in E\} \tag{3.6.16}$$

of E has measure 1, we get

$$d'_\epsilon(F(\xi_1), F(\eta_1)) \prec_{\xi_2, \eta_2, \rho} d_\epsilon(\xi_1, \eta_1)^{D/D'} \tag{3.6.17}$$

for (ξ_1, η_1) in a subset of full measure in $B_{\partial\Gamma}(\eta_2, \rho)^c \times B_{\partial\Gamma}(\xi_2, \rho)^c$. By the Fubini's theorem, (ξ_2, η_2) can be chosen from a set of measure 1, which is in particular dense. We may thus cover $\partial\Gamma^2$ by finitely many sets of the form $B_{\partial\Gamma}(\eta_2, \rho)^c \times B_{\partial\Gamma}(\xi_2, \rho)^c$, and the estimate (3.6.17) actually holds uniformly on a subset $E' \subseteq \partial\Gamma^2$ of measure 1.

Now, denote by E'' the set consisting of $\xi \in \partial\Gamma$ such that the section $E'_\xi = \{\eta : (\xi, \eta) \in E'\}$ has measure 1. For $\xi, \eta \in E''$ and $\zeta \in E'_\xi \cap E'_\eta$ we have

$$\begin{aligned} d'_\epsilon(F(\xi), F(\eta)) &\le d'_\epsilon(F(\xi), F(\zeta)) + d'_\epsilon(F(\eta), F(\zeta)) \\ &\prec d_\epsilon(\xi, \zeta)^{D/D'} + d_\epsilon(\eta, \zeta)^{D/D'}. \end{aligned} \tag{3.6.18}$$

If we let ζ converge to η from within the dense set $E'_\xi \cap E'_\eta$, we get a Hölder estimate

$$d'_\epsilon(F(\xi), F(\eta)) \prec d_\epsilon(\xi, \eta)^{D/D'} \tag{3.6.19}$$

for F, satisfied on the subset $E'' \subseteq \partial\Gamma$ of full measure. This implies that F is equal a.e. to a continuous map $H : \partial\Gamma \to \partial\Gamma$. By symmetry, from F^{-1} we may construct a continuous inverse of H, so it is a homeomorphism. Equivariance is clear, and by the remark in Section 3.1.4, H is in fact the identity map. Estimate (3.6.13) follows from (3.6.19) and symmetry of d_ϵ and d'_ϵ. □

Now we are ready to state and prove the equivalence result.

Theorem 3.6.5. *Let $d, d' \in \mathcal{D}(\Gamma)$ give rise to Patterson-Sullivan measures μ and μ', and boundary representations π and π'. In the conditions below, (1) implies (2), and (2) implies (3). If the measures in question are doubly ergodic, then the conditions are equivalent*

(1) d and d' are roughly similar,

(2) μ and μ' are equivalent,

(3) π and π' are unitarily equivalent.

Proof. For the metrics d and d' being roughly similar means exactly that $d \approx Ad'$. If this is satisfied, the visual metrics d_ϵ and $d'_{A\epsilon}$ are bi-Lipschitz equivalent. Hence, the corresponding Hausdorff measures are equivalent, and the boundary representations are equivalent by the discussion in Section 3.2.3.

For the last implication from (3) to (1), we use Lemma 3.6.4 to get the estimate (3.6.13) on the visual metrics, which implies that the Hausdorff measures μ and μ' are equivalent with Radon-Nikodym derivatives bounded away from 0 and ∞. Together with their Γ-quasi-conformality with respect to the corresponding metrics on Γ, this yields the estimate

$$\omega^{2(g,\xi)-|g|} \asymp \omega'^{2(g,\xi)'-|g|'} \tag{3.6.20}$$

uniformly in g and ξ. By taking the logarithms of both sides, and then suprema over ξ, using Lemma 3.3.1 we obtain

$$|g| \approx \frac{\log \omega'}{\log \omega} |g|' = \frac{D'}{D} |g|', \tag{3.6.21}$$

which ends the proof. □

Remark 3.6.6. It might be tempting to try to extend the class of boundary representations even further, by allowing d to be a pseudometric. For instance, if Γ acts properly and cocompactly on a space X then the orbit map in general induces a pseudometric on Γ. However, pseudometrics do not lead to any new representations. In fact, any such pseudometric d is roughly isometric to a metric

$$d^+(g,h) = \begin{cases} d(g,h) + 1 & \text{for } g \neq h \\ 0 & \text{for } g = h \end{cases} \tag{3.6.22}$$

yielding the same boundary representation.

3.7 Examples

In this section we apply the obtained results to some classes of groups appearing in nature. It is less self-contained than the preceding ones—for more details the reader is referred to the appropriate literature.

3.7.1 Fundamental groups of negatively curved manifolds

The following setting was studied by Bader and Muchnik, who established irreducibility of boundary representations of fundamental groups of negatively curved manifolds in [3]. Let M be a closed Riemannian manifold with strictly negative curvature. Its universal cover $\widetilde{M}$ is then a hyperbolic metric space on which $\Gamma = \pi_1(M)$ acts freely and cocompactly by isometries. This action thus extends to an action of Γ on the Gromov boundary $\partial\widetilde{M}$, and yields a unitary representation defined by formula (3.2.9). Any orbit map $\Gamma \to \widetilde{M}$ induces a hyperbolic metric $d \in \mathcal{D}(\Gamma)$, and since we may identify $\partial\Gamma$ with $\partial\widetilde{M}$, the resulting representation is actually the boundary representation of Γ associated to the metric d.

The group Γ may appear as the fundamental group of many non-isometric Riemannian manifolds, and any such realization leads to a potentially different representation. The main theorems of [3] state that all these representations are irreducible, and that they are equivalent if and only if the marked length spectra of the corresponding manifolds are proportional. By the marked length spectrum of M we understand the function $\ell\colon \pi_1(M) \to (0,\infty)$, which to every $[\gamma] \in \pi_1(M)$ assigns the length of the unique geodesic loop freely homotopic to γ.

The irreducibility of the Bader-Muchnik representations is a special case of Theorem 3.5.2. To conclude the equivalence condition of [3] using Theorem 3.6.5, it is enough to observe that proportionality of the marked length spectra is equivalent to rough similarity of the induced metrics on $\pi_1(M)$. In one direction it is trivial—the length of the shortest geodesic loop in the free homotopy class of $g \in \pi_1(M)$ is the translation length of g acting on $\widetilde{M}$, and can be expressed in terms of the metric as

$$\ell(g) = \lim_{n\to\infty} \frac{|g^n|}{n}. \tag{3.7.1}$$

For the other direction we may resort to [28, Theorem 2.2], which states that the marked length spectrum determines the cross-ratio on the boundary. Hence, proportional marked length spectra lead to cross-ratios satisfying $[\cdot]' = [\cdot]^\alpha$, and thus they arise from roughly similar metrics, by the arguments from the proofs of Lemma 3.6.4 and Theorem 3.6.5.

3.7.2 Green metrics and Poisson boundaries

Let Γ be a non-elementary hyperbolic group, and let ν be a symmetric probability measure on Γ, whose support generates Γ. Such a measure gives rise to a random walk on Γ; denote by $F(g,h)$ the probability that starting at g, it ever reaches h. Assume that ν has *exponential moment*, i.e. there exists $\lambda > 0$ such that

$$\sum_{g\in\Gamma} e^{\lambda|g|}\nu(g) < \infty, \tag{3.7.2}$$

where the length is taken with respect to any word metric on Γ, and that for any r there exists a constant $C(r)$ for which

$$F(x,y) \leq C(r)F(x,v)F(v,y) \tag{3.7.3}$$

for v within distance r from a geodesic (again, with respect to some fixed word metric) joining x and y. These two conditions hold for any finitely supported measure [8, Corollary 1.2], and ensure that the *Green metric*

$$d_G(g,h) = -\log F(g,h), \tag{3.7.4}$$

which was introduced in [6], and studied further in [7], belongs to the class $\mathcal{D}(\Gamma)$.

Trajectories of the random walk with law ν almost surely converge to a point in $\partial\Gamma$, and the hitting probability defines the *harmonic measure* $\hat{\nu}$ on $\partial\Gamma$ associated with ν. By [8, Theorems 1.1(ii) and 1.5], it turns out that $\hat{\nu}$ is equivalent to the Patterson-Sullivan measure associated with the Green metric d_G, and thus yields the same quasi-regular representation. On the other hand, a measure with exponential moment has finite first moment, and by [24, Theorem 7.4], the Gromov boundary with the harmonic measure $(\partial\Gamma,\hat{\nu})$ is actually isomorphic to the Poisson boundary of (Γ,ν). By Theorem 3.5.2, we therefore obtain the following new result.

Theorem 3.7.1. *Let ν be a symmetric measure on Γ, satisfying conditions* (3.7.2) *and* (3.7.3). *Then the quasi-regular representation associated with the Poisson boundary $(\partial\Gamma,\hat{\nu})$ is irreducible.*

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