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Plactic Algebras of Low Rank

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PLACTIC ALGEBRAS OF LOW RANK

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Abstract. We state general ring-theoretic properties of the plactic algebra $K[M_n]$ of rank $n \geq 1$ over an arbitrary field K. In particular, we compute $\operatorname{GKdim} K[M_n]$ and we prove that $K[M_n]$ is neither left nor right noetherian and does not satisfy any nontrivial polynomial identity in case $n \geq 2$. For $n \leq 2$ we also prove that $K[M_n]$ is prime and semiprimitive. We conclude with open problems motivated by the above results and concerning plactic algebras of arbitrary rank. In particular, they include: construction of irreducible representations of $K[M_n]$, description of minimal prime ideals of $K[M_n]$, and the question whether the Jacobson radical $J(K[M_n])$ and the Baer radical $B(K[M_n])$ of $K[M_n]$ coincide, and whether they are nilpotent and finitely generated.

The plactic monoid M_n of rank $n \geq 1$ was discovered by Knuth, who used an operation given by Schensted in his study of the longest increasing subsequence of a permutation. It was named and systematically studied by Lascoux and Schützenberger (cf. [9]), who allowed any totally ordered alphabet in the definition. It is known that the elements of M_n can be written in the canonical form, and in this form can be identified with some type of the Young tableaux (cf. [3, 8, 9, 12]). Because of this the plactic monoid became a very important tool in many branches of representation theory and algebraic combinatorics. The first significant application of the plactic monoid was to provide a complete proof of the Littlewood-Richardson rule, a combinatorial algorithm which allows to decompose tensor product of representations of unitary groups. Subsequent applications, also connected with group theory, physics and geometry, include a combinatorial description of the Kostka-Foulkes polynomials, which arise as entries of the character table of the finite linear groups $\mathrm{GL}_n(\mathbb{F}_q)$. The plactic monoid appeared also in the theory of modular representations of the symmetric group, in quantum groups, via the representation theory of quantum enveloping algebras, and even in language theory.

Our aim in this paper is to discuss ring-theoretic properties of the plactic algebra $K[M_n]$ over a field K.

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1. Preliminaries

The bicyclic monoid is one of the most fundamental semigroups, and it is known to have several remarkable properties (cf. [2]). For instance, the bicyclic monoid is completely determined by its lattice of subsemigroups. It is the simplest member of an extensive class of bisimple inverse monoids, it is also an "elementary inverse semigroup". The bicyclic monoid is an example of a simple semigroup containing a non-primitive idempotent, and it is known that any such a semigroup contains a bicyclic monoid. Moreover, the bicyclic monoid has also a number of applications to topics outside semigroup theory.

Definition 1.1. The *bicyclic monoid* B is the monoid defined by the presentation

$$B = \langle p, q : pq = 1 \rangle .$$

As can be easily verified, $qp \neq 1$, but we have the following result.

Theorem 1.2 (cf. [2, Lemma 1.31])**.** *Let $M = \langle a, b \rangle$ be a monoid in which $ab = 1$ and $ba \neq 1$. Then every element of the monoid M is uniquely expressible in the form $b^n a^m$ with $n, m \geq 0$, and hence M is isomorphic with the bicyclic monoid B.*

Theorem 1.2 implies that every element of the bicyclic monoid B can be written uniquely in the form $q^n p^m$ with $n, m \geq 0$. The bicyclic monoid has also the property that any homomorphic image of B is either cyclic, or it is an isomorphic copy of B.

Corollary 1.3 (cf. [2, Corollary 1.32])**.** *Let a semigroup S be a homomorphic image of the bicyclic monoid B. Then either S is isomorphic to B or else S is a cyclic group.*

In practice, this means that the bicyclic monoid can be found in many different contexts, e.g., in ring theory.

For $i \geq 1$ let

$$e_i = q^i p^i .$$

Then $\{e_i : i \geq 1\}$ is a chain of idempotents of B. These idempotents can be used to prove the following proposition.

Proposition 1.4. *Let $K[B]$ be the bicyclic algebra over a field K. Then $K[B]$ is neither left nor right noetherian.*

Proof. Observe that $e_{i+1} = e_i e_{i+1} e_i$ for $i \geq 1$, so we have an ascending chain of left ideals

$$K[B](1 - e_1) \subseteq K[B](1 - e_2) \subseteq K[B](1 - e_3) \subseteq \cdots$$

of the bicyclic algebra $K[B]$. These inclusions are, in fact, strict because if $1 - e_{i+1} = r(1 - e_i)$ for some $i \geq 1$ and $r \in K[B]$, then

$$e_i - e_{i+1} = (1 - e_{i+1})e_i = r(1 - e_i)e_i = 0,$$

so $e_i = e_{i+1}$, which implies $qp = 1$, a contradiction. A symmetric argument shows also that $K[B]$ is not right noetherian. □

Definition 1.5. A ring R is said to be *Dedekind-finite* if for all $a, b \in R$, $ab = 1$ implies $ba = 1$.

So these are the rings in which right-invertibility of elements implies left-invertibility. Many rings satisfying some form of "finiteness conditions" can be shown to be Dedekind-finite, but there do exist non-Dedekind-finite rings. If R is such a ring then there exist elements $a, b \in R$ such that $ab = 1$ but $e = ba \neq 1$. Then $e^2 = e$, so e is a non-trivial idempotent. For $i, j \geq 1$ let

$$e_{ij} = b^i(1-e)a^j.$$

Then $\{e_{ij} : i, j \geq 1\}$ is a set of matrix units in the sense that

$$e_{ij}e_{kl} = \delta_{jk}e_{il},$$

where δ_{jk} are the Kronecker deltas. In particular, if an algebra A over a field K is non-Dedekind-finite, then A contains, for any $n \geq 1$, an isomorphic copy of the matrix algebra $\mathrm{M}_n(K)$. Because of this A does not satisfy any non-trivial polynomial identity. To be more precise let us introduce the following definition.

Definition 1.6. We say that an algebra A over a field K *satisfies a polynomial identity* of degree $d \geq 1$ if there exists a polynomial f of degree d in the free algebra $K\langle x_1, \ldots, x_n \rangle$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$.

Proposition 1.7. *Let A be an algebra over a field K. Then if A satisfies a non-trivial polynomial identity then A is Dedekind-finite.*

Proof. It is known (cf. [11, Theorem 1.41] and comments below) that the matrix algebra $\mathrm{M}_n(K)$ does not satisfy any non-trivial polynomial identity of degree $< 2n$. If A is non-Dedekind-finite then A contains, for any $n \geq 1$, an isomorphic copy of $\mathrm{M}_n(K)$. Hence A does not satisfy any non-trivial polynomial identity, a contradiction. □

Corollary 1.8. *Let $K[B]$ be the bicyclic algebra over a field K. Then $K[B]$ does not satisfy any non-trivial polynomial identity.*

Definition 1.9. A ring R is said to be *left* (resp. *right*) *primitive* if R has a faithful simple left (resp. right) R-module. An ideal $P \subseteq R$ is said to be *left* (resp. *right*) *primitive* if the quotient ring R/P is left (resp. right) primitive.

Consider a countably generated vector space V over a field K with a basis $\{e_i : i \geq 1\}$. Let $f, g \in \mathrm{End}_K(V)$ be defined by

$$f(e_1) = 0, \qquad f(e_i) = e_{i-1} \qquad \text{for } i \geq 2,$$
$$g(e_i) = e_{i+1} \qquad \text{for } i \geq 1,$$

and let R be the K-subalgebra of $\mathrm{End}_K(V)$ generated by f, g and the identity on V. In [7, Example 11.22] it is proved that V is a faithful simple left R-module, hence R is left primitive. It can be shown, using the same method, that R is also right primitive. Moreover, simple calculations show that R is isomorphic to $K[B]$, so we have the following result.

Proposition 1.10. *Let $K[B]$ be the bicyclic algebra over a field K. Then $K[B]$ is left and right primitive.*

Now, let us recall the notion of the Jacobson and Baer radicals.

Definition 1.11. The *Jacobson radical* of a ring R, denoted by $J(R)$, is defined to be the intersection of all the left primitive ideals of R.

Although the above definition is not left-right symmetric, it can be shown that $J(R)$ is also equal to the intersection of all the right primitive ideals of R.

Definition 1.12. A ring R is said to be *semiprimitive* (or *Jacobson semisimple*) if $J(R) = 0$. An ideal $I \subseteq R$ is said to be *semiprimitive* if the quotient ring R/I is semiprimitive.

Observe that a ring R is semiprimitive if and only if it is a subdirect product of left (or right) primitive rings, i.e., if and only if there exists a family $\{P_i\}$ of left (or right) primitive ideals of R such that $\bigcap_i P_i = 0$.

Definition 1.13. A ring R is said to be *prime* if for ideals $I, J \subseteq R$, $IJ = 0$ implies $I = 0$ or $J = 0$. An ideal $P \subseteq R$ is said to be *prime* if $P \neq R$ and the quotient ring R/P is prime. The set of all prime ideals of R, denoted by $\mathrm{Spec}(R)$, is called the *prime spectrum* of R.

It is well known (cf. [7, Proposition 11.6]) that any left (or right) primitive ring is both semiprimitive and prime. As a consequence of this we obtain the following result.

Corollary 1.14. *Let $K[B]$ be the bicyclic algebra over a field K. Then $K[B]$ is prime and semiprimitive.*

Definition 1.15. The *Baer radical* (or *prime radical*) of a ring R, denoted by $B(R)$, is defined to be the intersection of all the prime ideals of R.

The definition of the Baer radical leads to another important class of rings, i.e., the class of semiprime rings.

Definition 1.16. A ring R is said to be *semiprime* if $B(R) = 0$. An ideal $I \subseteq R$ is said to be *semiprime* if the quotient ring R/I is semiprime.

It is easy to see that any nilpotent ideal of R is contained in $B(R)$, so R is semiprime if and only if R has no non-zero nilpotent ideal.

At the end of this section let us recall the notion of the classical Krull dimension and the Gelfand-Kirillov dimension (cf. [5, 6, 10]).

Definition 1.17. Let R be a ring. We define sets $\mathrm{Spec}_\alpha(R)$ of prime ideals of R for each ordinal α as follows. Let $\mathrm{Spec}_{-1}(R) = \varnothing$ and, for $\alpha \geq 0$, let $\mathrm{Spec}_\alpha(R)$ be the set of these $P \in \mathrm{Spec}(R)$ for which each $Q \in \mathrm{Spec}(R)$ properly containing P is in $\mathrm{Spec}_\beta(R)$ for some $\beta < \alpha$. In the case where $\mathrm{Spec}_\alpha(R) = \mathrm{Spec}(R)$ for some α, the *classical Krull dimension* of R, denoted by $\mathrm{clKdim}\, R$, is defined by

$$\mathrm{clKdim}\, R = \inf\{\alpha \geq -1 : \mathrm{Spec}_\alpha(R) = \mathrm{Spec}(R)\}.$$

Otherwise, we say that $\mathrm{clKdim}\, R$ does not exists.

When $\mathrm{clKdim}\, R$ is finite it is clearly the length of a maximal chain of prime ideals of R, and this characterization will be crucial for us.

Now, we shall introduce the Gelfand-Kirillov dimension, which is extremely useful in the theory of algebras over a field. This dimension is defined in a more combinatorial way.

Definition 1.18. Let A be an affine algebra over a field K (i.e., a finitely generated K-algebra). A finite dimensional K-subspace $V \subseteq A$ is said to be *generating subspace* of A if V generates A as an algebra. If V is such a subspace, let $V^0 = K$ and, for $n \geq 1$, let V^n be the subspace of A spanned by all the monomials $a_1 \cdots a_n$, where $a_1, \ldots, a_n \in V$. Finally, let $A_n = V^0 + \cdots + V^n$ for $n \geq 0$, and define the *growth function* of A by $d_A(n) = \dim_K A_n$ for $n \geq 0$. The *Gelfand-Kirillov dimension* of A, denoted by $\mathrm{GKdim}\, A$, is defined by

$$\mathrm{GKdim}\, A = \limsup_{n\to\infty} \frac{\log d_A(n)}{\log n}.$$

In general, if R is any (not necessarily affine) K-algebra, then the *Gelfand-Kirillov dimension* of R, denoted by $\mathrm{GKdim}\, R$, is defined by

$$\mathrm{GKdim}\, R = \sup\{\mathrm{GKdim}\, A : A \text{ is an affine } K\text{-subalgebra of } R\}.$$

It can be shown (cf. [6, 10]) that $\mathrm{GKdim}\, A$ does not depend on the choice of the generating subspace $V \subseteq A$.

2. Plactic algebras

In this section, we aim to present some general properties of the plactic monoid M_n and the plactic algebra $K[M_n]$ over a field K. Next, we shall concentrate on plactic algebras of low rank.

Definition 2.1. The *plactic monoid* of rank n is the finitely presented monoid $M_n = \langle a_1, \ldots, a_n\rangle$ defined by the *Knuth relations*

$$a_i a_k a_j = a_k a_i a_j \qquad \text{for } i \leq j < k,$$
$$a_j a_i a_k = a_j a_k a_i \qquad \text{for } i < j \leq k.$$

The semigroup algebra $K[M_n]$ is called the *plactic algebra* of rank n.

Firstly notice that the plactic algebra $K[M_n]$ is a finitely presented algebra defined by homogeneous semigroup relations. It also can be shown that the defining relations of M_n lead to the canonical form of elements of M_n (cf. [3, 8, 9, 12]). Namely, by a *row* in M_n we mean an element of the form $a_{i_1}\cdots a_{i_r}$, where $r \geq 1$ and $i_1 \leq \cdots \leq i_r$. We say that a row $v = a_{i_1}\cdots a_{i_r}$ dominates a row $w = a_{j_1}\cdots a_{j_s}$ if $r \leq s$ and $i_k > j_k$ for $k = 1,\dots,r$. We write $v \rhd w$ in this case. A *tableau* is a word $w_1\cdots w_t$ such that all $w_1,\dots,w_t$ are rows and $w_1 \rhd \cdots \rhd w_t$. Then every element in M_n is equal in M_n to a unique tableau. This tableau presentation of words implies that the growth function of the plactic monoid M_n is polynomially bounded (cf. [8]). Therefore the Gelfand-Kirillov dimension of the plactic algebra $K[M_n]$ is finite. Another consequence of homogeneity of Knuth relations is a natural gradation on $K[M_n]$, which can be defined via the length of words in the plactic monoid M_n.

Proposition 2.2 (cf. [1, Lemma 3]). *The following conditions are satisfied in the plactic monoid M_n:*

1. *the center $Z(M_n)$ of M_n is equal to the cyclic monoid $\langle z\rangle$, where $z = a_n\cdots a_1 \in M_n$,*
2. *if $zv = zw$ for some $v, w \in M_n$ then $v = w$,*
3. *if $w \in M_n$ then $w \in zM_n$ if and only if $w = v_n a_n \cdots v_1 a_1 v_0$ for some $v_0,\dots,v_n \in M_n$.*

The above result is an easy consequence of the defining relations and the canonical form of elements in M_n. Combining Proposition 2.2 with a simple induction on the length of elements, and reducing the proof to the case of a homogeneous element, we obtain the following result.

Proposition 2.3 (cf. [1, Proposition 4]). *Let $z = a_n\cdots a_1 \in M_n$. Then the center of $K[M_n]$ is $Z(K[M_n]) = K[z]$.*

If we denote by M_n' the quotient of M_n modulo its center $\langle z\rangle$, i.e.,

$$M_n' = M_n/(z = 1),$$

then we get a useful isomorphism.

Proposition 2.4 (cf. [1, Lemma 5]). *Let $z = a_n\cdots a_1 \in M_n$. Then $M_n\langle z\rangle^{-1} \cong \mathbb{Z}\times M_n'$. In particular, $K[M_n]\langle z\rangle^{-1} \cong K[x,x^{-1}][M_n']$.*

Proof. If we denote by $d(w)$ the degree of the word $w \in M_n\langle z\rangle^{-1}$ with respect to a_n, and by w' the image of the word $w \in M_n\langle z\rangle^{-1}$ in M_n', then we obtain a monoid epimorphism

$$M_n\langle z\rangle^{-1} \ni w \mapsto (d(w), w') \in \mathbb{Z}\times M_n'.$$

Moreover, if $d(w_1) = d(w_2)$ and $w_1' = w_2'$ for some $w_1, w_2 \in M_n\langle z\rangle^{-1}$, then $z^{n_1}w_1 = z^{n_2}w_2$ for some $n_1, n_2 \in \mathbb{Z}$. Since

$$n_1 + d(w_1) = d(z^{n_1}w_1) = d(z^{n_2}w_2) = n_2 + d(w_2),$$

we have $n_1 = n_2$, and Proposition 2.3 ends the proof. □

In the similar manner, we can consider the central localization

$$K[M_n](K[z] \setminus \{0\})^{-1},$$

and get the following isomorphism.

Proposition 2.5 (cf. [1, Lemma 6]). *Let $z = a_n \cdots a_1 \in M_n$. Then $K[M_n](K[z] \setminus \{0\})^{-1} \cong K(x)[M'_n]$.*

Proof. It is easy to see that the natural extension of the K-algebra monomorphism $K[M_n]\langle z\rangle^{-1} \to K(x)[M'_n]$, obtained from Proposition 2.4, to $K[M_n](K[z] \setminus \{0\})^{-1}$ is an isomorphism of K-algebras. □

Let us recall that Proposition 2.5 allows us to study some properties of $K[M_n]$ via the properties of the algebra $K(x)[M'_n]$.

A first question that we can ask is whether the plactic algebra is prime or semiprime. In fact we have some negative result. Let $n \geq 3$ and define

$$\alpha = (a_{n-1} \cdots a_1)(a_n \cdots a_2) - (a_n \cdots a_1)(a_{n-1} \cdots a_2),$$

$$\beta = a_1 a_n - a_n a_1, \qquad \gamma = a_2 a_n - a_n a_2.$$

Using Proposition 2.2 and the canonical form of words in M_n one can check that $\alpha K[M_n]\beta = 0$. Moreover, if $n \geq 4$, then $\alpha\gamma \neq 0$ but $\alpha\gamma K[M_n]\alpha\gamma = 0$. Putting this together we obtain the following result.

Theorem 2.6 (cf. [1, Theorem 9]). *Let $K[M_n]$ be the plactic algebra of rank n. Then*

(1) *if $n \geq 3$ then $K[M_n]$ is not prime,*
(2) *if $n \geq 4$ then $K[M_n]$ is not semiprime.*

By Proposition 1.4 we immediately obtain the following result.

Theorem 2.7. *Let $K[M_n]$ be the plactic algebra of rank $n \geq 2$. Then $K[M_n]$ is neither left nor right noetherian.*

Proof. Sending the generators $a_3, \ldots, a_n$ (if $n \geq 3$) to zero in $K[M_2]$ we get the epimorphism $K[M_n] \to K[M_2]$. But we also have an epimorphism $K[M_2] \to K[B]$. Composing these two epimorphisms we obtain an epimorphism $K[M_n] \to K[B]$. If the plactic algebra $K[M_n]$ is left or right noetherian, we conclude that $K[B]$ inherits this property, a contradiction with Proposition 1.4. □

Similarly, Corollary 1.8 implies the following result.

Theorem 2.8. *Let $K[M_n]$ be the plactic algebra of rank $n \geq 2$. Then $K[M_n]$ does not satisfy any non-trivial polynomial identity.*

Proof. As in the proof of Proposition 2.7 we have an epimorphism $K[M_n] \to K[B]$. By Corollary 1.8 the bicyclic algebra $K[B]$ does not satisfy any non-trivial polynomial identity, so $K[M_n]$ also has this property. □

As mentioned at the beginning of this section, the canonical form of elements in M_n ensures that the growth of the plactic monoid M_n is polynomially bounded. This quickly leads to the conclusion that the Gelfand-Kirillov dimension of the plactic algebra $K[M_n]$ is finite. In the following theorem we give an explicit formula.

Theorem 2.9. $\operatorname{GKdim} K[M_n] = n(n+1)/2$.

Proof. Let $V = Ka_1 + \cdots + Ka_n$ be the generating subspace of $K[M_n]$. By [8, Corollary 6.3.11] we know that $\dim_K V^k$, the cardinality of the set of words of degree k in M_n, is equal to the coefficient of z^k in

$$\frac{1}{(1-z)^n} \cdot \frac{1}{(1-z^2)^{n(n-1)/2}}.$$

Because $1/(1-z) = \sum_{i \geq 0} z^i$ and $1/(1-z^2) = \sum_{j \geq 0} z^{2j}$, we conclude that

$$(\dim_K V^k) z^k = \sum_{\substack{i_1, \ldots, i_n, j_1, \ldots, j_{n(n-1)/2} \geq 0 \\ i_1 + \cdots + i_n + 2j_1 + \cdots + 2j_{n(n-1)/2} = k}} z^{i_1} \cdots z^{i_n} z^{2j_1} \cdots z^{2j_{n(n-1)/2}}.$$

Thus

$$d_{K[M_n]}(k) = \left| \{ (a_{ij})_{1 \leq i \leq j \leq n} \in \mathbb{N}^{n(n+1)/2} : \sum_{1 \leq i \leq n} a_{ii} + 2 \sum_{1 \leq i < j \leq n} a_{ij} \leq k \} \right|.$$

Let

$$d(k) = \left| \{ (a_{ij})_{1 \leq i \leq j \leq n} \in \mathbb{N}^{n(n+1)/2} : \sum_{1 \leq i \leq j \leq n} a_{ij} \leq k \} \right|.$$

By [6, Example 1.6] we know that $d(k)$ is the growth function of the polynomial algebra $K[x_1, \ldots, x_{n(n+1)/2}]$ in $n(n+1)/2$ variables. Therefore $d(k)$ behaves like $Ck^{n(n+1)/2}$ for sufficiently large k, where C is a positive constant. Moreover, as can be easily verified

$$d_{K[M_n]}(k) \leq d(k) \leq d_{K[M_n]}(2k) \qquad \text{for } k \geq 0,$$

hence we have

$$\begin{aligned} \operatorname{GKdim} K[M_n] &= \limsup_{k \to \infty} \frac{\log d_{K[M_n]}(k)}{\log k} \\ &= \limsup_{k \to \infty} \frac{\log d(k)}{\log k} \\ &= \limsup_{k \to \infty} \frac{\log Ck^{n(n+1)/2}}{\log k} \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

□

2.1. **Plactic algebra of rank 1.** In this case $K[M_1] \cong K[x]$ the polynomial algebra in one variable. It is well known that this algebra is commutative, noetherian, prime and semiprimitive. Also the prime spectrum of $K[M_1]$ is pretty well understood. It consists of zero ideal and maximal ideals, which are in one-to-one correspondence with irreducible polynomials in $K[x]$. In particular, $\operatorname{clKdim} K[M_1] = 1$. Moreover, by Theorem 2.9, we also have $\operatorname{GKdim} K[M_1] = 1$.

2.2. **Plactic algebra of rank 2.** To prove primeness of the plactic algebra $K[M_2]$, we firstly prove a simple fact about central localization.

Lemma 2.10. *Let Q be a central localization of a ring R. Then if Q is a prime ring then R is also prime.*

Proof. Let $Q = RS^{-1}$ be a central localization of R with respect to a multiplicatively closed set $S \subseteq Z(R)$ consisting of regular elements. Suppose that R is not prime. Then there exist ideals $0 \neq I, J \subseteq R$ such that $IJ = 0$. Thus we have

$$IQJ = I(RS^{-1})J = (IRJ)S^{-1} = 0,$$

which contradicts the hypothesis on Q. □

As a consequence of Propositions 1.10, 2.5 and Lemma 2.10 we get the following result.

Theorem 2.11 (cf. [1, Theorem 10])**.** *The plactic algebra $K[M_2]$ is prime.*

Proof. Since $M_2' \cong B$, Proposition 1.10 implies that $K(x)[M_2']$ is left primitive, hence prime. By Proposition 2.5 the central localization of $K[M_2]$ is prime, and Lemma 2.10 ends the proof. □

To prove semiprimitivity of the plactic algebra $K[M_2]$ we construct a suitable family of left (and, in fact, right) primitive ideals of $K[M_2]$.

Theorem 2.12 (cf. [1, Theorem 11])**.** *Let $P_f = K[M_2]f(a_2a_1)K[M_2]$ for every irreducible monic polynomial $x \neq f(x) \in K[x]$. Then $\{P_f\}$ is a family of left and right primitive ideals of $K[M_2]$ such that $\bigcap_f P_f = 0$. In particular, the plactic algebra $K[M_2]$ is semiprimitive.*

Proof. Let $K\langle x_1, x_2, z\rangle$ be the free K-algebra on three generators. Consider the K-algebra epimorphism $K\langle x_1, x_2, z\rangle \to K[M_2]/P_f$ defined by

$$x_1 \mapsto a_1 + P_f, \qquad x_2 \mapsto a_2 + P_f, \qquad z \mapsto a_2a_1 + P_f.$$

It can be easily verified that the ideal

$$(x_1z - zx_1, x_2z - zx_2, x_2x_1 - z, f(z))$$

is the kernel of the above epimorphism. Therefore

$$\begin{aligned}K[M_2]/P_f &\cong K\langle x_1, x_2, z\rangle/(x_1 z - z x_1, x_2 z - z x_2, x_2 x_1 - z, f(z))\\ &\cong K[z]\langle x_1, x_2\rangle/(x_2 x_1 - z, f(z))\\ &\cong (K[z]/K[z]f(z)K[z])\langle x_1, x_2\rangle/(x_2 x_1 - \bar{z})\\ &\cong L[B],\end{aligned}$$

where $L = K[z]/K[z]f(z)K[z]$ is a field, and $\bar{z}$ is the image of z in L. By Proposition 1.10 the algebra $K[M_2]/P_f$ is left and right primitive, so P_f is left and right primitive ideal of $K[M_2]$.

Notice that if $x \neq f_1(x), f_2(x) \in K[x]$ are different irreducible monic polynomials, then they are comaximal in $K[x]$. Since $a_2 a_1 \in Z(K[M_2])$ it follows that P_{f_1} and P_{f_2} are comaximal ideals of $K[M_2]$. Thus we conclude that $P_{f_1} \cap P_{f_2} = P_{f_1} P_{f_2} = P_{f_1 f_2}$. This implies that any non-zero element of $\bigcap_f P_f$ has arbitrarily long word in its support. This contradiction completes the proof of the theorem. □

The structure of the prime spectrum of the plactic algebra $K[M_2]$ is also quite well understood. It can be shown (cf. [1, Theorem 12 and comment before]) that, for any height one prime ideal P in $K[M_2]$, there are only three possibilities:

(1) if $P \cap M_2 \neq \varnothing$, then $P = K[M_2]a_1 K[M_2]$ or $P = K[M_2]a_2 K[M_2]$ and $K[M_2]/P \cong K[x]$ the polynomial algebra in one variable;
(2) if $P \cap M_2 = \varnothing$ but $P \cap K[z] \neq 0$, where $z = a_2 a_1 \in M_2$, then $P = (P \cap K[z])K[M_2]$;
(3) while if $P \cap K[z] = 0$, then P can be viewed as a prime ideal of $K[M_2](K[z] \setminus \{0\})^{-1} \cong K(x)[B]$.

This allows us to compute classical Krull dimension of $K[M_2]$. For example, let

$$P = K[M_2](a_1 a_2 - a_2 a_1)K[M_2].$$

Then we have $K[M_2]/P \cong K[x, y]$ the polynomial algebra in two variables. In particular, $\operatorname{clKdim} K[M_2] \geq 3$ but, in fact, the above characterization of height one prime ideals in $K[M_2]$ implies that the equality $\operatorname{clKdim} K[M_2] = 3$ holds. Moreover, by Theorem 2.9, we also have $\operatorname{GKdim} K[M_2] = 3$.

2.3. **Plactic algebra of rank 3.** Although Theorem 2.6 implies that the plactic algebra $K[M_3]$ is not prime it is still semiprimitive.

Theorem 2.13 (cf. [1, Theorem 18])**.** *The plactic algebra $K[M_3]$ is semiprimitive.*

Proof of Theorem 2.13 is based on several observations. Firstly notice that $J(K[M_3])$ is a homogeneous ideal. This implies that if there exists $0 \neq \alpha \in J(K[M_3])$, then we may assume that α is homogeneous with respect to the gradation on $K[M_3]$ defined by the length of words in M_3, and also by the degree in each of the generators. Next, it can be shown (cf. [1, Theorem 17]) that any non-zero homogeneous (with respect to the degree in each of the generators) ideal $I \subseteq K[M_3]$ contains an element $\alpha \neq 0$ such that $|\mathrm{Supp}(\alpha)| \leq 2$. Putting this together and using more technical lemmas (cf. [1, Lemmas 14, 15, 16]), which give some description of homogeneous elements not belonging to $J(K[M_3])$, and characterize when $a_3a_2v = a_3a_2w$ and $va_2a_1 = wa_2a_1$ for different words $v, w \in M_3$, we get the desired result.

In case of rank 3 plactic algebra the complete characterization of the prime spectrum is still not known. However, one can show that there are only two minimal prime ideals

$$P_1 = K[M_3](a_1a_3 - a_3a_1)K[M_3],$$
$$P_2 = K[M_3](a_2a_1a_3a_2 - a_3a_2a_1a_2)K[M_3].$$

These ideals are principal, and each of them is determined by a homogeneous congruence on M_3. Since $K[M_3]$ is semiprime, it follows that $P_1 \cap P_2 = 0$. In particular, $K[M_3]$ is a subdirect product of $K[M_3]/P_1$ and $K[M_3]/P_2$, hence we have an embedding

$$M_3 \to M_3/(a_1a_3 = a_3a_1) \times M_3/(a_2a_1a_3a_2 = a_3a_2a_1a_2).$$

Moreover, by Theorem 2.9, we also have $\mathrm{GKdim}\, K[M_3] = 6$.

3. Open problems

The results of this paper contribute to the general program of studying finitely presented algebras defined by homogeneous semigroup relations. Recall that an algebra A (associative with unity) over a field K is defined by homogeneous semigroup relations, if it is isomorphic to the monoid algebra $K[M]$, where M is the monoid defined by the relations of the form $v = w$, where v and w are words of equal lengths in the corresponding free monoid.

Certain important classes of such algebras, and of the underlying monoids, have been recently considered (cf. [4, 5]). Clearly, the plactic algebra $K[M_n]$, and also the related Chinese algebra, is of this type. These algebras are defined by homogeneous semigroup relations of degree 3.

For certain important constructions of algebras defined by homogeneous semigroup relations it was shown (cf. [4, 5]) that the minimal prime ideals have a very special form, which proved to have far reaching consequences for the properties of the algebra. One might expect that this is a general phenomenon occurring in this class of algebras. There are also other results showing that the class of algebras defined by homogeneous semigroup relations has very special properties (cf. [5]). In consideration of algebras of this type also the irreducible representations should play a crucial role.

Our aim is to consider problems of this type for the class of plactic algebras. Since these problems are not solved in general, we started with the simplest cases, namely plactic algebras of low ranks. However, these results motivate the research in the general case of plactic algebras of higher ranks. We state the main problems that should be explored in the future.

Problem 3.1. Determine the irreducible representations of $K[M_n]$.

Problem 3.2. Describe the minimal prime ideals of $K[M_n]$. Are they homogeneous? Is the number of such ideals finite?

Problem 3.3. Do we have $J(K[M_n]) = B(K[M_n])$? Is it nilpotent? Is it finitely generated?

As mentioned, the solution of these problems can provide answers to many important questions related with the plactic monoid M_n, and the algebra $K[M_n]$. For example, by embedding M_n in $K[M_n]/J(K[M_n])$ (as in the case of M_3), we might obtain a new description of the plactic monoid M_n. On the other hand, the equality $J(K[M_n]) = B(K[M_n])$ would guarantee local nilpotency of the Jacobson radical $J(K[M_n])$, and thus would confirm the famous Zelmanov's conjecture in the class of plactic algebras.

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