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# Diamond Lemma and Gröbner Bases for Plactic Algebras

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# DIAMOND LEMMA AND GRÖBNER BASES FOR PLACTIC ALGEBRAS

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Abstract. We sketch the general Gröbner-Shirshov bases theory for the associative free algebra over an arbitrary field. In particular, we mention the concept of reductions and the notion of the generalized $S$-polynomial. We also introduce the Bergman's diamond lemma together with the closely related concept of ambiguities, and discuss its connections with the Gröbner-Shirshov bases. Finally, we construct finite Gröbner-Shirshov bases for plactic algebras of rank $n \leq 3$, and we prove that plactic algebras of rank $n \geq 4$ do not have finite Gröbner-Shirshov bases associated to the natural degree-lexicographic order on the corresponding free monoid.

## 1. General Gröbner-Shirshov bases and Bergman's diamond lemma

Commutative Gröbner bases were introduced by Buchberger in his doctoral dissertation in 1965 (cf. [1, 2, 11, 18, 19, 20]). Today they are well-known and widely applied to many problems in mathematics, computer science and engineering. For example, the word problem for commutative algebra presentations can be solved by Gröbner bases theory. In 1978 Bergman introduced his diamond lemma for ring theory (cf. [3]). As Mora has pointed out (cf. [19, 20]), Bergman's diamond lemma essentially contains a generalization of commutative Gröbner bases theory to the general non-commutative polynomial rings, which are also associative free algebras. Because in the non-commutative case similar methods were used by Shirshov (cf. [4]), the non-commutative Gröbner bases will be called the Gröbner-Shirshov bases.

1.1. **General Gröbner-Shirshov bases.** Let $\langle X \rangle$ denotes the free monoid generated by a set $X$. A typical element $1 \neq w = x_1 \cdots x_n$ ($x_i \in X$) of $\langle X \rangle$ is called a *word* (or *monomial*), and $n \geq 1$ is called the *degree* of the word $w$, denoted by $\deg(w)$. The degree of the identity in $\langle X \rangle$ is defined to be zero. Moreover, if $w = lvr$ for some $l, r, v, w \in \langle X \rangle$, then $w$ is said to be a *multiple* of $v$ or we say that $v$ *divides* $w$, denoted by $v \mid w$.

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**Definition 1.1.** A well order $\leq$ on the free monoid $\langle X\rangle$ is said to be a *monomial order* on $\langle X\rangle$ if the following conditions are satisfied:

(1) $1 \leq w$ for all $w \in \langle X\rangle$,
(2) if $v \leq w$ then $lvr \leq lwr$ for all $l, r, v, w \in \langle X\rangle$.

The strict part of $\leq$, denoted by $<$, is defined by $v < w$ if $v \leq w$ and $v \neq w$. The inverse of $\leq$, denoted by $\geq$, is defined by $w \geq v$ if $v \leq w$. Similarly, we define the strict part of $\geq$, denoted by $>$.

Suppose $X$ is well-ordered by $\prec$. Then we may extend $\prec$ to the *lexicographic order* on the monoid $\langle X\rangle$, also denoted by $\prec$, as follows. For $v, w \in \langle X\rangle$ let $v \prec w$ if $w = vr$ for some $1 \neq r \in \langle X\rangle$ or $v = lx_1r_1$, $w = lx_2r_2$ for some $l, r_1, r_2 \in \langle X\rangle$ and $x_1, x_2 \in X$ with $x_1 \prec x_2$. Unfortunately, if $|X| \geq 2$, it is not a monomial order on $\langle X\rangle$. Indeed, if $x \prec y$ in $X$, then we have $x^{j+1}y \prec x^j y$ for all $j \geq 0$. However, the *degree-lexicographic order* on $\langle X\rangle$, defined by $v < w$ if $\deg(v) < \deg(w)$ or $\deg(v) = \deg(w)$ and $v \prec w$ lexicographically, is a monomial order on $\langle X\rangle$. This order will be crucial for us.

Now, let $K$ be a field. Denote by $K\langle X\rangle$ the monoid algebra over $K$ of the free monoid $\langle X\rangle$. This algebra is known to be unital associative free $K$-algebra generated by the set $X$. Furthermore, if $|X| = 1$, then $K\langle X\rangle$ is a commutative polynomial ring in one variable. If $|X| \geq 2$, then $K\langle X\rangle$ is a non-commutative polynomial ring.

A typical element $0 \neq f \in K\langle X\rangle$ is called a *polynomial*, and it can be written in the unique form $f = \sum_{i=1}^{n} \alpha_i w_i$, where $0 \neq \alpha_i \in K$, $w_i \in \langle X\rangle$ and $w_i \neq w_j$ for $i \neq j$. The set $\{w_1, \ldots, w_n\}$ is called the *support* of $f$, denoted by $\mathrm{Supp}(f)$. Recall that every $\alpha_i w_i$ is called a *term* of $f$, and $\alpha_i$ is called the *coefficient* of this term. In addition, if $\leq$ is a monomial order on $\langle X\rangle$, then we may assume $w_1 > \cdots > w_n$. In this case, $\alpha_1 w_1$ is called the *leading term* of $f$ with respect to $\leq$, denoted by $\mathrm{LT}(f)$; $w_1$ is called the *leading monomial* of $f$ with respect to $\leq$, denoted by $\mathrm{LM}(f)$; and $\alpha_1$ is called the *leading coefficient* of $f$ with respect to $\leq$, denoted by $\mathrm{LC}(f)$.

**Definition 1.2.** The *leading monomial ideal* $\mathrm{LM}(G)$ with respect to a monomial order on $\langle X\rangle$ of a set $G \subseteq K\langle X\rangle \setminus \{0\}$ is defined to be the ideal generated by the monomials $\mathrm{LM}(g)$ for $g \in G$. Moreover, a set $G$ is said to be a *Gröbner-Shirshov basis* for an ideal $I \subseteq K\langle X\rangle$ with respect to a monomial order $\leq$ if $G \subseteq I$ and $\mathrm{LM}(G) = \mathrm{LM}(I \setminus \{0\})$.

Unlike commutative Gröbner bases, Gröbner-Shirshov bases are allowed to be infinite. There are at least two reasons for this. Firstly, finite Gröbner-Shirshov bases do not exist for some ideals of $K\langle X\rangle$. Note that infinite Gröbner-Shirshov bases always exist. Secondly, there do exist infinite sets of non-commutative polynomials which can play the same role as finite Gröbner-Shirshov bases.

**Proposition 1.3** (cf. [11, Theorem 3.3.8])**.** *Let $I$ be an ideal in $K\langle X\rangle$, and $G$ be a Gröbner-Shirshov basis for $I$ with respect to a monomial order $\leq$ on the free monoid $\langle X\rangle$. Then $G$ generates $I$ as an ideal.*

Next, we shall define a crucial notion of a polynomial reduction, a reduced polynomial and a reduced form of a polynomial.

**Definition 1.4.** Let $0 \neq f, g \in K\langle X\rangle$. If $\mathrm{LM}(g)$ divides some monomial $w \in \mathrm{Supp}(f)$, then $w = l\,\mathrm{LM}(g)r$ for some $l, r \in \langle X\rangle$. Suppose $\alpha \in K$ is the coefficient of $f$ corresponding to $w$, and let

$$h = f - \frac{\alpha}{\mathrm{LC}(g)} lgr.$$

In this case we say that $f$ *reduces to $h$ modulo $g$*, denoted by $f \xrightarrow{g} h$. The above process is called a *polynomial reduction.* Moreover, if there is a finite sequence of polynomial reductions

$$f \xrightarrow{g_1} \cdots \xrightarrow{g_n} h_n,$$

where $h_i \in K\langle X\rangle$ and $g_i \in G \subseteq K\langle X\rangle \setminus \{0\}$, then we say that $f$ *reduces to $h_n$ modulo $G$*. For convenience, zero always reduces to zero.

It is easy to see that in the reduction $f \xrightarrow{g} h$ the term $\alpha w$ in $f$ is replaced by a linear combination of monomials $< w$.

**Definition 1.5.** Let $G \subseteq K\langle X\rangle \setminus \{0\}$. A polynomial $f \in K\langle X\rangle$ is said to be *$G$-reduced* if no monomial in $\mathrm{Supp}(f)$ is divisible by $\mathrm{LM}(g)$ for some $g \in G$. Moreover, we say that $f$ has a *reduced form* with respect to $G$ if $f$ reduces to $h \in K\langle X\rangle$ modulo $G$ and $h$ is $G$-reduced. When the reduced form of $f$ with respect to $G$ is unique, it is denoted by $R_G(f)$.

Notice that if $f$ is $G$-reduced then $f$ has a unique reduced form with respect to $G$ given by $R_G(f) = f$. Furthermore, we have the following result.

**Proposition 1.6** (cf. [11, Reduction Process 3.4.6])**.** *Let $f \in K\langle X\rangle$ and $G \subseteq K\langle X\rangle \setminus \{0\}$. Then $f$ has a reduced form with respect to $G$.*

The process mentioned above shows that $f$ has the following representation

$$f = \sum_{i=1}^{n} \alpha_i l_i g_i r_i + h,$$

where $\alpha_i \in K$, $l_i, r_i \in \langle X\rangle$, $g_i \in G$ and $h \in K\langle X\rangle$ is a reduced form of $f$ with respect to $G$. In particular, if $h = 0$, then the above representation is called a *standard representation* of $f$ with respect to $G$.

Now, we shall introduce the non-commutative generalization of the notion of Buchberger's $S$-polynomial (cf. [11, 18, 19, 20]), which allows us to provide a characterization of the Gröbner-Shirshov bases.

**Definition 1.7.** The *set of matches* of a pair $(w_1, w_2) \in \langle X \rangle^2$, denoted by $\mathrm{MS}(w_1, w_2)$, is the set of all ordered 4-tuples $(l_1, r_1; l_2, r_2) \in \langle X \rangle^4$, such that $l_1 w_1 r_1 = l_2 w_2 r_2$ and satisfying one of the following conditions:

(1) $l_1, r_2 = 1$, $r_1, l_2 \neq 1$ and $w_1 = l_2 w$, $w_2 = w r_1$ for some $w \neq 1$,
(2) $l_2, r_1 = 1$, $l_1, r_2 \neq 1$ and $w_1 = w r_2$, $w_2 = l_1 w$ for some $w \neq 1$,
(3) $l_1, r_1 = 1$ and $w_1 = l_2 w_2 r_2$,
(4) $l_2, r_2 = 1$ and $w_2 = l_1 w_1 r_1$.

Note that the set of matches $\mathrm{MS}(w_1, w_2)$ is necessarily finite. However, it may also be empty.

**Definition 1.8.** Let $0 \neq f_1, f_2 \in K\langle X \rangle$. The *S-polynomial* of $f_1$ and $f_2$ with respect to $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(f_1), \mathrm{LM}(f_2))$ is defined by

$$S(f_1, f_2; l_1, r_1; l_2, r_2) = \frac{1}{\mathrm{LC}(f_1)} l_1 f_1 r_1 - \frac{1}{\mathrm{LC}(f_2)} l_2 f_2 r_2.$$

**Theorem 1.9** (cf. [11, Theorem 3.6.1])**.** *Let $\leq$ be a monomial order on the free monoid $\langle X \rangle$. Let $I$ be an ideal in $K\langle X \rangle$, and $G \subseteq I \setminus \{0\}$. Then the following conditions are equivalent:*

(1) *$G$ is a Gröbner-Shirshov basis for $I$ with respect to $\leq$,*
(2) *for every $0 \neq f \in I$ there exists $g \in G$ such that $\mathrm{LM}(g) \mid \mathrm{LM}(f)$,*
(3) *for every $f \in K\langle X \rangle$, $f$ reduces to zero modulo $G$ if and only if $f \in I$,*
(4) *for every $f \in K\langle X \rangle$, $f$ has a standard representation with respect to $G$ if and only if $f \in I$,*
(5) *for every $f \in K\langle X \rangle$, $f$ has a unique reduced form with respect to $G$,*
(6) *as $K$-vector spaces, $K\langle X \rangle = I \oplus \mathrm{Span}_K \langle X \rangle \setminus \mathrm{LM}(G)$,*
(7) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, $S(g_1, g_2; l_1, r_1; l_2, r_2)$ reduces to zero with respect to $G$,*
(8) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, $S(g_1, g_2; l_1, r_1; l_2, r_2)$ has a standard representation with respect to $G$.*

1.2. **Bergman's diamond lemma.** Let $C$ be any unital commutative associative ring, and let $C\langle X \rangle$ denotes, as before, the unital associative free $C$-algebra generated by a set $X$. Notice that $C\langle X \rangle$ is also a free $C$-module with the free monoid $\langle X \rangle$ as a basis.

**Definition 1.10.** The *reduction system* $S$ for $C\langle X \rangle$ is defined to be a set of pairs $\sigma = (w_\sigma, f_\sigma)$ such that $w_\sigma \in \langle X \rangle$ and $f_\sigma \in C\langle X \rangle$ with $w_\sigma \notin \mathrm{Supp}(f_\sigma)$. The *reduction* $R_{l\sigma r}$ associated with $\sigma \in S$ and $l, r \in \langle X \rangle$ is defined to be a $C$-module endomorphism of $C\langle X \rangle$ which sends the monomial $l w_\sigma r$ to $l f_\sigma r$, but fixes all the other monomials. A reduction $R_{l\sigma r}$ is said to be *trivial* on $f \in C\langle X \rangle$ if $R_{l\sigma r}(f) = f$. Moreover, if every reduction is trivial on $f$, then $f$ is said to be *S-reduced.*

It is easy to see that the set of all $S$-reduced polynomials in $C\langle X\rangle$, denoted by $\mathrm{Red}_S(C\langle X\rangle)$, forms a $C$-submodule of $C\langle X\rangle$.

**Definition 1.11.** Let $f \in C\langle X\rangle$. If there is a finite sequence of reductions $R_1, \dots, R_n$ such that $R_n \cdots R_1(f)$ is $S$-reduced, then it is said to be a *reduced form* of $f$ with respect to $S$. If for any infinite sequence of reductions $(R_j)_{j=1}^{\infty}$ there exists $n \geq 1$ such that $R_{j+1}$ is trivial on $R_j \cdots R_1(f)$ for all $j \geq n$, then $f$ is said to be *$S$-reduction-finite*. Moreover, $f$ is said to be *$S$-reduction-unique* if it is $S$-reduction-finite and has a unique reduced form with respect to $S$, denoted by $R_S(f)$.

It can be easily verified that the set of all $S$-reduction-finite elements in $C\langle X\rangle$ forms a $C$-submodule of $C\langle X\rangle$. In addition, if a polynomial $f$ is $S$-reduction-finite, then $f$ has a $S$-reduced form.

**Lemma 1.12** (cf. [3, Lemma 1.1])**.** *Let $S$ be a reduction system for $C\langle X\rangle$. Then the set of all $S$-reduction-unique elements in $C\langle X\rangle$ forms a $C$-submodule of $C\langle X\rangle$, and $R_S$ is a $C$-linear map from this submodule to the submodule $\mathrm{Red}_S(C\langle X\rangle)$ of all $S$-reduced elements in $C\langle X\rangle$. Moreover, if $f, g, h \in C\langle X\rangle$ are such that $uvw$ is $S$-reduction-unique for all $u \in \mathrm{Supp}(f)$, $v \in \mathrm{Supp}(g)$ and $w \in \mathrm{Supp}(h)$, then for any finite composition of reductions $R$, $fR(g)h$ is $S$-reduction-unique and $R_S(fR(g)h) = R_S(fgh)$.*

**Definition 1.13.** Let $S$ be a reduction system for $C\langle X\rangle$. A 5-tuple $(\sigma, \tau, l, w, r)$ with $\sigma, \tau \in S$ and $1 \neq l, w, r \in \langle X\rangle$ such that $w_\sigma = lw$ and $w_\tau = wr$ is called an *overlap ambiguity* of $S$. Similarly, a 5-tuple $(\sigma, \tau, l, w, r)$ with $\sigma, \tau \in S$, $\sigma \neq \tau$ and $l, w, r \in \langle X\rangle$ such that $w_\sigma = w$ and $w_\tau = lwr$ is called an *inclusion ambiguity* of $S$. We say that the overlap (resp. inclusion) ambiguity $(\sigma, \tau, l, w, r)$ is *resolvable* if there exist finite compositions of reductions, $R$ and $R'$ say, such that $R(f_\sigma r) = R'(lf_\tau)$ (resp. $R(lf_\sigma r) = R'(f_\tau)$).

Here, we consider a weakening of the notion of a monomial order. By a *monomial partial order* on the free monoid $\langle X\rangle$ we mean a partial order $\leq$ on $\langle X\rangle$ such that $v \leq w$ implies $lvr \leq lwr$ for all $l, r, v, w \in \langle X\rangle$. We say that $\leq$ satisfies *descending chain condition* on $\langle X\rangle$ if there is no infinite properly descending chain in $\langle X\rangle$ with respect to $\leq$.

**Definition 1.14.** A monomial partial order $\leq$ on the free monoid $\langle X\rangle$ is said to be *compatible* with a reduction system $S$ for $C\langle X\rangle$ if $\mathrm{LM}(f_\sigma) < w_\sigma$ for all $\sigma \in S$.

**Lemma 1.15** (cf. [11, Lemma 4.2.6])**.** *Let $\leq$ be a monomial partial order on $\langle X\rangle$ compatible with a reduction system $S$ for $C\langle X\rangle$. If $\leq$ satisfies descending chain condition on $\langle X\rangle$, then every element of $C\langle X\rangle$ is $S$-reduction-finite.*

For a reduction system $S$ for $C\langle X\rangle$, let $\mathrm{Id}_S(C\langle X\rangle)$ denotes the ideal generated by the elements $w_\sigma - f_\sigma$ for $\sigma \in S$. In addition, if a monomial partial order $\leq$ on $\langle X\rangle$ is compatible with $S$, then for any $w \in \langle X\rangle$ we denote by $M_w$ the $C$-submodule of $C\langle X\rangle$ spanned by all elements $l(w_\sigma - f_\sigma)r$, where $\sigma \in S$ and $l, r \in \langle X\rangle$ with $lw_\sigma r < w$.

**Definition 1.16.** An overlap (resp. inclusion) ambiguity $(\sigma, \tau, l, w, r)$ of a reduction system $S$ for $C\langle X\rangle$ is said to be *resolvable relative to* $\leq$ if $f_\sigma r - lf_\tau \in M_{lwr}$ (resp. $lf_\sigma r - f_\tau \in M_{lwr}$).

Using Lemma 1.12 and Lemma 1.15 on can prove the following result.

**Theorem 1.17** (Bergman's diamond lemma (cf. [3, Theorem 1.2]))**.** *Let $S$ be a reduction system for $C\langle X\rangle$. Let $\leq$ be a monomial partial order on $\langle X\rangle$ compatible with $S$ and having descending chain condition on $\langle X\rangle$. Then the following conditions are equivalent:*

(1) *all ambiguities of $S$ are resolvable,*
(2) *all ambiguities of $S$ are resolvable relative to $\leq$,*
(3) *all elements of $C\langle X\rangle$ are $S$-reduction-unique,*
(4) *as $C$-modules, $C\langle X\rangle = \mathrm{Id}_S(C\langle X\rangle) \oplus \mathrm{Red}_S(C\langle X\rangle)$.*

*When this conditions hold, then the $C$-algebra $C\langle X\rangle / \mathrm{Id}_S(C\langle X\rangle)$ may be identified with the $C$-module $\mathrm{Red}_S(C\langle X\rangle)$ made a $C$-algebra by the multiplication $f \cdot g = R_S(fg)$ for $f, g \in \mathrm{Red}_S(C\langle X\rangle)$.*

1.3. **Relations between general Gröbner-Shirshov bases and Bergman's diamond lemma.** Let $K$ be a field and $\leq$ be a monomial order on the free monoid $\langle X\rangle$. Let $G$ be a subset of $K\langle X\rangle \setminus \{0\}$, and assume $\mathrm{LC}(g) = 1$ for all $g \in G$. In this case we define a reduction system $S$ for $K\langle X\rangle$ by

$$S = \{(\mathrm{LM}(g), \mathrm{LM}(g) - g) : g \in G\}.$$

Clearly $\leq$ is compatible with $S$ and each polynomial reduction modulo $G$ corresponds to a Bergman's endomorphism reduction with respect to $S$. This implies that we may translate the "Gröbner-Shirshov notions" to the "Bergman's notions" and vice-versa. In fact, most translations are obvious. For example, $G$-reduced is equivalent to $S$-reduced, $\mathrm{Span}_K \langle X\rangle \setminus \mathrm{LM}(G) = \mathrm{Red}_S(K\langle X\rangle)$ and $R_G(f) = R_S(f)$.

Let us consider the correspondence between $S$-polynomials and overlap (resp. inclusion) ambiguities. The $S$-polynomial of $g_1, g_2 \in G$ with respect to $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ is given by

$$S(g_1, g_2; l_1, r_1; l_2, r_2) = l_1 g_1 r_1 - l_2 g_2 r_2.$$

There are only four possibilities:

(1) $l_1, r_2 = 1$, $l_2, r_1 \neq 1$ and $\mathrm{LM}(g_1) = l_2 w$, $\mathrm{LM}(g_2) = wr_1$ for some $w \neq 1$. This case corresponds to the overlap ambiguity $(\sigma, \tau, l_2, w, r_1)$, where we have $\sigma = (\mathrm{LM}(g_1), \mathrm{LM}(g_1) - g_1)$ and $\tau = (\mathrm{LM}(g_2), \mathrm{LM}(g_2) - g_2)$. Notice that

$$f_\sigma r_1 - l_2 f_\tau = -S(g_1, g_2; l_1, r_1; l_2, r_2).$$

(2) $l_2, r_1 = 1$, $l_1, r_2 \neq 1$ and $\mathrm{LM}(g_1) = wr_2$, $\mathrm{LM}(g_2) = l_1 w$ for some $w \neq 1$. This case corresponds to the overlap ambiguity $(\sigma, \tau, l_1, w, r_2)$, where we have $\sigma = (\mathrm{LM}(g_2), \mathrm{LM}(g_2) - g_2)$ and $\tau = (\mathrm{LM}(g_1), \mathrm{LM}(g_1) - g_1)$. Notice that

$$f_\sigma r_2 - l_1 f_\tau = S(g_1, g_2; l_1, r_1; l_2, r_2).$$

(3) $l_1, r_1 = 1$ and $\mathrm{LM}(g_1) = l_2 \, \mathrm{LM}(g_2) r_2$. This case corresponds to the inclusion ambiguity $(\sigma, \tau, l_2, \mathrm{LM}(g_2), r_2)$, where we have $\sigma = (\mathrm{LM}(g_2), \mathrm{LM}(g_2) - g_2)$ and $\tau = (\mathrm{LM}(g_1), \mathrm{LM}(g_1) - g_1)$. Notice that

$$l_2 f_\sigma r_2 - f_\tau = S(g_1, g_2; l_1, r_1; l_2, r_2).$$

(4) $l_2, r_2 = 1$ and $\mathrm{LM}(g_2) = l_1 \, \mathrm{LM}(g_1) r_1$. This case corresponds to the inclusion ambiguity $(\sigma, \tau, l_1, \mathrm{LM}(g_1), r_1)$, where we have $\sigma = (\mathrm{LM}(g_1), \mathrm{LM}(g_1) - g_1)$ and $\tau = (\mathrm{LM}(g_2), \mathrm{LM}(g_2) - g_2)$. Notice that

$$l_1 f_\sigma r_1 - f_\tau = -S(g_1, g_2; l_1, r_1; l_2, r_2).$$

Conversely, given an overlap ambiguity $(\sigma, \tau, l, w, r)$, by the definition of $S$, $w_\sigma = \mathrm{LM}(g_1) = lw$ and $w_\tau = \mathrm{LM}(g_2) = wr$ for some $g_1, g_2 \in G$. Then $(1, r; l, 1) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ and notice that

$$f_\sigma r - l f_\tau = -S(g_1, g_2; 1, r; l, 1).$$

Similarly, given an inclusion ambiguity $(\sigma, \tau, l, w, r)$, by the definition of $S$, $w_\sigma = \mathrm{LM}(g_1) = w$ and $w_\tau = \mathrm{LM}(g_2) = lwr$ for some $g_1, g_2 \in G$. Then $(l, r; 1, 1) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ and notice that

$$l f_\sigma r - f_\tau = -S(g_1, g_2; l, r; 1, 1).$$

Although the above correspondences are not required to be one-to-one, they are sufficient for us to deduce the following equivalence.

**Proposition 1.18** (cf. [11, Claim 4.3.2]). *Let $\leq$ be a monomial order on the free monoid $\langle X \rangle$. Let $S$ be the reduction system obtained from a set $G \subseteq K\langle X \rangle \setminus \{0\}$. Then all ambiguities of $S$ are resolvable if and only if for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ there exist finite compositions of reductions, $R$ and $R'$ say, such that*

$$R(l_1(\mathrm{LM}(g_1) - g_1) r_1) = R'(l_2(\mathrm{LM}(g_2) - g_2) r_2).$$

*Moreover, all ambiguities of $S$ are resolvable relative to $\leq$ if and only if for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ we have*

$$S(g_1, g_2; l_1, r_1; l_2, r_2) = \sum_{i=1}^{n} \alpha_i v_i h_i w_i,$$

*where $\alpha_i \in K$, $v_i, w_i \in \langle X \rangle$, $h_i \in G$ and*

$$\mathrm{LM}(v_i h_i w_i) < l_1 \, \mathrm{LM}(g_1) r_1 = l_2 \, \mathrm{LM}(g_2) r_2.$$

In summary, combining Theorem 1.9, Theorem 1.17 and Proposition 1.18 we get a full characterization of the Gröbner-Shirshov bases in terms of reductions, $S$-polynomials and Bergman's diamond lemma.

**Theorem 1.19** (cf. [11, Theorem 4.3.4]). *Let $\leq$ be a monomial order on the free monoid $X$ and $G \subseteq K\langle X\rangle \setminus \{0\}$. Let $I$ be the ideal in $K\langle X\rangle$ generated by $G$, and $S$ be the reduction system obtained from a set $G$. Then the following conditions are equivalent:*

(1) *$G$ is a Gröbner-Shirshov basis for $I$ with respect to $\leq$,*
(2) *for every $0 \neq f \in I$ there exists $g \in G$ such that $\mathrm{LM}(g) \mid \mathrm{LM}(f)$,*
(3) *for every $f \in K\langle X\rangle$, $f$ reduces to zero modulo $G$ if and only if $f \in I$,*
(4) *for every $f \in K\langle X\rangle$, $f$ has a standard representation with respect to $G$ if and only if $f \in I$,*
(5) *for every $f \in K\langle X\rangle$, $f$ has a unique reduced form with respect to $G$,*
(6) *as $K$-vector spaces, $K\langle X\rangle = I \oplus \mathrm{Span}_K \langle X\rangle \setminus \mathrm{LM}(G)$,*
(7) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, $S(g_1, g_2; l_1, r_1; l_2, r_2)$ reduces to zero with respect to $G$,*
(8) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, $S(g_1, g_2; l_1, r_1; l_2, r_2)$ has a standard representation with respect to $G$,*
(9) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ there exist finite compositions of reductions with respect to $S$, $R$ and $R'$ say, such that*
$$R(l_1(\mathrm{LM}(g_1) - g_1)r_1) = R'(l_2(\mathrm{LM}(g_2) - g_2)r_2),$$
(10) *for all $g_1, g_2 \in G$ and $(l_1, r_1; l_2, r_2) \in \mathrm{MS}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ we have*
$$S(g_1, g_2; l_1, r_1; l_2, r_2) = \sum_{i=1}^{n} \alpha_i v_i h_i w_i,$$
*where $\alpha_i \in K$, $v_i, w_i \in \langle X\rangle$, $h_i \in G$ and*
$$\mathrm{LM}(v_i h_i w_i) < l_1 \,\mathrm{LM}(g_1) r_1 = l_2 \,\mathrm{LM}(g_2) r_2.$$

## 2. Gröbner-Shirshov bases for plactic algebras

The origin of the plactic monoid $M_n$ stems from Schensted's algorithm that was developed in order to determine the maximal length of a non-increasing subsequence in any finite sequence with elements in the set $\{1, \ldots, n\}$. Moreover, combinatorics of the plactic monoid was also thoroughly studied (cf. [10, 15, 16, 21]). In particular, it is known that elements of $M_n$ admit a canonical normal form, expressed in terms of the associated Young tableaux. Later, deep applications of the plactic monoid to problems in representation theory, algebraic combinatorics, theory of quantum groups and relations to some other important areas of mathematics were discovered (cf. [10, 15, 16, 17]).

Some ring-theoretic properties of the plactic algebra were also described (cf. [7, 13]).

Let $K[M_n]$ denotes the plactic algebra of rank $n \geq 1$ over a filed $K$. So $K[M_n]$ is the monoid algebra over $K$ of the plactic monoid $M_n$ of rank $n$, which is defined by the following finite presentation

$$M_n = \langle X : R \rangle ,$$

where $X = \{x_1, \ldots, x_n\}$ and $R$ is the set of Knuth relations of the form

$$\begin{aligned} x_k x_i x_j &= x_i x_k x_j \qquad \text{for } i \leq j < k, \\ x_j x_k x_i &= x_j x_i x_k \qquad \text{for } i < j \leq k. \end{aligned}$$

If $\langle X \rangle$ denotes the free monoid generated by the set $X$, then $\langle X \rangle$ is equipped with the degree-lexicographic order with respect to the following order $x_1 < \cdots < x_n$ on the set $X$. This natural order is inherent in the nature of the plactic monoid $M_n$, originally developed for the combinatorial problem mentioned above. Notice also that $K[M_n]$ may be viewed as $K\langle X \rangle / \operatorname{Id}(R)$, where $\operatorname{Id}(R)$ is the corresponding ideal for $K[M_n]$ defined by the Knuth relations, i.e., $\operatorname{Id}(R)$ is the ideal in $K\langle X \rangle$ generated by the elements

$$\begin{aligned} &x_k x_i x_j - x_i x_k x_j \qquad \text{for } i \leq j < k, \\ &x_j x_k x_i - x_j x_i x_k \qquad \text{for } i < j \leq k. \end{aligned}$$

A Gröbner-Shirshov basis for $K[M_n]$ is defined to be a Gröbner-Shirshov basis for the corresponding ideal $\operatorname{Id}(R)$ in the free algebra $K\langle X \rangle$.

Because $K[M_1]$ is the polynomial algebra in one variable, so the corresponding ideal of the free algebra is equal to zero. Moreover, for $K[M_2]$, the corresponding ideal of the free algebra $K\langle X \rangle$ is generated by elements $x_2x_2x_1 - x_2x_1x_2$ and $x_2x_1x_1 - x_1x_2x_1$. In this case we have only one ambiguity $(x_2x_2x_1)x_1 = x_2(x_2x_1x_1)$, which is resolvable. By Theorem 1.19 the above elements, viewed as elements of the free algebra $K\langle X \rangle$, form a Gröbner-Shirshov basis for $K[M_2]$.

Our first non-trivial result reads as follows.

**Theorem 2.1** (cf. [14, Theorem 1]). *Let $K[M_3]$ be the plactic algebra of rank 3 over a field $K$. Then $K[M_3]$ has a finite Gröbner-Shirshov basis, associated to the degree-lexicographic order on the corresponding free monoid $\langle X \rangle$. Namely, the following elements, viewed as elements of the free algebra $K\langle X \rangle$, form such a basis:*

(1) $x_3x_3x_2 - x_3x_2x_3$,
(2) $x_3x_2x_2 - x_2x_3x_2$,
(3) $x_3x_3x_1 - x_3x_1x_3$,
(4) $x_3x_1x_1 - x_1x_3x_1$,
(5) $x_2x_2x_1 - x_2x_1x_2$,
(6) $x_2x_1x_1 - x_1x_2x_1$,
(7) $x_2x_3x_1 - x_2x_1x_3$,

(8) $x_3x_1x_2 - x_1x_3x_2$,
(9) $x_3x_2x_1x_2 - x_2x_3x_2x_1$,
(10) $x_3x_2x_1x_3x_1 - x_3x_1x_3x_2x_1$,
(11) $x_3x_2x_3x_2x_1 - x_3x_2x_1x_3x_2$.

*Proof.* First, we list all ambiguities between two types of reductions that can occur in the process of bringing a word to a form that cannot be reduced anymore (notice that such a form exists because the defining relations of $M_3$ are homogeneous):

$(A_1)$ $(x_3x_3x_2)x_2 = x_3(x_3x_2x_2)$,
$(A_2)$ $(x_3x_3x_2)x_2x_1 = x_3x_3(x_2x_2x_1)$,
$(A_3)$ $(x_3x_3x_2)x_1x_1 = x_3x_3(x_2x_1x_1)$,
$(A_4)$ $(x_3x_3x_2)x_3x_1 = x_3x_3(x_2x_3x_1)$,
$(A_5)$ $(x_3x_3x_2)x_1x_2 = x_3(x_3x_2x_1x_2)$,
$(A_6)$ $(x_3x_3x_2)x_1x_3x_1 = x_3(x_3x_2x_1x_3x_1)$,
$(A_7)$ $(x_3x_3x_2)x_3x_2x_1 = x_3(x_3x_2x_3x_2x_1)$,
$(A_8)$ $(x_3x_2x_2)x_1 = x_3(x_2x_2x_1)$,
$(A_9)$ $(x_3x_2x_2)x_2x_1 = x_3x_2(x_2x_2x_1)$,
$(A_{10})$ $(x_3x_2x_2)x_1x_1 = x_3x_2(x_2x_1x_1)$,
$(A_{11})$ $(x_3x_2x_2)x_3x_1 = x_3x_2(x_2x_3x_1)$,
$(A_{12})$ $(x_3x_3x_1)x_1 = x_3(x_3x_1x_1)$,
$(A_{13})$ $(x_3x_3x_1)x_2 = x_3(x_3x_1x_2)$,
$(A_{14})$ $x_2(x_3x_1x_1) = (x_2x_3x_1)x_1$,
$(A_{15})$ $x_3x_2x_1(x_3x_1x_1) = (x_3x_2x_1x_3x_1)x_1$,
$(A_{16})$ $(x_2x_2x_1)x_1 = x_2(x_2x_1x_1)$,
$(A_{17})$ $x_3x_1(x_2x_2x_1) = (x_3x_1x_2)x_2x_1$,
$(A_{18})$ $x_3x_2x_1(x_2x_2x_1) = (x_3x_2x_1x_2)x_2x_1$,
$(A_{19})$ $x_3x_1(x_2x_1x_1) = (x_3x_1x_2)x_1x_1$,
$(A_{20})$ $x_3x_2x_1(x_2x_1x_1) = (x_3x_2x_1x_2)x_1x_1$,
$(A_{21})$ $x_3x_2x_3(x_2x_1x_1) = (x_3x_2x_3x_2x_1)x_1$,
$(A_{22})$ $(x_2x_3x_1)x_2 = x_2(x_3x_1x_2)$,
$(A_{23})$ $x_3x_1(x_2x_3x_1) = (x_3x_1x_2)x_3x_1$,
$(A_{24})$ $x_3x_2x_1(x_2x_3x_1) = (x_3x_2x_1x_2)x_3x_1$,
$(A_{25})$ $x_3x_2x_1(x_3x_1x_2) = (x_3x_2x_1x_3x_1)x_2$,
$(A_{26})$ $x_3x_2(x_3x_2x_1x_2) = (x_3x_2x_3x_2x_1)x_2$,
$(A_{27})$ $x_3x_2(x_3x_2x_1x_3x_1) = (x_3x_2x_3x_2x_1)x_3x_1$.

Next, we shall show that all ambiguities can be resolved, using the reductions provided by the elements (1)–(11) listed above:

$(A_1)$ $(x_3x_3x_2)x_2 \xrightarrow{(1)} x_3x_2x_3x_2$,
$x_3(x_3x_2x_2) \xrightarrow{(2)} x_3x_2x_3x_2$,
$(A_2)$ $(x_3x_3x_2)x_2x_1 \xrightarrow{(1)} x_3x_2x_3x_2x_1 \xrightarrow{(11)} x_3x_2x_1x_3x_2$,
$x_3x_3(x_2x_2x_1) \xrightarrow{(5)} x_3x_3x_2x_1x_2 \xrightarrow{(1)} x_3x_2x_3x_1x_2 \xrightarrow{(7)} x_3x_2x_1x_3x_2$,

$$
\begin{aligned}
(A_3)\quad & (x_3x_3x_2)x_1x_1 \xrightarrow{(1)} x_3x_2x_3x_1x_1 \xrightarrow{(7)} x_3x_2x_1x_3x_1 \xrightarrow{(10)} x_3x_1x_3x_2x_1,\\
& x_3x_3(x_2x_1x_1) \xrightarrow{(6)} x_3x_3x_1x_2x_1 \xrightarrow{(3)} x_3x_1x_3x_2x_1,\\
(A_4)\quad & (x_3x_3x_2)x_3x_1 \xrightarrow{(1)} x_3x_2x_3x_3x_1 \xrightarrow{(3)} x_3x_2x_3x_1x_3,\\
& x_3x_3(x_2x_3x_1) \xrightarrow{(7)} x_3x_3x_2x_1x_3 \xrightarrow{(1)} x_3x_2x_3x_1x_3,\\
(A_5)\quad & (x_3x_3x_2)x_1x_2 \xrightarrow{(1)} x_3x_2x_3x_1x_2 \xrightarrow{(7)} x_3x_2x_1x_3x_2,\\
& x_3(x_3x_2x_1x_2) \xrightarrow{(9)} x_3x_2x_3x_2x_1 \xrightarrow{(11)} x_3x_2x_1x_3x_2,\\
(A_6)\quad & (x_3x_3x_2)x_1x_3x_1 \xrightarrow{(1)} x_3x_2x_3x_1x_3x_1 \xrightarrow{(7)} x_3x_2x_1x_3x_3x_1\\
& \qquad \xrightarrow{(4)} x_3x_2x_1x_3x_1x_3 \xrightarrow{(10)} x_3x_1x_3x_2x_1x_3,\\
& x_3(x_3x_2x_1x_3x_1) \xrightarrow{(11)} x_3x_3x_1x_3x_2x_1 \xrightarrow{(3)} x_3x_1x_3x_3x_2x_1\\
& \qquad \xrightarrow{(1)} x_3x_1x_3x_2x_3x_1 \xrightarrow{(7)} x_3x_1x_3x_2x_1x_3,\\
(A_7)\quad & (x_3x_3x_2)x_3x_2x_1 \xrightarrow{(1)} x_3x_2x_3x_3x_2x_1 \xrightarrow{(1)} x_3x_2x_3x_2x_3x_1\\
& \qquad \xrightarrow{(7)} x_3x_2x_3x_2x_1x_3 \xrightarrow{(11)} x_3x_2x_1x_3x_2x_3,\\
& x_3(x_3x_2x_3x_2x_1) \xrightarrow{(11)} x_3x_3x_2x_1x_3x_2 \xrightarrow{(1)} x_3x_2x_3x_1x_3x_2\\
& \qquad \xrightarrow{(7)} x_3x_2x_1x_3x_3x_2 \xrightarrow{(1)} x_3x_2x_1x_3x_2x_3,\\
(A_8)\quad & (x_3x_2x_2)x_1 \xrightarrow{(2)} x_2x_3x_2x_1,\\
& x_3(x_2x_2x_1) \xrightarrow{(5)} x_3x_2x_1x_2 \xrightarrow{(9)} x_2x_3x_2x_1,\\
(A_9)\quad & (x_3x_2x_2)x_2x_1 \xrightarrow{(2)} x_2x_3x_2x_2x_1 \xrightarrow{(2)} x_2x_2x_3x_2x_1,\\
& x_3x_2(x_2x_2x_1) \xrightarrow{(5)} x_3x_2x_2x_1x_2 \xrightarrow{(2)} x_2x_3x_2x_1x_2 \xrightarrow{(9)} x_2x_2x_3x_2x_1,\\
(A_{10})\quad & (x_3x_2x_2)x_1x_1 \xrightarrow{(1)} x_2x_3x_2x_1x_1,\\
& x_3x_2(x_2x_1x_1) \xrightarrow{(6)} x_3x_2x_1x_2x_1 \xrightarrow{(9)} x_2x_3x_2x_1x_1,\\
(A_{11})\quad & (x_3x_2x_2)x_3x_1 \xrightarrow{(2)} x_2x_3x_2x_3x_1 \xrightarrow{(7)} x_2x_3x_2x_1x_3,\\
& x_3x_2(x_2x_3x_1) \xrightarrow{(7)} x_3x_2x_2x_1x_3 \xrightarrow{(2)} x_2x_3x_2x_1x_3,\\
(A_{12})\quad & (x_3x_3x_1)x_1 \xrightarrow{(3)} x_3x_1x_3x_1,\\
& x_3(x_3x_1x_1) \xrightarrow{(4)} x_3x_1x_3x_1,\\
(A_{13})\quad & (x_3x_3x_1)x_2 \xrightarrow{(3)} x_3x_1x_3x_2,\\
& x_3(x_3x_1x_2) \xrightarrow{(8)} x_3x_1x_3x_2,\\
(A_{14})\quad & x_2(x_3x_1x_1) \xrightarrow{(4)} x_2x_1x_3x_1,\\
& (x_2x_3x_1)x_1 \xrightarrow{(7)} x_2x_1x_3x_1,\\
(A_{15})\quad & x_3x_2x_1(x_3x_1x_1) \xrightarrow{(4)} x_3x_2x_1x_1x_3x_1 \xrightarrow{(6)} x_3x_1x_2x_1x_3x_1\\
& \qquad \xrightarrow{(8)} x_1x_3x_2x_1x_3x_1 \xrightarrow{(10)} x_1x_3x_1x_3x_2x_1,\\
& (x_3x_2x_1x_3x_1)x_1 \xrightarrow{(10)} x_3x_1x_3x_2x_1x_1 \xrightarrow{(6)} x_3x_1x_3x_1x_2x_1\\
& \qquad \xrightarrow{(8)} x_3x_1x_1x_3x_2x_1 \xrightarrow{(4)} x_1x_3x_1x_3x_2x_1,
\end{aligned}
$$

$$
\begin{aligned}
(A_{16})\quad & (x_2x_2x_1)x_1 \xrightarrow{(5)} x_2x_1x_2x_1,\\
& x_2(x_2x_1x_1) \xrightarrow{(6)} x_2x_1x_2x_1,\\
(A_{17})\quad & x_3x_1(x_2x_2x_1) \xrightarrow{(5)} x_3x_1x_2x_1x_2 \xrightarrow{(8)} x_1x_3x_2x_1x_2,\\
& (x_3x_1x_2)x_2x_1 \xrightarrow{(8)} x_1x_3x_2x_2x_1 \xrightarrow{(5)} x_1x_3x_2x_1x_2,\\
(A_{18})\quad & x_3x_2x_1(x_2x_2x_1) \xrightarrow{(5)} x_3x_2x_1x_2x_1x_2\\
& \qquad \xrightarrow{(9)} x_2x_3x_2x_1x_1x_2 \xrightarrow{(6)} x_2x_3x_1x_2x_1x_2\\
& \qquad \xrightarrow{(7)} x_2x_1x_3x_2x_1x_2 \xrightarrow{(9)} x_2x_1x_2x_3x_2x_1,\\
& (x_3x_2x_1x_2)x_2x_1 \xrightarrow{(9)} x_2x_3x_2x_1x_2x_1\\
& \qquad \xrightarrow{(9)} x_2x_2x_3x_2x_1x_1 \xrightarrow{(6)} x_2x_2x_3x_1x_2x_1\\
& \qquad \xrightarrow{(7)} x_2x_2x_1x_3x_2x_1 \xrightarrow{(5)} x_2x_1x_2x_3x_2x_1,\\
(A_{19})\quad & x_3x_1(x_2x_1x_1) \xrightarrow{(6)} x_3x_1x_1x_2x_1 \xrightarrow{(4)} x_1x_3x_1x_2x_1,\\
& (x_3x_1x_2)x_1x_1 \xrightarrow{(8)} x_1x_3x_2x_1x_1 \xrightarrow{(6)} x_1x_3x_1x_2x_1,\\
(A_{20})\quad & x_3x_2x_1(x_2x_1x_1) \xrightarrow{(6)} x_3x_2x_1x_1x_2x_1 \xrightarrow{(6)} x_3x_1x_2x_1x_2x_1\\
& \qquad \xrightarrow{(8)} x_1x_3x_2x_1x_2x_1 \xrightarrow{(9)} x_1x_2x_3x_2x_1x_1\\
& \qquad \xrightarrow{(6)} x_1x_2x_3x_1x_2x_1 \xrightarrow{(8)} x_1x_2x_1x_3x_2x_1,\\
& (x_3x_2x_1x_2)x_1x_1 \xrightarrow{(9)} x_2x_3x_2x_1x_1x_1 \xrightarrow{(6)} x_2x_3x_1x_2x_1x_1\\
& \qquad \xrightarrow{(7)} x_2x_1x_3x_2x_1x_1 \xrightarrow{(6)} x_2x_1x_3x_1x_2x_1\\
& \qquad \xrightarrow{(8)} x_2x_1x_1x_3x_2x_1 \xrightarrow{(6)} x_1x_2x_1x_3x_2x_1,\\
(A_{21})\quad & x_3x_2x_3(x_2x_1x_1) \xrightarrow{(6)} x_3x_2x_3x_1x_2x_1 \xrightarrow{(7)} x_3x_2x_1x_3x_2x_1,\\
& (x_3x_2x_3x_2x_1)x_1 \xrightarrow{(11)} x_3x_2x_1x_3x_2x_1,\\
(A_{22})\quad & (x_2x_3x_1)x_2 \xrightarrow{(7)} x_2x_1x_3x_2,\\
& x_2(x_3x_1x_2) \xrightarrow{(8)} x_2x_1x_3x_2,\\
(A_{23})\quad & x_3x_1(x_2x_3x_1) \xrightarrow{(7)} x_3x_1x_2x_1x_3 \xrightarrow{(8)} x_1x_3x_2x_1x_3,\\
& (x_3x_1x_2)x_3x_1 \xrightarrow{(8)} x_1x_3x_2x_3x_1 \xrightarrow{(7)} x_1x_3x_2x_1x_3,\\
(A_{24})\quad & x_3x_2x_1(x_2x_3x_1) \xrightarrow{(7)} x_3x_2x_1x_2x_1x_3 \xrightarrow{(9)} x_2x_3x_2x_1x_1x_3\\
& \qquad \xrightarrow{(6)} x_2x_3x_1x_2x_1x_3 \xrightarrow{(7)} x_2x_1x_3x_2x_1x_3,\\
& (x_3x_2x_1x_2)x_3x_1 \xrightarrow{(9)} x_2x_3x_2x_1x_3x_1\\
& \qquad \xrightarrow{(10)} x_2x_3x_1x_3x_2x_1 \xrightarrow{(7)} x_2x_1x_3x_3x_2x_1\\
& \qquad \xrightarrow{(1)} x_2x_1x_3x_2x_3x_1 \xrightarrow{(7)} x_2x_1x_3x_2x_1x_3,\\
(A_{25})\quad & x_3x_2x_1(x_3x_1x_2) \xrightarrow{(8)} x_3x_2x_1x_1x_3x_2\\
& \qquad \xrightarrow{(6)} x_3x_1x_2x_1x_3x_2 \xrightarrow{(8)} x_1x_3x_2x_1x_3x_2,\\
& (x_3x_2x_1x_3x_1)x_2 \xrightarrow{(10)} x_3x_1x_3x_2x_1x_2 \xrightarrow{(9)} x_3x_1x_2x_3x_2x_1\\
& \qquad \xrightarrow{(8)} x_1x_3x_2x_3x_2x_1 \xrightarrow{(11)} x_1x_3x_2x_1x_3x_2,
\end{aligned}
$$

$$
\begin{aligned}
(A_{26})\ \ & x_3x_2(x_3x_2x_1x_2) \xrightarrow{(9)} x_3x_2x_2x_3x_2x_1 \\
& \qquad \xrightarrow{(2)} x_2x_3x_2x_3x_2x_1 \xrightarrow{(11)} x_2x_3x_2x_1x_3x_2, \\
& (x_3x_2x_3x_2x_1)x_2 \xrightarrow{(11)} x_3x_2x_1x_3x_2x_2 \\
& \qquad \xrightarrow{(2)} x_3x_2x_1x_2x_3x_2 \xrightarrow{(9)} x_2x_3x_2x_1x_3x_2, \\
(A_{27})\ \ & (x_3x_2x_3x_2x_1)x_3x_1 \xrightarrow{(10)} x_3x_2x_1x_3x_2x_3x_1 \xrightarrow{(7)} x_3x_2x_1x_3x_2x_1x_3, \\
& x_3x_2(x_3x_2x_1x_3x_1) \xrightarrow{(11)} x_3x_2x_3x_1x_3x_2x_1 \xrightarrow{(7)} x_3x_2x_1x_3x_3x_2x_1 \\
& \qquad \xrightarrow{(1)} x_3x_2x_1x_3x_2x_3x_1 \xrightarrow{(5)} x_3x_2x_1x_3x_2x_1x_3.
\end{aligned}
$$

The assertion now follows from Theorem 1.19. □

Notice that the above results may be reformulated to say that $M_n$, for $n \leq 3$, admits a complete rewriting system (cf. [5]).

We conclude with the following surprising observation.

**Theorem 2.2** (cf. [14, Theorem 3])**.** *Let $K[M_n]$ be the plactic algebra of rank $n \geq 4$ over a field $K$. Then every Gröbner-Shirshov basis for $K[M_n]$, associated to the degree-lexicographic order on the corresponding free monoid $\langle X \rangle$, is infinite.*

*Proof.* Consider the words $w_j = x_3x_2x_3^jx_4x_3x_1 \in \langle X \rangle$ for $j \geq 1$. Since we have the following equalities in $M_n$:

$$
\begin{aligned}
x_3x_2x_3 &= x_3x_3x_2, \\
x_3x_2x_1x_3 &= x_3x_3x_2x_1, \\
x_3x_2x_4x_3x_1 &= x_3x_4x_2x_3x_1 = x_3x_4x_2x_1x_3 \\
&= x_3x_2x_4x_1x_3 = x_3x_2x_1x_4x_3,
\end{aligned}
$$

it follows that $w_j = x_3x_2x_3^jx_4x_3x_1 = x_3x_2x_1x_3^jx_4x_3$ holds also in $M_n$. Let $a = x_3x_2x_3^jx_4x_3 \in \langle X \rangle$ and $b = x_2x_3^jx_4x_3x_1 \in \langle X \rangle$. We claim that $a$ is the minimal word in $\langle X \rangle$ among all words that represent $x_3x_2x_3^jx_4x_3$ as an element of $M_n$. Similarly, we claim that $b$ is the minimal word in $\langle X \rangle$ among all words that represent $x_2x_3^jx_4x_3x_1$ as an element of $M_n$. Then it is clear that, in order to reduce the word $x_3x_2x_3^jx_4x_3x_1$, we have to include the reduction $x_3x_2x_3^jx_4x_3x_1 \to x_3x_2x_1x_3^jx_4x_3$ to the set of allowed reductions. Since $j \geq 1$ is arbitrary, the set of reductions must be infinite, and the assertion follows.

In order to verify the claims, notice first that the defining relations for $M_n$ do not allow to rewrite $a$ in $M_n$ in the form $x_2v$ or in the form $wx_4$ for some $v, w \in \langle X \rangle$. It follows easily that $a$ is a minimal word in its class in $M_n$. Next, consider the word $b$ and suppose that $b$ can be rewritten in $M_n$ as $x_1v$ for some $v \in \langle X \rangle$. The only defining relations for $M_n$ that can be used to bring $x_1$ to the first position at a certain step of a rewriting process of $b$ in $M_n$ are $x_3x_1x_2 = x_1x_3x_2$, $x_4x_1x_2 = x_1x_4x_2$ and $x_4x_1x_3 = x_1x_4x_3$. Then the obtained presentation of $b$ has the form $x_1x_3x_2x_3^kx_4x_3^{j-k}$, $x_1x_4x_2x_3^{j+1}$ and $x_1x_4x_3x_3^kx_2x_3^{j-k}$ for some $0 \leq k \leq j$, respectively. Since the maximal length of a decreasing subsequence of the given word is an invariant of the plactic class of this word (cf. [15]), this implies $b = x_1x_4x_3x_3^kx_2x_3^{j-k}$ in $M_n$. However, the standard tableaux form of the latter element is easily seen to be $(x_4x_3x_1)x_2x_3^j$, while the standard tableaux form of $x_2x_3^jx_4x_3x_1$ is $(x_4x_2x_1)x_3^{j+1}$ (cf. [15]), a contradiction. Suppose $b$ can be rewritten in $M_n$ as $x_2x_1w$ for some $w \in \langle X \rangle$. Then $b = x_2x_1x_3^kx_4x_3^{j-k+1}$ in $M_n$ for some $0 \leq k \leq j+1$, again a contradiction because the latter has no decreasing subsequence of length three. So a minimal word that represents $b$ in $M_n$ should start with $x_2x_3$. Hence it must be of the form $x_2x_3^jx_4x_3x_1$, because it should have a decreasing subsequence of length three. The result follows. □

The above result is in contrast with the corresponding result for the strongly related class of monoids, also defined by homogeneous monoid presentations and with all defining relations of degree 3, the so-called Chinese monoids. The latter class was introduced and its combinatorial properties were studied by Cassaigne, Espie, Krob, Novelli and Hivert (cf. [6]). It was shown that the Gröbner-Shirshov basis for the Chinese algebra of any rank $n \geq 1$ is finite (cf. [8]). Notice that the Chinese algebra of rank $n$ has the same growth function (cf. [12]) as the plactic algebra of rank $n$ and its elements also admit a canonical form expressed in terms of certain tableaux (cf. [9]). Moreover, if $n \leq 2$, then the two algebras coincide.

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