Łukasz Kubat

Instytut Matematyczny PAN

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Opiekun pracy: Daniel Simson

ON MINOR DEGENERATIONS OF THE FULL MATRIX ALGEBRA

ŁUKASZ KUBAT

ABSTRACT. We state general properties of the minor degeneration algebras $\mathrm{M}_n^q(K)$ of the full matrix algebra $\mathrm{M}_n(K)$ over a field K. In particular, we give a criterion for the existence of a K-algebra isomorphism $\mathrm{M}_n^q(K) \cong \mathrm{M}_n^{q'}(K)$ in terms of an action of the algebraic group $G_n(K)$ on the algebraic variety $\mathrm{ST}_n(K)$ of all minor structure matrices. We also give conditions for q to be $\mathrm{M}_n^q(K)$ a Frobenius algebra. Moreover, in case the field K is algebraically closed and the minor degeneration $A_q = \mathrm{M}_n^q(K)$ is a Frobenius algebra satisfying $J(A_q)^3 = 0$, we state that A_q is representation-finite if and only if $n = 3$, A_q is representation-tame if and only if $n = 4$, and A_q is representation-wild if and only if $n \geq 5$.

Let $n \geq 2$ be an integer and K be a field. A minor structure matrix q is an n-block matrix of $n \times n$ matrices over K with certain properties. A minor q-degeneration $\mathrm{M}_n^q(K)$ of the full matrix algebra $\mathrm{M}_n(K)$ is an n^2-dimensional K-vector space with an associative multiplication defined by a minor structure matrix q.

One of the motivations for study minor degenerations of the full matrix algebra is the fact that we are able to treat this class of algebras by an elementary algebraic geometry technique, and study them in a deformation theory context. There is also another motivation coming from the fact that, given a prime p and an algebraically closed field K of characteristic zero, any Hopf K-algebra of dimension p^2 is semisimple or is isomorphic to the Taft Hopf algebra. In connection with this result and the facts that finite-dimensional Hopf algebras are Frobenius algebras, the existence of a Hopf algebra structure on a Frobenius minor degeneration algebra $\mathrm{M}_n^q(K)$ (of dimension n^2), seems to be a natural problem to solve.

1. PRELIMINARIES

Our first goal in this section is to remind some basic definitions concerning finite-dimensional algebras over a field. Next, we shall focus our attention on the class of Frobenius algebras.

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Throughout this section, all considered algebras are assumed to be associative, unital and finite-dimensional over a field K.

1.1. **A glance at finite-dimensional algebras.** We start with some definitions, fixing the terminology and the notation (cf. [1, 2, 3]).

Definition 1.1. Let A be a K-algebra. An element $e \in A$ is called an *idempotent* if $e^2 = e$. An idempotent $e \in A$ is called a *central idempotent* if $ea = ae$ for all $a \in A$. Two idempotents $e, e' \in A$ are called *orthogonal* if $ee' = e'e = 0$. Moreover, an idempotent $e \in A$ is said to be *primitive* if it cannot be written as a sum $e = e' + e''$, where e', e'' are non-zero orthogonal idempotents of A.

Notice that every algebra A has two trivial idempotents, 0 and 1. If the idempotent e of A is non-trivial, then $e' = 1-e$ is also a non-trivial idempotent, the idempotents e and e' are orthogonal, and there is a non-trivial right A-module direct sum decomposition $A = eA \oplus e'A$. Conversely, if $A = M_1 \oplus M_2$ is a non-trivial right A-module direct sum decomposition and $1 = e_1 + e_2$, where $e_1 \in M_1$ and $e_2 \in M_2$, then e_1, e_2 is a pair of orthogonal idempotents of A, and $M_i = e_iA$ for $i = 1, 2$ is indecomposable if and only if e_i is primitive.

If e is a central idempotent, then so is e' and hence $eA, e'A$ are two-sided ideals. Moreover, they are easily shown to be K-algebras with identity elements $e \in eA, e' \in e'A$, respectively. In this case, the direct sum decomposition $A = eA \oplus e'A$ is a direct product decomposition of the algebra A.

Definition 1.2. A right A-module M is called *simple* if it is non-zero and any submodule of M is either 0 or M. A right A-module M is said to be *indecomposable* if M is non-zero and M cannot be written as a direct sum $M = M' \oplus M''$, where M', M'' are non-zero right A-modules.

Because the algebra A is finite-dimensional, the right A-module A admits a direct sum decomposition $A = P_1 \oplus \cdots \oplus P_n$, where $P_1, \dots, P_n$ are indecomposable right ideals of A. It follows from the preceding discussion that $P_1 = e_1A, \dots, P_n = e_nA$, where $e_1, \dots, e_n$ are primitive pairwise orthogonal idempotents of A such that $1 = e_1 + \cdots + e_n$. Conversely, every set of idempotents with the preceding properties induces a right A-module direct sum decomposition $A = P_1 \oplus \cdots \oplus P_n$ with indecomposable right ideals $P_1 = e_1A, \dots, P_n = e_nA$.

Definition 1.3. Any right A-module direct sum decomposition of the form $A = e_1A \oplus \cdots \oplus e_nA$, where $e_1, \dots, e_n$ are primitive pairwise orthogonal idempotents of A, is called an *indecomposable decomposition* of A, and such a set $\{e_1, \dots, e_n\}$ is called a *complete set of primitive orthogonal idempotents* of A. Moreover, the algebra A is called *basic* if $e_iA \not\cong e_jA$ for all $i \neq j$.

Definition 1.4. We say that an algebra A is *connected* (or *indecomposable*) if A is not a direct product of two algebras, or equivalently, if 0 and 1 are the only central idempotents of A.

Next, let us recall the notions of the Jacobson radical, the socle and the top of a right A-module.

Definition 1.5. Assume that M is a right A-module. The *Jacobson radical* $\operatorname{Rad}(M)$ of M is defined to be the intersection of all the maximal submodules of M. The *top* $\operatorname{Top}(M)$ of M is defined to be the quotient module $\operatorname{Top}(M) = M/\operatorname{Rad}(M)$. Finally, the *socle* $\operatorname{Soc}(M)$ of M is defined to be the sum of all simple submodules of M.

Definition 1.6. The *Jacobson radical* $J(A)$ of an algebra A is defined to be the intersection of all the maximal right ideals in A.

It can be shown that $J(A)$ is the intersection of all the maximal left ideals in A. In particular, $J(A)$ is a two-sided ideal of A.

The notion of the Jacobson radical could lead to confusion if M is the right A-module A. Fortunately, it can be shown that the Jacobson radicals $J(A)$ and $\operatorname{Rad}(A)$ coincide.

It is worth mentioning that all finite-dimensional K-algebras may be split in two classes (representation types), according to whether the number of distinct (up to isomorphism) finite-dimensional indecomposable right modules over a given K-algebra is finite or not.

To be more precise, let us introduce the following definition.

Definition 1.7. An algebra A is defined to be *representation-finite* (or an algebra of *finite representation type*) if the number of the isomorphism classes of finite-dimensional indecomposable right A-modules is finite. An algebra A is called *representation-infinite* (or an algebra of *infinite representation type*) if A is not representation-finite.

It follows from the standard duality between the categories of finite-dimensional left and right A-modules that this definition is left-right symmetric. One can prove that if A is representation-finite then the number of the isomorphism classes of all indecomposable left (resp. right) A-modules is finite, or equivalently, that every indecomposable left (resp. right) A-module is finite-dimensional (cf. [1, 2, 3]).

It should be noticed that all finite-dimensional K-algebras may also be broadly classified into two other types: *representation-tame* algebras (or algebras of *tame representation type*), and *representation-wild* algebras (or algebras of *wild representation type*) (cf. [11]).

Intuitively, the tameness of a K-algebra A means that there is a classification of the isomorphism classes of the finite-dimensional indecomposable right A-modules in the sense that, for each integer $d \geq 1$, the finite-dimensional indecomposable right A-modules of dimension d form at most finitely many 1-parameter families. The wildness of A

means that the category of finite-dimensional right A-modules is very complicated, and if we are able to classify the indecomposable modules in this category, then we are also able to classify the indecomposable modules in other categories of finite-dimensional right modules over finite-dimensional K-algebras.

Moreover, any representation-finite algebra is representation-tame.

1.2. **Frobenius algebras.** Let A be a finite-dimensional K-algebra with a basis $\{a_1, \dots, a_n\}$. Then for each $a \in A$ we have two $n \times n$ matrices $L(a)$ and $R(a)$, with coefficients in K, such that

$$a(a_1, \dots, a_n) = (a_1, \dots, a_n)L(a),$$
$$(a_1, \dots, a_n)a = (a_1, \dots, a_n)R(a)^t.$$

The correspondences

$$L\colon A \to \mathrm{M}_n(K) \qquad \text{and} \qquad R\colon A \to \mathrm{M}_n(K)$$

are K-algebra homomorphisms called the *left* and the *right regular representations*, respectively.

Frobenius studied algebras for which the regular representations are equivalent, and gave a necessary and sufficient condition for this equivalence. Later, Brauer and Nesbitt pointed out the importance of the algebras studied by Frobenius and named them Frobenius algebras.

However, instead of following this approach, we shall give a slightly different definition of a Frobenius algebra.

First, we remark that the K-dual space $D(A) = \mathrm{Hom}_K(A, K)$ has the structure of an A-bimodule. The left and right A-actions on A are defined, respectively, by

$$(a \cdot \varphi)(b) = \varphi(ba), \quad (\varphi \cdot a)(b) = \varphi(ab) \quad \text{for } a, b \in A \text{ and } \varphi \in D(A).$$

It is of interest to compare, say, the right A-module $D(A)$ with the right A-module A.

This leads to the following definition (cf. [10, 12, 13]).

Definition 1.8. A finite-dimensional K-algebra A is called a *Frobenius algebra* if we have a right A-module isomorphism $D(A) \cong A$.

As can be shown (cf. [10]), $D(A)$ is always an injective left and right A-module. Therefore, we have the following result.

Proposition 1.9 (cf. [10, Proposition 3.14]). *Any Frobenius K-algebra A is left and right self-injective, i.e., the module A is injective as a left and right A-module.*

The next theorem provides other possible characterizations of Frobenius algebras.

Theorem 1.10 (cf. [10, Theorem 3.15]). *For any finite-dimensional K-algebra A the following conditions are equivalent.*

(1) *A is a Frobenius algebra.*

(2) *$D(A) \cong A$ as left A-modules.*
(3) *$D(A)$ is a cyclic module as a left or right A-module.*
(4) *There exists a non-singular bilinear form $(-,-)\colon A \times A \to K$ with the associative property $(ab, c) = (a, bc)$ for all $a, b, c \in A$.*
(5) *There exists a K-liner form $\varphi\colon A \to K$, which does not contain any non-zero right ideal of A in its kernel.*
(6) *There exists a K-liner form $\psi\colon A \to K$, which does not contain any non-zero left ideal of A in its kernel.*

Note that Frobenius algebras have many interesting ring-theoretic and homological properties, which for obvious reasons cannot be cited here. Instead, we shall give some examples.

Example 1.11. Each matrix algebra $\mathrm{M}_n(K)$ is a Frobenius algebra. In this case, a bilinear form defining a Frobenius structure on $\mathrm{M}_n(K)$ is given by

$$(a, b) = \mathrm{Tr}(ab) \qquad \text{for } a, b \in \mathrm{M}_n(K),$$

where Tr denotes the usual trace map on $\mathrm{M}_n(K)$.

Example 1.12. Any n-dimensional K-algebra A has a natural homomorphism to its own endomorphism K-algebra $\mathrm{End}_K(A) \cong \mathrm{M}_n(K)$. A bilinear form can be defined on A in the sense of Example 1.11. If this bilinear form is non-singular then it equips A with the structure of a Frobenius algebra.

Example 1.13. Each group algebra $K[G]$ of a finite group G is a Frobenius algebra. In this case, a bilinear form defining a Frobenius structure on $K[G]$ is given by

$$(a, b) = \sum_{g \in G} a_g b_{g^{-1}} \qquad \text{for } a = \sum_{g \in G} a_g g, b = \sum_{g \in G} b_g g \in K[G].$$

This is a special case of Example 1.12.

Example 1.14. Assume that X is an n-dimensional oriented compact manifold. Let $H^p(X)$ be the de Rham cohomology of X of degree p. Then $H^*(X) = \bigoplus_{p=0}^{n} H^p(X)$ is an $\mathbb{R}$-algebra under the wedge product. In this case integration

$$(\varphi, \psi) = \int_X \varphi \wedge \psi \qquad \text{for } \varphi \in H^p(X) \text{ and } \psi \in H^q(X)$$

provides a Frobenius structure on $H^*(X)$.

Finally, let us recall that a basic finite-dimensional K-algebra A, with a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$, is a Frobenius algebra if and only if each projective module e_iA for $i = 1, \ldots, n$ has a simple socle, and $\mathrm{Soc}(e_iA) \ncong \mathrm{Soc}(e_jA)$ for all $i \neq j$. In this case, there is a permutation $\sigma \in S_n$, called the *Nakayama permutation*, such that $\mathrm{Soc}(e_iA) \cong \mathrm{Top}(e_{\sigma(i)}A)$ (cf. [1, 2, 3]). Moreover, if A is a Frobenius algebra then $\mathrm{Soc}(A_A) = \mathrm{Soc}({}_AA)$ (cf. [12, 13]).

2. Minor degenerations of the full matrix algebra

In this section, we aim to present some general properties of the minor degenerations of the full matrix algebra $\mathrm{M}_n(K)$ over a field K.

Let us introduce the crucial notions of the minor structure matrix q, and the minor q-degeneration $\mathrm{M}_n^q(K)$ of the full matrix algebra $\mathrm{M}_n(K)$ (cf. [4, 5, 6]).

Definition 2.1. Assume that $n \geq 2$. A *minor structure matrix* of size $n \times n^2$, with coefficients in a field K, is the n-block matrix

$$q = (q^{(1)}| \cdots |q^{(n)}),$$

where $q^{(k)} = (q_{ij}^{(k)}) \in \mathrm{M}_n(K)$ for $k = 1, \ldots, n$ are $n \times n$ matrices with coefficients in K satisfying the following conditions:

(1) $q_{ik}^{(k)} = q_{kj}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$,
(2) $q_{ij}^{(k)} q_{kj}^{(l)} = q_{il}^{(k)} q_{ij}^{(l)}$ for all $i, j, k, l \in \{1, \ldots, n\}$.

We call q *basic* if, in addition, the following condition is satisfied

(3) $q_{ii}^{(k)} = 0$ for all $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, n\} \setminus \{k\}$.

The minor structure matrix q is called $(0,1)$-*matrix*, if each entry $q_{ij}^{(k)}$ is either 0 or 1. We denote by $\mathrm{ST}_n(K) \subseteq \mathrm{M}_{n \times n^2}(K)$ the set of all minor structure matrices q of size $n \times n^2$ with coefficients in the field K.

Definition 2.2. Assume that $q = (q^{(1)}| \cdots |q^{(n)})$ is a minor structure matrix in $\mathrm{ST}_n(K)$. A *q-degeneration* $\mathrm{M}_n^q(K)$ of the full matrix algebra $\mathrm{M}_n(K)$ is defined to be the K-vector space $\mathrm{M}_n(K)$ equipped with the q-multiplication

$$\cdot_q \colon \mathrm{M}_n(K) \otimes_K \mathrm{M}_n(K) \to \mathrm{M}_n(K)$$

defined by the formula

$$a \cdot_q b = (c_{ij}) \qquad \text{for } a = (a_{ij}), b = (b_{ij}) \in \mathrm{M}_n(K),$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} q_{ij}^{(k)} b_{kj} \qquad \text{for } i, j = 1, \ldots, n.$$

A straightforward computation shows that $\mathrm{M}_n^q(K)$ is a K-algebra with the same identity as $\mathrm{M}_n(K)$. It is also worth mentioning that if a n-block matrix $q = (q^{(1)}| \cdots |q^{(n)})$ satisfies

$$q_{ij}^{(k)} = 1 \qquad \text{for } i, j, k = 1, \ldots, n$$

then the conditions (1) and (2) in the Definition 2.1 are satisfied, but the condition (3) is not. In this case, $\mathrm{M}_n^q(K)$ is an ordinary matrix algebra $\mathrm{M}_n(K)$, because the multiplication given in the Definition 2.1 reduces to the usual matrix multiplication on $\mathrm{M}_n(K)$.

Furthermore, it turns out that, under a suitable choice of q, the algebra $\mathrm{M}_n^q(K)$ is a degeneration of $\mathrm{M}_n(K)$ (cf. [8]). We recall that given two K-algebras A_0 and A_1 (with an underlying K-space K^m) defined by the structure matrices γ_0 and γ_1, respectively, γ_0 and γ_1 are viewed as elements of the algebraic variety $\mathrm{Alg}(K^m)$ of associative unitary K-algebra structures on the vector space K^m. The general linear group $\mathrm{GL}_m(K)$ acts on $\mathrm{Alg}(K^m)$ by the transport of structures (cf. [9]). An algebra A_0 is said to be a *degeneration* of the algebra A_1 (or A_1 is a *deformation* of the algebra A_0), if γ_0 lies in the closure of the $\mathrm{GL}_m(K)$-orbit of γ_1 in $\mathrm{Alg}(K^m)$ (cf. [7, 8]). We note that the set $\mathrm{ST}_n(K) \subseteq \mathrm{M}_{n\times n^2}(K)$ of minor structure matrices of size $n \times n^2$ is an algebraic K-variety. Moreover, there is a variety embedding

$$\mathrm{ST}_n(K) \subseteq \mathrm{Alg}(\mathrm{M}_n(K)) = \mathrm{Alg}(K^{n^2})$$

defined by attaching to any minor structure matrix q in $\mathrm{ST}_n(K)$ the matrix of the multiplication $\cdot_q\colon \mathrm{M}_n(K) \otimes_K \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ in the matrix units basis. It can be verified that $\mathrm{ST}_n(K)$ is a locally closed subset of $\mathrm{Alg}(K^{n^2})$.

Example 2.3 (cf. [6, Example 2.8]). If q is a minor structure matrix in $\mathrm{ST}_n(K)$, then one can prove that $q_{ii}^{(j)} = q_{jj}^{(i)}$ for $i, j = 1, \ldots, n$. In particular, in case $n = 2$, it follows that the minor structure matrix $q = (q^{(1)}|q^{(2)})$ has the form

$$q = q_\alpha = \left(\begin{array}{cc|cc} 1 & 1 & \alpha & 1 \\ 1 & \alpha & 1 & 1 \end{array}\right)$$

for some $\alpha \in K$. Moreover, it is straightforward that q_α is basic if and only if $\alpha = 0$. Now, let $A_\alpha = \mathrm{M}_2^{q_\alpha}(K)$. It can be shown that each A_α is a degeneration of the full matrix algebra $A_1 = \mathrm{M}_2(K)$, and $A_\alpha \cong A_1$ if and only if $\alpha \neq 0$. Moreover, A_0 is non-semisimple and admits a Hopf algebra structure.

Another elementary properties of the minor degeneration algebra $\mathrm{M}_n^q(K)$ are collected in Theorem 2.4 below. In particular, it follows that $\mathrm{M}_n^q(K)$ is a non-semisimple basic K-algebra if q is basic.

Theorem 2.4 (cf. [6, Theorem 2.9]). *Assume that q is a minor structure matrix in $\mathrm{ST}_n(K)$. Let $A_q = \mathrm{M}_n^q(K)$.*

(1) *A_q is an associative K-algebra such that $e_{ij} \cdot_q e_{kl} = \delta_{jk} q_{il}^{(k)} e_{il}$ for $i, j, k, l = 1, \ldots, n$, where e_{ij} for $i, j = 1, \ldots, n$ are the matrix units in the full matrix algebra $\mathrm{M}_n(K)$.*

(2) *The standard matrix idempotents $e_1, \ldots, e_n$ of the full matrix algebra $\mathrm{M}_n(K)$ are pairwise orthogonal primitive idempotents of the algebra A_q. Moreover, there is a right ideal decomposition $A_q = e_1 A_q \oplus \cdots \oplus e_n A_q$, there are K-algebra isomorphisms $\mathrm{End}_{A_q}(e_i A_q) \cong e_i A_q e_i \cong K$ for $i = 1, \ldots, n$, there are K-vector space isomorphisms $\mathrm{Hom}_{A_q}(e_j A_q, e_i A_q) \cong e_i A_q e_j \cong K e_{ij}$ for $i \neq j$, and there is an isomorphism $e_i A_q \cong e_j A_q$ of right ideals if and only if $q_{jj}^{(i)} = q_{ii}^{(j)} \neq 0$.*

(3) *The algebra* A_q *is basic if and only if the matrix* q *is basic.*
(4) *If the algebra* A_q *is basic then it is also connected. Moreover:*
  (*a*) *the Jacobson radical* $J(A_q)$ *of the algebra* A_q *consists of all matrices* $a = (a_{ij})$ *with* $a_{11} = \cdots = a_{nn} = 0$*, and it satisfies* $J(A_q)^n = 0$*,*
  (*b*) *the group of units* $U(A_q)$ *of the algebra* A_q *consists of all matrices* $a = (a_{ij})$ *with* $a_{11} \cdots a_{nn} \neq 0$*,*
  (*c*) *every non-zero two-sided ideal of the algebra* A_q *is generated by a finite subset of the set* $\{e_{ij} : i, j = 1, \ldots, n\}$ *of the matrix units in the full matrix algebra* $\mathrm{M}_n(K)$*,*
  (*d*) *the global dimension of the algebra* A_q *is infinite.*

As an easy consequence of Theorem 2.4 we get the following result.

Corollary 2.5 (cf. [6, Corollary 2.11])**.** *Assume that* q *is a minor structure matrix in* $\mathrm{ST}_n(K)$*. Then there exists a* K*-algebra isomorphism* $\mathrm{M}_n^q(K) \cong \mathrm{M}_n(K)$ *if and only if* $q_{22}^{(1)} \cdots q_{nn}^{(1)} \neq 0$*.*

Now, we shall focus our attention on the problem of existence of a K-algebra isomorphism between $\mathrm{M}_n^q(K)$ and $\mathrm{M}_n^{q'}(K)$ for various minor structure matrices q and q' in $\mathrm{ST}_n(K)$.

Definition 2.6. Given a matrix $a = (a_{ij}) \in \mathrm{M}_n(K)$ and a permutation $\sigma \in S_n$ we denote by $\sigma * a = (a_{ij}^{\sigma})$ the matrix in $\mathrm{M}_n(K)$ with

$$a_{ij}^{\sigma} = a_{\sigma^{-1}(i)\sigma^{-1}(j)} \qquad \text{for } i, j = 1, \ldots, n.$$

Moreover, for a minor structure matrix $q = (q^{(1)} | \cdots | q^{(n)}) \in \mathrm{ST}_n(K)$ we set

$$\sigma * q = (\sigma * q^{(\sigma^{-1}(1))} | \cdots | \sigma * q^{(\sigma^{-1}(n))}).$$

It is clear that the map

$$S_n \times \mathrm{ST}_n(K) \ni (\sigma, q) \mapsto \sigma * q \in \mathrm{ST}_n(K)$$

defines an action of the symmetric group S_n on the K-variety $\mathrm{ST}_n(K)$ of all minor structure matrices q of size $n \times n^2$. Note that the subvarieties of $\mathrm{ST}_n(K)$ consisting of all basic minor structure matrices and of all basic minor structure $(0,1)$-matrices are S_n-invariant.

The following simple result is very useful.

Proposition 2.7 (cf. [6, Lemma 2.15])**.** *Let* q *be a basic minor structure matrix in* $\mathrm{ST}_n(K)$*. Then if* $\sigma \in S_n$ *is a permutation then* $\sigma * q$ *is a basic minor structure matrix in* $\mathrm{ST}_n(K)$*. Moreover, the map*

$$\mathrm{M}_n^q(K) \ni a \mapsto \sigma * a \in \mathrm{M}_n^{\sigma * q}(K)$$

defines an isomorphism of K*-algebras, which maps* e_{ij} *to* $e_{\sigma^{-1}(i)\sigma^{-1}(j)}$ *for* $i, j = 1, \ldots, n$*.*

Next, we shall extend the action of the symmetric group S_n on $\mathrm{ST}_n(K)$ to an action of the following semidirect product algebraic group

$$G_n(K) = T_n(K) \rtimes S_n,$$

where

$$T_n(K) = \{t = (t_{ij}) \in \mathrm{M}_n(K) : t_{ii} = 1 \text{ and } t_{ij} \neq 0 \text{ for } i, j = 1, \dots, n\}$$

is viewed as a group with the coordinatewise multiplication, and the multiplication in $G_n(K)$ is defined by the formula

$$(t, \sigma) \cdot (t', \sigma') = (t \cdot (\sigma * t'), \sigma\sigma') \qquad \text{for } t, t' \in T_n(K) \text{ and } \sigma, \sigma' \in S_n.$$

It is clear that the group $T_n(K)$ is isomorphic to the $n(n-1)$-dimensional K-torus $K^* \times \cdots \times K^*$ (the product of $n(n-1)$ copies of the multiplicative group $K^* = K \setminus \{0\}$ of K).

We define the algebraic group action

$$*\colon G_n(K) \times \mathrm{ST}_n(K) \to \mathrm{ST}_n(K)$$

by the formula

$$(t, \sigma) * q = (\hat{q}^{(1)} | \cdots | \hat{q}^{(n)}),$$

where

$$\hat{q}_{ij}^{(k)} = q_{\sigma^{-1}(i)\sigma^{-1}(j)}^{(\sigma^{-1}(k))} t_{ik}^{-1} t_{ij} t_{kj}^{-1} \qquad \text{for } i, j, k = 1, \dots, n.$$

As can be proved, the subvariety of $\mathrm{ST}_n(K)$ consisting of all basic minor structure matrices is $G_n(K)$-invariant.

The following result shows that the $G_n(K)$-orbits of $\mathrm{ST}_n(K)$ classify the isomorphism classes of the basic algebras $\mathrm{M}_n^q(K)$ of dimension n^2.

Theorem 2.8 (cf. [6, Theorem 2.18]). *Given two basic minor structure matrices q and q' in $\mathrm{ST}_n(K)$, the following statements are equivalent.*

1. *The K-algebras $\mathrm{M}_n^q(K)$ and $\mathrm{M}_n^{q'}(K)$ are isomorphic.*
2. *The matrices q and q' belong to the same $G_n(K)$-orbit.*
3. *There exist a permutation $\sigma \in S_n$ and a matrix $t = (t_{ij}) \in T_n(K)$ such that $t_{ik} q_{ij}'^{(k)} t_{kj} = q_{\sigma(i)\sigma(j)}^{(\sigma(k))} t_{ij}$ for all $i, j, k \in \{1, \dots, n\}$.*

As a consequence of Theorem 2.8 we get the following isomorphism criterion.

Corollary 2.9 (cf. [6, Corollary 2.19]). *Assume that q and q' are basic minor structure $(0,1)$-matrices in $\mathrm{ST}_n(K)$. Then the K-algebras $\mathrm{M}_n^q(K)$ and $\mathrm{M}_n^{q'}(K)$ are isomorphic if and only if q and q' are in the same S_n-orbit, i.e., if and only if there exists a permutation $\sigma \in S_n$ such that $q_{\sigma(i)\sigma(j)}^{(\sigma(k))} = q_{ij}'^{(k)}$ for all $i, j, k \in \{1, \dots, n\}$.*

3. Frobenius basic minor degenerations of the full matrix algebra

In this section, we shall be interested in such minor q-degenerations of $\mathrm{M}_n(K)$ that are Frobenius K-algebras, where K is a field.

The following definition is of particular importance (cf. [6]).

Definition 3.1. Let $A_q = \mathrm{M}_n^q(K)$ be a minor degeneration of $\mathrm{M}_n(K)$ with a minor structure matrix $q = (q^{(1)}|\cdots|q^{(n)})$ in $\mathrm{ST}_n(K)$. We define a $(0,1)$-*limit* of q to be the minor structure $(0,1)$-matrix

$$\bar{q} = (\bar{q}^{(1)}|\cdots|\bar{q}^{(n)}),$$

where the matrices $\bar{q}^{(k)} = (\bar{q}_{ij}^{(k)})$ for $k = 1,\ldots,n$ are defined by the formula

$$\bar{q}_{ij}^{(k)} = \begin{cases} 0 & \text{if } q_{ij}^{(k)} = 0, \\ 1 & \text{if } q_{ij}^{(k)} \neq 0 \end{cases} \qquad \text{for } i,j = 1,\ldots,n.$$

The algebra $\bar{A}_q = \mathrm{M}_n^{\bar{q}}(K)$ is called the $(0,1)$-*limit* of $A_q = \mathrm{M}_n^q(K)$.

Our first result in this section reads as follows.

Proposition 3.2 (cf. [6, Proposition 3.2])**.** *Let $A_q = \mathrm{M}_n^q(K)$ be a basic minor degeneration of $\mathrm{M}_n(K)$ and $\bar{A}_q = \mathrm{M}_n^{\bar{q}}(K)$ be its $(0,1)$-limit.*

(1) *A K-vector subspace V of $\mathrm{M}_n(K)$ is a two-sided ideal of A_q if and only if V is a two-sided ideal of $\bar{A}_q$. In particular, we have $J(A_q)^m = J(\bar{A}_q)^m$ for every $m \geq 1$.*

(2) *Assume that the field K is algebraically closed. Let $\{A_{q_\alpha}\}_{\alpha\in K}$ be a 1-parameter algebraic family (cf. [9]) of minor degenerations $A_{q_\alpha} = \mathrm{M}_n^{q_\alpha}(K)$ of the algebra $\mathrm{M}_n(K)$ such that $A_{q_0} = A_q$ and almost all K-algebras A_{q_α} are isomorphic. If the algebra A_q is representation-finite (resp. representation-tame) then A_{q_α} is representation-finite (resp. representation-tame) for almost all minor structure matrices q_α.*

Our next aim is to describe the socle $\mathrm{Soc}(A_A)$ of the basic minor degeneration $A = \mathrm{M}_n^q(K)$ of the full matrix algebra $\mathrm{M}_n(K)$. In particular, we shall show that $A = \mathrm{M}_n^q(K)$ is a Frobenius K-algebra if and only if its $(0,1)$-limit algebra $\bar{A} = \mathrm{M}_n^{\bar{q}}(K)$ is a Frobenius K-algebra.

Proposition 3.3 (cf. [6, Proposition 5.1])**.** *Let q be a basic minor structure matrix in $\mathrm{ST}_n(K)$ and $\bar{q}$ be its $(0,1)$-limit. Let $A = \mathrm{M}_n^q(K)$ and $\bar{A} = \mathrm{M}_n^{\bar{q}}(K)$ be the corresponding basic minor degenerations of the algebra $\mathrm{M}_n(K)$. Let $e_1,\ldots,e_n$ be the standard primitive matrix idempotents of A and $\bar{A}$.*

(1) *Given $i \in \{1,\ldots,n\}$, a right ideal $S \subseteq e_iA$ of A is simple if and only if S has the form $S = e_{ij}K \cong e_iA/e_iJ(A)$, where e_{ij} is a matrix unit such that $i \neq j$, and $q_{ik}^{(j)} = 0$ for all $k \neq j$.*

(2) *Given* $i \in \{1, \ldots, n\}$, $\mathrm{Soc}(e_i A) = \sum_{j \in \Lambda_i} e_{ij} K$, *where*

$$\begin{aligned} \Lambda_i &= \{j \in \{1, \ldots, n\} : q_{ik}^{(j)} = 0 \text{ for all } k \neq j\} \\ &= \{j \in \{1, \ldots, n\} : i \neq j \text{ and } e_{ij} \cdot_q J(A) = 0\}. \end{aligned}$$

(3) *Given* $i \in \{1, \ldots, n\}$, *if* S *and* S' *are two different simple submodules of* $e_i A$ *then* $S \ncong S'$.

(4) *The socle* $\mathrm{Soc}(A_A)$ *of the right* A*-module* A *is a two-sided ideal of* A *of the form*

$$\mathrm{Soc}(A_A) = \{a \in A : a \cdot_q J(A) = 0\} = \sum_{i=1}^{n} \sum_{j \in \Lambda_i} e_{ij} K.$$

(5) $\mathrm{Soc}(e_i A) = \mathrm{Soc}(e_i \bar{A})$ *for* $i = 1, \ldots, n$, *and* $\mathrm{Soc}(A_A) = \mathrm{Soc}(\bar{A}_{\bar{A}})$.

Now, we shall give necessary and sufficient conditions for a basic minor structure matrix q in $\mathrm{ST}_n(K)$ to be the K-algebra $\mathrm{M}_n^q(K)$ Frobenius.

Theorem 3.4 (cf. [6, Theorem 5.3])**.** *Assume that* q *is a basic minor structure matrix in* $\mathrm{ST}_n(K)$ *and* $\bar{q}$ *is its* $(0,1)$*-limit. Let* $A = \mathrm{M}_n^q(K)$ *and* $\bar{A} = \mathrm{M}_n^{\bar{q}}(K)$ *be the corresponding basic minor degenerations of* $\mathrm{M}_n(K)$*. Let* $e_1, \ldots, e_n$ *be the standard primitive matrix idempotents of* A *and* $\bar{A}$*. Then the following conditions are equivalent.*

(1) A *is a Frobenius* K*-algebra.*

(2) $\bar{A}$ *is a Frobenius* K*-algebra.*

(3) *The right simple ideals* $\mathrm{Soc}(e_1 A), \ldots, \mathrm{Soc}(e_n A)$ *of* A *are pairwise non-isomorphic and* $\dim_K \mathrm{Soc}(e_i A) = 1$ *for* $i = 1, \ldots, n$.

(4) *The right ideals* $e_1 \mathrm{Soc}(A_A), \ldots, e_n \mathrm{Soc}(A_A)$ *of* A *are pairwise non-isomorphic and* $\dim_K \mathrm{Soc}(A_A) = n$.

(5) *For every* $i \in \{1, \ldots, n\}$ *there exists a unique* $k \neq i$ *such that* $q_{ij}^{(k)} = 0$ *for all* $j \neq k$*. Moreover, for all* $i, j, k \in \{1, \ldots, n\}$ *such that* $i \neq j$ *and* $k \neq i, j$ *there exists* $l \in \{1, \ldots, n\}$ *such that* $l \neq k$ *and* $q_{il}^{(k)} \neq 0$ *or* $q_{jl}^{(k)} \neq 0$.

(6) *There exists a permutation* $\sigma \in S_n$ *such that* $\sigma(i) \neq i$ *for all* $i \in \{1, \ldots, n\}$, *and for all* $i, k \in \{1, \ldots, n\}$ *the equality* $q_{ij}^{(k)} = 0$ *holds for all* $j \neq k$ *if and only if* $k = \sigma(i)$.

(7) *There exists a permutation* $\sigma \in S_n$ *such that* $\sigma(i) \neq i$ *for all* $i \in \{1, \ldots, n\}$, *and* $q_{i\sigma(i)}^{(k)} \neq 0$ *for all* $i, k \in \{1, \ldots, n\}$.

Finally, we shall give a simple description of all basic minor structure matrices q in $\mathrm{ST}_n(K)$ such that the minor degeneration $A_q = \mathrm{M}_n^q(K)$ is a Frobenius algebra and $J(A_q)^3 = 0$. To present it, we associate to a given integer $n \geq 3$ and a permutation $\sigma \in S_n$ such that $\sigma(i) \neq i$ for $i = 1, \ldots, n$, the n-block matrix

$$q_\sigma = (q_\sigma^{(1)} | \cdots | q_\sigma^{(n)})$$

defined by the formula (cf. [4, 6])

$$(q_\sigma^{(k)})_{ij} = \begin{cases} 1 & \text{if } k \in \{i,j\} \text{ or } j = \sigma(i), \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i,j,k = 1,\dots,n.$$

It can be easily verified that the block matrix q_σ is a basic minor structure $(0,1)$-matrix in $\mathrm{ST}_n(K)$ (cf. [4, 5]).

Theorem 3.5 (cf. [6, Theorem 5.5]). *Assume that q is a basic minor structure matrix in $\mathrm{ST}_n(K)$ and $\bar{q}$ is its $(0,1)$-limit. Let $A = \mathrm{M}_n^q(K)$ and $\bar{A} = \mathrm{M}_n^{\bar{q}}(K)$ be the corresponding basic minor degenerations of $\mathrm{M}_n(K)$. Let $e_1,\dots,e_n$ be the standard primitive matrix idempotents of A and $\bar{A}$. Then the following conditions are equivalent.*

(1) *A is a Frobenius K-algebra and $J(A)^3 = 0$.*
(2) *$\bar{A}$ is a Frobenius K-algebra and $J(\bar{A})^3 = 0$.*
(3) *Either $n = 2$ and A is the algebra A_0 of Example 2.3 or $n \geq 3$ and A is a Frobenius K-algebra such that $J(A)^2 = \mathrm{Soc}(A)$.*
(4) *Either $n = 2$ and q is the minor structure matrix q_0 of Example 2.3 or $n \geq 3$ and there exists a permutation $\sigma \in S_n$ such that $\sigma(i) \neq i$ for $i = 1,\dots,n$, and $q_{ij}^{(k)} \neq 0$ if and only if $k \in \{i,j\}$ or $j = \sigma(i)$.*
(5) *Either $n = 2$ and q is the minor structure matrix q_0 of Example 2.3 or $n \geq 3$ and there exists a permutation $\sigma \in S_n$ such that $\sigma(i) \neq i$ for $i = 1,\dots,n$, and the $(0,1)$-limit $\bar{q} \in \mathrm{ST}_n(K)$ of the block matrix q has the form $\bar{q} = q_\sigma$.*

In this case, σ is the Nakayama permutation of A and of $\bar{A}$. Moreover, $A/J(A)^2 \cong \bar{A}/J(\bar{A})^2$.

Using Theorem 3.4 and Theorem 3.5 one can prove the following interesting classification result.

Proposition 3.6 (cf. [6, Corollary 5.6]). *Assume that $n \geq 3$, q is a basic minor structure matrix in $\mathrm{ST}_n(K)$ such that $A_q = \mathrm{M}_n^q(K)$ is a Frobenius algebra and $J(A_q)^3 = 0$.*

(1) *The algebra A_q is representation-finite if and only if $n = 3$.*
(2) *If the field K is algebraically closed then the algebra A_q is representation-tame if and only if $n = 4$.*
(3) *If the field K is algebraically closed then the algebra A_q is representation-wild if and only if $n \geq 5$.*

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Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warsaw, Poland
E-mail address: L.Kubat@impan.pl