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Polynomial functions on Young diagrams arising from bipartite graphs

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POLYNOMIAL FUNCTIONS ON YOUNG DIAGRAMS ARISING FROM BIPARTITE GRAPHS

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ABSTRACT. We study the class of functions on the set of (generalized) Young diagrams arising as the number of embeddings of bipartite graphs and their linear combinations. We give a criterion for checking when such a function is a polynomial function on Young diagrams (in the sense of Kerov and Olshanski) in terms of combinatorial properties of the corresponding linear combination of bipartite graphs. Our method involves development of a differential calculus of functions on the set of generalized Young diagrams.

1. INTRODUCTION

1.1. The algebra of polynomial functions on Young diagrams.

The character $\chi^\lambda(\pi)$ of the symmetric group is usually considered as a function of the permutation π, with the Young diagram λ fixed. It was observed by Kerov and Olshanski [KO94] that for several problems of the asymptotic representation theory it is convenient to do the opposite: keep the permutation π fixed and let the Young diagram λ vary. In this way it is possible to study the structure of the series of the symmetric groups $\mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots$ and their representations in a uniform way. In order for this idea to be successful it is convenient to replace the usual characters $\chi^\lambda(\pi)$ by, so called, *normalized characters* $\Sigma_\pi(\lambda)$.

Kerov and Olshanski [KO94] defined and studied the *algebra* $\mathcal{P}$ *of polynomial functions on the set* $\mathbb{Y}$ *of Young diagrams* which is spanned by the normalized characters. This concept became very fruitful because the algebra of polynomial functions turned out to be the 'right' object: it is neither too small nor too big, as well as it has several quite distinct facets, each emphasizing another feature of the symmetric groups and their representations. We will review briefly some of these facets in the following.

Firstly, we can view the elements of $\mathcal{P}$ as functions on the set $\mathbb{Y}$ of Young diagrams. In this approach there are several very natural algebraic bases of this algebra, each related to another aspect of the representation theory of the symmetric groups. The most prominent ones are the normalized characters corresponding to cycles, *free cumulants* [Ker00a, Bia03], various functionals of the shape of a Young diagram, and various functions in the alphabet of the contents of the boxes of a Young diagram.

Secondly, the algebra $\mathcal{P}$ turns out to be isomorphic to a subalgebra of the algebra of partial permutations of Ivanov and Kerov [IK99]. Therefore we can view the elements of the algebra $\mathcal{P}$ as partial permutations. Since the multiplication of polynomial functions on the set of Young diagrams corresponds to the convolution of central functions on partial permutations, we see that the algebra $\mathcal{P}$ turns out to be very closely related to the problems of computing connection coefficients and multiplication of conjugacy classes in the symmetric groups. It is remarkable that this subalgebra of the algebra of partial permutations (and hence the algebra $\mathcal{P}$) is isomorphic to a rather classic object: the algebra studied by Farahat and Higman in 1959 [FH59].

Thirdly, the algebra $\mathcal{P}$ is canonically isomorphic to the algebra of shifted symmetric functions and it is convenient to identify these two algebras [OO98, OO97a, OO97b].

The above collection of alternative ways of viewing the algebra of polynomial functions on the set of Young diagrams probably is not complete, but it already shows the richness of this structure. Several problems from the asymptotic representation theory of symmetric groups turned out to be equivalent to questions concerning the algebra $\mathcal{P}$ and relating various ways of viewing it—in particular, finding relations between its various algebraic bases [Bia02, IO02, Bia03, Śni06a, Śni06b, DFŚ10].

1.2. **Polynomial functions on Young diagrams and bipartite graphs.** It turns out that the polynomial functions on the set of Young diagrams can be represented as a linear combination of *the numbers of embeddings of bipartite graphs into the Young diagram*. We postpone the formal definition until Section 2; roughly speaking, an embedding of a bipartite graph into a Young diagram λ is a function which maps the edges of the graph to the boxes of λ with some additional properties. Let $\mathcal{G}$ be a formal sum of bipartite graphs and λ be a Young diagram. We will denote a linear combination of the numbers of embeddings of bipartite graphs corresponding to $\mathcal{G}$ into the Young diagram λ by $N_{\mathcal{G}}(\lambda)$. The family $(N_{\mathcal{G}})$ generates an algebra $\mathcal{QP}$ which contains the algebra $\mathcal{P}$ of polynomial functions on the set $\mathbb{Y}$.

1.2.1. *Application: asymptotics of characters.* This idea was initiated in [FŚ07], where the authors found explicitly representations in terms of numbers of embeddings of bipartite graphs into the Young diagram for the normalized characters $\Sigma_\pi(\lambda)$ as well as for the free cumulants of Young diagrams. The original motivation of the paper [FŚ07] was to give a conceptual reformulation of a formula conjectured by Stanley [Sta06] for the characters on multirectangular Young diagrams and which was proved by Féray

[Fér10]. The reformulation of Stanley-Féray character formula in terms of embeddings of bipartite graphs made it much easier to explore its implications: from a single formula they managed to find new, simple proofs of some old asymptotic estimates of the characters of symmetric groups as well as new, much stronger bounds [FŚ07].

1.2.2. *Application: Kerov polynomials and their variations.* Let us fix some nice algebraic basis of the algebra $\mathcal{P}$. We ask *what is the expansion of a given polynomial function F in this basis?* This kind of problem was stated by Kerov [Ker00a] who asked about the case when $F = \Sigma_\pi$ is the normalized character and when the basis is formed by free cumulants. In this case the expansion is called *Kerov polynomial*. We are also interested in the case of some other bases, for example *fundamental functionals of shape* [DFŚ10].

It turns out that if the representation of the function F as a linear combination of numbers of embeddings of bipartite graphs into the Young diagram is known, then the coefficients in the expansion of F in some bases can be easily recovered from the structure of the corresponding bipartite graphs. In the case when the expansion in the basis of free cumulants is considered, such an expansion is called (generalized) Kerov polynomial ([Fér09, DFŚ10]).

1.3. **The main result.** One can ask a following question:

Problem 1.1. *For a given interesting polynomial function F on the set of Young diagrams, how to find explicitly the expansion of F as a linear combination of the numbers of embeddings of bipartite graphs into the Young diagram?*

This problem is too ambitious and too general to be tractable. In this article we will tackle the following, more modest question.

Problem 1.2. *For which linear combinations of bipartite graphs $\mathcal{G}$ the numbers of embeddings of bipartite graphs into the Young diagram $N_\mathcal{G}$ are polynomial functions on the set of Young diagrams?*

Surprisingly, in some cases the answer to this more modest Problem 1.2 can be helpful in finding the answer to the more important Problem 1.1. The answer for a Problem 1.2 is given by a Theorem 5.2 which states a simple combinatorial condition about the underlying sum of bipartite graphs. In other words, if we have a function $N_\mathcal{G}$ as above there is a simple combinatorial condition which verifies if $N_\mathcal{G}$ is a polynomial function on the set of Young diagrams or not.

1.3.1. *Application: Jack polynomials.* There exists a family $(\mathcal{G})$ of linear combinations of bipartite graphs such that the family of functions $(N_{\mathcal{G}})$ generates the algebra $\mathcal{P}$. As we pointed out in Subsection 1.1, the algebra $\mathcal{P}$ is isomorphic to algebra of shifted symmetric functions. This object can be consider more generally, as an example of so-called *algebra of α-shifted symmetric functions* with $\alpha = 1$. We can evaluate functions $N_{\mathcal{G}}$ at so-called *α-anisotropic Young diagrams* introduced by Kerov [Ker00b] which are, roughly speaking, Young diagrams rescaled along the OX-axis by a parameter α. Then, in these more general frames, functions $\lambda \mapsto N_{\mathcal{G}}(\alpha\lambda)$ with the domain $\mathbb{Y}$ generate the algebra of α-shifted symmetric functions (see for example [Las09]). If we investigate the results of Féray and Śniady ([FŚ07, FŚ10]) we can see that functions N_G are very natural tools to attack some conjectures of Lassalle ([Las08] and [Las09]). Indeed, we can characterize a Jack shifted symmetric function with parameter α (up to a normalization factor) J^α_μ by the following conditions:

(a) $J^\alpha_\mu(\mu) \neq 0$ and for each Young diagram λ such that $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$ we have $J^\alpha_\mu(\lambda) = 0$;
(b) J^α_μ is a shifted symmetric function with parameter α or equivalently is a polynomial function on the set of α-anisotropic Young diagrams;
(c) J^α_μ has degree equal to $|\mu|$.

We can write Jack shifted symmetric functions in the form

$$J^{(\alpha)}_\mu(\lambda) = \sum_{\pi \vdash |\mu|} n^{(\alpha)}_\pi \, \Sigma^{(\alpha)}_\pi(\mu) \, \Sigma^{(\alpha)}_\pi(\lambda),$$

where $n^{(\alpha)}_\pi$ is some combinatorial factor which is out of scope of the current paper and where $\Sigma^{(\alpha)}_\pi$, called *Jack character*, is an α-anisotropic polynomial function on the set of Young diagrams. The problem is therefore reduced to finding the expansion in terms of functions N_G for Jack characters. It is tempting to solve this problem by guessing the right form of the expansion and then by proving that so defined $J^{(\alpha)}_\mu$ have the required properties.

We expect that verifying a weaker version of condition (a), namely:

(a') For each Young diagram λ such that $|\lambda| < |\mu|$ we have $J^{(\alpha)}_\mu(\lambda) = 0$

should not be too difficult; sometimes it does not matter if in the definition of $N_G(\lambda)$ we count all embeddings of the graph into the Young diagram or we count only injective embeddings in which each edge of the graph is mapped into a different box of λ. If this is the case then condition (a') holds trivially if all graphs G over which we sum have exactly $|\mu|$ edges. Also condition (c) would follow trivially. The true difficulty is to check that condition (b) is fulfilled and applying Theorem 5.2 would give an answer for this question.

1.3.2. *Application: New proofs of old results.* Special cases of Jack characters are normalized character and zonal character, which correspond to $\alpha = 1$ and $\alpha = 2$ respectively. We will express them in functions $N_{\mathcal{G}}$. This was done by Féray and Śniady in [FŚ07] and [FŚ10] to prove some conjectures of Lassalle, but the main result of this paper gives us much shorter and simpler proof of this expansion. In this point we can see that having an expression of Jack characters in terms of $N_{\mathcal{G}}$ provide us a quick answer for some conjectures of Lassalle, hence functions $N_{\mathcal{G}}$ seem to be very powerful and interesting tools.

1.4. **Overview of the paper.** The paper is organized as follows. Section 2 gives some basic definitions about Young diagrams and embedding function. Section 3 introduces differential methods for functions on the set of Young diagrams. Section 4 introduces some combinatorics of bipartite graphs. Section 5 shows the connection between combinatorics from Section 4 and differential methods from Section 3 and gives the proof of the main result. Section 6 gives a couple of applications of the main result of this paper.

2. Preliminaries

2.1. **Russian and French convention.** We will use two conventions for drawing Young diagrams: the *French* one in the $0xy$ coordinate system (presented on Figure 1) and the *Russian* one in the $0zt$ coordinate system (presented on Figure 2). Notice that the graphs in the Russian convention are created from the graphs in the French convention by rotating counterclockwise by $\frac{\pi}{4}$ and by scaling by a factor $\sqrt{2}$. Alternatively, this can be viewed as choice of two coordiate systems on the plane: $0xy$, corresponding to the French convention, and $0zt$, corresponding to the Russian convention. The coordinates in both systems are related to each other by

$$\begin{cases} z = x - y, \\ t = x + y, \end{cases}$$

$$\begin{cases} x = \dfrac{z+t}{2}, \\ y = \dfrac{t-z}{2}. \end{cases}$$

For a point on the plane we will define its *content* as its z-coordinate.

In the French coordinates will use the plane $\mathbb{R}^2$ equipped with the standard Lebesgue measure, i.e. the area of a unit square with vertices (x, y)

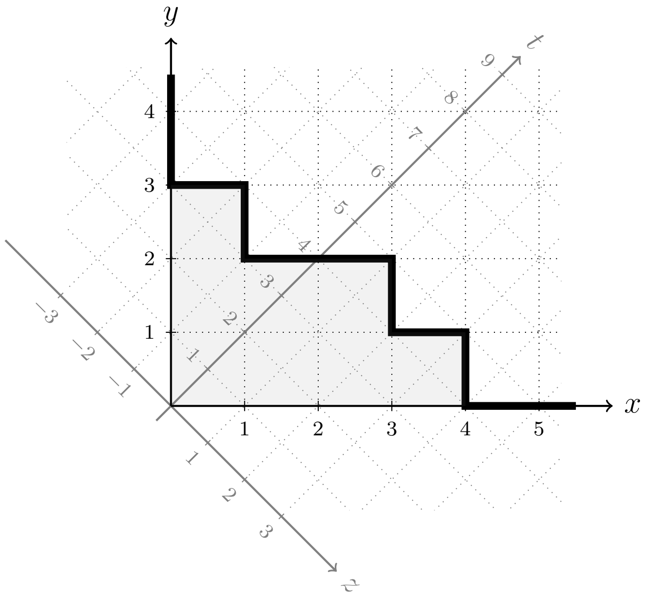

FIGURE 1. Young diagram $(4, 3, 1)$ shown in the French convention. The solid line represents the profile of the Young diagram. The coordinates system (z, t) corresponding to the Russian convention and the coordinate system (x, y) corresponding to the French convention are shown.

such that $x, y \in \{0, 1\}$ is equal to 1. This measure in the Russian coordinates correponds to a the Lebesgue measure on $\mathbb{R}^2$ multiplied by the factor 2, i.e. the area of a unit square with vertices (z, t) such that $z, t \in \{0, 1\}$ is equal to 2.

2.2. **Generalized Young diagrams.** Any Young diagram drawn in the French convention can be identified with its graph which is equal to the set $\{(x, y) : 0 \le x,\ 0 \le y \le f(x)\}$ for a suitably chosen function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the French convention)* as the set of bounded, non-increasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with a compact support; in this way any Young diagram can be regarded as a generalized Young diagram.

We can identify a Young diagram drawn in the Russian convention with its profile, see Figure 2. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the Russian convention)* as the set of functions $\omega : \mathbb{R} \to \mathbb{R}_+$ which fulfill the following two conditions:

- ω is a Lipschitz function with constant 1, i.e. $|\omega(z_1) - \omega(z_2)| \le |z_1 - z_2|$,

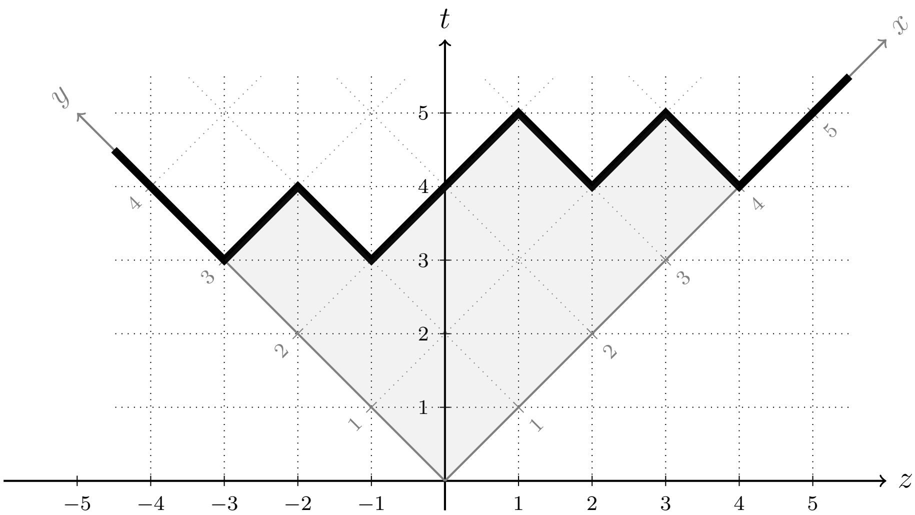

FIGURE 2. Young diagram $(4, 3, 1)$ shown in the Russian convention. The solid line represents the profile of the Young diagram. The coordinates system (z, t) corresponding to the Russian convention and the coordinate system (x, y) corresponding to the French convention are shown.

- $\omega(z) = |z|$ if $|z|$ is large enough.

We will define the support of ω (in the Russian convention) in a natural way:

$$\operatorname{supp}(\omega) = \overline{\{z \in \mathbb{R} : \omega(z) \neq |z|\}}.$$

At the first sight it might seem that we have defined the set $\mathbb{Y}$ of generalized Young diagrams in two different ways, but we prefer to think that these two definitions are just two conventions (French and Russian) for drawing the same object. This will not lead to confusions since it will be always clear from the context which of the two conventions is being used.

2.2.1. *α-anisotropic Young diagrams.* Following Kerov [Ker00b], for a given Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$ we define the α-anisotropic Young diagram $\alpha\lambda$ as the generalized Young diagram stretched anisotropically only along the OX axis, i.e.

$$\alpha\lambda = (\alpha\lambda_1, \alpha\lambda_2, \dots).$$

2.3. **Functionals of shape.** We define the fundamental functionals of shape for integers $k \geq 2$

$$S_k(\lambda) = (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2}\, dx\, dy = \frac{1}{2}(k-1) \iint_{(z,t)\in\lambda} z^{k-2}\, dz\, dt, \tag{1}$$

where the first integral is written in the French and the second in the Russian coordinates. Family $(S_k)_{k\geq 2}$ generates the algebra $\mathcal{P}$ [DFŚ10].

2.4. **Numbers of colorings of bipartite graphs.** The bipartite graphs considered in this article will have no isolated vertices and (if not stated otherwise) will not have multiple edges. The set of vertices will always be $V = V_1 \sqcup V_2$ with the elements of V_1 (respectively, V_2) referred to as white (respectively, black) vertices.

We consider a coloring h of the white vertices in V_1 by columns of the given Young diagram λ and of the black vertices in V_2 by rows of λ. Formally, a coloring is a function $h : V_1 \sqcup V_2 \to \mathbb{R}_+$ and we say that this coloring is *compatible* with Young diagram λ if $(h(v_1), h(v_2)) \in \lambda$ for each edge $(v_1, v_2) \in V_1 \times V_2$ of the underlying bipartite graph G. Alternatively, a coloring which is compatible with λ can be viewed as a function which maps the edges of the bipartite graph to points in λ (viewed in French convention as a subset of $\mathbb{R}_+^2$) with a property that if edges e_1, e_2 share a common white (respectively, black) vertex then $h(e_1)$ and $h(e_2)$ has the same x-coordinate (respectively, the same y-coordinate). We can think that such a coloring defines an embedding of a graph G into the Young diagram λ.

For a given bipartite graph G, we can think how many colorings of this graph are compatible with a Young diagram λ (which is the number of embeddings of G into λ by identification as above). If we fix an order of the vertices in $V = V_1 \sqcup V_2$, we can think of a coloring h as an element of $\mathbb{R}_+^{|V|}$. Then we can define the number of colorings of a bipartite graph G as a function on the set $\mathbb{Y}$ in the following way

$$N_G(\lambda) = \operatorname{vol}\{h \in \mathbb{R}_+^{|V|} : h \text{ compatible with } \lambda\}.$$

The linear combinations of functions of this form will be called *quasi-polynomial functions* on $\mathbb{Y}$. Quasi-polynomial functions on $\mathbb{Y}$ form an algebra denoted by $\mathcal{QP}$ which contains the algebra $\mathcal{P}$ and the main question about $\mathcal{QP}$ is how to characterize quasi-polynomial functions on $\mathbb{Y}$ which belong to $\mathcal{P}$.

Proposition 2.1. *Let G be a bipartite graph, and $G = \bigsqcup_{1\leq i\leq n} G_i$ be a decomposition of G into connected components. Then $N_G = \prod_{1\leq i\leq n} N_{G_i}$.*

Proof. It follows directly from the definition of N_G. □

3. Differential calculus of functions on Young diagrams

3.1. Technical preliminaries.

Lemma 3.1. *Let G be a bipartite graph and $C > 0$ be a number. For any sets $A_1, A_2 \subseteq [0, C]^2$*

$$|N_G(A_1) - N_G(A_2)| \leq |E|\, C^{|V|-2}\ \operatorname{Area}(A_1 \Delta A_2),$$

where $|E|$ denotes the number of edges of G and Δ is a symmetric difference symbol i.e. $A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

Proof. Suppose that the lemma holds true under an additional assumption that $A_2 \subseteq A_1$. Then the general case would follow; indeed it would be enough to apply the lemma twice for $A_1' := A_1 \cup A_2$ and $A_2' := A_2$ and for $A_1' := A_1 \cup A_2$ and $A_2' := A_1$. From the following on, we will assume that $A_2 \subseteq A_1$.

The difference $N_G(A_1) - N_G(A_2)$ can be interpreted as the volume of the colorings of the graph G which are compatible with the set A_1 and which are not compatible with the set A_2. For every such a coloring h there exists an edge e such that $h(e) \notin A_2$. Therefore

$$0 \leq N_G(A_1) - N_G(A_2) \leq \sum_e N_{G,e}(A_1, A_2),$$

where

$$N_{G,e}(A_1, A_2) = \big(\text{volume of the colorings of } G \\ \text{compatible with } A_1 \text{ and such that } h(e) \in A_1 \setminus A_2\big).$$

Every coloring which contributes to $N_{G,e}(A_1, A_2)$ can be constructed as follows: firstly we decide the value of $h(e) \in A_1 \setminus A_2$; then we decide the value of $h(v)$ for $v \notin e$. It follows that

$$N_{G,e}(A_1, A_2) \leq C^{|V|-2}\ \operatorname{Area}(A_1 \setminus A_2)$$

which finishes the proof. □

3.2. Content-derivatives.

Let F be a function on the set of generalized Young diagrams, let ω be a generalized Young diagram and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that

$$\partial_{C_z} F(\lambda) = f(z)$$

if for any $\epsilon > 0$ and $C > 0$ there exists $\delta > 0$ such that for any generalized Young diagrams ω_1, ω_2 such that $\|\omega - \omega_i\|_{L^1} < \delta$ for $i \in \{1, 2\}$ and such that $\operatorname{supp}(\omega_1), \operatorname{supp}(\omega_2) \subseteq [-C, C]$

$$\left| F(\omega_1) - F(\omega_2) - \frac{1}{2} \int_{\mathbb{R}} f(z)\big(\omega_1(z) - \omega_2(z)\big)\, dz \right| \leq \epsilon \, \|\omega_1 - \omega_2\|_{L^1}, \tag{2}$$

where $\|f\|_{L^1} = \int_{\mathbb{R}} |f(z)|\, dz$ is a standard L^1 norm. This definition is motivated by the Gâteaux derivative which is a classical tool in functional analysis [Gât13] and roughly speaking $\partial_{C_z} F(\lambda)$ measures how quickly the value of $F(\lambda)$ would change if we change the shape of λ by adding infinitesimal boxes with content equal to z. The strange constant $\frac{1}{2}$ in this definition appears, because of the fact that we are working with the Russian convention which rescaled the length and the height of the Young diagram by a factor $\sqrt{2}$, hence

$$\operatorname{Area}(\lambda) = \frac{1}{2} \int_{\mathbb{R}} \big(\omega(z) - |z|\big)\, dz.$$

Lemma 3.2. *If the derivative $\partial_{C_z} F(\lambda)$ exists, then it is unique.*

Proof. Let $z_0 \in \mathbb{R}$, $\epsilon > 0$. Let $\delta > 0$ be such that (2) is fulfilled and furthermore

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

We will construct ω_1, ω_2 such that $\|\omega - \omega_i\|_{L^1} < \delta$ for $i \in \{1, 2\}$,

$$\omega_1(z) - \omega_2(z) \geq 0 \qquad \text{for any } z \in \mathbb{R},$$

support of $\omega_1 - \omega_2$ is contained in the ϵ-neighborhood of z_0 and is non-empty. Then

$$f(z_0) - \epsilon \leq \inf_{|z - z_0| < \delta} f(z) \leq \frac{\int_{\mathbb{R}} f(z)\big(\omega_1(z) - \omega_2(z)\big)\, dz}{\int_{\mathbb{R}} \omega_1(z) - \omega_2(z)\, dz} \leq \\ \frac{2F(\omega_1) - 2F(\omega_2)}{\int_{\mathbb{R}} \omega_1(z) - \omega_2(z)\, dz} + 2\epsilon.$$

This, together with an analogous inequality shows that

$$\left| f(z_0) - \frac{2F(\omega_1) - 2F(\omega_2)}{\int_{\mathbb{R}} \omega_1(z) - \omega_2(z)\, dz} \right| < 3\epsilon$$

which shows that function f, if exists, must be unique.

It remains to show existence of ω_1, ω_2 with the required properties. Assume that $C > |z_0|$ is such that $\operatorname{supp}(\omega) \subseteq [-C, C]$. We choose $0 < \gamma < 1$ which is small enough that $\|\omega - \omega_1\| < \frac{\delta}{2}$, where

$$\omega_1(z) = \begin{cases} |z| & \text{for } |z| > C, \\ C + (\omega(z) - C)\,(1 - \gamma) & \text{for } |z| \leq C. \end{cases}$$

If $\epsilon < C - |z_0|$ then ω_1 is Lipschitz on the ϵ-neighborhood of z_0 with constant $1 - \gamma < 1$.

Let $s : \mathbb{R} \to \mathbb{R}_+$ be a Lipschitz function with compact support. Then for sufficiently small value of $a > 0$ and sufficietly big value of b

$$\omega_2(z) = \omega_1(z) + as\big(b(z - z_0)\big)$$

has the desired properties. □

3.3. Content-derivatives of polynomial functions.

Proposition 3.3.

(i) The Leibnitz rule holds, i.e. if F_1, F_2 are sufficiently smooth functions then

$$\partial_{C_z} F_1 F_2 = (\partial_{C_z} F_1)\, F_2 + F_1 \partial_{C_z} F_2.$$

(ii) For any integer $k \geq 2$

$$\partial_{C_z} S_k = (k-1) z^{k-2}.$$

Proof. The second part follows from the definition (1) and the first part is a classical calculation almost the same as in Gâteaux derivative case, hence we omit it. □

Proposition 3.4. *Let F be a polynomial function on the set of Young diagrams.*

(i) For any Young diagram λ

$$z \mapsto \partial_{C_z} F(\lambda)$$

is a polynomial.

(ii) For any $z_0 \in \mathbb{R}$

$$\lambda \mapsto \partial_{C_{z_0}} F(\lambda)$$

is a polynomial function on the set of Young diagrams.

(iii) For any integer $k \geq 0$

$$\lambda \mapsto [z^k] \partial_{C_z} F(\lambda)$$

is a polynomial function on the set of Young diagrams.

Proof. By the linearity it is enough to prove it for $F = \prod_{1 \leq i \leq n} S_{k_i}$. Then, thanks to the Proposition 3.3, we have that

$$\partial_{C_z} F = \sum_{1 \leq i \leq n} (k_i - 1) z^{k_i - 2} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} S_{k_j},$$

which is a polynomial in z for fixed λ, and which is a polynomial function on the set of Young diagrams for fixed $z = z_0$. Moreover $[z^k]\partial_{C_z} F(\lambda)$ is a linear combination of products of S_{k_i}, hence it is a polynomial function on the set of Young diagrams, which finishes the proof. □

The main result of this paper is that (in some sense) the opposite implication is true as well and thus it characterizes the polynomial functions on the set of Young diagrams.

3.4. **Changing shape of Young diagrams.**

Lemma 3.5. *Let $\mathbb{R} \ni t \mapsto \lambda_t$ be a sufficiently smooth trajectory in the set of Young diagrams and let F be a sufficiently smooth function on the set of Young diagrams. Then*

$$\frac{d}{dt} F(\lambda_t) = \int_{\mathbb{R}} \frac{1}{2} \frac{d\omega_t(z)}{dt}\, \partial_{C_z} F(\lambda_t)\, dz.$$

Proof. Strictly from the definition (2) we have the following:

$$\left(\int_{\mathbb{R}} \frac{1}{2} \frac{d\omega_t(z)}{dt}\, \partial_{C_z} F(\lambda_t)\, dz \right) - \epsilon \left\| \frac{d\omega_t(z)}{dt} \right\|_{L^1} \leq \tag{3}$$
$$\leq \frac{d}{dt} F(\lambda_t) \leq$$
$$\leq \left(\int_{\mathbb{R}} \frac{1}{2} \frac{d\omega_t(z)}{dt}\, \partial_{C_z} F(\lambda_t)\, dz \right) + \epsilon \left\| \frac{d\omega_t(z)}{dt} \right\|_{L^1}$$

for any $\epsilon > 0$. Taking $\epsilon \to 0$ we finish the proof. □

4. Derivatives on bipartite graphs

Let G be a bipartite graph. We denote

$$\partial_z G = \sum_e (G, e),$$

which is a formal sum (formal linear combination) running over all edges e of G. We will think about the pair (G, e) that it is the graph G with one edge e decorated with the symbol z. In the future we will also refer to this decorated edge as to the edge z. More generally, if $\mathcal{G}$ is a linear combination of bipartite graphs, this definition extends by linearity.

If G is a bipartite graph with one edge decorated by the symbol z, we define

$$\partial_x G = \sum_f G_{f \equiv z}$$

which is a formal sum running over all edges $f \neq z$ which share a common black vertex with the edge z. The symbol $G_{f \equiv z}$ denotes the graph G in which the edges f and z are glued together (which means that also the white vertices of f and z are glued together and that from the resulting graph all multiple edges are replaced by single edges). The edge resulting from gluing f and z will be decorated by z. More generally, if $\mathcal{G}$ is a linear combination of bipartite graphs, this definition extends by linearity.

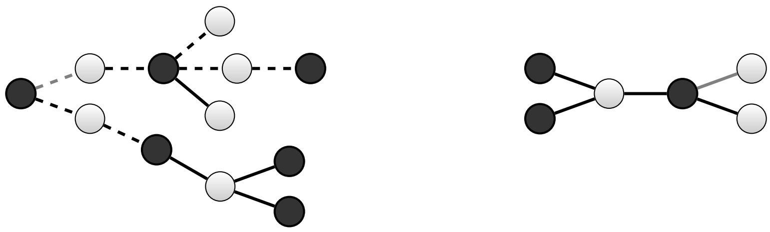

FIGURE 3. A pair of graphs $\mathcal{T}' \subseteq \mathcal{T}$ (draw on the left) with a common decorated edge by a gray color and a graph with one decorated edge by a gray color (draw on the right) obtained by collapsing a subgraph $\mathcal{T}'$ of a graph $\mathcal{T}$ to one decorated edge. A subgraph $\mathcal{T}' \subseteq \mathcal{T}$ is indicated on a picture by drawing dotted edges in $\mathcal{T}$.

We also define

$$\partial_y G = \sum_f G_{f \equiv z}$$

which is a formal sum which runs over all edges $f \neq z$ which share a common white vertex with the edge z.

Theorem 4.1. *Let $\mathcal{G}$ be a linear combination of bipartite graphs without a cycle of length 4 with a property that*

$$(\partial_x - (-\partial_y))\, \partial_z \mathcal{G} = 0. \tag{4}$$

Then for any integer $k \geq 1$

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = 0. \tag{5}$$

Proof. We will prove it by induction on k. For $k = 1$ there is nothing to do. Let $k = 2$. We will show that

$$(\partial_x \partial_y - \partial_y \partial_x)\, \partial_z \mathcal{G} = 0,$$

which implies that

$$\left(\partial_x^2 - (-\partial_y)^2\right) \partial_z \mathcal{G} = 0,$$

because of the assertion (4). Indeed, $\frac{1}{2}\partial_x\partial_y\partial_z\mathcal{G}$ and $\frac{1}{2}\partial_y\partial_x\partial_z\mathcal{G}$ can be both interpreted as taking all subtrees of $\mathcal{G}$ of type $\mathcal{T}_{[1,2]}^{[1,2]}$ (see Figure 4) and collapsing them to one decorated edge (see Figure 3). We assume that the statement holds for all $l < k$.

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = \left(\partial_x(-\partial_y)^{k-1} - \partial_y \partial_x^{k-1}\right) \partial_z \mathcal{G}$$

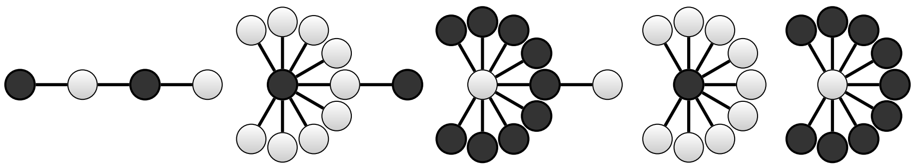

FIGURE 4. $\mathcal{T}^{[1,2]}_{[1,2]}$, $\mathcal{T}^{[1,k]}_{[1^{k-1},2]}$, $\mathcal{T}^{[1^{k-1},2]}_{[1,k]}$, $\mathcal{T}^{[k+1]}_{[1^{k+1}]}$ and $\mathcal{T}^{[1^{k+1}]}_{[k+1]}$ from the left to the right, where $\mathcal{T}^{[1,k]}_{[1^{k-1},2]}$, $\mathcal{T}^{[1^{k-1},2]}_{[1,k]}$, $\mathcal{T}^{[k+1]}_{[1^{k+1}]}$ and $\mathcal{T}^{[1^{k+1}]}_{[k+1]}$ have $k+1$ edges.

by the inductive assertion. If G is a bipartite graph with one edge decorated by the symbol z, we define

$$\partial_N(G,z) = \sum_{w,b} G_{w,b\equiv z}$$

which is a formal sum running over all edges $w \neq z$ which share a common white vertex with the edge z and over all edges $b \neq z$ which share a common black vertex with the edge z. We observe that

$$\frac{1}{k!}\partial_x \partial_y^{k-1} \partial_z \mathcal{G} = \frac{1}{(k-1)!(k-1)} \partial_N \partial_y^{k-2} \partial_z \mathcal{G}.$$

Indeed, both of them can be interpreted as taking all subtrees of $\mathcal{G}$ of type $\mathcal{T}^{[1^{k-1},2]}_{[1,k]}$ (see Figure 4) and collapsing them to one decorated edge. Similarly,

$$\frac{1}{k!}\partial_y \partial_x^{k-1} \partial_z \mathcal{G} = \frac{1}{(k-1)!(k-1)} \partial_N \partial_x^{k-2} \partial_z \mathcal{G}.$$

Finally,

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = \frac{k}{k-1} \partial_N \left((-\partial_y)^{k-2} - \partial_x^{k-2}\right) \partial_z \mathcal{G} = 0$$

by the inductive assertion, which finishes the proof. □

We are not able to omit the assumption about a cycle of length 4, however we believe in the following Conjecture:

Conjecture 4.2. *Let $\mathcal{G}$ be a linear combination of bipartite graphs with a property that*

$$\left(\partial_x - (-\partial_y)\right) \partial_z \mathcal{G} = 0.$$

Then for any integer $k \geq 1$

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = 0.$$

5. Characterization of functions arising from bipartite graphs which are polynomial

5.1. **Complete bipartite graphs.** By $K_{r,s}$ we will denote the complete bipartite graph with r white and s black vertices. By $\widetilde{K}_{r,s}$ we will denote the complete bipartite graph $K_{r,s}$ with one decorated edge.

Proposition 5.1.

(i)

$$N^{\lambda}_{K_{r,s}} = rs \iint_{(x,y)\in\lambda} x^{r-1} y^{s-1}\, dx\, dy$$

(ii)

$$S_k(\lambda) = N^{\lambda}_{\mathcal{G}}$$

for

$$\mathcal{G} = \frac{1}{k} \sum_{r+s=k} \binom{k}{r} (-1)^{s-1} K_{r,s}$$

(iii) for any integers $r, s \geq 1$

$$\begin{aligned} \partial_z K_{r,s} &= rs \widetilde{K}_{r,s}, \\ \partial_x \widetilde{K}_{r,s} &= (r-1) \widetilde{K}_{r-1,s}, \\ \partial_y \widetilde{K}_{r,s} &= (s-1) \widetilde{K}_{r,s-1} \end{aligned}$$

(iv)

$$N_{(\partial_x^i - (-\partial_y)^i)\partial_z \mathcal{G}} = 0$$

for

$$\mathcal{G} = \frac{1}{k} \sum_{r+s=k} \binom{k}{r} (-1)^{s-1} K_{r,s}$$

and for any $i \in \mathbb{N}_+$

Proof. Let V_1 and V_2 be the sets of black and white vertices of $K_{r,s}$ respectively, and let h be compatible with λ. Then

$$h(V_1) \times h(V_2) = \{(x_i, y_j) \in \lambda : 1 \leq i \leq r, 1 \leq j \leq s\}.$$

There exists an element $(x_i, y_j) \in h(V_1) \times h(V_2)$ such that $x_{i'} \leq x_i$ and $y_{j'} \leq y_j$ for all $(x_{i'}, y_{j'}) \in h(V_1) \times h(V_2)$, so for fixed (x_i, y_j) as before we can set the rest possible values of h in $x_i^{r-1} y_j^{s-1}$ places. Of course there is rs pairs (i, j) such that $1 \leq i \leq r, 1 \leq j \leq s$, hence

$$N^{\lambda}_{K_{r,s}} = \operatorname{vol}\{h \in \mathbb{R}_+^{r+s} : h \text{ compatible with } \lambda\} = rs \iint_{(x,y)\in\lambda} x^{r-1} y^{s-1}\, dx\, dy.$$

Moreover

$$S_k(\lambda) = \iint_{(x,y)\in\lambda} (k-1)(x-y)^{k-2}\, dx\, dy = \sum_{r+s=k-2} \frac{(k-1)!}{r!s!}(-1)^s \iint_{(x,y)\in\lambda} x^r y^s\, dx\, dy = \frac{1}{k}\sum_{r+s=k} \frac{k!}{r!s!}(-1)^{s-1} rs \iint_{(x,y)\in\lambda} x^{r-1}y^{s-1}\, dx\, dy = N_{\mathcal{G}}^{\lambda} \tag{6}$$

Part (iii) is obvious.

In order to prove (iv) we can assume that $i < k$, because

$$\partial_x^k \mathcal{G} = \partial_y^k \mathcal{G} = 0.$$

Then, the following equalities hold

$$\partial_x^i \partial_z \mathcal{G} = \frac{1}{k} \sum_{r+s=k; r-i>1} \binom{k}{r} (-1)^{s-1} sr \left((r-1)_i \widetilde{K}_{r-i,s} \right) = (k-1)! \sum_{r+s=k; r-i>0} \frac{(-1)^{s-1}}{(s-1)!(r-1-i)!)} \widetilde{K}_{r-i,s} = (k-1)! \sum_{r+s=k-i} \frac{(-1)^{s-1}}{(s-1)!(r-1)!)} \widetilde{K}_{r,s} = (k-1)! \sum_{r+s=k; s-i>0} \frac{(-1)^{s-1-i}}{(s-1-i)!(r-1)!)} \widetilde{K}_{r,s-i} = \frac{(-1)^i}{k} \sum_{r+s=k; s-i>1} \binom{k}{r} (-1)^{s-1} sr \left((s-1)_i \widetilde{K}_{r,s-i} \right) = (-\partial_y)^i \partial_z \mathcal{G}. \tag{7}$$

□

5.2. **The main result.**

Theorem 5.2. *Let $\mathcal{F}$ be a function on $\mathbb{Y}$. $\mathcal{F}$ is a polynomial function on $\mathbb{Y}$ iff there exists a linear combination of bipartite graphs $\mathcal{G}$ such that $\mathcal{F} = N_{\mathcal{G}}$ and $\mathcal{G}$ has a property that*

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = 0 \tag{8}$$

for any $k \in \mathbb{N}_+$.

5.3. **Proof of the main result.**

5.3.1. *Colorings of bipartite graphs with decorated edges.* Let a Young diagram λ and a bipartite graph G be given. If an edge of G is decorated by a real number z, we decorate its white end by the number $\frac{\omega(z)+z}{2}$ (which is the x-coordinate of the point at the profile of λ with contents equal to z) and we decorate its black end by the number $\frac{\omega(z)-z}{2}$ (which is the y-coordinate of the point at the profile of λ with contents equal to z). If edges are decorated by n real numbers $z_1, \dots, z_n$, then we require that decorated edges are vertex-disjoint and then we decorate white and black vertices as above.

We define $N_G(\lambda)$, the number of colorings of λ as the volume of the set of functions from undecorated vertices to $\mathbb{R}_+$ such that these functions extended by values of decorated vertices are compatible with λ.

Remark 5.3. For bipartite graph G with no decorated vertices $N_G(\lambda)$ is different than $N_G(\lambda)$, where G has some decorated vertices. We use the same symbol for these two expressions, but it should be clear from the context if a graph G has decorated vertices or not.

Lemma 5.4. *Let (G, z) be a pair of bipartite graph with one edge decorated by a real number z and decorated vertices of this edge as above. Then*

(i) $\partial_{C_z} N_G = N_{\partial_z G}$,

(ii) $\frac{d}{dz} N_{(G,z)}(\lambda) = \frac{\omega'(z)+1}{2} N_{\partial_x (G,z)}(\lambda) + \frac{\omega'(z)-1}{2} N_{\partial_y (G,z)}(\lambda)$.

Proof. In order to find the content-derivative of $F(\omega)$ it is enough to prove that for any $\epsilon > 0$ and $C > 0$ there exists $\delta > 0$ such that for any generalized Young diagrams ω_1, ω_2 such that $\|\omega - \omega_i\|_{L^1} < \delta$ for $i \in \{1, 2\}$, $\lambda_2 \subseteq \lambda_1$ and such that $\operatorname{supp}(\omega_1), \operatorname{supp}(\omega_2) \subseteq [-C, C]$ inequality (2) holds. Indeed, for any generalized Young diagrams λ_1, λ_2 we can construct a generalized Young diagram $\omega_3(z) = \min(\omega_1(z), \omega_2(z))$ such that $\lambda_3 \subseteq \lambda_1, \lambda_2$ and $\operatorname{supp}(\omega_3) \subseteq \operatorname{supp}(\omega_1) \cup \operatorname{supp}(\omega_2)$. If we pick some $\epsilon > 0$ and some $C > 0$ then for any generalized Young diagrams λ_1, λ_2 such that $\|\omega - \omega_i\|_{L^1} < \delta/2$ for $i \in \{1, 2\}$ (where δ is as above) and $\operatorname{supp}(\omega_1), \operatorname{supp}(\omega_2) \subseteq [-C, C]$ we know from triangle inequality that $\|\omega - \omega_i\|_{L^1} < \delta$ for $i \in \{1, 2, 3\}$ and we know that $\operatorname{supp}(\omega_3) \subseteq [-C, C]$. Hence

$$\left| F(\omega_1) - F(\omega_2) - \frac{1}{2} \int_{\mathbb{R}} f(z) \big(\omega_1(z) - \omega_2(z)\big) \, dz \right| \leq$$
$$\left| F(\omega_1) - F(\omega_3) - \frac{1}{2} \int_{\mathbb{R}} f(z) \big(\omega_1(z) - \omega_3(z)\big) \, dz \right| +$$
$$\left| F(\omega_2) - F(\omega_3) - \frac{1}{2} \int_{\mathbb{R}} f(z) \big(\omega_2(z) - \omega_3(z)\big) \, dz \right| \leq$$
$$\epsilon \ \left(\|\omega_2 - \omega_3\|_{L^1} + \|\omega_1 - \omega_3\|_{L^1} \right) = \epsilon \|\omega_1 - \omega_2\|_{L^1}.$$

Let $\lambda_2 \subseteq \lambda_1$ be generalized Young diagrams such that $\|\omega_1 - \omega_2\|_{L^1} < \epsilon/n$ for some $n \in \mathbb{N}$. Then

$$N_G(\lambda_1) - N_G(\lambda_2) = \sum_e N_{G,e}(\lambda_1, \lambda_2) + O((\epsilon/n)^2),$$

where we sum over all edges of G and

$$N_{G,e}(\lambda_1, \lambda_2) = \big(\text{volume of the colorings } h \text{ of } G \\ \text{compatible with } \lambda_2 \text{ and such that } h(e) \in \lambda_1 \setminus \lambda_2\big).$$

By Fubini's Theorem we can see that

$$\sum_e N_{G,e}(\lambda_1, \lambda_2) = \frac{1}{2} \int_{\mathbb{R}} N_{\partial_z G}\big(\omega_1(z) - \omega_2(z)\big)\, dz.$$

We have shown that

$$\left| N_G(\lambda_1) - N_G(\lambda_2) - \frac{1}{2} \int_{\mathbb{R}} N_{\partial_z G}\big(\omega_1(z) - \omega_2(z)\big)\, dz \right| = O((\epsilon/n)^2).$$

Taking n sufficiently large we conclude the proof of the part (i).

Let (G, z) be a bipartite graph with one decorated edge. We consider a function $f : \lambda \to \mathbb{R}$ such that $f(x_0, y_0)$ is a volume of a set of colorings of G which are compatible with λ and such that the black vertex $\bullet$ of an edge z is colored by x_0 and the white vertex $\circ$ of an edge z is colored by y_0. We observe that

$$f(x_0 - \epsilon, y_0) - f(x_0, y_0) = f(x_0, y_0, y_\epsilon) + O(\epsilon^2),$$

where $f(x, y, y_\epsilon)$ is a volume of a set of colorings of G which are compatible with λ and such that the black vertex $\bullet$ of an edge z is colored by $x - \epsilon$, the white vertex $\circ$ of an edge z is colored by y and exactly one white vertex which is connected to $\bullet$ is colored by some $y_\epsilon \in [h(x), h(x-\epsilon)]$, for h giving a generalized Young diagram λ in French convension. Hence, for $y_0 = h(x_0)$ we have that $\partial_x f(x_0, y_0) = f_x(x_0, y_0)$, where f_x is defined exactly as the function f but for $\partial_x(G, z)$ instead of (G, z). Similarly, we can show that for x_0 such that $y_0 = h(x_0)$ we have that $\partial_y f(x_0, y_0) = f_y(x_0, y_0)$, where f_y is defined exactly as the function f but for $\partial_y(G, z)$ instead of (G, z). We can write $N_{(G,z)}(\lambda) = f(x(z), y(z))$, where two variables $x(z)$ and $y(z)$ depend on z, hence

$$\frac{d}{dz} N_{(G,z)} = x'(z) \partial_x f(x(z), y(z)) + y'(z) \partial_y f(x(z), y(z)).$$

Taking $x(z) = \frac{\omega(z)+z}{2}$, $y(z) = \frac{\omega(z)-z}{2}$ we conclude the proof of (ii). □

Lemma 5.5. *Let the assumptions of Theorem 5.2 be fulfilled. Then* $z \mapsto N_{\partial_z G}(\lambda)$ *is a polynomial.*

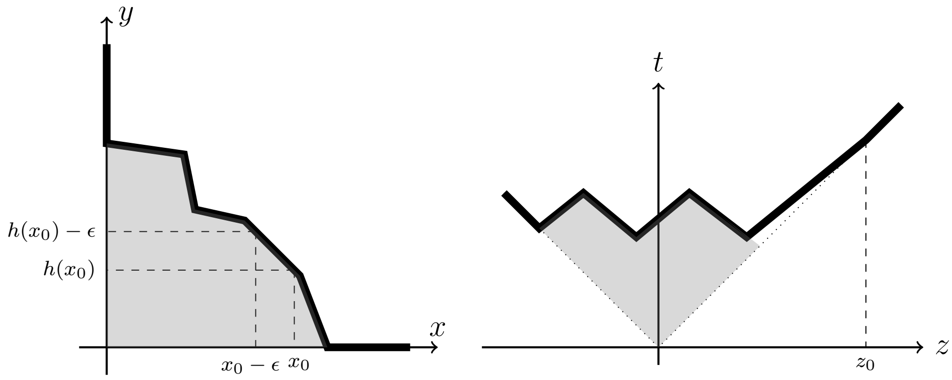

FIGURE 5. Function h which gives the Young diagram λ in French convension on the left. On the right we have a profile of the Young diagram $\lambda_{z_0,\epsilon}$ which is draw by a thick line and the Young diagram λ which is indicated by gray area.

Proof. By the assumption and by Theorem 4.1 we have that:

$$(\partial_x + \partial_y)(\partial_x - \partial_y)^i \partial_z \mathcal{G} = 2^i(\partial_x^{i+1} - (-\partial_y)^{i+1})\partial_z \mathcal{G} = 0.$$

Then, by Lemma 5.4(ii) we can calculate that:

$$\frac{d^i}{dz^i} N_{\partial_z \mathcal{G}} = \frac{N_{(\partial_x - \partial_y)^i \partial_z \mathcal{G}}}{2^i} = N_{\partial_x^i \partial_z \mathcal{G}} = (-1)^i N_{\partial_y^i \partial_z \mathcal{G}}.$$

Hence for sufficiently large $i > 0$ we have that $\frac{d^i}{dz^i} N_{\partial_z \mathcal{G}} = 0$ which shows that

$$z \mapsto N_{\partial_z \mathcal{G}}(\lambda)$$

is a polynomial. □

5.3.2. *Proof of the main result.*

Proof of Theorem 5.2. If $N_\mathcal{G}$ is a polynomial function on the set of Young diagrams, then $N_\mathcal{G}$ is a polynomial in functionals S_k. Let $\mathcal{G}_k$ be a formal sum of bipartite graphs such that $S_k = N_{\mathcal{G}_k}$. It is enough to prove that $N_{(\partial_x + \partial_y)\partial_z \mathcal{G}_k} = 0$ for any integer $k > 1$. Proposition 5.1(iv) finishes the proof of the first implication.

In order to prove the opposite implication we can assume without loss of generality that every graph which contributes to $\mathcal{G}$ has the same number of vertices, equal to m. Indeed, if this is not the case, we can write $\mathcal{G} = \mathcal{G}_2 + \mathcal{G}_3 + \cdots$ as a finite sum, where every graph contributing to $\mathcal{G}_i$ has i vertices; then clearly (8) is fulfilled for every $\mathcal{G}' := \mathcal{G}_i$. We will prove by induction on m that $N_\mathcal{G}$ is a polynomial function on the set $\mathbb{Y}$. If $m = 2$, then $\mathcal{G} = \alpha K_{1,1}$ where $\alpha \in \mathbb{R}$, hence $N_\mathcal{G} = \alpha S_2$ is a polynomial function on the set $\mathbb{Y}$. We assume that the statement is true for any $\mathcal{G}$ as above such that $\deg(\mathcal{G}) < m$ and we assume that $\deg(\mathcal{G}) = m$. For a Young diagram λ and for $t \in \mathbb{R}_+$ we will define a rescaled Young diagram $t\lambda$ which has a profile ω_t given by $\omega_t(z) = t\omega(z/t)$. We observe that a function $\mathbb{R}_+ \ni t \mapsto N_{\mathcal{G}}(t\lambda)$ is a homogeneous function of the degree equal to m, hence we can write that

$$N_{\mathcal{G}}(\lambda) = \frac{1}{m} \frac{d}{dt} N_{\mathcal{G}}(t\lambda)\bigg|_{t=1}.$$

Then, if we apply Lemma 3.5 we have that

$$N_G(\lambda) = \frac{1}{2m} \int_{\mathbb{R}} (\omega(z) - z\omega'(z)) \partial_{C_z} N_{\mathcal{G}}(\lambda) dz = \frac{1}{2m} \int_{\mathbb{R}} (\omega(z) - z\omega'(z)) N_{\partial_z \mathcal{G}}(\lambda) dz. \tag{9}$$

By Lemma 5.5 we know that $N_{\partial_z \mathcal{G}}$ is a polynomial in z. Let $\operatorname{supp}(\lambda) \subseteq [-C, C]$. For any $z_0 \in \mathbb{R}_+ \setminus [-C, C]$ we can construct $\lambda_{z_0,\epsilon} \in \mathbb{Y}$ (see Figure 5) such that:

(i) $\|\lambda_{z_0,\epsilon} - \lambda\|_{L^1} < \epsilon$,
(ii) $\lambda \subseteq \lambda_{z_0,\epsilon}$,
(iii) $z_0 = \sup(\operatorname{supp}(\lambda_{z_0,\epsilon}))$.

We know by Lemma 3.1 that

$$|N_{\partial_{z_0} G}(\lambda) - N_{\partial_{z_0} \mathcal{G}}(\lambda_{z_0,\epsilon})| < A_G\, \epsilon,$$

for some $A_G \in \mathbb{R}_+$ which depends only on the bipartite graph G. It means that

$$N_{\partial_{z_0} G}(\lambda) = \lim_{\epsilon \to 0} N_{\partial_{z_0} G}(\lambda_{z_0,\epsilon})$$

for any bipartite graph G. Let (G, z_0) be a graph which contributes to $\mathcal{G}$ with one edge decorated by z_0. If the black vertex $\bullet$ colored by z_0 has at least one white neighbor different than the already colored one, then this neighbor can be colored by at most $O(\epsilon)$ ways to obtain a compatible coloring with $\lambda_{z_0,\epsilon}$, hence $\lim_{\epsilon \to 0} N_{(G,z_0)}(\lambda_{z_0,\epsilon}) = 0$. Unless, we observe that each black vertex of degree one which is connected to white vertex $\circ$ colored by 0 can be colored by z_0 colors to obtain a compatible coloring with $\lambda_{z_0,\epsilon}$. The last part of this analysis is the observation that the colors of the remaining vertices are somehow independent of z_0. More precisely, let us consider the graph G' which is constructed from G by removing vertices $\bullet, \circ$ and all black vertices of degree one connected to $\circ$. Then:

$$N_{(G,z_0)}(\lambda) = z_0^{\text{number of black vertices of degree one connected to } \circ} N_{G'}(\lambda).$$

It means, that there exist linear combinations of bipartite graphs $\mathcal{G}_0, \cdots, \mathcal{G}_{m-2}$ such that $\deg(\mathcal{G}_k) = k$ and

$$N_{\partial_{z_0}\mathcal{G}}(\lambda) = \sum_{0 \le k \le m-2} {z_0}^{m-2-k} N_{\mathcal{G}_k}(\lambda).$$

The above equation holds for any sufficiently big z_0, hence we can conclude that

$$N_{\partial_z\mathcal{G}}(\lambda) = \sum_{0 \le k \le m-2} z^{m-2-k} N_{\mathcal{G}_k}(\lambda)$$

for any $z \in \mathbb{R}$, because of the fact that $N_{\partial_z\mathcal{G}}(\lambda)$ is a polynomial in z. We want to check that $\mathcal{G}_k$ satisfies the main assumption (8):

$$(\partial_x^i - (-\partial_y)^i)\partial_z\mathcal{G}_k = (\partial_{x_1}^i - (-\partial_{y_1})^i)\partial_{z_1}[z^i]N_{\partial_z\mathcal{G}} = \\ [z_i]\partial_{C_z} N_{(\partial_{x_1}^i - (-\partial_{y_1})^i)\partial_{z_1}\mathcal{G}} = [z^i]\partial_{C_z} 0 = 0.$$

Now, applying an inductive assertion we have that

$$N_{\partial_z\mathcal{G}} = \sum_{0 \le i \le m-2} z^i \mathcal{F}_i,$$

where $\mathcal{F}_i = N_{\mathcal{G}_{m-2-i}} \in \mathcal{P}$ for each $0 \le i \le m-2$. Finally,

(10)
$$N_{\mathcal{G}}(\lambda) = \frac{1}{2m} \sum_{1 \le i \le m-2} \mathcal{F}_i(\lambda)/(i+1) \int_{\mathbb{R}} (\omega(z) - z\omega'(z))(i+1) z^i dz = \\ \frac{1}{2m} \sum_{1 \le i \le m-2} \mathcal{F}_i(\lambda) S_{i+2}(\lambda)/(i+1),$$

which finishes the proof. □

6. APPLICATIONS

6.1. **Bipartite maps.** In order to state a theorem let us recall some facts about maps.

A labeled (bipartite) graph drawn on a surface will be called a *(bipartite) map*. If this surface is orientable and its orientation is fixed, then the underlying map is called *oriented*; otherwise the map is *unoriented*. We will always assume that the surface is minimal in the sense that after removing the graph from the surface, the latter becomes a collection of disjoint open discs. If we draw an edge of such a graph with a fat pen and then take its boundary, this edge splits into two *edge-sides*. In the above definition of the map, by *'labeled'* we mean that each edge-side is labeled with a number from the set $[2n]$ and each number from this set is used exactly once.

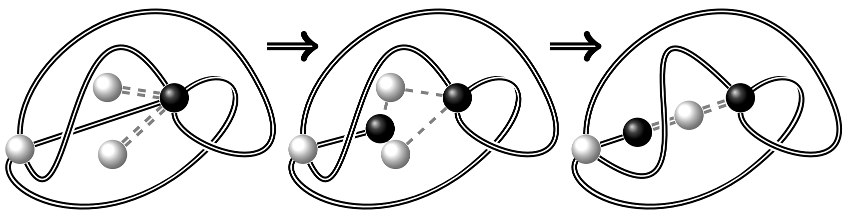

FIGURE 6. Example of a construction of a map and its subtree $(\tilde{\mathcal{M}}, \tilde{\mathcal{T}})$ (on the right) from a given map with its subtree $(\mathcal{M}, \mathcal{T})$ (on the left), such that $\tilde{\mathcal{M}}/\tilde{\mathcal{T}} = \mathcal{M}/\mathcal{T}$. Face type of maps is given by $\mu = (12)$. As pair-partitions we have $\mathcal{M} = \{\{1,7\},\{2,3\},\{4,6\},\{5,11\},\{8,9\},\{10,12\}\}$, $\mathcal{T} = \{\{2,3\},\{8,9\}\}$, $\tilde{\mathcal{M}} = \{\{1,7\},\{2,8\},\{3,9\},\{4,6\},\{5,11\},\{10,12\}\}$, $\tilde{\mathcal{T}} = \{\{2,8\},\{3,9\}\}$.

Each bipartite labeled map can be constructed by the following procedure. For a partition $\lambda \vdash n$ we consider a family of $\ell(\lambda)$ bipartite polygons with the number of edges given by partition $2\lambda = (2\lambda_1, \dots, 2\lambda_{\ell(\lambda)})$. Then we label the edges of the polygons by elements of $[2n]$ in such a way that each number is used exactly once. A *pair-partition* of $[2n]$ is defined as a family $P = \{V_1, \dots, V_n\}$ of disjoint sets called *blocks* of P, each containing exactly two elements and such that $\bigcup P = [2n]$. For a given pair-partition P we glue together each pair of edges of the polygons which is matched by P in such a way that a white vertex is glued with the other white one, and a black vertex with the other black one.

6.2. Normalized and zonal characters.

Theorem 6.1 (Féray, Śniady)**.** *Let Σ^{α}_{μ} denote the Jack character with parameter α. Then:*

(i) $\Sigma^1_\mu = \sum_{\mathcal{M}} (-1)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}$, *where the summation is over all labeled bipartite oriented maps with the face type μ,*

(ii) $\Sigma^2_\mu = \sum_{\mathcal{M}} (-2)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}$, *where the summation is over all rooted bipartite maps (not necessary oriented) with the face type μ.*

Proof. It suffices to show that the right hand sides of (i) and (ii) satisfies conditions (a), (b), (c) as it was mentioned in the Introduction.

Definition of N gives us property (c) immediately. Property (a) can be showed by proving that if we change the functionals N on the right hand sides of (i) and (ii) by some other functionals $\tilde{N}$, the equalities will still hold. The proof of that will be the same as in [FŚ10], hence we omit it.

The novelty in the current proof is showing the property (b). First, we notice that

$$\sum_{\mathcal{M}} (-2)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}(\lambda) = \sum_{\mathcal{M}} (-1)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}(2\lambda).$$

In the following we shall prove that condition (8) is fulfilled. Let us look at $\partial_x^k(\mathcal{M}, z)$ for some bipartite map $\mathcal{M}$ with one decorated edge by z. The procedure of derivation with respect to x can be viewed as taking all subtrees of $\mathcal{M}$ of type $\mathcal{T}_{[1^{k+1}]}^{[k+1]}$ (see Figure 4) with one edge decorated by z and collapsing them to one decorated edge. Let us choose such a subtree $\mathcal{T}$. We can do the following procedure with $\mathcal{T}$: we unglue all edges corresponding to $\mathcal{T}$ locally in a way that we create a bipartite $2k$-gon such that local orientation of each vertex is preserved; then we glue it again but in such a way that we glue together pairs of edges sharing a common white vertex. We obtained in this way a new bipartite map $\tilde{\mathcal{M}}$, such that $|V_b(\tilde{\mathcal{M}})| = |V_b(\mathcal{M})| + k$ and which contains a subtree $\tilde{\mathcal{T}}$ of type $\mathcal{T}_{[k+1]}^{[1^{k+1}]}$ (see Figure 6). Moreover, collapsing of $\mathcal{T}$ in $\mathcal{M}$ to one decorated edge gives us the same bipartite graph as collapsing of $\tilde{\mathcal{T}}$ in $\tilde{\mathcal{M}}$ to one decorated edge. We should check that this map has a face type μ, but this is clear from our construction. Of course we can do the same procedure if we start from $\partial_y^k(\mathcal{M}, z)$, because of the symmetry. These two procedures are inverses of each other hence (8) holds true. Applying the Main Theorem 5.2 to our case we obtain the property (b), which finishes the proof. □

6.3. **Maps of a fixed genus.** Let $\sigma \in \mathfrak{S}_\infty$ and let $\ell(\sigma)$ denote the length function on $\mathfrak{S}_\infty$ which counts the minimal number of transpositions $\tau_1, \dots,$ $\tau_k \in \mathfrak{S}_\infty$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_k$. Let us consider the pair $(\sigma_1, \sigma_2) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $\sigma_1 \circ \sigma_2 = (1, \dots, n)$ and such that $\ell(\sigma_1) + \ell(\sigma_2) = n - 1 + 2g$. There is a one to one correspondence between such pairs of permutations and between rooted bipartite maps of a genus g with n edges and one face. The correspondence is very natural and can be found in [GJ92].

We will introduce a series of polynomial functions on $\mathbb{Y}$ which generalize free cumulants and cumulants introduced by Goulden and Rattan [GR07] and which arises from labeled bipartite maps of a fixed genus.

Definition 6.2. Let $G_k^g = \sum (-1)^{|\mathcal{V}_b(\mathcal{M})|} \mathcal{M}$, where summation is over all labeled bipartite maps of a genus g with $k + 2g - 1$ edges and with one face. Then $R_k^g := N_{G_k^g}$. The family $R_2^g, R_3^g, \dots$ will be called a family of *genus g cumulants*.

We notice that this is a very natural generalization of free cumulants investigated by Kerov [Ker03] and Biane [Bia98] and of so-called C_k cumulants introduced by Goulden and Rattan [GR07]. More precisely, for each g the cumulant R_k^g describes exactly the homogeneous part of normalized character Σ_{k+2g-1} of degree k. Hence, for example $R_k^0 = R_k$, and $R_k^1 = \frac{(k+2)(k+1)k}{24} C_k$ for each integer $k \geq 1$.

Theorem 6.3. *For each integer* $0 \leq g \leq \left[\frac{k+1}{2}\right]$ *there exists a polynomial* $K_k^g \in \mathbb{Q}[R_2^g, \dots, R_k^g]$ *such that:*

$$\Sigma_k = \sum_{0 \leq j \leq g-1} R_k^j + K_k^g$$

Proof. It suffices to show that for each $g \geq 0$ there exist polynomials $f_k^g \in \mathbb{Q}[R_2^g, \dots, R_k^g]$ such that $R_k = f_k^g$. We will show it by induction on k. We know that for each $g > 0$ there exists $p(g) \in \mathbb{Q}$ such that $R_2 = p(g) R_2^g$. Let us assume that the assertion holds for some k. We know that for each $g \geq 0$ we can write $R_{k+1}^g = n(g) R_{k+1} + h_{k+1}^g$, where $n(g) \in \mathbb{N}_+$ and $h_{k+1}^g \in \mathbb{N}[R_2, \dots, R_k]$. Hence $R_{k+1} = \frac{1}{n(g)} \big(R_{k+1}^g + h_{k+1}^g (f_2^g, \dots, f_k^g) \big)$ which finishes the proof. □

For $g = 0$ the polynomial K_k^g is a Kerov polynomial. For $g = 1$ we obtain a Goulden-Rattan polynomial. It is an easy observation that if the series of polynomials $\{K_k^{g+1}\}_k$ has nonnegative coefficients, then the family $\{K_k^g\}_k$ has the same property. Hence it was very interesting for us how big the parameter g can be to obtain a nonnegativity of coefficients and if there are any combinatorial reasons for that. Unfortunately, the computer experiment showed us that K_{11}^3 has some nonnegative coefficients. It suggests that the Goulden-Rattan polynomials are somehow the best possible with the nonnegative property and it turns out them to be even more interesting.

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