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Structure of Kerov character polynomials: proof of a conjecture of lassalle

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STRUCTURE OF KEROV CHARACTER POLYNOMIALS: PROOF OF A CONJECTURE OF LASSALLE

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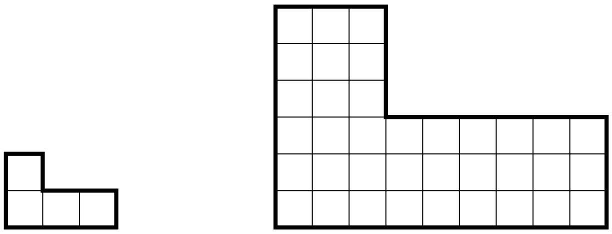
ABSTRACT. We study asymptotics of characters of the symmetric groups on a fixed conjugacy class. It was proved by Kerov that such a character can be expressed as a polynomial in free cumulants of the Young diagram (certain functionals describing the shape of the Young diagram). We show that for each genus there exists a universal symmetric polynomial which gives the coefficients of the part of Kerov character polynomials with the prescribed homogeneous degree. The existence of such symmetric polynomials was conjectured by Lassalle.

1. INTRODUCTION

1.1. **Asymptotic representation theory of symmetric groups.** *What can we say about the representations of the symmetric groups $\mathfrak{S}(n)$ in the limit $n \to \infty$?* This very general question is the subject of investigations of *the asymptotic representation theory of the symmetric groups*. Even though for almost any question of the representation theory of the symmetric groups the answer is known, usually this answer is given by a combinatorial algorithm (for example, Murnaghan-Nakayama rule or Littlewood-Richardson rule) involving manipulations with boxes of a Young diagram. As n, the number of boxes, tends to infinity, such combinatorial algorithms become very cumbersome and it is not easy to extract from them some reasonable asymptotic answer. For this reason one has to look for new, alternative approaches, which would less depend on the details of boxes of a given Young diagram, but rather on its 'global' features.

1.2. **Asymptotic shape of Young diagrams.** In this article we study the scaling of *balanced Young diagrams* which means that a Young diagram with n boxes is assumed to have at most $O(\sqrt{n})$ rows and columns. This scaling almost inevitably leads to the concept of *(asymptotic) shape* of a Young diagram: roughly speaking, we disregard the information about the number of boxes of a Young diagram and we are interested only how the Young diagram looks in large-scale perspective. More precisely, this concept of (asymptotic) shape of a Young diagram corresponds to the *dilated Young diagram* $\frac{1}{\sqrt{n}}\lambda$ which, roughly speaking, is obtained by replacing each



λ 3λ

FIGURE 1. Young diagram $\lambda = (3, 1)$ drawn in the french convention and its dilation 3λ.

unit box of a Young diagram by a box of dimensions $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$. Such a dilated Young diagram is usually no longer a Young diagram but is a *generalized Young diagram* and we should not regard it as a combinatorial object but rather as a geometric one. Since the area of the dilated Young diagram $\frac{1}{\sqrt{n}}\lambda$ is always equal to 1 (where n denotes the number of boxes of λ), this setup is very convenient for comparing shapes of Young diagrams with different number of boxes. In this way we get a unified framework which allows us to consider and compare Young diagrams with various number of boxes, all at the same time.

Probably the most celebrated result related to this scaling of Young diagrams is the one of Logan and Shepp [LS77] and Veršik and Kerov [VK77] who proved that a random Young diagram (distributed according to the Plancherel measure) will typically be very close to some explicit asymptotic shape.

In order to keep this paper as simple as possible and to avoid generalized Young diagrams, in the following we will consider only dilations of Young diagrams by factors which are positive integers. This operation can be easily described on a graphical representation of a Young diagram: we just dilate the picture of λ or, alternatively, we replace each box of λ by a grid of $s \times s$ boxes, see Figure 1. Note that if we fix a Young diagram λ then the sequence of dilated Young diagrams $(s\lambda)_{s=1,2,\dots}$ is an example of a collection of balanced Young diagrams. It follows that our rather vague plan of studying balanced Young diagrams can be made more concrete by studying the sequence of Young diagrams $(s\lambda)_{s=1,2,\dots}$ in the limit as $s \to \infty$.

1.3. **How to normalize the characters?** *How should we normalize the characters of the symmetric groups in order to obtain some meaningful asymptotic quantities?* The answer for this question was given by [Bia03] who gave the following definition. For any permutation $\pi \in \mathfrak{S}(k)$ and an irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ corresponding to the Young diagram λ we define the *normalized character*

$$\Sigma^{\lambda}_{\pi} = \begin{cases} \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^{\lambda}(\pi)}{\text{dimension of } \rho^{\lambda}} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

An interesting feature of this definition is that we do not require that k, the index of the symmetric group to which π belongs, must be equal to n, the number of boxes of λ. In order for $\rho^{\lambda}(\pi)$ to make sense, for $k < n$ we just declare that the permutation $\pi \in \mathfrak{S}(k)$ can be also regarded as an element of $\mathfrak{S}(n)$; we just add to π additional $n-k$ fixpoints. As it was pointed out by Scarabotti [Sca11], it would be more appropriate to use the name of *spherical function* instead of *character* for these objects, nevertheless we will stick to this old nomenclature.

Particularly interesting are the values of characters on cycles, therefore we will use the notation

$$\Sigma^{\lambda}_{k} = \Sigma^{\lambda}_{(1,2,\dots,k)}.$$

In this article we will study the problem: *for fixed value of* k*, what can we say about the normalized characters* Σ^{λ}_{k} *when a balanced Young diagram* λ *tends to infinity* or, in a slightly more concrete reformulation, *what can we say about the normalized characters* $\Sigma^{s\lambda}_{k}$ *related to dilated Young diagrams* $s\lambda$ *in the limit as* $s \to \infty$*?*

1.4. **How to describe the shape of a Young diagram?** A natural question arises: *how to choose parameters which describe the shape of a Young diagram in the most convenient way?* A very interesting answer for this question was proposed by Biane [Bia98] who for a (generalized) Young diagram λ defined a family of parameters $R^{\lambda}_{2}, R^{\lambda}_{3}, \dots$, called *free cumulants* of λ.

The original definition of free cumulants of λ given by Biane is quite involved (free cumulants of a Young diagram are Speicher's *free cumulants* [NS06] — related to Voiculescu's *free probability theory* [VDN92] — of Kerov's *transition measure* [Ker93] of the Young diagram λ), but it has an advantage that it is very explicit and allows an algorithmic computation of free cumulants in terms of the *shape* of a Young diagram.

From the results of Biane [Bia98] one can show a very surprising fact that for any integer $k \geq 1$ and any Young diagram λ the values of the normalized character on dilations of λ

$$\mathbb{N} \ni s \mapsto \Sigma^{s\lambda}_{k}$$

are given by a polynomial function of degree (at most) $k+1$. Furthermore, the leading coefficient is equal to one of the free cumulants:

$$R^{\lambda}_{k+1} = [s^{k+1}]\Sigma^{s\lambda}_k = \lim_{s\to\infty} \frac{\Sigma^{s\lambda}_k}{s^{k+1}}. \tag{2}$$

From the perspective of the Biane's work [Bia98] this is a highly nontrivial and very interesting result: it shows that free cumulants (which are viewed as concrete, algorithmically computable quantities) describe the first-order asymptotics of characters. For the purposes of this article we can reverse the optics and take (2) as a convenient (even if somewhat abstract) definition of free cumulants.

1.5. **Free cumulants.** We define the *free cumulants* $R^{\lambda}_2, R^{\lambda}_3, \dots$ as

$$R^{\lambda}_k = \lim_{s\to\infty} \frac{1}{s^k} \Sigma^{s\lambda}_{k-1},$$

in other words each free cumulant is asymptotically the dominant term of the character on a cycle of appropriate length in the limit when the Young diagram tends to infinity.

One of the reasons why free cumulants are so useful in asymptotic representation theory is that they are homogeneous with respect to dilations of the Young diagrams, namely

$$R^{s\lambda}_k = s^k R^{\lambda}_k.$$

1.6. **Kerov character polynomials.** It turns out that free cumulants can be used not only to provide asymptotic approximations for the characters of symmetric groups, but also for exact formulas. Kerov during a talk in Institut Henri Poincaré in January 2000 [Ker00] announced the following result (the first published proof was given by Biane [Bia03]): for each permutation π there exists a unique universal polynomial K_π with integer coefficients, called *Kerov character polynomial*, with a property that

$$\Sigma^{\lambda}_{\pi} = K_\pi(R^{\lambda}_2, R^{\lambda}_3, \dots)$$

holds true for any Young diagram λ. We say that the Kerov polynomial is universal because it does not depend on the choice of λ. In order to simplify notation we suppress the λ-dependence of characters and free cumulants, writing

$$\Sigma_\pi = K_\pi(R_2, R_3, \dots).$$

As usual, we are mostly concerned with the values of the characters on cycles, therefore we introduce special notation for such Kerov polynomials

$$\Sigma_k = K_k(R_2, R_3, \dots).$$

The first few Kerov polynomials K_k are as follows [Bia01]:

$$\Sigma_1 = R_2,$$
$$\Sigma_2 = R_3,$$
$$\Sigma_3 = R_4 + R_2,$$
$$\Sigma_4 = R_5 + 5R_3,$$
$$\Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$
$$\Sigma_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3.$$

The primary motivation for investigation of this subject is the asymptotic representation theory, namely a good understanding of Kerov character polynomials might in the future shed some light on asymptotics of characters, also in the most difficult scaling when the length of the permutation on which we evaluate the character grows with the number of boxes of the Young diagram, see [FŚ11].

The second motivation — which is key for the purposes of this article — is related to *algebraic combinatorics*. The last decade has seen a number of research papers which stated (sometimes conjecturally) several very surprising combinatorial properties of Kerov polynomials. For example, it was conjectured by Kerov [Ker00] that the coefficients of Kerov polynomials are non-negative integers. These papers showed not only the richness of the combinatorial and the analytic structures of the Kerov character polynomials but also the difficulty in fully understanding these polynomials. Since the results proved in most of these papers will be necessary for the purposes of this article we decided to postpone the presentation of these papers until they are needed. A more complete presentation of the history of the subject and bibliography can be found in the paper [DFŚ10].

1.7. **Genus expansion.** It is convenient to consider a gradation with respect to which the degree of the free cumulant R_k is equal to k. We denote by $K_{k,d}$ the homogeneous part of degree d of the Kerov character polynomial K_k. One of the results announced by Kerov [Ker00] was that the only non-zero polynomials $K_{k,d}$ are of the form $K_{k,k+1-2g}$ where $g \geq 0$ is an integer. It is possible to give some topological meaning to many calculations related to Kerov polynomials in which the integer g can be interpreted as the genus of the resulting two-dimensional surface. For this reason, studying the polynomials $K_{k,k+1-2g}$ for a fixed value of g is often called the *genus expansion*.

The form of the highest-degree term

$$K_{k,k+1} = R_{k+1}$$

was announced by Kerov [Ker00] and proved by Biane [Bia03]. The form of the next term $K_{k,k-1}$ was conjectured by Biane [Bia03] and proved by Śniady [Śni06]. Explicit but rather complicated formulas for the general genus $K_{k,k+1-2g}$ were found by Goulden and Rattan [GR07] (for a more elementary proof we refer to the work of Biane [Bia07]) and we shall discuss their result in Section 2.3.

1.8. **The main result: proof of some conjectures of Lassalle.** Lassalle announced as a conjecture [Las08] that there is an additional structure in the genus expansion of Kerov polynomials. He claimed that for a fixed genus g there exists a symmetric function f_g which describes polynomials $K_{k,k+1-2g}$ and which is independent of k. Before presenting his conjectures we need to prepare some notations.

A partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ is a finite weakly decreasing sequence of nonnegative integers. The non-zero μ_i in a partition μ are called the parts of μ. We will denote by $m_i(\mu)$ the number of parts of μ equal to i; by $l(\mu)$ the number of parts of μ; and denote $|\mu| = \mu_1 + \mu_2 + \cdots$. Following Lassalle, we define $R_1 = 0$ and for strictly positive integer i we define

$$\begin{aligned} \mathcal{R}_i =&(i-1)R_i, \\ \mathcal{R}_\mu =&\prod_i \frac{\mathcal{R}_i^{m_i(\mu)}}{m_i(\mu)!}, \\ Q_i =&\sum_{|\mu|=i} \big(l(\mu)-1\big)!\, \mathcal{R}_\mu, \\ \mathcal{Q}_\mu =&\prod_i \frac{Q_i^{m_i(\mu)}}{m_i(\mu)!}. \end{aligned} \tag{3}$$

As usual, we denote by e_i the elementary symmetric functions, by h_i the complete symmetric functions and by p_i the power-sum symmetric functions. For any partition μ, we denote by e_μ, h_μ or p_μ their product over the parts of μ, and by m_μ the monomial symmetric function — the sum of all distinct monomials whose exponent is a permutation of μ.

The main result of this article is a proof of the following results which were stated as the first and the sixth conjecture in the paper [Las08] by Lassalle.

Theorem 1.1. *For any $g \geq 1$ there exist inhomogeneous symmetric functions f_g and h_g, having maximal degree $4(g-1)$, such that*

$$K_{k,k+1-2g} = \binom{k+1}{3} \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\; f_g(\mu)\; \mathcal{R}_\mu \tag{4}$$

$$= \binom{k+1}{3} \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\; h_g(\mu)\; \mathcal{Q}_\mu, \tag{5}$$

where $f_g(\mu) = f_g(\mu_1,\mu_2,\dots)$ and $h_g(\mu_1,\mu_2,\dots)$. These symmetric functions are independent of k.

This result sheds some light on the structure of Kerov polynomials but it also leads to many new open problems, in particular the positivity conjectures of Lassalle [Las08] and his questions concerning combinatorial interpretations of the coefficients in the expansions of the above symmetric functions.

1.9. **General idea of the proof.** For a given positive integer g we define a symmetric function $k(\mu) := |\mu| + 2g - 1 = p_1(\mu) + 2g - 1$, therefore for a given symmetric functions f_g and h_g we can define symmetric functions $\tilde{f}_g(\mu) := k(\mu)\binom{k(\mu)+1}{3} f_g(\mu)$ and $\tilde{h}_g(\mu) := k(\mu)\binom{k(\mu)+1}{3} h_g(\mu)$. We notice that in equations (4) and (5) we sum over all partitions μ which satisfy $k(\mu) = k$. Therefore the following proposition is an immediate consequence of Theorem 1.1.

Proposition 1.2. *For any $g \geq 1$ there exist inhomogeneous symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$, having maximal degree $4g$, such that*

$$k\; K_{k,k+1-2g} = \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\; \tilde{f}_g(\mu) \mathcal{R}_\mu$$

$$= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\; \tilde{h}_g(\mu) \mathcal{Q}_\mu,$$

where $\tilde{f}_g(\mu) = \tilde{f}_g(\mu_1,\mu_2,\dots)$ and $\tilde{h}_g(\mu) = \tilde{h}_g(\mu_1,\mu_2,\dots)$. These symmetric functions are independent of k.

In fact, the opposite implication holds true as well and Theorem 1.1 is a consequence of Proposition 1.2: roughly speaking we will show that the symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$ are divisible by the polynomial $k\binom{k+1}{3}$. One can notice that Proposition 1.2 stated that $\tilde{f}_g$ and $\tilde{h}_g$ are independent of k, so the divisibility which we will show is a divisibility of symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$ by the symmetric function K which has a property that for any partition μ such that $|\mu| = k+1-2g$ we have that $K(\mu) = k\binom{k+1}{3}$. We shall explain precisely what this divisibility means and show in Section 3 that it holds indeed by studying the arithmetic properties of Kerov polynomials and their divisibility by prime numbers.

The remaining difficulty is to prove Proposition 1.2. We shall do it in Section 2 by analysis of the Goulden-Rattan formula.

Section 4 is a presentation of technical and complicated proofs of lemmas which are used in the previous sections.

2. GOULDEN-RATTAN FORMULA AND EXISTENCE OF SYMMETRIC POLYNOMIALS

2.1. **Power series P_λ.** Following Goulden and Rattan [GR07] we define

$$C(t) = \frac{1}{1 - \sum_{i\geq 2} \mathcal{R}_i t^i} = \sum_{\mu} t^{|\mu|} l(\mu)!\, \mathcal{R}_\mu. \tag{6}$$

Let $D = t\frac{d}{dt}$ and define for $m \geq 1$

$$P_m(t) = -\frac{1}{m!} C(t)\big(D+m-2\big)C(t)\cdots\big(D+1\big)C(t)DC(t).$$

For example, we have

$$\begin{aligned}
P_1(t) &= -C(t),\\
P_2(t) &= -\frac{1}{2} C(t)DC(t),\\
P_3(t) &= -\frac{1}{6} C(t)(D+1)C(t)DC(t)\\
&= -\frac{1}{6}\left[C(t)DC(t)DC(t) + C(t)^2 DC(t)\right]\\
&= -\frac{1}{6}\left[C(t)D\big(C(t)\cdot DC(t)\big) + C(t)^2 DC(t)\right]\\
&= -\frac{1}{6}\left[C(t)\big(DC(t)\big)^2 + C(t)^2 D^2 C(t) + C(t)^2 DC(t)\right].
\end{aligned}$$

Finally, for a partition λ, we write $P_\lambda(t) = \prod_{j=1}^{l(\lambda)} P_{\lambda_j}(t)$.

For $p = (p_0, \dots, p_l) \in \mathbb{N}^{l+1}$ we define

$$E(p) := C(t)^{p_0} DC(t)^{p_1} \cdots DC(t)^{p_l}.$$

Lemma 2.1.

(a) Let $p \in \mathbb{N}^{l+1}, q \in \mathbb{N}^{m+1}$ and denote by $|p| := p_0 + \cdots + p_l$ ($|q| = q_0 + \cdots q_m$ respectively). Then

$$E(p) \cdot E(q) = \sum_{\substack{r \in \mathbb{N}^{l+m+1},\\ |r| = |p|+|q|}} c_r^{p,q} E(r),$$

where $c_r^{p,q} \in \mathbb{Z}$.

(b) *For any partition* λ *the fraction* $\frac{P_\lambda(t)}{C(t)}$ *is a linear combination of terms* $E(p)$ *where* $p \in \mathbb{N}^k$ *such that* $|p| = |\lambda| - 1$ *and* $k \leq |\lambda| - l(\lambda) + 1$.

Proof. Let $p = (p_0, \dots, p_l) \in \mathbb{N}^{l+1}$, $q = (q_0, \dots, q_m) \in \mathbb{N}^{m+1}$. We will show part (a) by induction on l. It is obvious for $l = 0$. For $l = 1$ we have:

$$E(p) \cdot E(q) = E(p_0, p_1 + q_0, q_1, q_2, \dots, q_m) - E(p_0 + p_1, q_0, q_1, \dots, q_m)$$

by the Leibniz rule. Let us assume, that the inductive assertion holds for some $l \geq 1$ and let $p = (p_0, \dots, p_{l+1})$. Then by the Leibniz rule we have that

$$E(p) \cdot E(q) = C(t)^{p_0} D\left[E(p') \cdot E(q)\right] - C(t)^{p_0+p_1}\left[E(p'') \cdot E(q')\right], \tag{7}$$

where $p' = (p_1, \dots, p_{l+1})$, $p'' = (0, p_2, \dots, p_{l+1})$, $q' = (0, q_0, \dots, q_m)$. By the inductive assertion, the right hand side of (7) is equal to

$$\sum_{\substack{\alpha \in \mathbb{N}^{l+m+1}, \\ |\alpha| = |p| + |q| - p_0}} c_\alpha C(t)^{p_0} D E(\alpha) - \sum_{\substack{\beta \in \mathbb{N}^{l+m+2}, \\ |\beta| = |p| + |q|}} c_\beta E(\beta),$$

where $c_\alpha, c_\beta \in \mathbb{Z}$. But it means that

$$E(p) \cdot E(q) = \sum_{\substack{r \in \mathbb{N}^{l+m+2}, \\ |r| = |p| + |q|}} c_r^{p,q} E(r),$$

where $c_r^{p,q} \in \mathbb{Z}$ which finishes the proof of part (a).

For part (b) we notice that each function $P_m(t)$ is a linear combination of $E(p)$, where $p \in \mathbb{N}^k$, $|p| = m$ and $k \leq m - 1$ and we apply part (a). □

2.2. **Polynomial structure of coefficients of P_λ.**

Definition 2.2. If f is a symmetric function of degree d and $2g \geq 2$ is an integer then the formal power series

$$F(t) = \sum_\mu t^{|\mu|} \big(l(\mu) + 2g - 2\big)!\, f(\mu)\, \mathcal{R}_\mu \tag{8}$$

will be called *a power-sum of the first kind with degree* d *and genus* g and the formal power series

$$F(t) = \sum_\mu t^{|\mu|} (2g - 1)^{l(\mu)}\, f(\mu)\, \mathcal{Q}_\mu \tag{9}$$

will be called *a power-sum of the second kind with degree* d *and genus* g.

Lemma 2.3.

(a) $C(t)$ *is a power-sum of the first (respectively, second) kind with degree* 0 *and genus* 1.

(b) If $F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus g then $DF(t)$ is a power-sum of the first (respectively, second) kind with degree $d+1$ and genus g.
(c) If $F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus g then $C(t)F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus $g+\frac{1}{2}$.

Proof. In order to prove point (a) it suffices to notice that

$$C(t) = \sum_{\mu} t^{|\mu|} l(\mu)!\, \mathcal{R}_\mu = \sum_{\mu} t^{|\mu|} \mathcal{Q}_\mu.$$

In order to prove point (b) let $F(t)$ be in the form (8). Then

$$DF(t) = \sum_{\mu} t^{|\mu|} \big(l(\mu)+2g-2\big)!\ \big[(p_1(\mu) f(\mu)\big]\ \mathcal{R}_\mu$$

is again of the form (8).

If $F(t)$ is of the form (9), then

$$DF(t) = \sum_{\mu} t^{|\mu|} (2g-1)^{l(\mu)}\ \big[(p_1(\mu) f(\mu)\big]\ \mathcal{Q}_\mu$$

is again of the form (9) which shows part (b).

For part (c) we can assume that the symmetric function f is equal to monomial symmetric function m_λ for some partition λ.

We define

$$C_n = \sum_{|\mu|=n} l(\mu)!\, \mathcal{R}_\mu = \sum_{|\mu|=n} \mathcal{Q}_\mu,$$

$$\mathcal{C}_\mu = \prod_{i\geq 2} \frac{C_i^{m_i(\mu)}}{m_i(\mu)!}.$$

The correspondence between these three families (Q, R and C) is given by

$$Q_n = \sum_{|\mu|=n} (-1)^{l(\mu)} (l(\mu)-1)!\, \mathcal{C}_\mu,$$

$$-\mathcal{R}_n = \sum_{|\mu|=n} (-1)^{l(\mu)}\, \mathcal{Q}_\mu = \sum_{|\mu|=n} (-1)^{l(\mu)} l(\mu)!\, \mathcal{C}_\mu.$$

Following Lassalle, [Las08] we define the (formal) alphabet $\mathbb{A}$ by

$$\mathcal{R}_i = -h_i(\mathbb{A}), \quad Q_i = -p_i(\mathbb{A})/i, \quad C_i = (-1)^i e_i(\mathbb{A}).$$

Writing

$$u_\mu = l(\mu)!/\prod_{i\geq 1} m_i(\mu), \quad \epsilon_\mu = (-1)^{n-l(\mu)}, \quad z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!,$$

the previous relations can be understood in a frame of symmetric functions theory, and they are merely the classical properties [Mac95, pp. 25 and 33]

$$\begin{aligned} p_n &= -n \sum_{|\mu|=n} (-1)^{l(\mu)} u_\mu h_\mu / l(\mu) = -n \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu / l(\mu), \\ e_n &= \sum_{|\mu|=n} \epsilon_\mu u_\mu h_\mu = \sum_{|\mu|=n} \epsilon_\mu z_\mu^{-1} p_\mu, \\ h_n &= \sum_{|\mu|=n} z_\mu^{-1} p_\mu = \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu. \end{aligned}$$

Using this notation, it suffices to show that for any monomial symmetric function m_λ and any g we have

(10)
$$\left(\sum_\mu t^{|\mu|} m_\lambda(\mu) \frac{(l(\mu)+2g-2)!}{l(\mu)!} (-1)^{l(\mu)} u_\mu h_\mu \right) \left(\sum_\rho t^{|\rho|} (-1)^{l(\rho)} u_\rho h_\rho \right) = \left(\sum_\nu t^{|\nu|} \frac{m_\lambda(\nu)}{l(\lambda)+2g-1} \frac{(l(\nu)+2g-1)!}{l(\nu)!} (-1)^{l(\nu)} u_\nu h_\nu \right),$$

because the right hand side is a power-sum of the first kind with degree $|\lambda|$ and genus $g + \frac{1}{2}$, and it suffices to show, that for any monomial symmetric function m_λ and any g we have

(11)
$$\left(\sum_\mu t^{|\mu|} m_\lambda(\mu) (2g-1)^{l(\mu)} (-1)^{l(\mu)} z_\mu^{-1} p_\mu \right) \left(\sum_\rho t^{|\rho|} (-1)^{l(\rho)} z_\rho^{-1} p_\rho \right) = \left(\sum_\nu t^{|\nu|} \left(\frac{2g-1}{2g} \right)^{l(\lambda)} m_\lambda(\nu) (2g)^{l(\nu)} (-1)^{l(\nu)} z_\nu^{-1} p_\nu \right),$$

because the right hand side is a power-sum of the second kind with degree $|\lambda|$ and genus $g + \frac{1}{2}$. In order to prove (10) and (11) it is enough to use Lemma 4.1. □

The main result of this subsection is the following proposition.

Proposition 2.4. $\frac{P_\lambda(t)}{C(t)}$ *is a linear combination of power-sums of the first (respectively, second) kind of genus* $\frac{|\lambda|}{2}$ *and degree at most* $|\lambda| - l(\lambda)$.

Proof. It is enough to apply Lemma 2.1 and Lemma 2.3. □

2.3. **Goulden-Rattan formula.** For a partition λ let m_λ denote the monomial symmetric function in indeterminates $x_1, x_2, \dots$. In this paper we consider the particular evaluation of the monomial symmetric function at $x_i = i$, for $i = 1, \dots, k-1$, and $x_i = 0$, for $i \geq k$, and write this as $\hat{m}_\lambda$. Let $A(t)$ be a formal power series. We denote the coefficient of t^k in $A(t)$ by $[t^k]A(t)$.

Theorem 2.5 (Goulden and Rattan [GR07])**.** *For $g \geq 1$, $k \geq 2g-1$,*

$$\Sigma_{k,k+1-2g} = -\frac{1}{k}[t^{k+1-2g}] \sum_{|\lambda|=2g} \hat{m}_\lambda \frac{P_\lambda(t)}{C(t)}. \tag{12}$$

2.4. **Proof of Proposition 1.2.**

Proof of Proposition 1.2. Equation (12) can be written in the form

$$k\,\Sigma_{k,k+1-2g} = -[t^{k+1-2g}] \sum_{|\lambda|=2g} \hat{m}_\lambda \frac{P_\lambda(t)}{C(t)}.$$

The analogous to $\hat{m}_\lambda$ evaluation of the power-sum symmetric function

$$\hat{p}_s = 1^s + \cdots + (k-1)^s$$

is a polynomial in k of degree $s+1$; it follows immediately that the evaluation of the power-sum symmetric function $\hat{p}_\lambda$ is a polynomial in k of degree $|\lambda| + l(\lambda)$. The monomial symmetric function m_λ is a linear combination of power-sum symmetric functions p_μ, where each partition μ which appears in this linear combination is obtained from partition λ by gluing some of their parts (see for example [Mac95]). It means that for each such μ we have

$$|\lambda| + l(\lambda) \geq |\mu| + l(\mu),$$

and for this reason also $\hat{m}_\lambda$ is a polynomial in k of degree at most $|\lambda| + l(\lambda)$.

For any partition μ such that $|\mu| = k+1-2g$ we have $k = p_1(\mu) + 2g - 1$, where p_1 is a power symmetric function, hence there exists a symmetric function f_λ of degree $|\lambda| + l(\lambda)$ which doesn't depend on k such that $\hat{m}_\lambda = f_\lambda(\mu)$. Proposition 2.4 finishes the proof. □

3. Divisibility of polynomials

3.1. **Implications of divisibility.** At this step we proved Proposition 1.2. In order to prove Theorem 1.1 we would like to show that for each integer $g \geq 1$, functions $\tilde{f}_g$ and $\tilde{h}_g$ are divisible by the symmetric function

$$(p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!},$$

where p_1 denotes the power symmetric function. By word divisible we mean that there exist symmetric functions f_g and h_μ such that

$$\tilde{f}_g = (p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!} f_g$$

and

$$\tilde{h}_g = (p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!} h_g.$$

Observe that for fixed $g \geq 1$ and for any partition μ there exists number k such that $|\mu| = k + 1 - 2g$ and then

$$\tilde{f}_g(\mu) = k\binom{k+1}{3} f_g(\mu),$$

$$\tilde{h}_g(\mu) = k\binom{k+1}{3} h_g(\mu)(\mu).$$

The idea of showing divisibility of symmetric function by symmetric function of degree 1 is similar to the case of showing divisibility of some polynomial by some monomial. The main idea is dividing with a remainder and using a fact that if some integer is divisible by infinite number of primes then it has to be equal to zero. The remainning of this section is a formalisation of this idea.

The first difficulty is that we have to deal with polynomials in several variables. Hence, in order to show some generalisation of the "dividing with remainder" technique, we need the following technical lemma:

Lemma 3.1. *Let m be a fixed integer and f be a polynomial in variables $x_k, \dots, x_l$ with a property that $f(\mu_k, \dots, \mu_l) = 0$ for all integers $\mu_k, \dots, \mu_l \geq 1$ which fulfill the following equations:*

$$\sum_{k \leq j \leq l} \mu_j > m, \tag{13}$$

$$\mu_i > \mu_{i+1} + \cdots + \mu_l \tag{14}$$

for all values of i for which it makes sense. Then $f = 0$.

The proof of this lemma can be find in Section 4.

The next lemma is key for this section: it allows to translate information about arithmetic properties of Kerov polynomials into information about the polynomials governing the coefficients.

Lemma 3.2. *Let f, respectively h, be a symmetric function of degree at most d with rational coefficients, $g \geq 1$ be an integer; we define*

$$L_k = \sum_{|\mu| = k+1-2g} \big(l(\mu) + 2g - 2\big)!\; f(\mu)\; \mathcal{R}_\mu,$$

respectively,

$$L'_k = \sum_{|\mu|=k+1-2g} \big(2g-1\big)^{l(\mu)}\ h(\mu)\ \mathcal{Q}_\mu$$

and view it as a polynomial in $R_2, R_3, \dots$.

Assume that an integer Δ *has a property that all coefficients of* $L_{p+\Delta}$ *(respectively, all coefficients of* $L'_{p+\Delta}$*) are integers divisible by* p *for an infinite number of prime numbers* p*. Then there exists a symmetric function* $\tilde{f}$ *(respectively,* $\tilde{h}$*) with rational coefficients of degree at most* $d-1$ *such that*

$$L_k = (k-\Delta) \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}(\mu)\ \mathcal{R}_\mu,$$

respectively,

$$L'_k = (k-\Delta) \sum_{|\mu|=k+1-2g} \big(2g-1\big)^{l(\mu)}\ \tilde{h}(\mu)\ \mathcal{Q}_\mu.$$

Proof. For simplicity assume that the coefficients of f (respectively, h) are integer numbers; if this is not the case we multiply L_k and f (respectively, L'_k and h) by some common multiple of the denominators.

Let $\mu = (\mu_1, \mu_2, \dots)$ be a sequence of indeterminates. We use the notation $|\mu| = \mu_1+\mu_2+\cdots$ and define variable $z = |\mu|+2g-1-\Delta$. The family of indeterminates μ can be alternatively parametrized by $z, \mu_2, \mu_3, \dots$; we just use the substitution $\mu_1 = z+\Delta+1-2g-\mu_2-\mu_3-\cdots$. Now we can consider $f, h \in \Lambda[z]$ as polynomials in one variable z with coefficients in the ring Λ of symmetric functions in variables $\mu_2, \mu_3, \dots$ and we can divide f and h by z with a remainder. Hence

$$\begin{aligned} f(\mu) &= (|\mu|+2g-1-\Delta)\tilde{f}(\mu) + r(\mu_2, \mu_3, \dots), \\ h(\mu) &= (|\mu|+2g-1-\Delta)\tilde{h}(\mu) + s(\mu_2, \mu_3, \dots) \end{aligned}$$

for some $\tilde{f}, \tilde{h} \in \Lambda[z]$ and for some $r, s \in \Lambda$. Below we will show that $r = s = 0$. This would imply that

$$\begin{aligned} f(\mu) &= (|\mu|+2g-1-\Delta)\tilde{f}(\mu), \\ h(\mu) &= (|\mu|+2g-1-\Delta)\tilde{h}(\mu), \end{aligned}$$

where by substitution $z = |\mu|+2g-1-\Delta$ we view $\tilde{f}, \tilde{h} \in \Lambda[\mu_1]$ as polynomials in one variable μ_1 with coefficients in Λ. For any permutation π of the set of positive integers which moves only finitely many elements we have

$$(|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu_1, \mu_2, \dots) = f(\mu_1, \mu_2, \dots) = \\ f(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots) = (|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots)$$

hence from the cancellation property

$$\tilde{f}(\mu_1, \mu_2, \dots) = \tilde{f}(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots)$$

is a symmetric function. In an analogous way we show that $\tilde{h}$ is a symmetric function. The lemma follows now immediately.

It remains now to show that $r = s = 0$. From the following on let $\mu_2 > \cdots > \mu_l$ be fixed integers bigger than 1 which fulfill Equations (14) and (13) with $m = \Delta - 2g$. Define $\mu_1 = p + \Delta + 1 - 2g - \mu_2 - \mu_3 - \cdots - \mu_l$, where p is a prime number. Notice that $\mu_1 - 1 < p$, because we required that $\Delta + 1 - 2g - \mu_2 - \mu_3 - \cdots - \mu_l < 1$. We consider the integral vector $\mu = (\mu_1, \dots, \mu_l)$.

If p is big enough, parts of μ are all different and it follows that

$$[R_{\mu_1} R_{\mu_2} \cdots R_{\mu_l}] L_{p+\Delta} = (l + 2g - 2)!\; f(\mu)\; (\mu_1 - 1)(\mu_2 - 1) \cdots (\mu_l - 1).$$

Also, if prime number p is big enough then it does not divide $(l + 2g - 2)!\; (\mu_1 - 1)(\mu_2 - 1) \cdots (\mu_l - 1)$. It follows that for infinitely many prime numbers p the number

$$f(\mu) = p\tilde{f}(\mu) + r(\mu_2, \dots, \mu_l)$$

is divisible by p. We proved in this way that $r(\mu_2, \dots, \mu_l, 0, \dots)$ is an integer which is divisible by an infinite number of primes hence $r(\mu_2, \dots, \mu_l, 0, \dots) = 0$. Finaly Lemma 3.1 shows that $r = 0$.

If p is big enough then condition (14) holds true for all $1 \leq i \leq l - 1$ therefore every partition resulting from μ by gluing together some of its parts cannot be obtained by gluing the parts of μ in some other way (in fact, this property is the main reason of introducing Equation (14)). From (3) it follows that

$$[R_{\mu_1} R_{\mu_2} \cdots] L'_{p+\Delta} = \sum_{\nu \geq \mu} (2g - 1)^{l(\nu)} \big(l(\nu) - 1\big)!\; h(\nu),$$

where $\nu \geq \mu$ means that partition ν can be obtained from partition μ by gluing some parts of μ. We also know that for infinitely many prime numbers

p the following number

$$\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu)-1\big)!\, h(\nu) =$$

$$p \left[\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu)-1\big)!\, \tilde{h}(\nu) \right] + \sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu)-1\big)!\, s(\nu')$$

is divisible by p, where for $\nu = (\nu_1, \nu_2, \dots)$ we denote $\nu' = (\nu_2, \nu_3, \dots)$. Notice that the set of values of ν' which contribute to the right hand side does not depend on the choice of p, because only ν_1 depends on the choice of p. Thus we proved that for infinitely many prime numbers p the second summand on the right hand side does not depend on the choice of p and is a fixed integer divisible by all these prime numbers, hence

$$\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu)-1\big)!\, s(\nu') = 0. \tag{15}$$

We will use induction over k to show that $s(x_2, \dots, x_k, 0, \dots)$ is equal to the zero polynomial for any $k > 1$, which proves that $s = 0$. Indeed, assume, that $s(x_2, \dots, x_k, 0, \dots)$ is equal to the zero polynomial for $k < l$. From the induction hypothesis it follows that all summands on the left-hand side of (15) vanish, except for $\nu = \mu$, which shows that $s(\mu') = 0$. We use Lemma 3.1 to show that $s = 0$ as claimed. □

3.2. **Divisibility.** In order to prove Theorem 1.1 we would like to apply Lemma 3.2 to Kerov polynomials. For this, we need some interesting arithmetic properties of coefficients of Kerov polynomials. The next lemma, which was formulated as a conjecture by Światosław Gal [Gal08], shows some properties of these kind.

Lemma 3.3. *If p is an odd prime number then*

(a) $\frac{\Sigma_p - R_{p+1} + 2R_2}{p}$,
(b) $\frac{\Sigma_{p-1} - R_p}{p}$,
(c) $\frac{\Sigma_{p+1} - R_{p+2} + R_3}{p}$

are polynomials in free cumulants $R_2, R_3, \dots$ with nonnegative integer coefficients.

We will prove this Lemma in Section 4, because the proof is very technical. Finally, we can prove the main result.

3.3. **Proof of the main result.**

Proof of Theorem 1.1. We know by Proposition 1.2 that for any integer $g \geq 1$ there exist inhomogeneous symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$, having maximal degree $4g$, such that

$$\begin{aligned} L_k := L'_k := k\ K_{k,k+1-2g} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}_g(\mu)\mathcal{Q}_\mu. \end{aligned}$$

By applying Lemma 3.2 for $\Delta = 0$ we obtain that

$$\begin{aligned} k\ K_{k,k+1-2g} =& k \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}'_g(\mu)\mathcal{R}_\mu \\ =& k \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}'_g, \tilde{h}'_g$ are symmetric functions of degree at most $4g-1$.

Let

$$\begin{aligned} L_k := L'_k := K_{k,k+1-2g} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}'_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'_g(\mu)\mathcal{Q}_\mu. \end{aligned}$$

Lemma 3.3(a) shows that Lemma 3.2 can be applied for $\Delta = 0$, thus

$$\begin{aligned} K_{k,k+1-2g} =& k \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}''_g(\mu)\mathcal{R}_\mu \\ =& k \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}''_g, \tilde{h}''_g$ are symmetric functions of degree at most $4g-2$.

Let

$$\begin{aligned} L_k := L'_k := \frac{AK_{k,k+1-2g}}{k} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!A\ \tilde{f}''_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}A\ \tilde{h}''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where A is the common multiple of the denominators of coefficients of $\tilde{h}''_g$ and $\tilde{f}''_g$. We know that $L_{p-1} = L'_{p-1}$ has integer coefficients as polynomial in $R_2, R_3, \dots$ and we know, thanks to Lemma 3.3(b), that for infinitely many prime numbers p the coefficients of $(p-1)L_{p-1} = (p-1)L'_{p-1}$ are divisible by p, hence coefficients of $L_{p-1} = L'_{p-1}$ are also divisible by p, because $p-1$ and p are coprime. Then we can apply Lemma 3.2 for $\Delta = -1$ and we obtain that there exist symmetric functions $\tilde{f}'''_g, \tilde{h}'''_g$ of degree at most $4g-3$ such that

$$\begin{aligned} K_{k,k+1-2g} =& k\ (k+1) \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}'''_g(\mu)\mathcal{R}_\mu \\ =& k\ (k+1) \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}''_g, \tilde{h}''_g$ are symmetric functions of degree at most $4g-3$.

Similarly as before, thanks to Lemma 3.3(c) and thanks to the fact that for prime number $p>2$ the numbers p and $p+1$ are coprime and the numbers $p+2$ and p are coprime, we can apply Lemma 3.2 for $\Delta = 1$ for

$$\begin{aligned} L_k := L'_k := \frac{BK_{k,k+1-2g}}{k(k+1)} =& \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!B\ \tilde{f}'''_g(\mu)\mathcal{R}_\mu \\ =& \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)} B\ \tilde{h}'''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where B is the common multiple of the denominators of coefficients of $\tilde{h}'''_g$ and $\tilde{f}'''_g$ and we obtain that there exist symmetric functions $\tilde{f}''''_g, \tilde{h}''''_g$ of degree at most $4g-4$ such that

$$\begin{aligned} K_{k,k+1-2g} =& (k-1)\ k\ (k+1) \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}''''_g(\mu)\mathcal{R}_\mu \\ =& (k-1)\ k\ (k+1) \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}''''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

which finishes the proof. □

4. Technical lemmas

In this Section we prove all technical lemmas we used in this article.

4.1. Identities on symmetric functions.

Lemma 4.1. *The following abstract equalities hold:*

(16)
$$\sum_{\mu \cup \rho = \nu} m_\lambda(\mu) \frac{\big(l(\mu)+2g-2\big)!}{l(\mu)!} u_\mu u_\rho = \frac{m_\lambda(\nu)}{l(\lambda)+2g-1} \frac{\big(l(\nu)+2g-2\big)!}{l(\nu)!} u_\nu;$$

(17)
$$\sum_{\mu \cup \rho = \nu} m_\lambda(\mu)(2g-1)^{l(\mu)} z_\mu^{-1} z_\rho^{-1} = \left(\frac{2g-1}{2g}\right)^{l(\lambda)} m_\lambda(\nu)(2g)^{l(\nu)} z_\nu^{-1}.$$

Proof. From the definition of the monomial symmetric function we know that $m_\lambda(\mu) = 0$ for all partitions μ such that $l(\mu) < l(\lambda)$. We use an identity that for every integer n such that $l(\lambda) \leq n \leq l(\nu)$ we have

$$\binom{l(\nu) - l(\lambda)}{n - l(\lambda)} m_\lambda(\nu) = \sum_{\substack{\mu \cup \rho = \nu, \\ l(\mu) = n}} m_\lambda(\mu) \left(\prod_{i \geq 1} \frac{m_i(\nu)!}{m_i(\mu)!\, m_i(\rho)!} \right). \tag{18}$$

Indeed,

$$\sum_{\substack{\mu \cup \rho = \nu, \\ l(\mu) = n}} m_\lambda(\mu) \left(\prod_{i \geq 1} \frac{m_i(\nu)!}{m_i(\mu)!\, m_i(\rho)!} \right) = \sum_{\substack{\mu \subset \nu, \\ l(\mu) = n}} m_\lambda(\mu) =$$

$$\sum_{\substack{\mu' \subset \mu \subset \nu, \\ l(\mu) = n, l(\mu') = l(\lambda)}} m_\lambda(\mu') = \sum_{\substack{\mu' \subset \nu, \\ l(\mu') = l(\lambda)}} \sum_{\substack{\mu' \subset \mu \subset \nu, \\ l(\mu) = n}} m_\lambda(\mu') =$$

$$\binom{l(\nu) - l(\lambda)}{n - l(\lambda)} \sum_{\substack{\mu' \subset \nu, \\ l(\mu') = l(\lambda)}} m_\lambda(\mu') = \binom{l(\nu) - l(\lambda)}{n - l(\lambda)} m_\lambda(\nu),$$

where $\sum_{\substack{\mu \subset \nu, \\ l(\mu) = n}}$ means that $\mu = (\nu_{\sigma(1)}, \dots, \nu_{\sigma(n)})$ for some $\sigma \in \mathfrak{S}(l(\nu))$ such that $\sigma(i) < \sigma(i+1)$ for all $i \in \{1, \dots, n-1\}$ and we are summing over all such permutations σ. Now, we can write the left hand side of (16) in the following way:

$$\sum_{l(\lambda) \leq n \leq l(\nu)} \frac{(n + 2g - 2)!}{n!} \left(\sum_{\substack{\mu \cup \rho = \nu, \\ l(\mu) = n}} m_\lambda(\mu) \prod_{i \geq 1} \frac{m_i(\nu)!}{m_i(\mu)\, !m_i(\rho)!} \right) \times$$

$$u_\nu \frac{n!\, (l(\nu) - n)!}{l(\nu)!} =$$

$$\sum_{l(\lambda) \leq n \leq l(\nu)} \frac{(n + 2g - 2)!}{n!} \frac{(l(\nu) - l(\lambda))!}{(l(\nu) - n)!(n - l(\lambda))!} m_\lambda(\nu) \times$$

$$u_\nu \frac{n!\, (l(\nu) - n)!}{l(\nu)!} =$$

$$\frac{(l(\nu) - l(\lambda))!}{l(\nu)!} \sum_{l(\lambda) \leq n \leq l(\nu)} \frac{(n + 2g - 2)!}{(n - l(\lambda))!} m_\lambda(\nu) u_\nu$$

and using the equality

$$\sum_{0\leq i\leq b} \binom{a+i}{i} = \binom{a+b+1}{b}$$

we have

$$\frac{(l(\nu)-l(\lambda))!}{l(\nu)!} \sum_{l(\lambda)\leq n\leq l(\nu)} \frac{(n+2g-2)!}{(n-l(\lambda))!} =$$

$$\frac{(l(\nu)-l(\lambda))!(l(\lambda)+2g-2)!}{l(\nu)!} \sum_{0\leq n\leq l(\nu)-l(\lambda)} \frac{(n+l(\lambda)+2g-2)!}{n!(l(\lambda)+2g-2)!} =$$

$$\frac{(l(\nu)-l(\lambda))!(l(\lambda)+2g-2)!}{l(\nu)!} \sum_{0\leq n\leq l(\nu)-l(\lambda)} \binom{l(\lambda)+2g-2+n}{n} =$$

$$\frac{(l(\nu)-l(\lambda))!\ (l(\lambda)+2g-2)!\ (l(\nu)+2g-1)!}{l(\nu)!\ (l(\nu)-l(\lambda))!\ (l(\lambda)+2g-1)!} = \frac{(l(\nu)+2g-1)!}{l(\nu)!(l(\lambda)+2g-1)}$$

which finishes the proof of (16).

Using (18) we can write the left hand side of (17) in the following form:

$$\sum_{l(\lambda)\leq n\leq l(\nu)} (2g-1)^n \left(\sum_{\substack{\mu\cup\rho=\nu,\\ l(\mu)=n}} m_\lambda(\mu) \prod_{i\geq 1} \frac{m_i(\nu)!}{m_i(\mu)\ !m_i(\rho)!} \right) z_\nu^{-1} =$$

$$\left(\sum_{l(\lambda)\leq n\leq l(\nu)} (2g-1)^n \binom{l(\nu)-l(\lambda)}{n-l(\lambda)} \right) m_\lambda(\nu) z_\nu^{-1} =$$

$$(2g-1)^{l(\lambda)} \left(\sum_{0\leq n\leq l(\nu)-l(\lambda)} \binom{l(\nu)-l(\lambda)}{n} (2g-1)^n \right) m_\lambda(\nu) z_\nu^{-1} =$$

$$\left(\frac{2g-1}{2g}\right)^{l(\lambda)} m_\lambda(\nu)(2g)^{l(\nu)} z_\nu^{-1},$$

where the last equality holds because of the binomial identity:

$$\sum_{0\leq n\leq m} \binom{m}{n} a^n = (a+1)^m,$$

which finishes the proof. □

4.2. **Proof of Lemma 3.1.**

Proof of Lemma 3.1. Let l be fixed; we will use backward induction over k. For $k=l$ we know that $f(\mu_k)=0$ for infinitely many choices of μ_k.

In other words, polynomial f has infinitely many zeros hence $f = 0$, as claimed.

Let us assume that the inductive assertion holds for some $k \leq l$. We can write

$$f(x_{k-1}, \dots, x_l) = \sum_{0 \leq i \leq N} x_{k-1}^i \, f_i(x_k, \dots, x_l)$$

for some N. Let us fix integers $\mu_k, \dots, \mu_l$ bigger than 1 which satisfy (14) and (13). Then we can find infinitely many integer numbers μ_{k-1} for which the vector $(\mu_{k-1}, \dots, \mu_l)$ satisfies both (14) and (13); for each such a number we have $f(\mu_{k-1}, \dots, \mu_l) = 0$ therefore the polynomial $x_{k-1} \mapsto f(x_{k-1}, \mu_k, \dots, \mu_l)$ has infinitely many zeros hence it is the zero polynomial and $f_i(\mu_k, \dots, \mu_l) = 0$. This shows that the inductive assertion can be applied to the polynomial $f_i(x_k, \dots, x_l)$ and therefore $f_i(x_k, \dots, x_l) = 0$. This finishes the proof. □

4.3. **Arithmetic properties of Kerov polynomials.**

4.3.1. *Auxiliary results.* We present two theorems we need to prove Lemma 3.3.

Theorem 4.2 (Dołęga, Féray, Śniady [DFŚ10]). *Let $k \geq 1$ and let $s_2, s_3, \dots$ be a sequence of non-negative integers with only finitely many non-zero elements. The coefficient of $R_2^{s_2} R_3^{s_3} \cdots$ in the Kerov polynomial K_k is equal to the number of triples (σ_1, σ_2, q) with the following properties:*

(a) *σ_1, σ_2 is a factorization of the cycle; in other words $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ are such that $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$;*
(b) *the number of cycles of σ_2 is equal to the number of factors in the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_2)| = s_2 + s_3 + \cdots$;*
(c) *the total number of cycles of σ_1 and σ_2 is equal to the degree of the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$;*
(d) *$q : C(\sigma_2) \to \{2, 3, \dots\}$ is a coloring of the cycles of σ_2 with a property that each color $i \in \{2, 3, \dots\}$ is used exactly s_i times (informally, we can think that q is a map which to cycles of $C(\sigma_2)$ associates the factors in the product $R_2^{s_2} R_3^{s_3} \cdots$);*
(e) *for every set $A \subset C(\sigma_2)$ which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq C(\sigma_2)$) there are more than $\sum_{i \in A} \big(q(i) - 1\big)$ cycles of σ_1 which intersect $\bigcup A$.*

We say that a partition Π of the set $[k] = \{1, \dots, k\}$ is a *pushing partition* if any pair of neighboring elements of $[k]$ with respect to the cyclic order (i.e. i and $i + 1$ are a pair of neighboring elements for any $1 \leq i \leq k - 1$ as well as 1 and k) does not belong to the same block of Π.

The cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on the set of all partitions (respectively, the set of pushing partitions) of the set $[k]$ as follows: for a partition Π of $[k]$ and $i \in \mathbb{Z}/k\mathbb{Z}$ we define $i+\Pi$ as the partition of $[k]$ with a property that a, b belong to the same block of Π if and only if a', b' belong to the same block of $i + \Pi$ for all $a, b, a', b' \in [k]$ such that $a + i \equiv a' \pmod{k}$, $b + i \equiv b' \pmod{k}$.

For any pushing partition Π it is possible (see [Śni06]) to define the normalized character Σ_Π. It has a property that $\Sigma_\Pi = \Sigma_\pi$, where the right-hand side should be understood as in (1) for some $\pi \in \mathfrak{S}(l)$, $l \geq 1$. So defined partition-indexed character has the following properties:

Theorem 4.3 (Proposition 4.4, Claim 3.1, Proposition 3.2 in [Śni06])**.**

- *The map $\Pi \mapsto \Sigma_\Pi$ is constant on the orbits of the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on the set of pushing partitions of $[k]$.*
- *For any integer $k \geq 2$*

$$R_k = \sum_{\Pi} I_\Pi \, \Sigma_\Pi, \tag{19}$$

  where the sum runs over pushing partitions of $[k]$ and $I_\Pi \in \mathbb{Z}$, called free index, is constant on the orbits of the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on the set of pushing partitions of $[k]$.
- *For the minimal partition $\Pi = \big\{\{1\}, \dots, \{k\}\big\}$ the corresponding character is given by*

$$\Sigma_{\{\{1\},\dots,\{k\}\}} = \Sigma_{k-1}.$$

4.3.2. *Proof of Lemma 3.3.*

Proof of Lemma 3.3. In the following we shall prove that the coefficients are integer numbers. Their nonnegativity would follow from Theorem 4.2.

In order to prove that the coefficients of $\frac{\Sigma_p - R_{p+1} + 2R_2}{p}$ are integer we consider the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on the set of triples (σ_1, σ_2, q) which contribute to Theorem 4.2 defined by conjugation

$$\psi(i)(\sigma_1, \sigma_2, q) = \big(c^i \sigma_1 c^{-i}, c^i \sigma_2 c^{-i}, q'\big),$$

where $c = (1, 2, \dots, k)$ is the cycle and $q'(a) = q(c^{-i} a c^i)$ for $a \in C(\sigma_2)$. All orbits of this action consist of p elements except for the fixpoints of this action which are of the form $\sigma_1 = c^a$, $\sigma_2 = c^{1-a}$. These fixpoints contribute to the monomial R_{p+1} (with multiplicity 1) and to the monomial R_2 (with multiplicity $p-2$). This finishes the proof of the integrality of coefficients of (a).

We apply Theorem 4.3 in the case when $k = p$ is a prime number. The right-hand side of (19) is constant on each orbit of the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. Each orbit of this action consists of p elements, except for the fixpoints. The only pushing partition of $[p]$ which is invariant under the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ is the minimal partition $\big\{\{1\},\dots,\{p\}\big\}$. In this way we proved that

$$R_p = \Sigma_{p-1} + p\,\big(\text{linear combination of the characters } \Sigma_\pi \text{ for } \pi \in \mathfrak{S}(l),\, l \geq 1 \text{ with integer coefficients}\big).$$

Thanks to Kerov polynomials, each Σ_π can be written as a polynomial in free cumulants with integer coefficients. This shows part (b).

In the following we shall use the notations and results presented in the paper of Biane [Bia03]. In order to prove part (c) we consider the formal power series

$$H(z) = z - \sum_{j\geq 1} B_{j+1} z^{-j}$$

where B_j are Boolean cumulants. Biane showed that

$$(-p-1)\Sigma_{p+1} = [z^{-1}]H(z)H(z-1)\cdots H(z-p) \tag{20}$$

and

$$(-p-1)R_{p+2} = [z^{-1}]H(z)^{p+1}.$$

We know from [Bia03] that B_j is a polynomial in free cumulants $R_2, R_3, \dots$ with integer coefficients as well as Σ_{p+1} is a polynomial in Boolean cumulants $B_2, B_3, \dots$ with integer coefficients; hence it suffices to show that $(-p-1)(\Sigma_{p+1} - R_{p+2} + R_3)$ is a polynomial in Boolean cumulants with all coefficients divisible by p. It is equivalent to show that $\Sigma_{p+1} - R_{p+2} + R_3 = 0$ under additional assumption that all coefficients of the power series are taken from a field of characteristic p, hence all formulas are considered in a field of characteristic p from now.

From (20) it follows that

$$[B_3]\Sigma_{p+1} = \frac{1}{p+1} \sum_{0\leq z\leq p} \frac{1}{2}\frac{d^2}{dz^2}\big[z(z-1)\cdots(z-p)\big].$$

From Fermat's little theorem (see for example [GKP94]) it follows that in the field of characteristic p

$$z(z-1)\cdots(z-p) = z(z^p - z) = z^{p+1} - z^2$$

hence

$$[B_3]\Sigma_{p+1} = \frac{1}{p+1}\sum_{0\leq z\leq p}(-1) = -1. \tag{21}$$

We define $B_0 = -1$ and $B_1 = 0$; then using binomial formula we have

$$H(z-i) = \sum_{j\geq -1}\sum_{k\geq 0} (-1)^{k+1}\binom{-j}{k} i^k B_{j+1} z^{-(j+k)},$$

hence

$$(22)\quad -\frac{1}{p+1} H(z)H(z-1)\cdots H(z-p) = \\ -\frac{1}{p+1} \sum_{k_0,\dots,k_p\geq 0} \sum_{j_0,\dots,j_p\geq -1} (-1)^{k_1+\cdots+k_p+p+1} \\ \left(\prod_{0\leq i\leq p} \binom{-j_i}{k_i} i^{k_i} B_{j_i+1} \right) z^{-(j_0+\cdots+j_p+k_1+\cdots+k_p)}.$$

For any $a \in \mathbb{Z}/p\mathbb{Z}$ such that $a \neq 0$ the map $x \mapsto ax$ is a bijection of the multiset $(0,1,\dots,p) \subset \mathbb{Z}/p\mathbb{Z}$ (notice that $0 = p$ appears twice in this multiset) therefore the left-hand side of (22) is equal to

$$(23)\quad -\frac{1}{p+1} H(z)H(z-a)\cdots H(z-pa) = \\ -\frac{1}{p+1} \sum_{k_0,\dots,k_p\geq 0} \sum_{j_0,\dots,j_p\geq -1} (-1)^{k_1+\cdots+k_p+p+1} a^{k_1+\cdots+k_p} \\ \left(\prod_{0\leq i\leq p} \binom{-j_i}{k_i} i^{k_i} B_{j_i+1} \right) z^{-(j_0+\cdots+j_p+k_1+\cdots+k_p)}.$$

The coefficient of z^{-1} in (23) can be viewed as a polynomial in a; we shall denote it by $P(a)$. In the following we will study its coefficients of highest degrees. We are interested only in the summands for which $j_0 + \cdots + j_p + k_1 + \cdots + k_p = 1$; since $j_0 + \cdots + j_p \geq -p-1$ therefore $k_1 + \cdots + k_p \leq p+2$ and the degree of $P(a)$ is at most $p+2$.

However, $k_1 + \cdots + k_p = p+2$ would correspond to the case $j_0 = \cdots = j_p = -1$ which is equivalent to setting $B_2 = B_3 = \cdots = 0$; therefore $[a^{p+2}]P(a) = 0$.

For $k_1 + \cdots + k_p = p+1$ there is no summand for which $j_0, \dots, j_p \neq 0$ hence $[a^{p+1}]P(a) = 0$.

For $k_1+\cdots+k_p = p$ every summand which contributes is of the following form: one of the numbers $j_0, \dots, j_p$ is equal to 1 and all the others are equal to -1. This shows that $[a^p]P(a)$ viewed as a polynomial in $B_2, B_3, \dots$ contains only one monomial, namely a multiple of B_2; also $[B_2]P(a)$ viewed as a polynomial in a contains only one monomial namely a multiple of a^p. Therefore $[a^p]P(a)$ is a multiple of B_2 and the value of the coefficient of B_2 fulfills:

$$[B_2][a^p]P(a) = [B_2]P(1) = [B_2]\Sigma_{p+1}.$$

Since p is odd, the expansion of Σ_{p+1} into Boolean cumulants contains only summands which are of odd degree [Bia03]; it follows that $[a^p]P(a) = 0$.

In an analogous way we prove that $[a^{p-1}]P(a)$ is a multiple of B_3 and

$$[B_3][a^{p-1}]P(a) = [B_3]\Sigma_{p+1} = -1$$

from (21).

In this way we proved that $P(a)$ is a polynomial of degree $p-1$ which takes the same value for all $a \in \{1, \dots, p-1\}$. Polynomial

$$\widetilde{P}(a) = -B_3 a^{p-1} + P(0)$$

has the same properties. It follows that $P - \widetilde{P}$ has degree at most $p-2$ which takes the same value for all $a \in \{1, \dots, p-1\}$ hence it must be equal to the constant. It follows that $\widetilde{P} = P$.

Therefore

$$\Sigma_{p+1} = P(1) = -B_3 + R_{p+2}.$$

Observation that $B_3 = R_3$ finishes the proof for the third expression. □

It is interesting that for the first two expressions we managed to find combinatorial proofs while for the last expression there seems to be no natural candidate for a combinatorial approach.

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