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Jack characters and combinatorics of bipartite maps

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JACK CHARACTERS AND COMBINATORICS OF BIPARTITE MAPS

MACIEJ DOŁĘGA

ABSTRACT. We consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by M. Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that these objects are strongly connected to combinatorics of bipartite maps. Our combinatorial methods allow us to prove some part of conjectures of M. Lassalle

1. INTRODUCTION

1.1. **Jack polynomials.** In a seminal paper [Jac71], H. Jack introduced a family of symmetric functions $J_\lambda^{(\alpha)}$ depending on an additional parameter α. These functions are now called *Jack polynomials*. For some special values of α, they coincide with some established families of symmetric functions. Namely, up to multiplicative constants, for $\alpha = 1$ Jack polynomials coincide with Schur polynomials, for $\alpha = 2$ they coincide with zonal polynomials, for $\alpha = \frac{1}{2}$ they coincide with symplectic zonal polynomials, for $\alpha = 0$ we recover the elementary symmetric functions and finally their highest degree component in α are the monomial symmetric functions. Moreover, some other specializations appear in different contexts: the case $\alpha = 1/k$, where k is an integer, has been considered by Kadell in relation with generalizations of Selberg's integral [Kad97]. In addition, Jack polynomials for $\alpha = -(k+1)/(r+1)$ verify some interesting annihilation conditions [FJMM02] and this property makes them useful in some statistical physics models.

Over the time it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials (Section (VI,10) of I.G. Macdonald's book [Mac95] gives a few results of this kind), therefore Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions.

1.2. **Dual approach.** We will later define *Jack character* to be equal (up to some simple normalization constants) to the coefficient $[p_\mu]J_\lambda$ in the expansion of the Jack polynomial J_λ in the basis of power-sum symmetric functions. The idea of dual approach is to consider Jack characters as a function of λ and not as a function of μ as usual. In more concrete words, we would like to express the Jack character as a sum of some quantities depending on λ over some combinatorial set depending on μ (in Stanley's results, it is roughly the opposite).

Inspired by the case $\alpha = 1$ (which corresponds to the usual characters of the symmetric groups), Lassalle [Las09] suggested to express Jack characters in terms of, so called, free cumulants of the transition measure of the Young diagram λ.

This expression, called *Kerov polynomials for Jack characters*, involves rational functions in α, which are conjecturally polynomials with non-negative coefficients in α and $\beta = 1 - \alpha$ (we refer to this as Lassalle's conjecture). This suggests the existence of a combinatorial interpretation. A result of this type holds true in the case $\alpha = 1$, see [DFŚ10].

In this paper, we prove a part of Lassalle's conjecture, that is we present a top-degree part of Kerov polynonomials for Jack characters. More precisely, recalling that $(R_k)_{k\geq 2}$ is an algebraic basis of the ring generated by Jack characters, where R_k is a free cumulant, we can define a gradation on this ring by choosing arbitrarily the degree of each of the generators M_k. In this section, we will use the most natural one:

$$\deg_1(R_k) = k \qquad \text{for } k \geq 2.$$

Theorem 1.1. *For $k \geq 1$, one has*

$$(1)\quad K_k = R_{k+1} + \gamma \frac{k}{2} \sum_{|\mu|=k} (\ell(\mu)-1)!\tilde{R}_\mu + \sum_{|\mu|=k-1} \left(\frac{1}{4}\binom{k+1}{3} + \gamma^2 k \frac{3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{24} \right) \ell(\mu)!\tilde{R}_\mu + \text{terms of lower degree with respect to } \deg_1,$$

where $\tilde{R}_i = (i-1)R_i$ and $\tilde{R}_\mu = \prod_i \frac{\tilde{R}_i^{m_i(\mu)}}{m_i(\mu)!}$.

This theorem is proved in Section 4. Our methods are based on combinatorial analysis of bipartite maps, which has a great impact on understanding of structure of Jack characters.

1.3. **Outline of the paper.** The paper is organized as follows. Section 2 gives all necessary definitions and background; in particular we recall the notions of free cumulants and Kerov polynomials. In Section 3 we introduce bipartite maps and we analize their structure. Section 4 is devoted to the computation of some coefficients of Kerov polynomials for Jack characters by using their connection to bipartite maps.

2. Jack characters and Kerov polynomials

2.1. **Jack characters.**

2.1.1. *The case $\alpha = 1$: recent developments on characters of symmetric group.* In the case $\alpha = 1$ Jack polynomials correspond, up to some normalization constants, to Schur symmetric functions. The coefficients of the latter in the basis of the power-sum symmetric functions are known to be equal to the irreducible characters of the symmetric groups.

For a Young diagram λ we denote by ρ^λ the corresponding irreducible representation of the symmetric group $\mathfrak{S}_n$ with $n = |\lambda|$. Any partition μ such that $|\mu| = n$ can be viewed as a conjugacy class in $\mathfrak{S}_n$. Let $\pi_\mu \in \mathfrak{S}_n$ be any permutation from this conjugacy class; we will denote by $\operatorname{Tr}\rho^\lambda(\mu) := \operatorname{Tr}\rho^\lambda(\pi_\mu)$ the value of the corresponding irreducible character. If $m \leq n$, any permutation $\pi \in \mathfrak{S}_m$ can be also viewed as an element of $\mathfrak{S}_n$, we just have to add $n-m$ additional fixpoints to π; for this reason

$$\operatorname{Tr}\rho^\lambda(\mu) := \operatorname{Tr}\rho^\lambda\left(\mu\, 1^{|\lambda|-|\mu|}\right)$$

makes sense also when $|\mu| \leq |\lambda|$.

Normalized characters of the symmetric group were defined by Kerov and Olshanski [KO94] as follows:

$$\operatorname{Ch}^{(1)}_\mu(\lambda) = \underbrace{n(n-1)\cdots(n-|\mu|+1)}_{|\mu| \text{ factors}} \frac{\operatorname{Tr}\rho^\lambda(\mu)}{\text{dimension of } \rho^\lambda} \tag{2}$$

(the superscript in the notation $\operatorname{Ch}^{(1)}_\mu(\lambda)$ is related to the fact that we consider the case $\alpha = 1$). The novelty of the idea was to view the character as a function $\lambda \mapsto \operatorname{Ch}^{(1)}_\mu(\lambda)$ on the set of Young diagrams (of any size) and to keep the conjugacy class fixed. The normalization constants in (2) were chosen in such a way that the normalized characters $\lambda \mapsto \operatorname{Ch}^{(1)}_\mu(\lambda)$ form a linear basis (when μ runs over the set of all partitions) of the algebra $\Lambda^\star$ of so-called *shifted symmetric functions*, which is very rich in structure (this property is, for example, the key point in a recent approach to study asymptotics of random Young diagrams under Plancherel measure [IO02]). In addition, recently a combinatorial description of the quantity (2) has been given [Sta06, Fér10], which is particularly suitable for study of asymptotics of character values [FŚ11a].

Using Frobenius' formula for characters of the symmetric groups [Fro00], definition (2) can be rephrased using Schur functions (see [Las08] for details). We expand the Schur polynomial s_λ in the base of the power-sum symmetric functions (p_ρ) as follows:

$$\frac{n!\, s_\lambda}{\dim(\lambda)} = \sum_{\substack{\rho:\\ |\rho|=|\lambda|}} \theta^{(1)}_\rho(\lambda)\, p_\rho \tag{3}$$

for some numbers $\theta^{(1)}_\rho(\lambda)$. Then

$$\operatorname{Ch}^{(1)}_\mu(\lambda) = \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{(1)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda), \tag{4}$$

where

$$z_\mu = \mu_1\mu_2\cdots\; m_1(\mu)!\, m_2(\mu)!\cdots$$

and $m_i(\mu)$ denotes the multiplicity of i in the partition μ.

2.1.2. *Back to the general case.* In this section we will define analogues of the quantity $\operatorname{Ch}^{(1)}_\mu(\lambda)$ via Jack polynomials. First of all, as there are several of them, we have to fix a normalization for Jack polynomials. In our context, the best is to use the functions denoted by J in the book of Macdonald [Mac95, VI, (10.22)]. With this normalization, one has

$$J^{(1)}_\lambda = \frac{n!\, s_\lambda}{\dim(\lambda)},$$
$$J^{(2)}_\lambda = Z_\lambda,$$

where Z_λ are zonal polynomials.

If in (3) we replace the left-hand side by the Jack polynomial:

$$J^{(\alpha)}_\lambda = \sum_{\substack{\rho:\\ |\rho|=|\lambda|}} \theta^{(\alpha)}_\rho(\lambda)\, p_\rho \tag{5}$$

then in analogy to (4) we can define

$$\mathrm{Ch}^{(\alpha)}_\mu(\lambda) = \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{(\alpha)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda).$$

These quantities are called *Jack characters*. Notice that for $\alpha = 1$, we recover the usual normalized character values of the symmetric groups.

Jack characters have been first considered by M. Lassalle in [Las08]. They have also been studied in papers [Las09], [FŚ11b] (the latter deals with the case $\alpha = 2$ which corresponds to zonal polynomials) and [DF12]. Note that the normalization used here is the same as in the last of the previous papers and is the most convenient in our opinion.

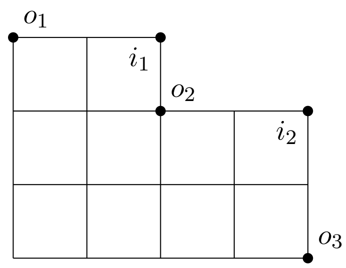
2.2. **Polynomial functions on the set of Young diagrams.** The ring $\Lambda^\star$ of *polynomial functions on the set of Young diagrams* (briefly: ring of *polynomial functions*) has been introduced by S. Kerov and G. Olshanski in order to study irreducible character values of symmetric groups [KO94].

The first characterization of $\Lambda^\star$ is the following: it is the ring of *shifted symmetric functions* in $\lambda_1, \lambda_2, \dots$. By definition, a shifted symmetric function F is a collection of polynomials $F_h \in \mathbb{Q}[\lambda_1, \dots, \lambda_h]$ such that each F_h is symmetric in variables $\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_h - h$ and such that the compatibility relation

$$F_{h+1}(\lambda_1, \dots, \lambda_h, 0) = F_h(\lambda_1, \dots, \lambda_h)$$

holds true for all values of h. For any partition μ, the function $\mathrm{Ch}^{(1)}_\mu$ is a shifted symmetric function of λ. Moreover, the family of such functions forms a linear basis of $\Lambda^\star$. We refer to [**?**] or to the case $\alpha = 1$ of [Las08, Proposition 2] for a proof of this fact.

Another equivalent description can be given using Kerov's interlacing coordinates of a Young diagram. Recall that the *content* of a box of a Young diagram is $j - i$, where j is its column index and i its row index and, more generally, the content of a point of a plane is the difference of its x-coordinate and its y-coordinate. We denote by $\mathbb{I}_\lambda$ the sets of contents of the *inner corners* of λ, that is corners, at which a box can be removed from λ to obtain a new diagram of size $|\lambda| - 1$. Similarly, the set $\mathbb{O}_\lambda$ is defined as the contents of the *outer corners*, that is corners at which a box could be added to λ to obtain a new diagram of size $|\lambda| + 1$.



$$\mathbb{I}_\lambda = \{-1, 2\}$$
$$\mathbb{O}_\lambda = \{-3, 0, 4\}$$

FIGURE 1. A Young diagram with its inner and outer corners (marked respectively with i and o).

An example is given on Figure 1 (we use the french convention to draw Young diagrams).

If k is a positive integer, one can consider the power sum symmetric function p_k, evaluated on the difference of alphabets $\mathbb{O}_\lambda - \mathbb{I}_\lambda$. By definition, it is a function on Young diagrams given by:

$$\lambda \mapsto p_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda) := \sum_{o \in \mathbb{O}_\lambda} o^k - \sum_{i \in \mathbb{I}_\lambda} i^k.$$

As any symmetric function can be written (uniquely) in terms of p_k, we can define $f(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$ for any symmetric function f as follows: if $f = \sum_\rho a_\rho p_{\rho_1} \cdots p_{\rho_\ell}$, then by definition

$$f(\mathbb{O}_\lambda - \mathbb{I}_\lambda) = \sum_\rho a_\rho p_{\rho_1}(\mathbb{O}_\lambda - \mathbb{I}_\lambda) \cdots p_{\rho_\ell}(\mathbb{O}_\lambda - \mathbb{I}_\lambda).$$

V. Ivanov and G. Olshanski [IO02, Corollary 2.8] have shown that the functions $(\lambda \mapsto p_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda))_{k \geq 2}$ forms an algebraic basis of $\Lambda^\star$ (for all diagrams λ, one has $p_1(\mathbb{O}_\lambda - \mathbb{I}_\lambda) = 0$). In other terms, $\Lambda^\star$ is the ring of symmetric functions evaluated in the difference of alphabets $\mathbb{O}_\lambda - \mathbb{I}_\lambda$.

2.3. **Transition measure and free cumulants.** S. Kerov [**?**] introduced the notion of *transition measure* of a Young diagram. This probability measure μ_λ associated to λ is defined by its Cauchy transform

$$G_{\mu_\lambda}(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z - x} = \frac{\prod_{i \in \mathbb{I}_\lambda} z - i}{\prod_{o \in \mathbb{O}_\lambda} z - o}.$$

In particular, it is supported on $\mathbb{O}_\lambda$. Besides, its moments are $h_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$, where h_k is the complete symmetric function of degree k. In particular, they are polynomial functions on the set of Young diagrams; we will denote them by $M_k^{(1)}$. The family $(M_k^{(1)})_{k \geq 2}$ forms an algebraic basis of polynomial functions on the set of Young diagrams ($M_1^{(1)}$ is the null function).

In Voiculescu's free probability it is very convenient to associate to aknop probability measure μ a sequence of numbers $(R_k(\mu))_{k \geq 1}$ called *free cumulants* [Voi86, Spe94]. The free cumulants of the transition measure of Young diagrams appeared

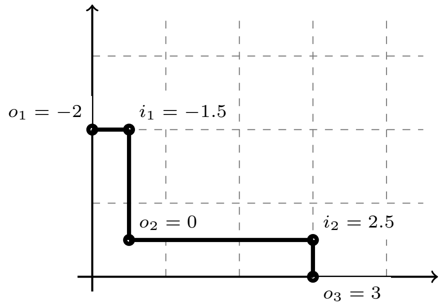


FIGURE 2. A generalized Young diagram L with the corresponding set $\mathbb{O}_L$ and $\mathbb{I}_L$.

first in the work of P. Biane [Bia98] and play an important role in asymptotic representation theory. As explained by M. Lassalle (look at the case $\alpha = 1$ of [Las09, Section 5]), they can be expressed as

$$R_k^{(1)}(\lambda) := R_k(\mu_\lambda) = e_k^\star(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$$

for some homogeneous symmetric function $e_k^\star$ of degree k. Note also that $e_k^\star$ is an algebraic basis of symmetric functions and, hence $(R_k^{(1)})_{k\geq 2}$ is an algebraic basis of ring of polynomial functions on the set of Young diagrams ($R_1^{(1)}$ is the null function).

2.4. **Generalized Young diagrams.** The second description of $\Lambda^\star$ is interesting because it shows that the value of a polynomial function is defined on more general objects than just Young diagrams.

Indeed, let us consider a zigzag line L going from a point $(0, y)$ on the y-axis to a point $(x, 0)$ on the x-axis. We assume that every piece is either an horizontal segment from left to right or a vertical segment from top to bottom. A Young diagram can be seen as such a zigzag line: just consider its border. Therefore, we call these zigzag lines *generalized Young diagrams*. The notions of inner and outer corners can be easily adapted to generalized Young diagrams, as well as the sets $\mathbb{I}_L$ and $\mathbb{O}_L$ of their contents. It is illustrated on Figure 2.

Any polynomial function F on the set of Young diagrams corresponds to the function

$$\lambda \mapsto f(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$$

for some symmetric function f. Hence, F can be canonically extended to generalized Young diagrams by setting

$$F(L) = f(\mathbb{I}_L - \mathbb{O}_L).$$

We will be in particular interested in the following generalized Young diagrams. Let λ be a (generalized) Young diagram and s and t two positive real numbers. We denote $T_{s,t}(\lambda)$ the broken line obtained by stretching λ horizontally by a factor s and vertically by a factor t (see Figure 3). These *anisotropic* Young diagrams have been introduced by S. Kerov in [**?**].

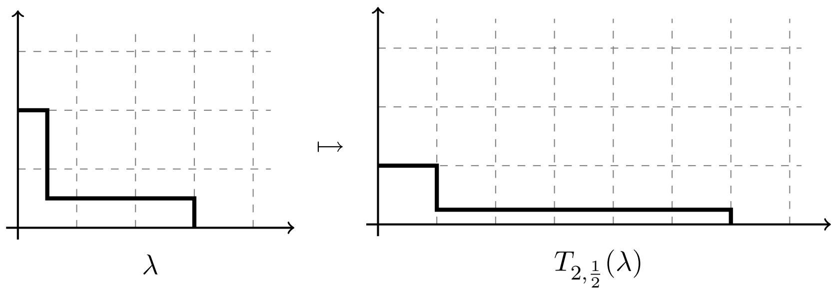


FIGURE 3. Example of stretched Young diagram.

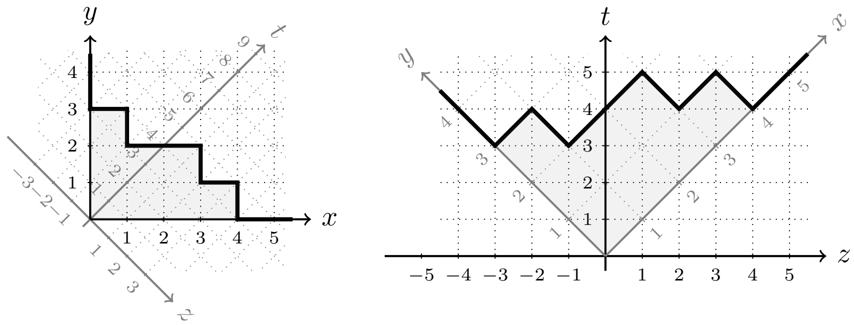


FIGURE 4. Young diagram $\lambda = (4, 3, 1)$ and the graph of the associated function ω_λ.

In the case $s = t$, we denote by $D_s(\lambda) = T_{s,s}(\lambda)$ the diagram obtained from λ by applying a homothetic transformation of ratio s. It is easy to check that the sets $\mathbb{I}_{D_s(\lambda)}$ and $\mathbb{O}_{D_s(\lambda)}$ are obtained from $\mathbb{I}_\lambda$ and $\mathbb{O}_\lambda$ by multiplying all values by s. In particular, if F is a polynomial function such that the corresponding symmetric function f is homogeneous of degree d, then

$$\lambda \mapsto F(D_s(\lambda)) = f(\mathbb{I}_{D_s(\lambda)} - \mathbb{O}_{D_s(\lambda)}) = s^d f(\mathbb{I}_\lambda - \mathbb{O}_\lambda) = s^d F(\lambda)$$

is also a polynomial function. Finally, for any fixed $s > 0$, F is a polynomial function if and only if $\lambda \mapsto F(D_s(\lambda))$ is a polynomial function.

A generalized Young diagram can also be seen as a function on the real line. Indeed, if one rotates the zigzag line counterclockwise by $45°$ and scale it by a factor $\sqrt{2}$ (so that the new x-coordinate corresponds to contents), then it can be seen as the graph of a piecewise affine continuous function with slope ± 1. We denote this function $\omega(\lambda)$. This definition is illustrated on Figure 4.

2.5. α-polynomial functions.

Definition 2.1. We say that F is an *α-polynomial function* on the set of (generalized) Young diagrams if

$$\lambda \mapsto F(T_{\alpha^{-1},1}(\lambda))$$

is a polynomial function. The set of α-polynomial functions is an algebra which will be denoted by $\Lambda^\star_{(\alpha)}$.

Using the characterization *via* shifted symmetric function, this means that the polynomial $F(\alpha^{-1}\lambda_1, \cdots, \alpha^{-1}\lambda_h)$ is symmetric in $\lambda_1 - 1, \ldots, \lambda_h - h$. Equivalently (by a change of variables), F is symmetric in $\alpha\lambda_1 - 1, \ldots, \alpha\lambda_h - h$ or in

$$\lambda_1 - \frac{1}{\alpha}, \ldots, \lambda_h - \frac{h}{\alpha}.$$

The last characterization is the definition of what is usually called an α-shifted symmetric function [OO97, Las08].

It would be equivalent to ask in the definition of α-polynomial functions that

$$\lambda \mapsto F\left(T_{\sqrt{\alpha}^{-1},\sqrt{\alpha}}(\lambda)\right)$$

is a polynomial function. In particular, the α-anisotropic free cumulants defined by

$$R_k^{(\alpha)}(\lambda) = R_k^{(1)}\left(T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)\right)$$

are α-polynomial. Moreover, the family $(R_k^{(\alpha)})_{k\geq 2}$ is an algebraic basis of the algebra $\Lambda^\star_{(\alpha)}$ of α-polynomial functions.

M. Lassalle has shown that Jack characters $\mathrm{Ch}_\mu^{(\alpha)}$ form a basis of the algebra of α-polynomial functions (see Section 3 and in particular Proposition 2 of [Las08]). In particular, they are α-polynomial functions and can be expressed in terms of the free cumulants.

Definition-Proposition 2.2. *Let μ be a partition and $\alpha > 0$ a fixed real number. There exists polynomial $K_\mu^{(\alpha)}$ such that, for every λ,*

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = K_\mu^{(\alpha)}\left(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \cdots\right).$$

The polynomials $K_\mu^{(\alpha)}$ have been introduced by S. Kerov in the case $\alpha = 1$ [Ker00, Bia03] and by M. Lassalle in the general case[Las09]. Once again, our normalizations are different from his.

From now on, when it does not create any confusion, we suppress the superscript (α).

We present a few examples of polynomials K_μ (in particular the case of a one-part partition μ of length lower than 6). This data has been computing using the one given in [Las09, page 2230]

$$\begin{aligned}
K_{(1)} &= R_2, \\
K_{(2)} &= R_3 + \gamma R_2, \\
K_{(3)} &= R_4 + 3\gamma R_3 + (1+2\gamma^2) R_2, \\
K_{(4)} &= R_5 + \gamma(6R_4 + R_2^2) + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2, \\
K_{(5)} &= R_6 + \gamma(10R_5 + 5R_3R_2) + 15R_4 + 5R_2^2 + \gamma^2(35R_4 + 10R_2^2) \\
&\quad + (55\gamma + 50\gamma^3) R_3 + (8 + 46\gamma^2 + 24\gamma^4) R_2, \\
K_{(2,2)} &= R_3^2 + 2\gamma R_3 R_2 - 4R_4 + (\gamma^2 - 2) R_2^2 - 10\gamma R_3 - (6\gamma^2 + 2) R_2,
\end{aligned}$$

where we set $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$.

A few striking facts appear on these examples:

- All coefficients are polynomials in the auxiliary parameter γ: we prove this fact in the next section with explicit bounds on the degrees.
- For one part partition, polynomials $K_{(r)}$ have non-negative coefficients. We are unfortunately unable to prove this statement, which is a more precise version of [Las09, Conjecture 1.2]. A similar conjecture holds for several part partitions, see [Las09, Conjecture 1.2].

3. Enumeration of bipartite maps

3.1. **Maps.** A labeled graph drawn on a surface (compact, connected, 2-dimensional manifold) will be called a *map*. For a given map $\mathcal{M}$ we define its Euler characteristic $\chi(\mathcal{M})$ as an Euler characteristic of an underlying surface. We will always assume that the surface is minimal in the sense that after removing the graph from the surface, the latter becomes a disjoint collection of open discs. If we draw an edge of such a graph with a fat pen and then take its boundary, this edge splits into two *edge-sides*. In the above definition of the map, by *labeled*, we mean that each edge-side is labeled with a number from the set $[2n]$ and each number from this set is used exactly once.

In the next part of this article we will consider only maps with one face. Each map of this form, which has n edges can be considered as a $2n$-gon with pair-partition P which identify their edges. If a map is bipartite, then our $2n$-gon is bipartite, so the pair-partition P tells us in a unique way which edges we are glueing according to the orientation and which adges we are glueing conversly to the orientation (because black vertices are identified with each other and white vertices are identified with each other). If not, then we have to define which elements of pair-partitions are twists. For a map $\mathcal{M}$, we denote the set of black (white respectively) vertices by $V_\bullet(\mathcal{M})$ ($V_\circ(\mathcal{M})$ respectively) and we denote the set of edges by $E(\mathcal{M})$. Euler characteristic of such map is equal to

$$\chi(\mathcal{M}) = 1 - n + |V(\mathcal{M})|,$$

where $V(\mathcal{M}) = V_\bullet(\mathcal{M}) \cup V_\circ(\mathcal{M})$. For more informations about bipartite maps we refer to [FŚ11b].

3.2. **Skeleton.**

Definition 3.1. $\mathrm{NOrMaps}_i$ is a set of pairs $(\mathcal{M}, q)$ such that:

- $\mathcal{M}$ is a labeled bipartite map with $\chi(\mathcal{M}) = i$;
- $q : V_\bullet(\mathcal{M}) \to \mathbb{N}_+$;
- for every set $A \subset V_\bullet(\mathcal{M})$ which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq V_\bullet(\mathcal{M})$) there are more than $\sum_{j \in A} \big(q(j) - 1\big)$ white vertices in $\mathcal{M}$ which have a neighbour in A.

$\widetilde{\mathrm{NOrMaps}}_i$ is a set of pairs $(\mathcal{M}, q)$ such that:

- $\mathcal{M}$ is a map with $\chi(\mathcal{M}) = i$;
- $q : V(\mathcal{M}) \to \mathbb{N}_+$.

Definition 3.2. For $i < 1$ we define an operator $\mathcal{D}_i : \mathrm{NOrMaps}_i \to \widetilde{\mathrm{NOrMaps}}_i$, which gives us a skeleton $\mathcal{D}_i(\mathcal{M}, q)$ of a pair $(\mathcal{M}, q)$, where $\mathcal{D}_i(\mathcal{M}, q)$ is constructed in the following way:

(1) We erase labellings of $\mathcal{M}$,
(2) We delete from $\mathcal{M}$ white vertices of degree 1 with corresponding edges.
(3) Then we define a function $q' : V(\mathcal{M}) \to \mathbb{N}_+$ such, that

$$q'(v) = \begin{cases} |\deg(v)| + 1 - q(v) & \text{if } v \in V_\bullet(\mathcal{M}), \\ 1 & \text{if } v \in V_\circ(\mathcal{M}). \end{cases}$$

(4) For each vertex v_1 of degree 2 we glue it with one of his neighbours v_2 along the edge, which connect them. For a vertex v constructed like that we define $q'(v) = \max(q'(v_1), q'(v_2))$.
(5) We define $\mathcal{D}_i(\mathcal{M}, q) = (\mathcal{M}', q')$, where $\mathcal{M}'$ is a map we obtained, and q' is defined above (see an example on Figure 5).

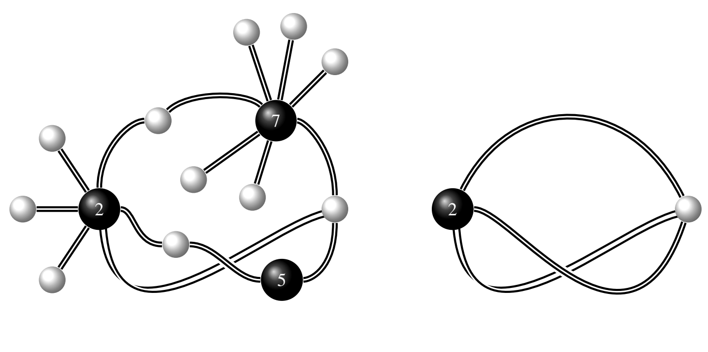


FIGURE 5. $(\mathcal{M}, q) \in \mathrm{NOrMaps}_0$ on the left and their skeleton $\mathcal{D}_0(\mathcal{M}, q)$ on the right.

3.3. **Counting.** Let $i < 2$ be an integer and μ be an integer partition. We are interested in counting the cardinality of the finite subset $a_i(\mu) \subset \mathrm{NOrMaps}_i$ which consists of pairs $(\mathcal{M}, q)$ such that $\mathcal{M}$ has $\ell(\mu)$ black vertices and $|\mu|$ vertices.

We have a natural action of $2\mathbb{Z}_{2n} \cong \mathbb{Z}_n$ on the set of labeled maps with n edges by action on labelings. More precisely, if P is a pair-partition which represent a map $\mathcal{M}$, then for $x \in \mathbb{Z}_n$ a map $x\mathcal{M}$ is given by pair partition $P + 2x$, where $P+2x$ is a pair-partitions with blocks obtained by adding $2x$ to blocks of P modulo $2n$. The following lemma explains the connection between this natural action and symmetries of a given map.

Lemma 3.3. *Let us consider a labeled bipartite map $\mathcal{M}$ with n edges. Erase labels and erase white vertices of degree 1 and corresponding edges. Then we put labels on our new map such that we obtain a bippartite map $\mathcal{M}'$ with k edges for $k \leq n$. Then we have a following equality*

$$|\mathbb{Z}_n(\mathcal{M})| = n\frac{A}{|\operatorname{Stab}(\mathcal{M}')|}, \tag{6}$$

where A is the number of possibilities for drawing white vertices of degree 1 and connecting them to black vertices from a map $\mathcal{M}'$ to construct a map which can be labeled to obtain a map $\mathcal{M}$.

Proof. Let us count the number of pairs $(\mathcal{M}_1, \mathcal{M}_2)$, where $\mathcal{M}_1 \subset \mathcal{M}_2$ is a submap of a map $\mathcal{M}_2$ constructed from a map $\mathcal{M}_2$ by erasing white vertices of degree 1 and corresponding edges. Moreover $\mathcal{M}_2$ is labeled such, that $\mathcal{M}_2 = \mathcal{M}$, and $\mathcal{M}_1$ is labeled such, that $\mathcal{M}_1 = \mathcal{M}''$ for some $\mathcal{M}'' \in \mathbb{Z}_n(\mathcal{M}')$. These two labellings are independent. We can count these pairs in two ways. From the one hand, we can draw $\mathcal{M}_1$, we can add edges to obtain a map which can be labeled such as a map $\mathcal{M}$ (we can do it in A ways) and we can label a map $\mathcal{M}_2$ in n ways by action of elements of $\mathbb{Z}_n$ which was described above. Hence, we have An pairs. From the other hand we can draw a map $\mathcal{M}_2$ at the beginning and we can label their edges in $|\mathbb{Z}_n(\mathcal{M})|$ ways. After that, we can label the edges of a submap $\mathcal{M}_1$ obtained by removing proper edges and vertices in $|\operatorname{Stab}(\mathcal{M}')|$ ways. After that we have $|\mathbb{Z}_n(\mathcal{M})||\operatorname{Stab}(\mathcal{M}')|$ pairs. Comparing these two ways of counting the same thing we have $nA = |\mathbb{Z}_n(\mathcal{M})| \cdot |\operatorname{Stab}(\mathcal{M}')|$, which finishes the proof. □

Lemma 3.4. *Let us consider $(\mathcal{M}', q') \in \widetilde{\mathrm{NOrMaps}}_j$ with one marked side of one edge (randomly chosen) and let a map $\mathcal{M}'' \in \mathcal{D}_j^{-1}((\mathcal{M}', q'))$ be fixed. For $b \in V_\bullet(\mathcal{M}')$ let $f_{(\mathcal{M}', q')}(b, i)$ denotes the number of pairs $(\mathcal{M}, q) \in \mathrm{NOrMaps}_j$ such, that $D_j(\mathcal{M}, q) = (\mathcal{M}', q')$, $q(b) = i$, and label of a marked side is equal to 1, and after proceeding 1 and 2, but only to vertices connected with b by edge, we obtain $\mathcal{M}''$. Then the following equality holds.*

$$f_{(\mathcal{M}', q')}(b, i) = \binom{i + q'(b) - 2}{\deg(b) - 1}. \tag{7}$$

Proof. Let $(\mathcal{M}, q) \in \mathrm{NOrMaps}_g$ be a pair as in the statement of lemma. The number of white vertices of degree 1, connected to the vertex b by an edge, is equal to

$$\deg_{\mathcal{M}}(b) - \deg(b) = q(b) + q'(b) - 1 - \deg(b).$$

All vertices added to a map $\mathcal{M}'$ to obtain a map $\mathcal{M}$ (all, except white vertices of degree 1 linked with the vertex b) are added and colored in a unique way. Also labelling is uniquelly determined. Hence, the only possibility for constructing different maps $\mathcal{M}$ is by adding white vertices of degree 1 to a vertex b in a different ways. Let $\{h_1, \dots, h_m\}$ be a set of half-edges (each edge of a locally orientable map is a pair of two half-edges determined by the vertices which are connected by this edge) of b such, that h_{i+1} is a succecor of h_i (so $m = \deg(b)$). Let then x_i denotes the number of edges we putted between h_i and h_{i+1}, to connect x_i added white vertices of degree 1 to the vertex b. There is a correspondence between the number of ways of connecting all these white vertices to the vertex b and the number of solutions of the following equation:

$$x_1 + \cdots + x_m = l \tag{8}$$

in nonnegative integers for $l = q(b) + q'(b) - 1 - \deg(b)$. This number is equal to $\binom{m+l-1}{m-1}$. Setting $m = \deg(b)$ we have:

$$\binom{m+l-1}{m-1} = \binom{q(b)+q'(b)-2}{\deg(b)-1} = f_{(\mathcal{M}',q')}(b,i) \tag{9}$$

thanks to the equality of $q(b) = i$. □

Lemma 3.5. *Let $(\mathcal{M}, q) \in \mathrm{NOrMaps}_j$ have a one face and $\mathcal{D}_j(\mathcal{M}, q) = (\mathcal{M}', q')$, where $j < 1$. Then the number of pairs, which belong to $a_j(\mu) \cap \mathcal{D}_j^{-1}((\mathcal{M}', q'))$ is equal to:*

$$\frac{(|\mu| - j + 1)\left(l(\mu) - j + |V_\circ(\mathcal{M}')|\right)!}{\left(|V(\mathcal{M}')| - j\right)!\,|\operatorname{Stab}(\mathcal{M}')|\,\prod_{i\geq 2} m_i(\mu)!} \times \sum_{h \in A_{1-1}^{V_\bullet(\mathcal{M}')}} \left(\prod_{i \in A \setminus h(V_\bullet(\mathcal{M}'))} (\mu_i - 1) \prod_{b \in V_\bullet(\mathcal{M}')} f_{(\mathcal{M}',q')}(b, \mu_{h(b)}) \right), \tag{10}$$

where $A = \{1, \dots, \ell(\mu)\}$, $A_{1-1}^{V_\bullet(\mathcal{M}')}$ is a set of injective functions from the set $V_\bullet(\mathcal{M}')$ into the set A and $f_{(\mathcal{M}',q')}(b, \mu_{i_b})$ is defined by equality (7).

Proof. For simplicity we assume, that $|\operatorname{Stab}(\mathcal{M}')| = 1$, which means that our two-colored map is not symmetric (otherwise we have to divide by rank of stabilizer, as in Lemma 3.3). Let us analyse an inversion of a procedure described in 3.2 to count the rank of a intersection of a set $\mathcal{D}_j^{-1}((\mathcal{M}', q'))$ with a set of pairs contributing to $a_j(\mu)$.

(1) For each pair of vertices $a, b \in V_\bullet(\mathcal{M}')$ we draw a white vertex on the edge connecting these two vertices.
   (a) If the edge e connecting a and b is not a twist, then both edges, we obtain by dividing edge e by drawing a white vertex on it, are not twists;

(b) If the edge e connecting a and b is a twist, then both edges, we obtain by dividing edge e by drawing a white vertex on it, are twists.

Let $b \in V_\bullet(\mathcal{M}')$ and we assume, that it was colored by a number $\mu_{h(b)}$, where $h(b) \in \{1, \dots, \ell(\mu)\}$. Then we have to add some number of white vertices of degree 1 connected with a vertex b and by Lemma 3.4 we can do it in $f_{(\mathcal{M}',q')}(b, \mu_{h(b)})$ ways.

(2) Other black vertices, which can appear in our map have the property, that after removing their white neighbours of degree 1, each black vertex will have degree 2. We can put these black vertices in our two-colored map in a following way. We fix the order of the vertices of a map $\mathcal{M}'$ and we fix the order of the edges of this map. White vertices are successors of black vertices in this order. We put an orientation on each edge. $|E| - |V| = 1 - j$, by Euler characteristic. Let us take an ordered sequence of $\ell(\mu) - j + |V_\circ(\mathcal{M}')|$ variables and we choose a subsequence $\mathcal{A}$ of $|V(\mathcal{M}')| - j$ variables. We can do it in

$$\binom{l(\mu) - j + |V_\circ(\mathcal{M}')|}{|V(\mathcal{M}')| - j}$$

ways. We are putting $\mu_1, \dots, \mu_{\ell(\mu)}$ for all variables except $|V_\circ(\mathcal{M}')| - j$ last variables from the subsequence $\mathcal{A}$. We can do it in

$$\frac{\ell(\mu)!}{\prod_{i \geq 2} m_i(\mu)!}$$

ways. For such choice the corresponding map has as many black vertices on the edge with number l as we have variables between the l-th and the $l+1$-th element of the subsequence $\mathcal{A}$. Moreover, each black vertex on this edge is colored by the proper number of the form μ_r chosen according to the order of this vertex with respect to the orientation of the edge. For the black vertex of degree greater then 2 we put the color of the proper variable from the subsequence $\mathcal{A}$ according to the order of the black vertices of the two-colored map $\mathcal{M}'$, and according to the natural order of the elements of the subsequence $\mathcal{A}$ (example in 6). Let $\mathcal{M}''$ be a map corresponding to such choice. For each black vertex b of degree 2 we can adjoin to it missing white vertices of degree 1 in $\mu_{h(b)} - 1$ ways. Hence, for fixed choice of $\mathcal{A}$ and for fixed coloring of the $|V_\bullet(\mathcal{M}')|$ first variables in $\mathcal{A}$, after coloring the rest variables in all possible ways, and after summing over all injective functions $h \in A_{1-1}^{V_\bullet(\mathcal{M}')}$, which "code" what are the numbers putting in the $|V_\bullet(\mathcal{M}')|$ first variables in the subsequence $\mathcal{A}$, and after summing over all possibilities of choosing colors from the set $A \setminus h(V_\bullet(\mathcal{M}'))$, we obtain, that the number of pairs contributing to the $a_j(\mu)$ and belonging to the set

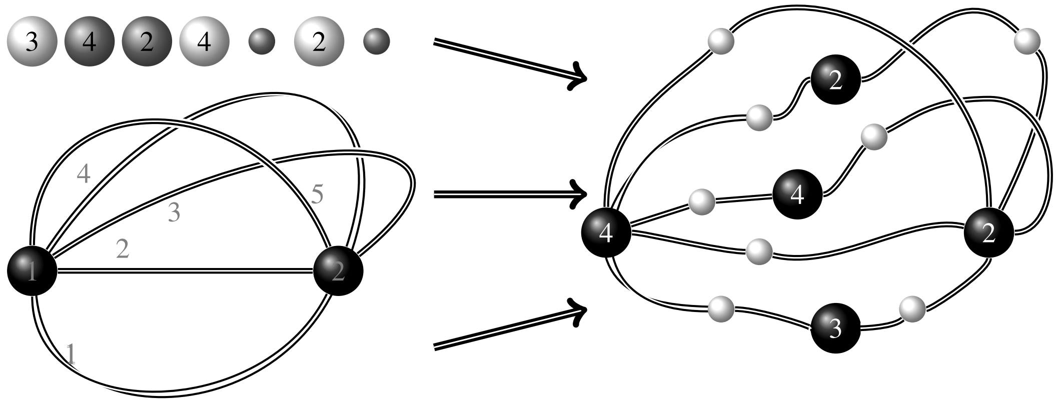


FIGURE 6. Example of a procedure described in the lemma 3.3 for $\mu = \{4,4,3,2,2\}$ and $j = -2$. Grey balls correspond to the choice of a subsequence of $|V_\bullet(\mathcal{M}')| - j$ elements.

$\mathcal{D}_j^{-1}((\mathcal{M}', q'))$ is equal to:

$$\binom{l(\mu) - j + |V_\circ(\mathcal{M}')|}{|V(\mathcal{M}')| - j} \frac{(l(\mu) - |V_\bullet(\mathcal{M}')|)!}{\prod_{i\geq 2} m_i(\mu)!} \times \sum_{h \in A_{1-1}^{V_\bullet(\mathcal{M}')}} \left(\prod_{i \in A \setminus h(V_\bullet(\mathcal{M}'))} (\mu_i - 1) \prod_{b \in V_\bullet(\mathcal{M}')} f_{(\mathcal{M}', q')}(b, \mu_{h(b)}) \right),$$

where $A = \{1, \dots, l(\mu)\}$, and $A_{1-1}^{V_\bullet(\mathcal{M}')}$ is a set of injective functions going from the set $V_\bullet(\mathcal{M}')$ into the set A. Thanks to the equality $|V_\bullet(\mathcal{M})| + |V_\circ(\mathcal{M})| = |V(\mathcal{M})|$, the following holds:

$$\binom{\ell(\mu) - j + |V_\circ(\mathcal{M}')|}{|V(\mathcal{M}')| - j} \frac{(\ell(\mu) - |V_\bullet(\mathcal{M}')|)!}{\prod_{i\geq 2} m_i(\mu)!} = \frac{(\ell(\mu) - j + |V_\circ(\mathcal{M}')|)!}{(|V(\mathcal{M}')| - j)! \prod_{i\geq 2} m_i(\mu)!}, \tag{11}$$

which finishes the proof.

□

Let $\mathfrak{h}_\pi(\mu)$ denote the monomial symmetric function indexed by π evaluated in variables μ_1,$\mu_2, \dots$. For example,

$$\mathfrak{h}_{1^2}(\mu) = \sum_{i<j} \mu_i \mu_j.$$

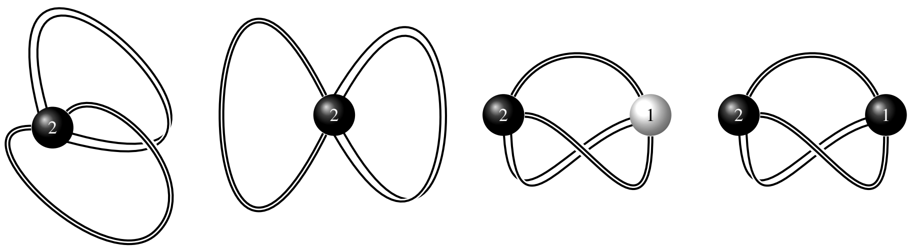


FIGURE 7. All skeletons $(\mathcal{M}_1, q_1)$, $(\mathcal{M}_2, q_2)$, $(\mathcal{M}_3, q_3)$, $(\mathcal{M}_4, q_4)$ of nonorientable maps of Euler characteristic 0.

Theorem 3.6. *The following equalities hold:*

$$|a_0(\mu)| = \frac{k_0 \ell(\mu)!}{4} \prod_{i \geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!} \left(\frac{k_0^2 - 1 + 3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{6} \right),$$

$$|a_1(\mu)| = \frac{k_1(\ell(\mu)-1)!}{2} \prod_{i \geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!},$$

where $k_0 = |\mu| + 1$ *and* $k_1 = |\mu|$.

Proof. At the beginning we would like to investigate all skeletons of nonorientable maps of Euler characteristic equal to 0. We have, that $|\mathcal{D}_0(\mathrm{NOrMaps}_0)| = 4$ and all these skeletons are shown on a Figure 7. We write them as $(\mathcal{M}_1, q_1), \dots, (\mathcal{M}_4, q_4)$ respectively (notation is the same as in Figure 7).

We have, that

$$(12) \quad a_0(\mu) = \left(\mathcal{D}_0^{-1}((\mathcal{M}_1, q_1)) \cap a_0(\mu)\right) \cup \cdots \cup \left(\mathcal{D}_0^{-1}((\mathcal{M}_4, q_4)) \cap a_0(\mu)\right).$$

We can apply Lemma 3.3 to count $|\mathcal{D}_j^{-1}((\mathcal{M}_i, q_i)) \cap a_j(\mu)|$ for i, j which make sense.

Firstly, $|\operatorname{Stab}(\mathcal{M}_1)| = |\operatorname{Stab}(\mathcal{M}_2)| = 2$ and $|\operatorname{Stab}(\mathcal{M}_3)| = |\operatorname{Stab}(\mathcal{M}_4)| = 1$. Moreover $f_{(\mathcal{M}_1,q_1)}(b, \mu_i) = f_{(\mathcal{M}_2,q_2)}(b', \mu_i) = \binom{\mu_i}{3} = (\mu_i - 1)\frac{\mu_i^2 - 2\mu_i}{6}$, where b

(respectively b') is a black vertex of $\mathcal{M}_1$ (respectively $\mathcal{M}_2$). It gives us:

$$|\mathcal{D}_0^{-1}((\mathcal{M}_1, q_1)) \cap a_0(\mu)| = |\mathcal{D}_0^{-1}((\mathcal{M}_2, q_2)) \cap a_0(\mu)| = \frac{(|\mu|+1)\ell(\mu)!}{2\prod_{i\geq 2} m_i(\mu)!} \times \prod_{1\leq i\leq \ell(\mu)} (\mu_i - 1) \frac{\mathfrak{h}_2(\mu) - 2\mathfrak{h}_1(\mu)}{6} \tag{13}$$

Secondly, $f_{(\mathcal{M}_3,q_3)}(b, \mu_i) = \binom{\mu_i}{2} = (\mu_i - 1)\frac{\mu_i}{2}$, where b is a black vertex of $\mathcal{M}_3$ and $f_{(\mathcal{M}_4,q_4)}(b_1, \mu_i) = \binom{\mu_i}{2} = (\mu_i - 1)\frac{\mu_i}{2}$, $f_{(\mathcal{M}_4,q_4)}(b_2, \mu_j) = \binom{\mu_j - 1}{2} = (\mu_j - 1)\frac{\mu_j - 2}{2}$, where b_1 is a black vertex of $\mathcal{M}_4$ with $q_4(b_1) = 2$ and b_2 is a black vertex of $\mathcal{M}_4$ with $q_4(b_2) = 1$. Applying formula 10 to these maps we obtain:

$$|\mathcal{D}_0^{-1}((\mathcal{M}_3, q_3)) \cap a_0(\mu)| = \frac{(|\mu|+1)(\ell(\mu)+1)!}{2\prod_{i\geq 2} m_i(\mu)!} \times \prod_{1\leq i\leq \ell(\mu)} (\mu_i - 1) \frac{\mathfrak{h}_1(\mu)}{2}$$

and

$$|\mathcal{D}_0^{-1}((\mathcal{M}_3, q_3)) \cap a_0(\mu)| = \frac{(|\mu|+1)\ell(\mu)!}{2\prod_{i\geq 2} m_i(\mu)!} \times \prod_{1\leq i\leq \ell(\mu)} (\mu_i - 1) \frac{2\mathfrak{h}_{1^2}(\mu) - 2(\ell(\mu)-1)\mathfrak{h}_1(\mu)}{4} \tag{14}$$

and adding these two expressions we have

$$|\mathcal{D}_0^{-1}((\mathcal{M}_3, q_3)) \cap a_0(\mu)| + |\mathcal{D}_0^{-1}((\mathcal{M}_3, q_3)) \cap a_0(\mu)| = \frac{(|\mu|+1)\ell(\mu)!}{2\prod_{i\geq 2} m_i(\mu)!} \times \prod_{1\leq i\leq \ell(\mu)} (\mu_i - 1) \frac{\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{2}. \tag{15}$$

Finally, putting

$$a_0(\mu) = \left(\mathcal{D}_0^{-1}((\mathcal{M}_1, q_1)) \cap a_0(\mu)\right) \cup \cdots \cup \left(\mathcal{D}_0^{-1}((\mathcal{M}_4, q_4)) \cap a_0(\mu)\right),$$

we have that

$$|a_0(\mu)| = \frac{(|\mu|+1)\ell(\mu)!}{\prod_{i\geq 2} m_i(\mu)!} \times \prod_{1\leq i\leq \ell(\mu)} (\mu_i - 1) \frac{2\mathfrak{h}_2(\mu) + 3\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{12}, \tag{16}$$

and putting

$$\frac{k_0(k_0^2 - 1)}{24} = \frac{(|\mu|+1)\mathfrak{h}_1(\mu)(\mathfrak{h}_1(\mu)+2)}{24} = \frac{(|\mu|+1)(\mathfrak{h}_2(\mu) + 2\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu))}{24}$$

we have that

$$|a_0(\mu)| = \frac{\ell(\mu)! k_0}{4} \prod_{i\geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!} \left(\frac{k_0^2 - 1 + 3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{6} \right).$$

Now, we want to count $|a_1(\mu)|$. Let us take any pair $(\mathcal{M}, q) \in a_1(\mu)$ and let us remove all labels, all vertices of degree 1 and associated edges from $\mathcal{M}$. What we obtain is a bipartite cycle $\mathcal{M}'$ with $\ell(\mu)$ black and $\ell(\mu)$ white vertices. It is an easy consequence of the fact that $\chi(\mathcal{M}) = 1$ which means that $\mathcal{M}$ has the same number of edges and vertices. We can easily see that there are

$$\frac{\ell(\mu)!}{\prod_{i\geq 2} m_i(\mu)!}$$

ways of putting numbers $\mu_1, \dots, \mu_{\ell(\mu)}$ on black vertices of $\mathcal{M}'$, by the same argument as in the proof of Lemma . By Lemma 3.4, we can adjoin i white vertices to any black vertex from $\mathcal{M}'$ in $i-1$ ways. We can see $\mathcal{M}'$ as a pair partition $\{\{1, 2\ell(\mu)+1\}, \dots, \{j, 2\ell(\mu)+j\}, \dots, \{2\ell(\mu), 4\ell(\mu)\}\}$ and this description easily implies that $|\operatorname{Stab}(\mathcal{M}')| = 2\ell(\mu)$. Applying Lemma 3.3 with $n = |\mu| = k_1$ we obtain that:

$$|a_1(\mu)| = \frac{k_1 \ell(\mu)!}{2\ell(\mu)} \prod_{i\geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!} = \frac{k_1(\ell(\mu)-1)!}{2} \prod_{i\geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!},$$

which finishes the proof.

□

4. High degree terms of Kerov polynomials for $\deg_1$

The highest degree term of K_μ for $\deg_1$ is easy to compute. We know that K_μ has at most degree $|\mu| + \ell(\mu)$ (this has also been proved by M. Lassalle [Las09, Proposition 9.2 (ii)]). Moreover, its component of degree $|\mu| + \ell(\mu)$ does not depend on α. As this dominant term is known in the case $\alpha = 1$ (see for example [Śni06, Theorem 4.9]), one obtains the following result (which extends [Las09, Theorem 10.2]):

$$K_\mu = \prod_{i \leq \ell(\mu)} R_{\mu_i + 1} + \text{smaller degree terms.}$$

In this section, we give explicit formulas for more terms, confirming Lassalle's conjectural datas.

4.1. **Explicit formulas for smaller degrees.**

We introduce the notation $\tilde{R}_i = (i-1)R_i$ and $\tilde{R}_\mu = \prod_i \frac{\tilde{R}_i^{m_i(\mu)}}{m_i(\mu)!}$.

Theorem 4.1. *For $k \geq 1$, one has*

$$(17)\quad K_k = R_{k+1} + \gamma \frac{k}{2} \sum_{|\mu|=k} (\ell(\mu)-1)! \tilde{R}_\mu +$$
$$\sum_{|\mu|=k-1} \left(\frac{1}{4}\binom{k+1}{3} + \gamma^2 k \frac{3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{24} \right) \ell(\mu)! \tilde{R}_\mu +$$

terms of lower degree with respect to $\deg_1$.

Proof. Let us write:

$$K_k = \sum_{\mu} c_\mu R_\mu.$$

By [DF12, Proposition 3.7], c_μ is a polynomial in γ of degree at most $k+1-|\mu|$, hence c_μ is a polynomial in γ of degree at most 2 for $|\mu| \geq k-1$. Moreover, we know explicitly how to express K_k in terms of free cumulants for $\alpha = \frac{1}{2}, 1, 2$ (which corresponds to $\gamma = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ respectively). Indeed, the case $\alpha = 1$ has been solved separately in papers [GR07, Śni06], while the cases $\alpha = 1/2$ and 2 follows from the combinatorial interpretation given in [FŚ11b], hence we can compute c_μ for $|\mu| \geq k-1$.

For $|\mu| = k+1$, we know that $c_\mu = 0$ for all $\mu \neq k+1$ and $c_{k+1} = 1$.

For $|\mu| = k$ we know, that $c_\mu(0) = 0$ and $c_\mu(\frac{1}{\sqrt{2}}) = |a_1(\mu)|$, where $a_i(\mu)$ is described in Subsection 3.3

Finally, for $|\mu| = k-1$ we know that:

$$c_\mu(0) = \ell(\mu)! \binom{k+1}{3} \frac{1}{4} \prod_{i \geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!},$$

$$c_\mu(-\frac{1}{\sqrt{2}}) = c_\mu(0) + |a_0(\mu)|, \quad c_\mu(-\frac{1}{\sqrt{2}}) = \frac{c_\mu(\frac{1}{\sqrt{2}})}{4}.$$

Easy calculations show that $c_\mu = c_\mu(0) + \left(|a_0(\mu)| - c_\mu(\frac{1}{\sqrt{2}})\right)\gamma^2$. Applying formulas from Theorem 3.6:

$$|a_0(\mu)| = \frac{\ell(\mu)!k}{4} \prod_{i \geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!} \left(\frac{k^2 - 1 + 3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{6} \right),$$

$$|a_1(\mu)| = \frac{(\ell(\mu)-1)!k}{2} \prod_{i \geq 2} \frac{(i-1)^{m_i(\mu)}}{m_i(\mu)!},$$

finishes the proof. □

Remark. One can notice, that the explicit formulas for c_μ with $|\mu| \geq k$ were also proved by Lassalle [Las09, Theorems 10.2 and 10.3]. Moreover, our calculations for c_μ with $|\mu| = k-1$ are consistent with Lassalle's computer experiments [Las09, p. 2257], which provide a new evidence of Conjecture 11.2 from Lassalle's paper [Las09].

4.2. **Characters for more complicated partitions.** In the previous section we have focused on character values on a single cycle. Let us consider now the case of more complicated conjugacy classes, that is the functions $\mathrm{Ch}_{k_1,\dots,k_l}$.

In fact, it turns out to be more convenient to consider *cumulant of character values* $\kappa^{\mathrm{id}}(\mathrm{Ch}_{k_1}, \dots, \mathrm{Ch}_{k_l})$. Precise definition of these quantities can be found in [Śni06], for the purpose of this article it is enough to know that their relation to the characters $\mathrm{Ch}_{k_1,\dots,k_l}$ is analogous to the relation between classical cumulants of random variables and their moments. As pointed out in [Śni06], the above quantities with $\alpha = 1$, i.e. $\kappa^{\mathrm{id}}(\mathrm{Ch}_r^{(1)}, \mathrm{Ch}_s^{(1)}, \dots)$ are very useful in the study of fluctuations of random Young diagrams; in fact this is also the case for any parameter α. Let $\tilde{K}_\mu$ denote the polynomial, which express cumulant $\kappa^{\mathrm{id}}(\mathrm{Ch}_{\mu_1}, \dots, \mathrm{Ch}_{\mu_{\ell(\mu)}})$ in terms of free cumulants. One can found a following bound from above for degree of polynomial $\tilde{K}_\mu$:

Theorem 4.2. $\deg_1(\tilde{K}_\mu) \leq \max(|\mu| + 2 - \ell(\mu), |\mu| + \ell(\mu) - 3)$.

Proof. Let us write:

$$\tilde{K}_\mu = \sum_\rho c_{\mu,\rho} R_\rho.$$

We know that $c_{\mu,\rho}$ are polynomials in γ of degree at most $|\mu| + \ell(\mu) - |\rho|$, hence for $|\rho| \geq |\mu| + \ell(\mu) - 2$ they are polynomials in γ of degree at most 2. Moreover, it is a simple exercice on Euler characteristic, using the combinatorial interpretations given in [DFŚ10, FŚ11b], to show that $\deg_1(\tilde{K}_\mu^\alpha) = |\mu| + 2 - \ell(\mu)$ for $\alpha = \frac{1}{2}, 1, 2$. In other words, $c_{\mu,\rho}(\gamma) = 0$ for $\gamma = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ and for $|\rho| > |\mu| + 2 - \ell(\mu)$. This implies that $c_{\mu,\rho} \equiv 0$ for $|\rho| > \max(|\mu| + 2 - \ell(\mu), |\mu| + \ell(\mu) - 3)$ which finishes the proof. □

In our opinion this is not the optimal bound for the degree. We conjecture that the bound established for $\alpha = 1, 1/2, 2$ still holds for a general parameter α and we conclude our paper by conjecture:

Conjecture 4.3. $\deg_1(\tilde{K}_\mu) = |\mu| + 2 - \ell(\mu)$.

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