Kombinatoryka asymptotycznej teorii reprezentacji grup permutacji

Rozprawa doktorska

Autor
Maciej Dołęga

Promotor
prof. dr hab. Piotr Śniady
Promotor pomocniczy
dr Światosław Gal

Uniwersytet Wrocławski
Wrocław
Czerwiec 2013

Combinatorics of asymptotic representation theory of the symmetric groups

A Ph. D. dissertation presented
by
Maciej Dołęga
under the supervision of
prof. dr hab. Piotr Śniady
and secondary advisor
dr Światosław Gal

Uniwersytet Wrocławski
Wrocław
June 2013

Promotor: prof. dr hab. Piotr Śniady Maciej Dołęga

Kombinatoryka asymptotycznej teorii reprezentacji grup permutacji

STRESZCZENIE

W niniejszej pracy badam strukturę wielomianów Kerova, będących jednym z najważniejszych narzędzi w asymptotycznej teorii reprezentacji grup permutacji. Opis tej struktury polega między innymi na ukazaniu związku z mapami dwudzielnymi i badaniu kombinatoryki tych map. W pracy badana jest również struktura uogólnionych wielomianów Kerova pochodzących od charakterów Jacka. W szczególności przedstawione zostaną dowody hipotez Lassalle'a opisujących tę strukturę, jak również zastosowanie otrzymanych wyników do opisania typowych zachowań losowych diagramów Younga względem miary Jacka. Struktura rozprawy przedstawia się następująco.

Rozdział 1 jest krótkim wstępem do niniejszej pracy. Opisuje on w sposób ogólny dziedzinę badań niniejszej pracy.

Rozdział 2 jest wprowadzeniem szczegółowym do tematyki rozprawy. Opisuje on w szczególności historię omawianych przez nas zagadnień, oraz wprowadza wszelkie niezbędne definicje.

W Rozdziale 3 badamy asymptotykę charakterów grup permutacji na ustalonej klasie sprzężoności. Kerov dowiódł, że takie charaktery wyrazić można jako wielomian w wolnych kumulantach diagramu Younga (są to pewne funkcjonały opisujące kształt diagramu Younga). Pokażemy, że dla dowolnego genusu istnieje pewna uniwersalna funkcja symetryczna wyrażająca współczynniki jednorodnej części wielomianu Kerova stopnia zdeterminowanego przez genus. Istnienie takiej funkcji zostało postawione jako hipoteza przez Lassalle'a [Las08a]. Wyniki tego rozdziału zostały opublikowane w pracy [DŚ12].

W Rozdziale 4 rozważamy pewną deformację wielomianów Kerova związaną z wielomianami Jacka. Deformacja ta została wprowadzona ostatnio przez Lassalle'a [Las08b, Las09], który sformułował wiele opisujących ją hipotez. Sugerują one istnienie pewnej ukrytej struktury kombinatorycznej pozwalającej w pełni tę deformacją zrozumieć. Jednym z wyników tego rozdziału jest dowód pewnej części tych hipotez, mianowicie dowodzimy, że pewne wartości opisujące wprowadzone przez Lassalle'a wielomiany Kerova są wielomianami zmiennej Jacka α o ustalonym stopniu. Jako prosty wniosek dowodzimy kilku hipotez postawionych przez

Lassalle'a. Wynik ten ma kilka interesujących konsekwencji idących w różnych kierunkach. Po pierwsze, podamy nowy dowód tego, że współczynniki w rozwinięciu wielomianów Jacka w bazie jednomianowej lub potęgowej zależą wielomianowo od α. Po drugie, pewna część Hipotezy o sparowaniach Jacka postawionej przez Gouldena i Jacksona zostanie udowodniona. W końcu, jako główny wniosek udowodnimy Prawo Wielkich Liczb oraz Centralne Twierdzenie Graniczne dla miary Jacka, będącej jednoparametrową deformacją miary Plancherela. Ten wynik uogólnia słynne twierdzenie Vershika-Kerova o kształcie granicznym oraz Centralne Twierdzenie Graniczne Kerova i udowodniony jest przy pomocy wielowymiarowej wersji metody Steina. Fragmenty tego rozdziału pojawiają się w preprincie [DF12].

W ostatnim Rozdziale 5 badamy kombinatoryczną strukturę charakterów Jacka. Formułujemy hipotezę dotyczącą istnienia pewnych wag na zbiorze map (tzn. na zbiorze grafów narysowanych na powierzchniach), pozwalających wyrazić charaktery Jacka jako sumy z wagami pewnych funkcji określonych na zbiorze map, mających bardzo prostą strukturę. Podajemy przykład takich wag, który daje pozytywną odpowiedź na naszą hipotezę w kilku, acz nie wszystkich przypadkach. Waga przez nas opisana mierzy nieorientowalność danej mapy. Dowodzimy również, że nasza hipoteza implikuje pewne hipotezy dotyczące struktury charakterów Jacka, postawione przez Lassalle'a. Wyniki znajdujące się w tym rozdziale można znaleźć w preprincie [DFŚ13].

Thesis advisor: prof. dr hab. Piotr Śniady Maciej Dołęga

Combinatorics of asymptotic representation theory of the symmetric groups

Abstract

We investigate the structure of the Kerov characters polynomial, which is one of the main tools in the asymptotic representation theory of the symmetric groups. Description of this structure involves a connection to bipartite maps and investigation of their combinatorial structure. The structure of Jack deformation of Kerov character polynomials is also investigated in this dissertation. In particular, the proofs of conjectures of Lassalle describing this structure are presented. Moreover, several applications of these results, concerning behaviour of the random Young diagrams under Jack measure, are shown. The structure of this dissertation is the following.

Chapter 1 is a short introduction to the main subject of this dissertation.

In Chapter 2 we provide the whole background for the problems which appear in this thesis. In particular, in this chapter the reader can find the history of the problems as well as all necessary definitions.

In Chapter 3 we study asymptotics of characters of the symmetric groups on a fixed conjugacy class. It was proved by Kerov that such a character can be expressed as a polynomial in free cumulants of the Young diagram (certain functionals describing the shape of the Young diagram). We show that for each genus there exists a universal symmetric polynomial which gives the coefficients of the part of Kerov character polynomials with the prescribed homogeneous degree. The existence of such symmetric polynomials was conjectured by Lassalle [Las08a]. This result was published in the paper [DŚ12].

In Chapter 4 we consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that some quantities describing the structure of Kerov polynomials are polynomials with prescribed degree in the Jack parameter α. As a consequence we prove some of the conjectures of Lassalle. Our result has several interesting consequences in various

directions. Firstly, we give a new proof of the fact that the coefficients of Jack polynomials expanded in the monomial or power-sum basis depend polynomially in α. Secondly, a small part of Matching Jack conjecture from Goulden and Jackson is proved. Finaly, the last and main consequence is a proof of Law of Large Numbers and Central Limit Theorem for random Young diagrams under Jack measure, which is a one-parameter deformation of Plancherel measure. This result is a generalization of celebrated Vershik-Kerov's limit shape and Kerov's Central Limit Theorem and is proved using multivariate Stein's method. Some parts of this chapter can be found in the preprint [DF12].

In the last Chapter 5 we study combinatorial structure of Jack characters. We conjecture existence of a weight on maps (i.e., graphs drawn on surfaces), allowing to express Jack characters as weighted sums of some simple functions indexed by maps. We provide a candidate for this weight which gives a positive answer to our conjecture in some, but unfortunately not all, cases. This candidate weight measures non-orientability of a given map. We also show how our conjecture implies some of the conjectures of Lassalle. This chapter can be found as a preprint [DFŚ13].

Contents

Listing of figures

To my lovely wife and wonderful kids.
They are bottomless source of inspiration.

Acknowledgments

FIRST AND FOREMOST, I WOULD LIKE TO THANK MY SUPERVISOR, PROF. PIOTR ŚNIADY, WHO WAS EXTREMELY HELPFUL AND OFFERED INVALUABLE ASSISTANCE, SUPPORT AND GUIDANCE.

Secondly, I have learnt a great deal from numerous people over the years and gratefully acknowledge my debt to them, especially to: prof. Marek Bożejko, prof. Ewa Damek, dr hab. Krzysztof Dębicki, dr hab. Jan Dymara, dr Tomasz Elsner, dr Światosław Gal, prof. Paweł Głowacki, prof. Waldemar Hebisch, prof. Andrzej Hulanicki, prof. Tadeusz Januszkiewicz, prof. Grzegorz Karch, dr hab. Piotr Kowalski, dr Jan Kraszewski, prof. Paweł Krupski, prof. Ludomir Newelski, prof. Janusz Pawlikowski, dr hab. Tadeusz Pezda, prof. Grzegorz Plebanek, prof. Tomasz Rolski, prof. Ryszard Szwarc, prof. Jacek Świątkowski, dr Paweł Zawiślak, dr hab. Jacek Zienkiewicz - all from Wrocław University, and also to dr Mariusz Kraus - my high school teacher. Specifically, I would like to express my gratitude for Światosław Gal and Valentin Féray; Światosław for numerous discussions, comments and advices and Valentin for our joint work - it was a pleasure and great lesson for me.

Last but not least, very special thanks to my family for their patience and support, as well as to my parents and friends.

The research project was supported by National Science Centre funds for science - grant UMO-2011/03/N/ST1/00117.

1

Introduction

1.1 Representation theory

The idea of the representation theory of groups is to see the elements of a given abstract group as linear transformations. Hence, the representation theory is a powerful tool because it reduces problems in abstract algebra to problems in linear algebra, a subject that is well understood. Representation theory is also important in physics because, for example, it describes how the symmetry group of a physical system affects the solutions of equations describing that system.

Another motivation comes from the harmonic analysis. One of the most efficient tools for analysis and probability on the real line $\mathbb{R}$ is the Fourier transform. It is not so obvious how to define the Fourier transform if we replace the real line $\mathbb{R}$ by a finite group G. It turns out that representations of the group G are the right tool to define such an analogue which is linked with the *characters* of a given representation. For any homomorphism ρ of a given group G into the group of endomorphisms of a finite-dimensional linear space V (such homomorphism is called a *representation* of a group G), we define its *character* $\chi_\rho(g)$ by $\chi_\rho(g) := \mathrm{Tr}(\rho(g))$, where $g \in G$ and Tr denotes the trace. A significant part of the representation theory is devoted to investigation of such characters. At first sight it might appear that changing the focus from representations to characters might cause some loss of information because a matrix contains much more data than just its trace. Surprisingly, it is not the case as almost all natural questions of the representation theory can be reformulated in the language of characters.

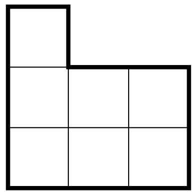


Figure 1.2.1: Young diagram $(3, 3, 1)$ drawn in the French convention.

1.2 ASYMPTOTIC REPRESENTATION THEORY

Asymptotic representation theory is an area of mathematics which investigates asymptotic properties of representations. Let us look on an example.

Problem 1.2.1. *Let $G_0 < G_1 < \dots$ be a sequence of finite groups and $(\rho_0, \rho_1, \dots)$ be a sequence of their representations. What can we say about ρ_n, when $n \to \infty$?*

One of the most basic and important finite groups is the symmetric group $\mathfrak{S}_n$. Representation theory of the symmetric group $\mathfrak{S}_n$ is quite old. One of the most fundamental facts in this area is that irreducible representations of $\mathfrak{S}_n$ are indexed by very nice geometric objects, namely Young diagrams of size n. Young diagram of size n represents an integer partition of n, i.e., a weakly decreasing sequence of nonnegative integers summing up to n (see Figure 1.2.1). Even though for almost any question of the representation theory of the symmetric groups the answer is known, usually this answer is given by a combinatorial algorithm (for example, Murnaghan-Nakayama rule or Littlewood-Richardson rule) involving manipulations with boxes of a Young diagram. As n, the number of boxes, tends to infinity, such combinatorial algorithms become very cumbersome and it is not easy to extract from them some reasonable asymptotic answers. For this reason, in order to study asymptotic problems for $n \to \infty$, one has to look for new, alternative approaches, which would less depend on the details of boxes of a given Young diagram, but rather on its 'global' features.

Recently, new tools have been used to study this kind of asymptotic problems. Instead of studying the characters $\chi_\lambda(\mu)$, where μ is a permutation and λ is Young diagram defining an irreducible representation, we would rather study quantities $\mathrm{Ch}_\mu(\lambda)$ called *normalized characters*. These quantities defined by Ivanov and Kerov [IK99] are just characters normalized in some special way. The difference seems very subtle, but actually, is really significant, because typically characters are considered as functions of μ, and λ is viewed as a parameter. Contrary to that, one can change the perspective and fix a favorite group element μ and treat Ch_μ as a function defined on the set $\mathbb{Y}$ of Young diagrams. There are plenty of advantages of this treatment from the asymptotic point of view. Let us give some examples.

First of all, the normalized characters can be expressed in terms of some very nice functionals of the Young diagram λ. These functionals are easy to compute from the shape of a given Young diagram. In particular, with proper normalization, they can be quite effectively estimated for very large Young diagrams. Moreover, normalized characters span some algebra $\Lambda_\star$ of functions on the set of Young diagrams. This algebra is canonically isomorphic to the *algebra of shifted symmetric functions* — an object which is very rich in structure and which was intensively studied in the context of the symmetric functions theory. The concept of using normalized characters and the algebra of shifted symmetric functions turned out to be really fruitful. It allowed to answer several big questions of the asymptotic representation theory of the symmetric groups such as describing fluctuations around the limit shape of the random Young diagrams [Ker93a, IO02], or describing the asymptotic behaviour of the characters when the corresponding, rescaled, Young diagrams converge to some prescribed shape [Bia98].

To conclude this subsection, let me give some (among many others) motivations for studying asymptotic representation theory:

- Representation theory may be used as a non-commutative analogue of the Fourier transform for analysis on groups, in particular to describe a random walk on a group. One can show that the speed at which the distribution of a random walk converges towards the uniform distribution can be estimated thanks to the group characters. These kinds of questions were considered by Diaconis and Shahshahani [DS81] for symmetric groups $\mathfrak{S}_n$ in the limit $n \to \infty$. Answers for these questions require non-trivial asymptotic estimates on the characters [Roi96].

- In the recent years surprising connections between the representation theory, random matrix theory and algebraic geometry were found. The importance which the mathematical community attributes to this discovery can be seen in the fact that the Fields medals in 2006 were granted (among others) to Andrei Okounkov and Terence Tao who were studying such connections. Speaking very briefly, the statistical properties of group representations coincide with their counterparts in the random matrix theory [Oko00]; many aspects of this coincidence remain still unclear.

- Young diagrams appearing in the representation theory of symmetric groups are simple discrete objects which can be easily geometrically rescaled and hence are a perfect model in the statistical mechanics to describe the thermodynamics of continuous media [KSW96a, KSW96b].

1.3 Generalizations

In a seminal paper [Jac71], Jack introduced a family of symmetric polynomials — which are now known as *Jack polynomials* $J_\mu^{(\alpha)}$ — indexed by an additional deformation parameter α.

From the contemporary viewpoint probably the main motivation for studying Jack polynomials comes from the fact that they are a special case of the celebrated *Macdonald polynomials* which *"have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics"* [GR05]. Indeed, some surprising features of Jack polynomials [Sta89] have led in the past to the discovery of Macdonald polynomials [Mac95] and Jack polynomials have been regarded as a relatively easy case [LV95] which later allowed understanding of the more difficult case of Macdonald polynomials [LV97]. A brief overview of Macdonald polynomials (and their relationship to Jack polynomials) is given in [GR05]. Jack polynomials are also interesting on their own, for instance in the context of Selberg integrals [Kad97] and in theoretical physics [FJMM02, BH08].

For some special choices of the deformation parameter α, Jack polynomials coincide (up to some simple normalization constants) with some very established families of symmetric polynomials. In particular, the case $\alpha = 1$ corresponds to *Schur polynomials*, $\alpha = 2$ corresponds to *zonal polynomials* and $\alpha = \frac{1}{2}$ corresponds to *symplectic zonal polynomials*; see [Mac95, Chapter 1 and Chapter 7] for more information about these functions. For these special values of the deformation parameter, Jack polynomials are particularly nice because they have some additional structures and features (usually related to algebra and representation theory) and for this reason they are much better understood.

Lassalle [Las08b, Las09] initiated investigation of a kind of *dual approach* to Jack polynomials. Roughly speaking, it is the investigation (as a function of λ, with μ being fixed) of the coefficient standing at $p_{\mu,1,1,\dots}$ in the expansion of Jack symmetric polynomial $J_\lambda^{(\alpha)}$ in the basis of *power-sum symmetric functions*. The motivation for studying such quantities comes from the observation that in the important special case of Schur polynomials ($\alpha = 1$) one recovers in this way the normalized characters $\mathrm{Ch}_\mu^{(1)} = \mathrm{Ch}_\mu$ which already proved to have a rich and fascinating structure. The cases of zonal polynomials and symplectic polynomials give rise to some new quantities called *zonal characters* $\mathrm{Ch}_\mu^{(2)}$ and *symplectic zonal characters* $\mathrm{Ch}_\mu^{(1/2)}$ respectively [FŚ11b]. For general Jack polynomials we obtain general *Jack characters* $\mathrm{Ch}_\mu^{(\alpha)}$.

Another motivation for studying Jack characters $\mathrm{Ch}_\mu^{(\alpha)}$ comes from the observation that they form a linear basis of the algebra $\Lambda_\star^{(\alpha)}$ of α-*shifted symmetric functions* (which is a simple deformation of the algebra of shifted symmetric functions $\Lambda_\star$ mentioned above). This fact is far from being trivial and was established by Lassalle [Las08b, Proposition 2].

Last, but not least, there is a natural way, using Jack polynomials, to define a probability measure $\mathbb{P}_n^{(\alpha)}$ on the set of Young diagrams of size n, called *Jack measure*. For $\alpha = 1$ this measure coincides with the Plancherel measure for the symmetric group, which was in the center of attention of representation theorists in the last fourty years [KV77, LS77, Ker93a, Oko00, BOO00], hence general Jack measure can be treated as a continuous deformation of the Plancherel measure. This deformation appeared recently in several research papers [Ful04, Mat08, Ols10, Mat10] and it was found that, surprisingly, it has many connections to different fields of mathematics such as algebraic geometry or random matrix theory. It is presented as an important area of research in Okounkov's survey on random partitions [Oko03, Section 3.3].

1.4 Outline of this thesis

In Chapter 2 we provide the whole background for the problems which appear in this thesis. In particular, in this chapter the reader can find the history of the problems as well as all necessary definitions.

In Chapter 3 we study asymptotics of characters of the symmetric groups on a fixed conjugacy class. It was proved by Kerov that such a character can be expressed as a polynomial in free cumulants of the Young diagram (certain functionals describing the shape of the Young diagram). We show that for each genus there exists a universal symmetric polynomial which gives the coefficients of the part of Kerov character polynomials with the prescribed homogeneous degree. The existence of such symmetric polynomials was conjectured by Lassalle [Las08a]. This result was published in the paper [DŚ12].

In Chapter 4 we consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that some quantities describing the structure of Kerov polynomials are polynomials with prescribed degree in the Jack parameter α. As a consequence we prove some of the conjectures of Lassalle. Our result has several interesting consequences in various directions. Firstly, we give a new proof of the fact that the coefficients of Jack polynomials expanded in the monomial or power-sum basis depend polynomially in α. Secondly, a small part of Matching Jack conjecture from Goulden and Jackson is proved. Finaly, the last and main consequence is a proof of Law of Large Numbers and Central Limit Theorem for random Young diagrams under Jack measure, which is a one-parameter deformation of Plancherel measure. This result is a generalization of celebrated Vershik-Kerov's limit shape and Kerov's Central Limit Theorem and is proved using multivariate Stein's method. Some parts of this chapter can be found in the preprint [DF12].

In the last Chapter 5 we study combinatorial structure of Jack characters. We conjecture existence of a weight on maps (i.e., graphs drawn on surfaces), allowing to express Jack characters as weighted sums of some simple functions indexed by maps. We provide a candidate for this weight which gives a positive answer to our conjecture in some, but unfortunately not all, cases. This candidate weight measures non-orientability of a given map. We also show how our conjecture implies some of the conjectures of Lassalle. This chapter can be found as a preprint [DFŚ13].

2

Preliminaries

2.1 Young diagrams

2.1.1 Partitions and Young diagrams

A *partition* $\pi = (\pi_1, \dots, \pi_l)$ is defined as a weakly decreasing finite sequence of positive integers. If $\pi_1 + \cdots + \pi_l = n$ we also say that π *is a partition of* n and denote it by $\pi \vdash n$. We will use the following notations: $|\pi| := \pi_1 + \cdots + \pi_l$; furthermore $\ell(\pi) := l$ denotes the *number of parts of* π and

$$m_i(\pi) := \big|\{k : \pi_k = i\}\big|$$

denotes the *multiplicity* of $i \geq 1$ in the partition π. When dealing with partitions we will use the shorthand notation

$$k^l := (\underbrace{k, \dots, k}_{l \text{ times}}).$$

Any partition can be alternatively viewed as a *Young diagram*.

2.1.2 Russian and French convention

We will use two conventions for drawing Young diagrams: the *French* one in the $0xy$ coordinate system and the *Russian* one in the $0zt$ coordinate system (presented in Figure 2.1.1). Notice that the graphs in the Russian convention are created from the graphs in the French convention by rotating counterclockwise by $\frac{\pi}{4}$ and by scaling by a factor $\sqrt{2}$. Alternatively, this can be viewed

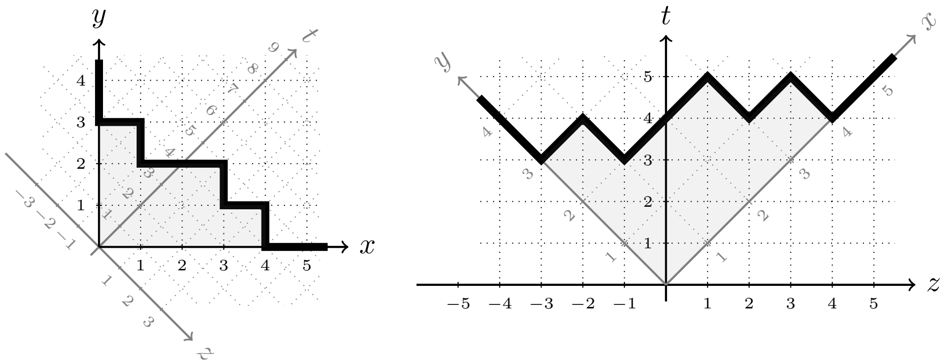


Figure 2.1.1: Young diagram $(4, 3, 1)$ shown in the French and Russian conventions. The solid line represents the *profile* of the Young diagram. The coordinates system $0zt$ corresponding to the Russian convention and the coordinate system $0xy$ corresponding to the French convention are shown.

as choice of two coordinate systems on the plane: $0xy$, corresponding to the French convention, and $0zt$, corresponding to the Russian convention. The coordinates in both systems are related to each other by

$$\begin{cases} z = x - y, \\ t = x + y, \end{cases} \qquad \begin{cases} x = \dfrac{z+t}{2}, \\ y = \dfrac{t-z}{2}. \end{cases}$$

For a point on the plane we will define its *content* as its z-coordinate.

In the French coordinates will use the plane $\mathbb{R}^2$ equipped with the standard Lebesgue measure, i.e., the area of a unit square with vertices (x, y) such that $x, y \in \{0, 1\}$ is equal to 1. This measure in the Russian coordinates corresponds to a the Lebesgue measure on $\mathbb{R}^2$ multiplied by the factor 2, i.e. , the area of a unit square with vertices (z, t) such that $z, t \in \{0, 1\}$ is equal to 2.

2.1.3 Stanley coordinates

If $\mathbf{p} = (p_1, \ldots, p_\ell)$, $\mathbf{q} = (q_1, \ldots, q_\ell)$ are sequences of non-negative integers such that $q_1 \geq q_2 \geq \cdots \geq q_\ell$, we consider the *multirectangular* Young diagram

$$\mathbf{p} \times \mathbf{q} = (\underbrace{q_1, \ldots, q_1}_{p_1 \text{ times}}, \ldots, \underbrace{q_\ell, \ldots, q_\ell}_{p_\ell \text{ times}}).$$

This concept is illustrated in Figure 2.1.2. Quantities $\mathbf{p}$ and $\mathbf{q}$ are referred to as *Stanley coordinates* of multirectangular Young diagram $\mathbf{p} \times \mathbf{q}$ [Sta06].

A special case of multirectangular Young diagrams is a *rectangular* Young diagram which

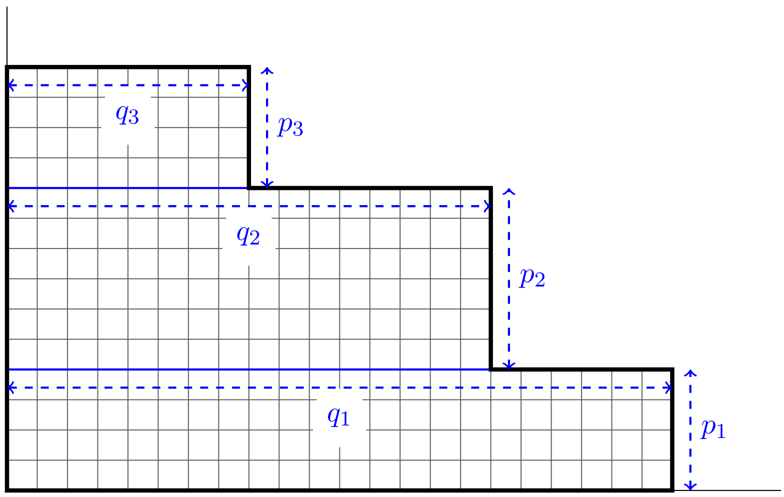


Figure 2.1.2: Multirectangular Young diagram $\mathbf{p} \times \mathbf{q}$.

is simply

$$p \times q = \underbrace{q, \dots, q}_{p \text{ times}}.$$

2.1.4 Generalized Young diagrams and anisotropic Young diagrams

Any Young diagram drawn in the French convention can be identified with its graph which is equal to the set $\{(x, y) : 0 \le x,\ 0 \le y \le f(x)\}$ for a suitably chosen function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the French convention)* as the set of bounded, non-increasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with a compact support; in this way any Young diagram can be regarded as a generalized Young diagram.

We can identify a Young diagram drawn in the Russian convention with its *profile*, see Figure 2.1.1. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the Russian convention)* as the set of functions $\omega : \mathbb{R} \to \mathbb{R}_+$ which fulfill the following two conditions:

- ω is a Lipschitz function with constant 1, i.e., $|\omega(z_1) - \omega(z_2)| \le |z_1 - z_2|$,
- $\omega(z) = |z|$ if $|z|$ is large enough.

We will define the *support* of ω (in the Russian convention) in a natural way:

$$\operatorname{supp}(\omega) = \overline{\{z \in \mathbb{R} : \omega(z) \neq |z|\}}.$$

At the first sight it might seem that we have defined the set $\mathbb{Y}$ of generalized Young diagrams in two different ways, but we prefer to think that these two definitions are just two conventions

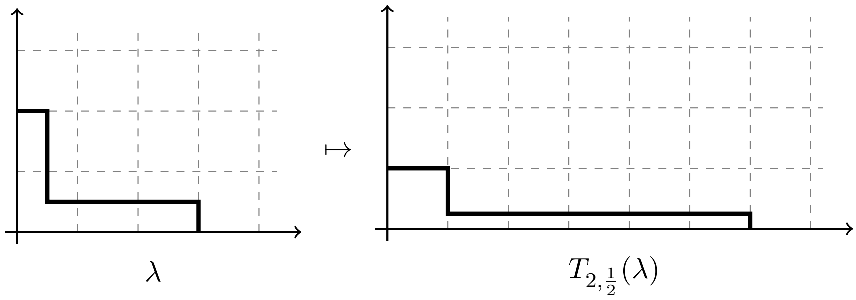


Figure 2.1.3: Example of a Young diagram λ on the left and a stretched Young diagram $T_{2,\frac{1}{2}}(\lambda)$ on the right.

(French and Russian) for drawing the same object. This will not lead to confusions since it will be always clear from the context which of the two conventions is being used.

We will be in particular interested in the following generalized Young diagrams. Let λ be a (generalized) Young diagram and s and t two positive real numbers. We denote $T_{s,t}(\lambda)$ the generalized Young diagram obtained by stretching λ horizontally by a factor s and vertically by a factor t in French convention (see Figure 2.1.3). These *anisotropic* Young diagrams have been introduced by Kerov in [Ker00b].

In the case $s = t$, we denote by $D_s(\lambda) = T_{s,s}(\lambda)$ the diagram obtained from λ by applying a homothetic transformation of ratio s and we will call it *dilated Young diagram*. In the case $s = t^{-1} = \sqrt{\alpha}$ for some $\alpha \in \mathbb{R}_+$ we denote by $\lambda^{(\alpha)} = T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)$ the diagram obtained from λ by stretching it horizontally by a factor $\sqrt{\alpha}$ and vertically by a factor $\sqrt{\alpha}^{-1}$. We call it *α-anisotropic Young diagram.*

2.2 Symmetric functions

In this section, we provide an introduction to the theory of symmetric functions. For more details we refer to [Mac95].

2.2.1 Definition

Let $x_1, \dots, x_n$ be indeterminates. The group $\mathfrak{S}_n$ acts on the ring $\mathbb{Z}[x_1, \dots, x_n]$ by permuting the indeterminates. Let

$$\mathrm{Sym}_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

be a ring of polynomials invariant under this action. We call such polynomials *symmetric*. For any $f \in \mathrm{Sym}_n$, we can expand:

$$f = \sum_{r \geq 0} f^{(r)},$$

where $f^{(r)}$ is the homogenous component of f of degree r. Since $f^{(r)}$ is symmetric, it means that Sym_n is a ring with gradation:

$$\mathrm{Sym}_n = \bigoplus_{r\geq 0} \mathrm{Sym}_n^r,$$

where Sym_n^r is the additive group of homogenous symmetric polynomials of degree r in $x_1, \dots, x_n$ (by convention, 0 is homogenous of every degree).

For any $n \in \mathbb{N}$ we have a surjective homomorphism of graded rings:

$$\mathrm{Sym}_{n+1} \to \mathrm{Sym}_n$$

defined by setting $x_{n+1} = 0$. Let

$$\mathrm{Sym}^r = \varprojlim_n \mathrm{Sym}_n^r$$

for any $r \geq 0$, and

$$\mathrm{Sym} = \bigoplus_{r\geq 0} \mathrm{Sym}^r .$$

By the definition of inverse (or projective) limits, an element of Sym^r is a sequence $(f_n)_{n\geq 0}$, where $f_n \in \mathrm{Sym}_n^r$ for each n and f_n is obtained from f_{n+1} by setting $x_{n+1} = 0$. We can, therefore, regard f_n as the partial sums of an infinite series f of monomials of degree r in infinitely many indeterminates $x_1, x_2, \dots$. Thus the elements of Sym are no longer polynomials, and traditionally called *symmetric functions*. For any commutative ring R we shall write

$$\mathrm{Sym}_R = \mathrm{Sym} \otimes_{\mathbb{Z}} R, \quad \mathrm{Sym}_{n,R} = \mathrm{Sym}_n \otimes_{\mathbb{Z}} R$$

for the ring of symmetric functions (respectively symmetric polynomials in n indeterminates) with coefficients in R. In the following by the symmetric functions ring we mean the ring of symmetric functions with real coefficients, i. e., $\mathrm{Sym} := \mathrm{Sym}_{\mathbb{R}}$.

2.2.2 Prominent examples of linear bases of symmetric functions

Any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ defines a monomial

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots .$$

Monomial symmetric function $\mathfrak{h}_\lambda$ is the sum of all distinct monomials that can be obtained from x^λ by permutations of indeterminates. For example

$$\mathfrak{h}_{(3,2)} = \sum_{i\neq j} x_i^3 x_j^2, \quad \mathfrak{h}_{(2,2)} = \sum_{i<j} x_i^2 x_j^2.$$

As λ runs through the partitions with at most n parts, $(\mathfrak{h}_\lambda(x_1, \dots, x_n))$ form a linear basis of Sym_n. As λ runs through all partitions, $(\mathfrak{h}_\lambda)$ form a linear basis of Sym.

For each $r \geq 1$ we define the rth *complete symmetric function* h_r:

$$h_r = \sum_{\lambda \vdash r} \mathfrak{h}_\lambda.$$

The family $(h_1, h_2, \dots)$ forms an algebraic basis of Sym and as λ runs through all partitions, complete symmetric functions h_λ defined by

$$h_\lambda = \prod_i h_{\lambda_i}$$

form a linear basis of Sym.

For each $r \geq 1$ we define the rth *elementary symmetric function* e_r:

$$e_r = \mathfrak{h}_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

The family $(e_1, e_2, \dots)$ forms an algebraic basis of Sym and as λ runs through all partitions, complete symmetric functions e_λ defined by

$$e_\lambda = \prod_i e_{\lambda_i}$$

form a linear basis of Sym.

For each $r \geq 1$ we define the rth *power-sum symmetric function* p_r:

$$p_r = \sum_i x_i^r.$$

The family $(p_1, p_2, \dots)$ forms an algebraic basis of Sym and as λ runs through all partitions, power-sum symmetric functions p_λ defined by

$$p_\lambda = \prod_i p_{\lambda_i}$$

form a linear basis of Sym.

2.3 Normalized characters and Jack characters

2.3.1 Normalized characters

Let $\lambda \vdash n$ be a Young diagram and ρ^λ be the corresponding irreducible representation of $\mathfrak{S}_n$. The *irreducible character* $\chi^\lambda(\pi) = \operatorname{Tr} \rho^\lambda(\pi)$ of the symmetric group is usually considered as a function of the permutation π, with the Young diagram λ fixed. It was a brilliant observation of Kerov and Olshanski [KO94] that for several problems in asymptotic representation theory it is convenient to do the opposite: keep the permutation π fixed and let the Young diagram λ vary. It should be stressed that the Young diagram λ is arbitrary, in particular there are no restrictions on the number of boxes of λ. In this way it is possible to study the structure of the series of the symmetric groups $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ and their representations in a uniform way. This concept is sometimes referred to as *dual combinatorics* of the characters of the symmetric groups.

Since characters are conjugacy invariant, we can do the following: any partition μ such that $|\mu| = n$ can be viewed as a conjugacy class in $\mathfrak{S}_n$. Let $\pi_\mu \in \mathfrak{S}_n$ be any permutation from this conjugacy class; we will denote by $\operatorname{Tr}\rho^\lambda(\mu) := \operatorname{Tr}\rho^\lambda(\pi_\mu)$ the value of the corresponding irreducible character. If $m \leq n$, any permutation $\pi \in \mathfrak{S}_m$ can be also viewed as an element of $\mathfrak{S}_n$, we just have to add $n-m$ additional fixpoints to π; for this reason

$$\operatorname{Tr}\rho^\lambda(\mu) := \operatorname{Tr}\rho^\lambda\left(\mu\, 1^{|\lambda|-|\mu|}\right)$$

makes sense also when $|\mu| \leq |\lambda|$.

Normalized characters of the symmetric group were defined by Kerov and Olshanski [KO94] as follows:

$$\operatorname{Ch}_\mu(\lambda) = \underbrace{n(n-1)\cdots(n-|\mu|+1)}_{|\mu| \text{ factors}} \frac{\operatorname{Tr}\rho^\lambda(\mu)}{\text{dimension of } \rho^\lambda}, \tag{2.1}$$

and, by convention, the right hand side is equal to 0, when $|\lambda| < |\mu|$. As we said before, the novelty of the idea was to view the character as a function $\lambda \mapsto \operatorname{Ch}_\mu(\lambda)$ on the set of Young diagrams (of any size) and to keep the conjugacy class fixed. The normalization constants in (2.1) were chosen in such a way that the normalized characters $\lambda \mapsto \operatorname{Ch}_\mu(\lambda)$ form a linear basis (when μ runs over the set of all partitions) of the algebra $\Lambda^\star$ of *polynomial functions on the set of Young diagrams* (see Section 2.4.1 for more details), which is very rich in structure (this property is, for example, the key point in a recent approach to study asymptotics of random Young diagrams under Plancherel measure [IO02]). In addition, recently a combinatorial description of the quantity (2.1) has been given [Sta06, Fér10b], which is particularly suitable for study of asymptotics of character values [FŚ11a].

Using Frobenius' formula for characters of the symmetric groups [Fro00], definition (2.1) can be rephrased using Schur functions (see [Las08b] for details). We expand the Schur polynomial s_λ in the base of the power-sum symmetric functions (p_ρ) as follows:

$$\frac{n!\, s_\lambda}{\dim(\lambda)} = \sum_{\substack{\rho: \\ |\rho|=|\lambda|}} \theta_\rho^{(1)}(\lambda)\, p_\rho \tag{2.2}$$

for some numbers $\theta^{(1)}_\rho(\lambda)$. Then

$$\mathrm{Ch}_\mu(\lambda) = \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu \, \theta^{(1)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda), \tag{2.3}$$

where

$$z_\mu = \mu_1\mu_2 \cdots \; m_1(\mu)! \, m_2(\mu)! \cdots$$

and $m_i(\mu)$ denotes the multiplicity of i in the partition μ.

2.3.2 Jack characters

In this section we will define analogues of the quantity $\mathrm{Ch}_\mu(\lambda)$ via *Jack polynomials*.

In a seminal paper [Jac71], Jack introduced a family of symmetric polynomials — which are now known as *Jack polynomials* $J^{(\alpha)}_\mu$ — indexed by an additional deformation parameter α. From the contemporary viewpoint probably the main motivation for studying Jack polynomials comes from the fact that they are a special case of the celebrated *Macdonald polynomials* which *"have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics"* [GR05]. Indeed, some surprising features of Jack polynomials [Sta89] have led in the past to the discovery of Macdonald polynomials [Mac95] and Jack polynomials have been regarded as a relatively easy case [LV95] which later allowed understanding of the more difficult case of Macdonald polynomials [LV97]. A brief overview of Macdonald polynomials (and their relationship to Jack polynomials) is given in [GR05]. Jack polynomials are also interesting on their own, for instance in the context of Selberg integrals [Kad97] and in theoretical physics [FJMM02, BH08].

As there are several normalizations of Jack polynomials, we have to fix one we use. In our context, the best is to use the functions denoted by J in the book of Macdonald [Mac95, VI, (10.22)]. With this normalization, one has

$$J^{(1)}_\lambda = \frac{n! \, s_\lambda}{\dim(\lambda)},$$
$$J^{(2)}_\lambda = Z_\lambda,$$

where s_λ are Schur polynomials and Z_λ are zonal polynomials.

If in (2.2) we replace the left-hand side by the Jack polynomial:

$$J^{(\alpha)}_\lambda = \sum_{\substack{\rho: \\ |\rho|=|\lambda|}} \theta^{(\alpha)}_\rho(\lambda) \, p_\rho \tag{2.4}$$

then in analogy to (2.3) we can define

$$\mathrm{Ch}^{(\alpha)}_\mu(\lambda) = \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu \, \theta^{(\alpha)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda).$$

These quantities are called *Jack characters*. Notice that for $\alpha = 1$, we recover the usual normalized character values of the symmetric groups, i. e., $\mathrm{Ch}^{(1)}_\mu(\lambda) = \mathrm{Ch}_\mu(\lambda)$.

Jack characters have been first considered by Lassalle in [Las08b]. They have also been studied in papers [Las09] and [FŚ11b] (the latter deals with the case $\alpha = 2$ which corresponds to zonal polynomials). Note that the normalization used here is different that the one of these papers. The reason of this new choice of normalization will be clear later.

2.3.3 Relationship to Lassalle's normalization

For reader's convenience, since we are going to refer to Lassalle's results quite often, we provide below the relationship between quantities used by Lassalle (in boldface) and the ones used by us:

$$\boldsymbol{\vartheta}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu\cup 1^{n-|\mu|}}}(\boldsymbol{\alpha}) = \binom{|\lambda| - |\mu| + m_1(\mu)}{m_1(\mu)} z_\mu\, \theta^{\lambda}_{\mu\cup 1^{|\lambda|-|\mu|}}(\alpha),$$
$$\mathrm{Ch}^{(\alpha)}_\mu(\lambda) = \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \boldsymbol{\vartheta}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu\cup 1^{n-|\mu|}}}(\boldsymbol{\alpha}). \tag{2.5}$$

Our convention has the advantage of being compatible with the symmetry $(\alpha, \lambda) \leftrightarrow (\alpha^{-1}, \lambda')$, where λ' is the transpose diagram of λ [Mac95, Section 1.1]. Namely,

$$\mathrm{Ch}^{(\alpha)}_\mu(\lambda) = (-1)^{|\mu|-\ell(\mu)}\, \mathrm{Ch}^{(1/\alpha)}_\mu(\lambda'). \tag{2.6}$$

2.4 Polynomial functions on the set of Young diagrams

2.4.1 Description of the polynomial functions on the set of Young diagrams

The ring $\Lambda^\star$ of *polynomial functions on the set of Young diagrams* (briefly: the ring of *polynomial functions*) has been introduced by Kerov and Olshanski in order to study irreducible character values of symmetric groups [KO94].

The first characterization of $\Lambda^\star$ is the following: it is the ring of *shifted symmetric functions* in $\lambda_1, \lambda_2, \dots$. This ring was first considered by Knop and Sahi [KS96] in a more general context. By definition, a shifted symmetric function F is a collection of polynomials $F_h \in \mathbb{Q}[\lambda_1, \dots, \lambda_h]$ such that each F_h is symmetric in variables $\lambda_1 - 1, \lambda_2 - 2, \dots, \lambda_h - h$ and such that the compatibility relation

$$F_{h+1}(\lambda_1, \dots, \lambda_h, 0) = F_h(\lambda_1, \dots, \lambda_h)$$

holds true for all values of h. For any partition μ, the function $\mathrm{Ch}^{(1)}_\mu$ is a shifted symmetric function of λ. Moreover, the family of such functions forms a linear basis of $\Lambda^\star$. We refer to [KO94] or to the case $\alpha = 1$ of [Las08b, Proposition 2] for a proof of this fact.

Another equivalent description can be given using Kerov's interlacing coordinates of a Young diagram. Recall that the *content* of a box of a Young diagram is $j - i$, where j is its

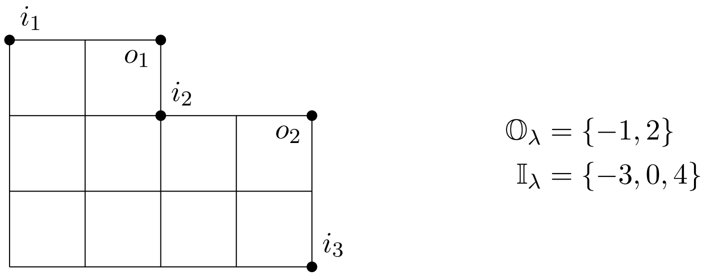


Figure 2.4.1: A Young diagram with its inner and outer corners (marked respectively with i and o).

column index and i its row index and, more generally, the content of a point of a plane is the difference of its x-coordinate and its y-coordinate. We denote by $\mathbb{I}_\lambda$ the sets of contents of the *inner corners* of λ, that is corners, at which a box could be added to λ to obtain a new diagram of size $|\lambda| + 1$. Similarly, the set $\mathbb{O}_\lambda$ is defined as the contents of the *outer corners*, that is corners at which a box can be removed from λ to obtain a new diagram of size $|\lambda| - 1$. An example is given on Figure 2.4.1 (we use the French convention to draw Young diagrams).

If k is a positive integer, one can consider the power-sum symmetric function p_k, evaluated on the difference of alphabets $\mathbb{I}_\lambda - \mathbb{O}_\lambda$. By definition, it is a function on Young diagrams given by:

$$\lambda \mapsto p_k(\mathbb{I}_\lambda - \mathbb{O}_\lambda) := \sum_{i \in \mathbb{I}_\lambda} i^k - \sum_{o \in \mathbb{O}_\lambda} o^k.$$

As any symmetric function can be written (uniquely) in terms of p_k, we can define $f(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$ for any symmetric function f as follows: if $f = \sum_\rho a_\rho p_{\rho_1} \cdots p_{\rho_\ell}$, then by definition

$$f(\mathbb{I}_\lambda - \mathbb{O}_\lambda) = \sum_\rho a_\rho p_{\rho_1}(\mathbb{I}_\lambda - \mathbb{O}_\lambda) \cdots p_{\rho_\ell}(\mathbb{I}_\lambda - \mathbb{O}_\lambda).$$

Ivanov and Olshanski [IO02, Corollary 2.8] have shown that the functions $(\lambda \mapsto p_k(\mathbb{I}_\lambda - \mathbb{O}_\lambda))_{k\geq 2}$ form an algebraic basis of $\Lambda^\star$ (for all diagrams λ, one has $p_1(\mathbb{I}_\lambda - \mathbb{O}_\lambda) = 0$). In other terms, $\Lambda^\star$ is the ring of symmetric functions evaluated in the difference of alphabets $\mathbb{I}_\lambda - \mathbb{O}_\lambda$ (see [Las03] for more on λ-ring notations).

Moreover, the algebra $\Lambda_\star$ turns out to be isomorphic to a subalgebra of the algebra of partial permutations of Ivanov and Kerov [IK99]. Therefore we can view the elements of the algebra $\Lambda_\star$ as partial permutations. Since the multiplication of polynomial functions on the set of Young diagrams corresponds to the convolution of central functions on partial permutations, we see that the algebra $\Lambda_\star$ turns out to be very closely related to the problems of computing connection coefficients and multiplication of conjugacy classes in the symmetric groups. It is remarkable that this subalgebra of the algebra of partial permutations (and hence the algebra $\Lambda_\star$) is isomorphic to a rather classic object: the algebra studied by Farahat and Higman in 1959 [FH59].

The above collection of alternative ways of viewing the algebra of polynomial functions on the set of Young diagrams probably is not complete, but it already shows the richness of this structure. Several problems from the asymptotic representation theory of symmetric groups turned out to be equivalent to questions concerning the algebra $\Lambda_\star$ and relating various ways of viewing it—in particular, finding relations between its various algebraic bases [Bia02, IO02, Bia03, Śni06a, Śni06b, DFŚ10].

2.4.2 Prominent examples of bases of the algebra $\Lambda_\star$

Transition measure, moments and free cumulants

Kerov [Ker93b] introduced the notion of *transition measure* of a Young diagram. This is a probability measure μ_λ on the real line $\mathbb{R}$ which is associated to λ and is defined by its Cauchy transform

$$G_{\mu_\lambda}(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{z-x} = \frac{\prod_{o\in\mathbb{O}_\lambda} z-o}{\prod_{i\in\mathbb{I}_\lambda} z-i}.$$

In particular, transition measure is supported on $\mathbb{I}_\lambda$. Besides, its moments, i e.,

$$M_k(\lambda) := \int_{\mathbb{R}} z^k d\mu_\lambda,$$

are equal to $h_k(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$, where h_k is the complete symmetric function of degree k. In particular, they are polynomial functions on the set of Young diagrams. The family $(M_k)_{k\geq 2}$ forms an algebraic basis of polynomial functions on the set of Young diagrams (M_1 is the null function) and is called simply a family of *moments*.

In Voiculescu's free probability it is very convenient to associate to a probability measure μ on $\mathbb{R}$, having all moments, a sequence of numbers $(R_k(\mu))_{k\geq 1}$ called *free cumulants* [Voi86, Spe94]. The free cumulants of the transition measure of Young diagrams appeared first in the work of Biane [Bia98] and play an important role in asymptotic representation theory. As explained by Lassalle [Las09, Section 5, $\alpha = 1$ case], they can be expressed as

$$R_k(\lambda) := R_k(\mu_\lambda) = e_k^\star(\mathbb{I}_\lambda - \mathbb{O}_\lambda) \tag{2.7}$$

for some homogeneous symmetric function $e_k^\star$ of degree k. Functions $e_k^\star$ form an algebraic basis of symmetric functions and, hence $(R_k)_{k\geq 2}$ is an algebraic basis of ring of polynomial functions on the set of Young diagrams (R_1 is the null function).

From the results of Biane [Bia98] one can show the very surprising fact that for any integer $k \geq 1$ and any Young diagram λ the values of the normalized character on dilations of λ on a cycle

$$\mathbb{N} \ni s \mapsto \mathrm{Ch}_{(k)}(D_s(\lambda))$$

are given by a polynomial function of degree (at most) $k+1$. Furthermore, the leading coefficient is equal to one of the free cumulants:

$$R_{k+1}(\lambda) = [s^{k+1}]\,\mathrm{Ch}_{(k)}(D_s(\lambda)) = \lim_{s\to\infty} \frac{\mathrm{Ch}_{(k)}(D_s(\lambda))}{s^{k+1}}. \tag{2.8}$$

This is a highly nontrivial and very interesting result: it shows that free cumulants (which are viewed as concrete, algorithmically computable quantities) describe the first-order asymptotics of characters.

One can treat equation (2.8) as a definition of free cumulants, from which it is clear that free cumulants should be interesting for investigations of the asymptotics of characters of the symmetric groups. Another reason why free cumulants are so useful in the asymptotic representation theory is that they are homogeneous with respect to dilations of the Young diagrams, namely

$$R_k(D_s(\lambda)) = s^k R_k(\lambda);$$

in other words the degree of the free cumulant R_k is equal to k. This property is an immediate consequence of (2.8) but it also follows from the equation (2.7).

Fundamental functionals of shape

Fundamental functionals of shape [DFŚ10] of a generalized Young diagram λ are defined by the following formula:

$$S_k(\lambda) = (k-1)\iint_{(x,y)\in\lambda} (x-y)^{k-2}\,dx\,dy = \frac{1}{2}(k-1)\iint_{(z,t)\in\lambda} z^{k-2}\,dz\,dt, \tag{2.9}$$

where the first integral is written in the French and the second in the Russian coordinates. Clearly, each functional S_k is a homogeneous function of degree k with respect to dilations of the Young diagram.

Fundamental functionals of shape $(S_2, S_3, \dots)$ form an algebraic basis of the algebra $\Lambda_\star$ and their advantage is that they are easily computable strictly from the definition. Moreover they have a significant meaning in studying combinatorial properties of the algebra $\Lambda_\star$. For more details, we refer to [DFŚ10].

2.5 α-polynomial functions

As it was explained by Biane [Bia98, Section 1.2], polynomial functions can be evaluated not only on Young diagrams, but also on generalized Young diagrams. Here, we explain how to evaluate any polynomial function on the class which is strictly contained in the class of generalized Young diagrams, but which contains all Young diagrams.

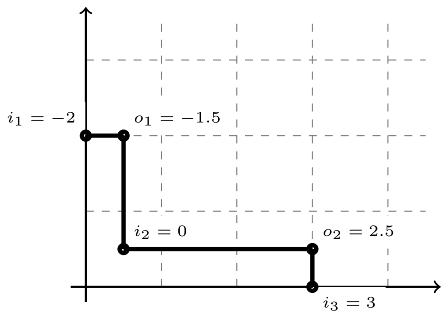


Figure 2.5.1: A generalized Young diagram L with the corresponding sets $\mathbb{O}_L = \{o_1, o_2\}$ and $\mathbb{I}_L = \{i_1, i_2, i_3\}$.

Let us consider a zigzag line L going from a point $(0, y)$ on the y-axis to a point $(x, 0)$ on the x-axis. We assume that every piece is either a horizontal segment from left to right or a vertical segment from top to bottom. Any Young diagram can be seen as such a zigzag line: just consider its border. Therefore, we call these zigzag lines *regular generalized Young diagrams*. The notions of inner and outer corners can be easily adapted to generalized Young diagrams, as well as the sets $\mathbb{I}_L$ and $\mathbb{O}_L$ of their contents. It is illustrated in Figure 2.5.1.

Any polynomial function F on the set of Young diagrams corresponds to the function

$$\lambda \mapsto f(\mathbb{I}_\lambda - \mathbb{O}_\lambda)$$

for some symmetric function f. Hence, F can be canonically extended to regular generalized Young diagrams by setting

$$F(L) = f(\mathbb{I}_L - \mathbb{O}_L).$$

We will be in particular interested in the α-anisotropic Young diagrams $\lambda^{(\alpha)}$.

Definition 2.5.1. We say that F is an *α-polynomial function* on the set of (generalized) Young diagrams if

$$\lambda \mapsto F(\lambda^{(\alpha^{-1})})$$

is a polynomial function. The set of α-polynomial functions is an algebra which will be denoted by $\Lambda^\star_{(\alpha)}$.

Using the characterization *via* shifted symmetric function, this means that the polynomial $F(\alpha^{-1}\lambda_1, \cdots, \alpha^{-1}\lambda_h)$ is symmetric in $\lambda_1 - 1$, …, $\lambda_h - h$. Equivalently (by a change of variables), F is symmetric in $\alpha\lambda_1 - 1, \dots, \alpha\lambda_h - h$ or in

$$\lambda_1 - \frac{1}{\alpha}, \dots, \lambda_h - \frac{h}{\alpha}.$$

The last characterization is the definition of what is usually called an α-shifted symmetric function [OO97, Las08b].

In particular, the α-anisotropic moments, free cumulants and fundamental functionals of shape defined by

$$M_k^{(\alpha)}(\lambda) = M_k\left(\lambda^{(\alpha)}\right),$$
$$R_k^{(\alpha)}(\lambda) = R_k\left(\lambda^{(\alpha)}\right),$$
$$S_k^{(\alpha)}(\lambda) = S_k\left(\lambda^{(\alpha)}\right),$$

are α-polynomial. Moreover, the families $(M_k^{(\alpha)})_{k\geq 2}$, $(R_k^{(\alpha)})_{k\geq 2}$ and $(S_k^{(\alpha)})_{k\geq 2}$ are algebraic bases of the algebra $\Lambda^\star_{(\alpha)}$ of α-polynomial functions.

Lassalle has shown that Jack characters $\mathrm{Ch}_\mu^{(\alpha)}$ form a linear basis of the algebra of α-polynomial functions (see Section 3 and in particular Proposition 2 of [Las08b]). In particular, they are α-polynomial functions and can be expressed in terms of the algebraic bases above.

Definition-Proposition 2.5.2. *Let μ be a partition and $\alpha > 0$ a fixed real number. There exist polynomials $L_\mu^{(\alpha)}$ and $K_\mu^{(\alpha)}$ such that, for every λ,*

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = L_\mu^{(\alpha)}\left(M_2^{(\alpha)}(\lambda), M_3^{(\alpha)}(\lambda), \cdots\right);$$
$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = K_\mu^{(\alpha)}\left(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \cdots\right).$$

The polynomials $K_\mu^{(\alpha)}$ have been introduced by Kerov in the case $\alpha = 1$ [Ker00a, Bia03] and by Lassalle in the general case [Las09] and they are called *Kerov polynomials*.

2.6 Kerov polynomials for normalized characters and for Jack characters

For $\alpha = 1$, we explained that free cumulants provide asymptotic approximation for the characters of the symmetric groups. The non-trivial fact that free cumulants can be used for exact formulas also was announced by Kerov during a talk in Institut Henri Poincaré in January 2000 [Ker00a]. Moreover, Kerov formulated during his talk the following conjecture:

Conjecture 2.6.1 ([Ker00a])**.** *The coefficients of Kerov polynomial $K_{(k)}$ for $k \geq 1$ are non-negative integers.*

Biane [Bia03] stated a very interesting conjecture that the underlying reason for positivity of the coefficients of Kerov polynomials is that they are equal to cardinalities of some combinatorial objects. Biane provided also some heuristics what these combinatorial objects could be. During last decade there was a big progress in understanding a combinatorial structure of Kerov polynomials ([Sta02, Bia03, Śni06a, GR07, Fér10b, DFŚ10]). However, their structure is still mysterious in some aspects and seems to be much richer than is known.

For arbitrary parameter α, the existence of Kerov polynomials was proved by Lassalle [Las09]. We present a few examples of them. This data has been computed using the data given in [Las09, page 2230]:

$$
\begin{aligned}
K_{(1)}^{(\alpha)} &= R_2^{(\alpha)}, \\
K_{(2)}^{(\alpha)} &= R_3^{(\alpha)} + \gamma R_2^{(\alpha)}, \\
K_{(3)}^{(\alpha)} &= R_4^{(\alpha)} + 3\gamma R_3^{(\alpha)} + (1+2\gamma^2) R_2^{(\alpha)}, \\
K_{(4)}^{(\alpha)} &= R_5^{(\alpha)} + \gamma(6R_4^{(\alpha)} + (R_2^{(\alpha)})^2) + (5+11\gamma^2) R_3^{(\alpha)} + (7\gamma + 6\gamma^3) R_2^{(\alpha)}, \\
K_{(5)}^{(\alpha)} &= R_6^{(\alpha)} + \gamma(10R_5^{(\alpha)} + 5R_3^{(\alpha)} R_2^{(\alpha)}) + 15R_4^{(\alpha)} + 5(R_2^{(\alpha)})^2 \\
&\quad + \gamma^2(35R_4^{(\alpha)} + 10(R_2^{(\alpha)})^2) + (55\gamma + 50\gamma^3) R_3^{(\alpha)} + (8 + 46\gamma^2 + 24\gamma^4) R_2^{(\alpha)}, \\
K_{(2,2)}^{(\alpha)} &= (R_3^{(\alpha)})^2 + 2\gamma R_3^{(\alpha)} R_2^{(\alpha)} - 4R_4^{(\alpha)} \\
&\quad + (\gamma^2 - 2)(R_2^{(\alpha)})^2 - 10\gamma R_3^{(\alpha)} - (6\gamma^2 + 2) R_2^{(\alpha)},
\end{aligned}
$$

where we set $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$.

A few striking facts appear on these examples:

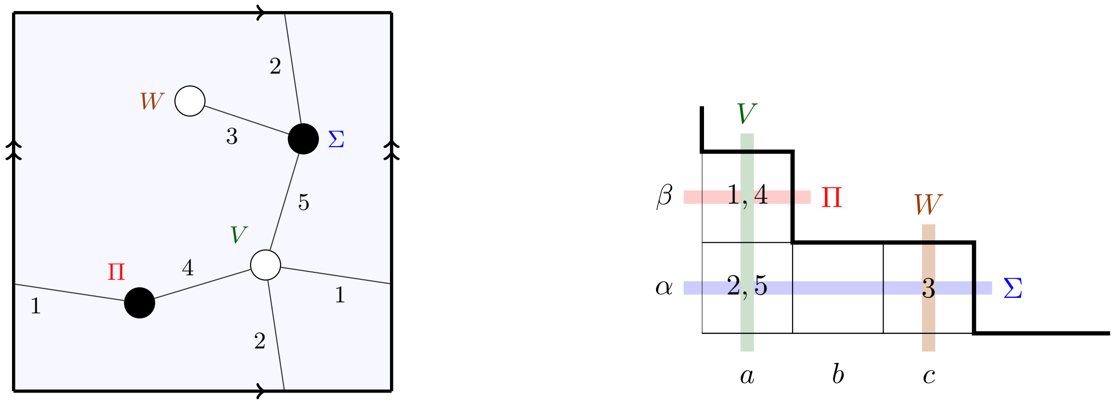
- All coefficients are polynomials in the auxiliary parameter γ.
- For one part partition, polynomials $K_{(r)}$ have non-negative coefficients. This is a more precise version of [Las09, Conjecture 1.2]. A similar conjecture holds for several part partitions, see [Las09, Conjecture 1.2].

2.7 Embeddings of bipartite graphs into the Young diagram

It turns out that any α-polynomial function on the set of Young diagrams can be represented as a linear combination of *the numbers of embeddings of bipartite graphs into the Young diagram*.

2.7.1 Numbers of embeddings of bipartite graphs into the Young diagram

An *embedding h of a bipartite graph G to a Young diagram λ* is a function which maps the set $V_\circ(G)$ of white vertices of G to the set of columns of λ, which maps the set $V_\bullet(G)$ of black vertices of G to the set of rows of λ, and maps the edges of G to boxes of λ, see Figure 2.7.1. We also require that an embedding preserves the relation of *incidence*, i.e., a vertex V and an incident edge E should be mapped to a row or column $h(V)$ which contains the box $h(E)$. We denote by $N_G(\lambda)$ the number of such embeddings of G to λ.



(a) (b)

Figure 2.7.1: (a) Example of a bipartite graph (drawn on the torus) and (b) an example of its embedding $h(\Sigma) = \alpha$, $h(\Pi) = \beta$, $h(V) = a$, $h(W) = c$. $h(1) = h(4) = (a\beta)$, $h(2) = h(5) = (a\alpha)$, $h(3) = (c\alpha)$. The columns of the Young diagram were indexed by small Latin letters, the rows by small Greek letters.

We can easily extend this definition to generalized Young diagrams. Let h be a function $h : V_\circ(G) \sqcup V_\bullet(G) \to \mathbb{R}_+$. We call it a *coloring of a bipartite graph* G and we say that this coloring is an *embedding* h *of a bipartite graph* G *to a generalized Young diagram* λ if $(h(v_1), h(v_2)) \in \lambda$ for each edge $(v_\circ, v_\bullet) \in V_\circ(G) \times V_\bullet(G)$. Alternatively, it can be viewed as a function which maps the edges of the bipartite graph to points in λ with a property that if edges e_1, e_2 share a common white (respectively, black) vertex then $h(e_1)$ and $h(e_2)$ have the same x-coordinate (respectively, the same y-coordinate).

For a given bipartite graph G, we can think what is the number of embeddings of G into λ in the following way. If we fix an order of the vertices in $V(G) = V_\circ(G) \sqcup V_\bullet(G)$, we can think of a coloring h as an element of $\mathbb{R}_+^{|V(G)|}$. Then we can define the number of embeddings of a bipartite graph G to λ as

$$N_G(\lambda) = \operatorname{vol}\{h \in \mathbb{R}_+^{|V|} : h \text{ is an embedding of } G \text{ to } \lambda\}.$$

Clearly, if λ is a Young diagram, this definition of $N_G(\lambda)$ is consistent with the previous one.

2.7.2 Stanley character formula

The problem of *understanding the structure of Jack characters* $\mathrm{Ch}_\mu^{(\alpha)}$ can be formulated concretely as finding some constants $m_G^{(\alpha)}$ with a property that for any Young diagram λ

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_G \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(G)|} \left(\sqrt{\alpha}\right)^{|V_\circ(G)|} m_G^{(\alpha)}\, N_G(\lambda), \tag{2.10}$$

where the sum is over some finite family of bipartite graphs which depends only on μ. We will refer to this kind of result as *Stanley character formula*. Since functions N_G are, in general, not linearly independent, this problem does not have a unique solution; therefore one can try *to find Stanley formula of a particularly nice form*.

In the following we will discuss the special cases $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$ for which Stanley character formula is already well-understood and we will discuss some hints about its structure for generic values of the deformation parameter α.

Roughly speaking, an *oriented (respectively, non-oriented) map* M is defined as a bipartite graph drawn on an oriented (respectively, non-oriented and possibly non-orientable) surface. Face-type of a map is a certain combinatorial property which describes this map. For a precise definition we refer to Section 2.8.

It has been observed in [FŚ11a] that a formula conjectured by Stanley [Sta06, Fér10b] for the normalized characters of the symmetric groups can be expressed as the sum

$$\mathrm{Ch}_\mu^{(1)}(\lambda) = (-1)^{\ell(\mu)} \sum_M (-1)^{|V_\bullet(M)|} N_M(\lambda) \tag{2.11}$$

over all *oriented bipartite maps* M *with the face-type* μ. Here and throughout this thesis, N_M denotes the function N indexed by the underlying graph of map M. Clearly, this formula fits nicely into the template (2.10).

In [FŚ11b] it has been observed that

$$\mathrm{Ch}_\mu^{(2)}(\lambda) = (-1)^{\ell(\mu)} \sum_M \left(-\frac{1}{\sqrt{2}}\right)^{|V_\bullet(M)|} \left(\sqrt{2}\right)^{|V_\circ(M)|} \cdot \left(-\frac{1}{\sqrt{2}}\right)^{|\mu|+\ell(\mu)-|V(M)|} N_M(\lambda), \tag{2.12}$$

$$\mathrm{Ch}_\mu^{(1/2)}(\lambda) = (-1)^{\ell(\mu)} \sum_M \left(-\sqrt{2}\right)^{|V_\bullet(M)|} \left(\frac{1}{\sqrt{2}}\right)^{|V_\circ(M)|} \cdot \left(\frac{1}{\sqrt{2}}\right)^{|\mu|+\ell(\mu)-|V(M)|} N_M(\lambda), \tag{2.13}$$

where the sums run over all *non-oriented maps* M *with the face-type specified by* μ. Clearly, also these formulas fit nicely into the template (2.10).

The main theme of Chapter 5 will be an attempt of finding formula (2.10) as precisely as possible. As we can observe from this section, the main combinatorial tool for that will be the set of non-oriented maps.

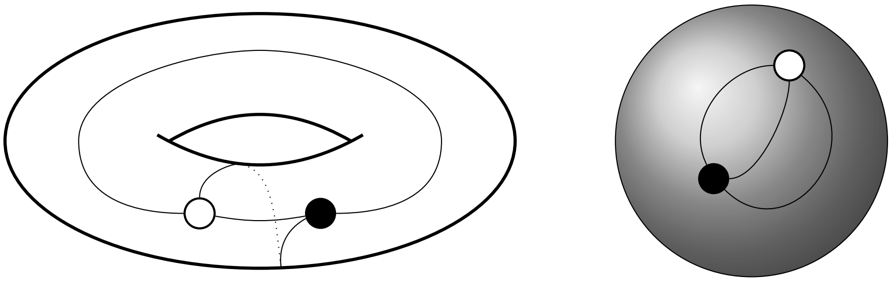


Figure 2.8.1: The same bipartite graph can be drawn on a torus and on a sphere to obtain two different maps.

2.8 MAPS

2.8.1 TOPOLOGICAL MAPS

Roughly speaking, a *map* is a graph drawn on a surface in a way that no edges cross each other. This very basic concept, inspired by the planar graphs case, can be defined as follows:

Definition 2.8.1. An *embedding* $i : G \to \Sigma$ of a graph G in a surface Σ is a continuous one-to-one function from the graph into the surface. The components of the complement of $i(G)$ are the *faces* of the embedding. If every face is homeomorphic to an open disc, then the embedding is a *cellular embedding*. A (bipartite) *map* is a cellular embedding of a (bipartite) graph in a surface. The map is *orientable* if the surface is orientable, otherwise it is *non-orientable*. All maps are locally orientable and we refer to both orientable and non-orientable maps as *non-oriented maps*.

Definition 2.8.2. Two maps $i : G \to \Sigma_1$ and $j : H \to \Sigma_2$ are *equivalent*, if there is a homeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h(i(G)) = j(H)$ and $h(i(V(G))) = j(V(H))$. If such a homeomorphism exists, then it follows that G and H are isomorphic graphs.

Definition 2.8.3. The *degree* of a face is the length of a closed walk passing once around its boundary. In a map with n edges, the degrees of the faces form an integer partition of $2n$. This partition, denoted by λ, is the *face-degree* partition of the map. Notice, that for a bipartite map the face-degree has a particular form: each part of λ is an even number. In that case, we define a *face-type* of bipartite map as a partition $T_{1/2,1}(\lambda) = (\lambda_1/2, \dots, \lambda_k/2)$, where $\lambda = (\lambda_1, \dots, \lambda_k)$ is a corresponding face-degree.

Example 2.8.4. The same bipartite graph can be used to define two different maps which is illustrated on Figure 2.8.1. On the left hand side there is a map drawn on a torus. This map has two vertices, three edges, and its face-type is equal to (3). On the right hand side, there is a map which has also two vertices and three edges, but its face-type is equal to (1^3) and this map is drawn on a sphere.

It turns out that sometimes it is more convenient to study some combinatorial object associated to a given map. Since there is a one–to–one correspondence between maps and these combinatorial structures, we call these structures *combinatorial maps*.

2.8.2 COMBINATORIAL MAPS

PAIRINGS AND POLYGONS

A *set-partition* of a set S is a set $\{I_1, \cdots, I_r\}$ of pairwise disjoint non-empty subsets whose union is S.

A *pairing* (or, alternatively, *pair-partition*) of S is a set-partition into pairs. If s is an element of S and P is a pairing of S, the *partner* of s in P is defined as the unique element $t \in S$ such that $\{s, t\}$ is a pair of P.

For instance, for any integer $n \geq 1$,

$$P = \big\{\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\big\}$$

is a pairing of $[2n]$ (we use the standard notation $[n] := \{1, \cdots, n\}$). Note that the existence of a pairing of S clearly implies that $|S|$ is even.

Let us consider now two pairings $\mathcal{B}, \mathcal{W}$ of the same set S consisting of $2n$ elements. We consider the following bipartite edge-labeled graph $\mathcal{L}(\mathcal{B}, \mathcal{W})$:

- it has n black vertices indexed by the two-element sets of $\mathcal{B}$ and n white vertices indexed by the two-element sets of $\mathcal{W}$;
- its edges are labeled with the elements of S. The extremities of the edge labeled i are the pairs of $\mathcal{B}$ and $\mathcal{W}$ containing i.

Note that each vertex has degree 2 and each edge has one white and one black extremity. Besides, if we erase the indices of the vertices, it is easy to recover them from the labels of the edges (the index of a vertex is the set of the two labels of the edges leaving this vertex). Thus, we forget the indices of the vertices and view $\mathcal{L}(\mathcal{B}, \mathcal{W})$ as an edge-labeled graph.

As every vertex has degree 2, the graph $\mathcal{L}(\mathcal{B}, \mathcal{W})$ is a collection of polygons. Moreover, because of the proper bicoloration of the vertices, all polygons have even length. Let $2\ell_1 \geq 2\ell_2 \geq \cdots$ be the ordered lengths of these polygons. The partition $(\ell_1, \ell_2, \ldots)$ is called the *type* of $\mathcal{L}(\mathcal{B}, \mathcal{W})$ or the *type* of the couple $(\mathcal{B}, \mathcal{W})$.

Special role will be played by polygons having exactly 2 edges. Such a polygon will be referred to as *bigon*.

Example 2.8.5. For partitions

$$\mathcal{B} = \big\{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{A,B\}, \{C,D\}\big\},$$
$$\mathcal{W} = \big\{\{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,1\}, \{B,C\}, \{D,A\}\big\}$$

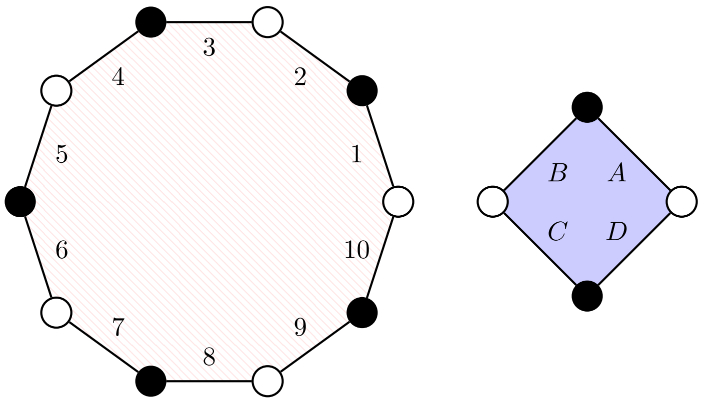


Figure 2.8.2: Polygons obtained from a couple of pairings from Example 2.8.5.

the corresponding polygons $\mathcal{L}(\mathcal{B}, \mathcal{W})$ are shown in Figure 2.8.2.

Let s_1 and s_2 be two elements of S that belong to the same polygon of $\mathcal{L}(P_1, P_2)$. Fix an arbitrary orientation of this polygon. Then, one can consider the number of elements of S between s_1 in s_2 in the polygon. We say that s_1 and s_2 are in an *even (respectively, odd) position* if this number is even (respectively, odd). As all polygons have even size, this definition does not depend on the choice of the orientation.

NON-ORIENTED MAPS

The central combinatorial object in this thesis is the following.

Definition 2.8.6. A *combinatorial map* is a triplet $(\mathcal{B}, \mathcal{W}, \mathcal{E})$ of pairings of the same set S.

As we mentioned before, the terminology comes from the fact that it is possible to represent such a triplet of pair-partitions as a bipartite graph embedded in a non-oriented (and possibly non-connected) surface. Let us explain how this works.

First, one can consider the union of polygons $\mathcal{L}(\mathcal{B}, \mathcal{W})$ defined in Section 2.8.2. The edges of these polygons, that is the elements of the set S are called *edge-sides*.

We consider the union of the interiors of these polygons as a (possibly disconnected) surface with a boundary. If we consider two edge-sides, we can *glue* them: that means that we identify their white extremities, their black extremities and the edge-side themselves.

For any pair in the pairing $\mathcal{E}$, we glue the two corresponding edge-sides. Doing that, we obtain a (possibly disconnected, possibly non-orientable) surface Σ without boundary. After the gluing, the polygons form a bipartite graph G embedded in the surface. For instance, with the pairings $\mathcal{B}$ and $\mathcal{W}$ from Example 2.8.5 and

$$\mathcal{E} = \big\{\{1,3\},\{2,10\},\{4,9\},\{5,D\},\{6,C\},\{7,B\},\{8,A\}\big\}, \tag{2.14}$$

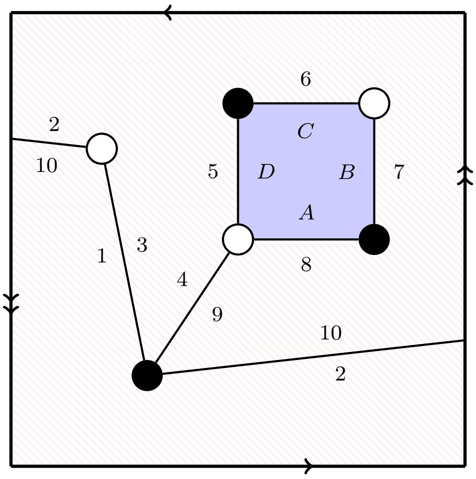


Figure 2.8.3: Example of a *non-oriented map* drawn on the projective plane. The left side of the square should be glued with a twist to the right side, as well as bottom to top (also with a twist), as indicated by arrows. This map has been obtained by gluing the edge-sides of the polygon of Figure 2.8.2 according to the pair-partition given by Eq. (2.14).

we get the graph from Figure 2.8.3 embedded in the projective plane.

In general, the graph G has as many connected components as the surface Σ. Besides, $\Sigma \setminus G$ corresponds to the interiors of the collection of polygons we are starting from. In particular, each connected component of $\Sigma \setminus G$ is homeomorphic to an open disc, hence these connected components are just faces of the underlying map. This makes the link with Definition 2.8.1.

Definition 2.8.7. Let $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ be a map.

- Elements of $\mathcal{B}$ (respectively, $\mathcal{W}$) are called black (respectively, white) *corners*.
- Elements of $\mathcal{E}$ are called *edges*; we use the notation $\mathcal{E}(M)$ for the set of edges of the map M (that is the third element of the triplet defining the map).
- The polygons $\mathcal{L}(\mathcal{B}, \mathcal{W})$ corresponding to the couple of pairings $(\mathcal{B}, \mathcal{W})$ are called *faces*; the set of faces will be denoted $F(M)$. The *face-type* of the map is the type of the couple $(\mathcal{B}, \mathcal{W})$, as defined in Section 2.8.2.
- The polygons $\mathcal{L}(\mathcal{B}, \mathcal{E})$ (respectively, $\mathcal{L}(\mathcal{W}, \mathcal{E})$) of the couple of pairings $(\mathcal{B}, \mathcal{E})$ (respectively, $(\mathcal{W}, \mathcal{E})$) are called *black vertices* (respectively, *white vertices*); their set is denoted $V_\bullet(M)$ (respectively, $V_\circ(M)$).
- A *leaf* of the map M is a vertex of M of degree 1, that is a bigon of $\mathcal{L}(\mathcal{B}, \mathcal{E})$ or $\mathcal{L}(\mathcal{W}, \mathcal{E})$. In other terms, a leaf is a pair of edge-sides which belongs to both $\mathcal{E}$ and $\mathcal{B}$ or which belongs to both $\mathcal{E}$ and $\mathcal{W}$.
- The *connected components* of the map M correspond to the connected components of the graph G constructed above. Formally, they are the equivalence classes of the transitive

closure of the relation: $x \sim y$ if x is the partner of y in $\mathcal{E}$, $\mathcal{B}$ or $\mathcal{W}$.

Note that our maps have labeled edge-sides and each element of S is used exactly once as a label.

The pairing $\mathcal{B}$ (respectively, $\mathcal{W}$) indicates which edge-sides share the same corner around a black (respectively, white) vertex. This explains the names of these pairings.

This encoding of (non-oriented) maps by triplets of pairings is of course not new. It can for instance be found in [GJ96b]; the presentation in that paper is nevertheless a bit different as the authors consider there *connected monochromatic* maps.

Summation over maps with a specified face-type (such as in (2.12) and (2.13)) should be understood as follows: we *fix* a couple of pairings $(\mathcal{B}, \mathcal{W})$ of type π and consider all pairings $\mathcal{E}$ of the same ground set; we sum over the resulting collection of maps $(\mathcal{B}, \mathcal{W}, \mathcal{E})$. The *set of maps with a specified face-type* should be understood in an analogous way.

Remark 2.8.8. Since topological maps and combinatorial maps are just two ways of describing the same object, we are going to use both of them and we will simply call these objects maps. It should be clear from the context which definition we are using.

2.8.3 Properties of maps

Euler characteristic

We define the *Euler characteristic* of the map M by a following formula:

$$\chi(M) := |V(M)| - |\mathcal{E}(M)| + |F(M)|.$$

Notice that whenever map M is connected, the number $\chi(M)$ is the Euler characteristic of the underlying surface.

We define also the *Euler genus* of the map:

$$d(M) := 2(\text{number of connected components of } M) - \chi(M).$$

Again, notice that whenever map M is connected, the number $\chi(M)$ is equal to the Euler genus of the underlying surface.

Three kinds of edges

Let a map M with some selected edge $E = \{s_1, s_2\}$ be given. We distinguish three cases (a schematic description and an example of each case are given in Figures 2.8.4 and 2.8.5):

- Both edge-sides s_1 and s_2 belong to the same face F *and* are in an even position (see the definition at the end of Section 2.8.2)

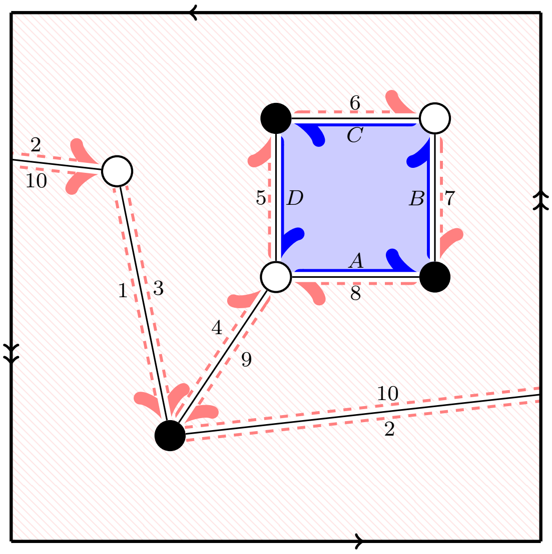
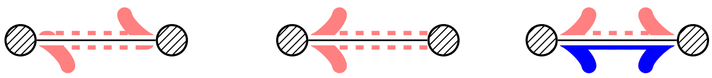


Figure 2.8.4: The non-oriented map from Figure 2.8.3. On the boundary of each face some arbitrary orientation was chosen, as indicated by arrows. Edge $\{4, 9\}$ is an example of a straight edge, edge $\{1, 3\}$ is an example of a twisted edge, edge $\{6, C\}$ is an example of an interface edge.



(a) (b) (c)

Figure 2.8.5: Three possible kinds of edges in a map (see Figure 2.8.4): (a) *straight edge*: both edge-sides of the edge belong to the same face and have opposite orientations, (b) *twisted edge*: both edge-sides of the edge belong to the same face and have the same orientation, (c) *interface edge*: the edge-sides of the edge belong to two different faces; their orientations are not important. In all three cases the colors of the vertices are not important.

Graphically, this means that if we travel along the boundary of the face F then we visit the edge E twice *and* the directions in which we travel twice along the edge E are *opposite*, see Figure 2.8.5a.

In this case the edge E is called *straight*.

- Both edge-sides s_1 and s_2 belong to the same face F *and* are in an odd position.

  Graphically, this means that if we travel along the boundary of the face F, we visit the edge E twice *and* the directions in which we travel twice along the edge E are *the same*, see Figure 2.8.5b.

  In this case the edge E is called *twisted*.

- Edge-sides s_1 and s_2 belong to different faces of the map, see Figure 2.8.5c.

  In this case the edge E is called *interface*.

Lemma 2.8.9. *If at least one extremity of an edge is a leaf, then this edge is straight.*

The proof is a simple exercise.

Remark 2.8.10. Notice that map M has twisted edge iff M is non-orientable.

3

Asymptotics of characters of symmetric groups: structure of Kerov character polynomials

Abstract

We study asymptotics of characters of the symmetric groups on a fixed conjugacy class. It was proved by Kerov that such a character can be expressed as a polynomial in free cumulants of the Young diagram (certain functionals describing the shape of the Young diagram). We show that for each genus there exists a universal symmetric polynomial which gives the coefficients of the part of Kerov character polynomials with the prescribed homogeneous degree. The existence of such symmetric polynomials was conjectured by Lassalle.

3.1 Introduction

Since in this chapter we are studying normalized characters of the symmetric groups defined on cycles and the corresponding Kerov polynomials, we are going to simplify notation, i. e. we denote:

$$\mathrm{Ch}_k := \mathrm{Ch}_{(k)}, \quad K_k := K_{(k)}.$$

3.1.1 GENUS EXPANSION

It is convenient to consider a gradation with respect to which the degree of the free cumulant R_k is equal to k. We denote by $K_{k,d}$ the homogeneous part of degree d of the Kerov character polynomial K_k. One of the results announced by Kerov [Ker00a] was that the only non-zero polynomials $K_{k,d}$ are of the form $K_{k,k+1-2g}$ where $g \geq 0$ is an integer. It is possible to give some topological meaning to many calculations related to Kerov polynomials in which the integer g can be interpreted as the genus of the resulting two-dimensional surface. For this reason, studying the polynomials $K_{k,k+1-2g}$ for a fixed value of g is often called the *genus expansion*.

The form of the highest-degree term

$$K_{k,k+1} = R_{k+1}$$

was announced by Kerov [Ker00a] and proved by Biane [Bia03]. The form of the next term $K_{k,k-1}$ was conjectured by Biane [Bia03] and proved by Śniady [Śni06a]. Explicit but rather complicated formulas for the general genus $K_{k,k+1-2g}$ were found by Goulden and Rattan [GR07] (for a more elementary proof we refer to the work of Biane [Bia07]) and we shall discuss their result in Section 3.2.3.

3.1.2 THE MAIN RESULT: PROOF OF SOME CONJECTURES OF LASSALLE

Lassalle announced as a conjecture [Las08c] that there is an additional structure in the genus expansion of Kerov polynomials. He claimed that for a fixed genus g there exists a symmetric function f_g which describes polynomials $K_{k,k+1-2g}$ and which is independent of k. Before presenting his conjectures we need to prepare some notations.

A partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ is a weakly decreasing sequence of nonnegative integers with finitely many non-zero elements. The non-zero μ_i in a partition μ are called the parts of μ. We will denote by $m_i(\mu)$ the number of parts of μ equal to i; by $l(\mu)$ the number of parts of μ; and denote $|\mu| = \mu_1 + \mu_2 + \cdots$. Following Lassalle, we define $R_1 = 0$ and for a strictly positive integer i we define

$$\begin{aligned}
\mathcal{R}_i &= (i-1)R_i, \\
\mathcal{R}_\mu &= \prod_i \frac{\mathcal{R}_i^{m_i(\mu)}}{m_i(\mu)!}, \\
Q_i &= \sum_{|\mu|=i} (l(\mu)-1)!\, \mathcal{R}_\mu, \\
\mathcal{Q}_\mu &= \prod_i \frac{Q_i^{m_i(\mu)}}{m_i(\mu)!}.
\end{aligned} \tag{3.1}$$

We recall, that we denote by e_i the elementary symmetric functions, by h_i the complete symmetric functions and by p_i the power-sum symmetric functions. For any partition μ, we denote by e_μ, h_μ or p_μ their product over the parts of μ, and by $\mathfrak{h}_\mu$ the monomial symmetric function — the sum of all distinct monomials whose exponent is a permutation of μ (see Section 2.2).

The main result of this article is a proof of the following results which were stated as the first and the sixth conjecture in the paper [Las08c] by Lassalle.

Theorem 3.1.1. *For any $g \geq 1$ there exist inhomogeneous symmetric functions f_g and h_g, having maximal degree $4(g-1)$, such that*

$$K_{k,k+1-2g} = \binom{k+1}{3} \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\; f_g(\mu)\; \mathcal{R}_\mu \tag{3.2}$$

$$= \binom{k+1}{3} \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\; h_g(\mu)\; \mathcal{Q}_\mu, \tag{3.3}$$

where $f_g(\mu) = f_g(\mu_1, \mu_2, \dots)$ and $h_g(\mu_1, \mu_2, \dots)$. These symmetric functions are independent of k.

This result sheds some light on the structure of Kerov polynomials but it also leads to many new open problems, in particular the positivity conjectures of Lassalle [Las08c] and his questions concerning combinatorial interpretations of the coefficients in the expansions of the above symmetric functions.

3.1.3 General idea of the proof

For a given positive integer g we define a symmetric function $k(\mu) := |\mu| + 2g - 1 = p_1(\mu) + 2g - 1$, therefore for a given symmetric functions f_g and h_g we can define symmetric functions $\tilde{f}_g(\mu) := k(\mu)\binom{k(\mu)+1}{3} f_g(\mu)$ and $\tilde{h}_g(\mu) := k(\mu)\binom{k(\mu)+1}{3} h_g(\mu)$. We notice that in equations (3.2) and (3.3) we sum over all partitions μ which satisfy $k(\mu) = k$. Therefore the following proposition is an immediate consequence of Theorem 3.1.1.

Proposition 3.1.2. *For any $g \geq 1$ there exist inhomogeneous symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$, having maximal degree $4g$, such that*

$$\begin{aligned} k\; K_{k,k+1-2g} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\; \tilde{f}_g(\mu) \mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\; \tilde{h}_g(\mu) \mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}_g(\mu) = \tilde{f}_g(\mu_1, \mu_2, \dots)$ and $\tilde{h}_g(\mu) = \tilde{h}_g(\mu_1, \mu_2, \dots)$. These symmetric functions are independent of k.

In fact, the opposite implication holds true as well and Theorem 3.1.1 is a consequence of Proposition 3.1.2: roughly speaking we will show that the symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$ are divisible by the polynomial $k\binom{k+1}{3}$. One can notice that Proposition 3.1.2 stated that $\tilde{f}_g$ and $\tilde{h}_g$ are independent of k, so the divisibility which we will show is a divisibility of symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$ by the symmetric function K which has a property that for any partition μ such that $|\mu| = k + 1 - 2g$ we have that $K(\mu) = k\binom{k+1}{3}$. We shall explain precisely what this divisibility means and show in Section 3.3 that it holds indeed by studying the arithmetic properties of Kerov polynomials and their divisibility by prime numbers.

The remaining difficulty is to prove Proposition 3.1.2. We shall do it in Section 3.2 by analysis of the Goulden-Rattan formula.

Section 3.4 is a presentation of technical and complicated proofs of lemmas which are used in the previous sections.

3.2 Goulden-Rattan formula and existence of symmetric polynomials

3.2.1 Power series P_λ

Following Goulden and Rattan [GR07] we define

$$C(t) = \frac{1}{1 - \sum_{i\geq 2} \mathcal{R}_i t^i} = \sum_{\mu} t^{|\mu|} l(\mu)!\, \mathcal{R}_\mu. \tag{3.4}$$

Let $D = t\frac{d}{dt}$ and define for $m \geq 1$

$$P_m(t) = -\frac{1}{m!} C(t)\big(D + m - 2\big)C(t) \cdots \big(D + 1\big)C(t)DC(t).$$

For example, we have

$$\begin{aligned}
P_1(t) &= -C(t),\\
P_2(t) &= -\frac{1}{2}C(t)DC(t),\\
P_3(t) &= -\frac{1}{6}C(t)(D+1)C(t)DC(t)\\
&= -\frac{1}{6}\left[C(t)DC(t)DC(t) + C(t)^2DC(t)\right]\\
&= -\frac{1}{6}\left[C(t)D\big(C(t)\cdot DC(t)\big) + C(t)^2DC(t)\right]\\
&= -\frac{1}{6}\left[C(t)\big(DC(t)\big)^2 + C(t)^2D^2C(t) + C(t)^2DC(t)\right].
\end{aligned}$$

Finally, for a partition λ, we write $P_\lambda(t) = \prod_{j=1}^{l(\lambda)} P_{\lambda_j}(t)$.

For $p = (p_0, \dots, p_l) \in \mathbb{N}^{l+1}$ we define

$$E(p) := C(t)^{p_0} DC(t)^{p_1} \cdots DC(t)^{p_l}.$$

Lemma 3.2.1.

(a) Let $p \in \mathbb{N}^{l+1}$, $q \in \mathbb{N}^{m+1}$ and denote by $|p| := p_0 + \cdots + p_l$ ($|q| = q_0 + \cdots q_m$ respectively). Then

$$E(p) \cdot E(q) = \sum_{\substack{r \in \mathbb{N}^{l+m+1}, \\ |r| = |p| + |q|}} c_r^{p,q} E(r),$$

where $c_r^{p,q} \in \mathbb{Z}$.

(b) For any partition λ the fraction $\frac{P_\lambda(t)}{C(t)}$ is a linear combination of terms $E(p)$ where $p \in \mathbb{N}^k$ such that $|p| = |\lambda| - 1$ and $k \leq |\lambda| - l(\lambda) + 1$.

Proof. Let $p = (p_0, \dots, p_l) \in \mathbb{N}^{l+1}$, $q = (q_0, \dots, q_m) \in \mathbb{N}^{m+1}$. We will show part (a) by induction on l. It is obvious for $l = 0$. For $l = 1$ we have:

$$E(p) \cdot E(q) = E(p_0, p_1 + q_0, q_1, q_2, \dots, q_m) - E(p_0 + p_1, q_0, q_1, \dots, q_m)$$

by the Leibniz rule. Let us assume, that the inductive assertion holds for some $l \geq 1$ and let $p = (p_0, \dots, p_{l+1})$. Then by the Leibniz rule we have that

$$E(p) \cdot E(q) = C(t)^{p_0} D \left[E(p') \cdot E(q)\right] - C(t)^{p_0 + p_1} \left[E(p'') \cdot E(q')\right], \tag{3.5}$$

where $p' = (p_1, \dots, p_{l+1})$, $p'' = (0, p_2, \dots, p_{l+1})$, $q' = (0, q_0, \dots, q_m)$. By the inductive assertion, the right hand side of (3.5) is equal to

$$\sum_{\substack{\alpha \in \mathbb{N}^{l+m+1}, \\ |\alpha| = |p| + |q| - p_0}} c_\alpha C(t)^{p_0} DE(\alpha) - \sum_{\substack{\beta \in \mathbb{N}^{l+m+2}, \\ |\beta| = |p| + |q|}} c_\beta E(\beta),$$

where $c_\alpha, c_\beta \in \mathbb{Z}$. But it means that

$$E(p) \cdot E(q) = \sum_{\substack{r \in \mathbb{N}^{l+m+2}, \\ |r| = |p| + |q|}} c_r^{p,q} E(r),$$

where $c_r^{p,q} \in \mathbb{Z}$ which finishes the proof of part (a).

For part (b) we notice that each function $P_m(t)$ is a linear combination of $E(p)$, where $p \in \mathbb{N}^k$, $|p| = m$ and $k \leq m - 1$ and we apply part (a). $\square$

3.2.2 Polynomial structure of coefficients of P_λ

Definition 3.2.2. If f is a symmetric function of degree d and $2g \geq 2$ is an integer then the formal power series

$$F(t) = \sum_{\mu} t^{|\mu|} \big(l(\mu) + 2g - 2\big)!\, f(\mu)\, \mathcal{R}_\mu \tag{3.6}$$

will be called *a power-sum of the first kind with degree d and genus g* and the formal power series

$$F(t) = \sum_{\mu} t^{|\mu|} (2g-1)^{l(\mu)}\, f(\mu)\, \mathcal{Q}_\mu \tag{3.7}$$

will be called *a power-sum of the second kind with degree d and genus g.*

Lemma 3.2.3.

(a) $C(t)$ is a power-sum of the first (respectively, second) kind with degree 0 and genus 1.

(b) If $F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus g then $DF(t)$ is a power-sum of the first (respectively, second) kind with degree $d+1$ and genus g.

(c) If $F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus g then $C(t)F(t)$ is a power-sum of the first (respectively, second) kind with degree d and genus $g + \frac{1}{2}$.

Proof. In order to prove point (a) it suffices to notice that

$$C(t) = \sum_{\mu} t^{|\mu|} l(\mu)!\, \mathcal{R}_\mu = \sum_{\mu} t^{|\mu|} \mathcal{Q}_\mu.$$

In order to prove point (b) let $F(t)$ be in the form (3.6). Then

$$DF(t) = \sum_{\mu} t^{|\mu|} \big(l(\mu) + 2g - 2\big)!\, \big[(p_1(\mu) f(\mu)\big]\, \mathcal{R}_\mu$$

is again of the form (3.6).

If $F(t)$ is of the form (3.7), then

$$DF(t) = \sum_{\mu} t^{|\mu|} (2g-1)^{l(\mu)}\, \big[(p_1(\mu) f(\mu)\big]\, \mathcal{Q}_\mu$$

is again of the form (3.7) which shows part (b).

For part (c) we can assume that the symmetric function f is equal to monomial symmetric function $\mathfrak{h}_\lambda$ for some partition λ.

We define

$$C_n = \sum_{|\mu|=n} l(\mu)!\, \mathcal{R}_\mu = \sum_{|\mu|=n} \mathcal{Q}_\mu,$$
$$\mathcal{C}_\mu = \prod_{i\geq 2} \frac{C_i^{m_i(\mu)}}{m_i(\mu)!}.$$

The correspondence between these three families (Q, R and C) is given by

$$Q_n = \sum_{|\mu|=n} (-1)^{l(\mu)} (l(\mu)-1)!\, \mathcal{C}_\mu,$$
$$-\mathcal{R}_n = \sum_{|\mu|=n} (-1)^{l(\mu)}\, \mathcal{Q}_\mu = \sum_{|\mu|=n} (-1)^{l(\mu)} l(\mu)!\, \mathcal{C}_\mu.$$

Following Lassalle, [Las08c] we define the (formal) alphabet $\mathcal{A}$ by

$$\mathcal{R}_i = -h_i(\mathcal{A}), \quad Q_i = -p_i(\mathcal{A})/i, \quad C_i = (-1)^i e_i(\mathcal{A}).$$

Writing

$$u_\mu = l(\mu)!/\prod_{i\geq 1} m_i(\mu), \quad \epsilon_\mu = (-1)^{n-l(\mu)}, \quad z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!,$$

the previous relations can be understood in a frame of symmetric functions theory, and they are merely the classical properties [Mac95, pp. 25 and 33]

$$p_n = -n \sum_{|\mu|=n} (-1)^{l(\mu)}\, u_\mu h_\mu / l(\mu) = -n \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu / l(\mu),$$
$$e_n = \sum_{|\mu|=n} \epsilon_\mu u_\mu h_\mu = \sum_{|\mu|=n} \epsilon_\mu z_\mu^{-1} p_\mu,$$
$$h_n = \sum_{|\mu|=n} z_\mu^{-1} p_\mu = \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu.$$

Using this notation, it suffices to show that for any monomial symmetric function $\mathfrak{h}_\lambda$ and any g we have

$$\left(\sum_\mu t^{|\mu|} \mathfrak{h}_\lambda(\mu) \frac{(l(\mu)+2g-2)!}{l(\mu)!} (-1)^{l(\mu)} u_\mu h_\mu\right) \left(\sum_\rho t^{|\rho|} (-1)^{l(\rho)} u_\rho h_\rho\right) = \left(\sum_\nu t^{|\nu|} \frac{\mathfrak{h}_\lambda(\nu)}{l(\lambda)+2g-1} \frac{(l(\nu)+2g-1)!}{l(\nu)!} (-1)^{l(\nu)} u_\nu h_\nu\right), \quad (3.8)$$

because the right hand side is a power-sum of the first kind with degree $|\lambda|$ and genus $g+\frac{1}{2}$, and it suffices to show, that for any monomial symmetric function $\mathfrak{h}_\lambda$ and any g we have

$$\left(\sum_\mu t^{|\mu|}\mathfrak{h}_\lambda(\mu)(2g-1)^{l(\mu)}(-1)^{l(\mu)}z_\mu^{-1}p_\mu\right)\left(\sum_\rho t^{|\rho|}(-1)^{l(\rho)}z_\rho^{-1}p_\rho\right) = \left(\sum_\nu t^{|\nu|}\left(\frac{2g-1}{2g}\right)^{l(\lambda)}\mathfrak{h}_\lambda(\nu)(2g)^{l(\nu)}(-1)^{l(\nu)}z_\nu^{-1}p_\nu\right), \quad (3.9)$$

because the right hand side is a power-sum of the second kind with degree $|\lambda|$ and genus $g+\frac{1}{2}$. In order to prove (3.8) and (3.9) it is enough to use Lemma 3.4.1. □

The main result of this subsection is the following proposition.

Proposition 3.2.4. $\frac{P_\lambda(t)}{C(t)}$ *is a linear combination of power-sums of the first (respectively, second) kind of genus* $\frac{|\lambda|}{2}$ *and degree at most* $|\lambda|-l(\lambda)$.

Proof. It is enough to apply Lemma 3.2.1 and Lemma 3.2.3. □

3.2.3 Goulden-Rattan formula

In this chapter we consider the particular evaluation of the monomial symmetric function $\mathfrak{h}_\lambda$ at $x_i = i$, for $i = 1,\dots,k-1$, and $x_i = 0$, for $i \geq k$, and write this as $\hat{\mathfrak{h}}_\lambda$. Let $A(t)$ be a formal power series. We denote the coefficient of t^k in $A(t)$ by $[t^k]A(t)$.

Theorem 3.2.5 (Goulden and Rattan [GR07])**.** *For* $g \geq 1$, $k \geq 2g-1$,

$$\mathrm{Ch}_{k,k+1-2g} = -\frac{1}{k}[t^{k+1-2g}]\sum_{|\lambda|=2g}\hat{\mathfrak{h}}_\lambda\frac{P_\lambda(t)}{C(t)}. \quad (3.10)$$

3.2.4 Proof of Proposition 3.1.2

Proof of Proposition 3.1.2. Equation (3.10) can be written in the form

$$k\ \mathrm{Ch}_{k,k+1-2g} = -[t^{k+1-2g}]\sum_{|\lambda|=2g}\hat{\mathfrak{h}}_\lambda\frac{P_\lambda(t)}{C(t)}.$$

The evaluation of the power sum symmetric function

$$\hat{p}_s = 1^s + \cdots + (k-1)^s$$

analogous to that for $\hat{\mathfrak{h}}_\lambda$ is a polynomial in k of degree $s+1$; it follows immediately that the evaluation of the power-sum symmetric function $\hat{p}_\lambda$ is a polynomial in k of degree $|\lambda|+l(\lambda)$. The monomial symmetric function $\mathfrak{h}_\lambda$ is a linear combination of power-sum symmetric functions p_μ, where each partition μ which appears in this linear combination is obtained from partition λ by gluing some of their parts (see for example [Mac95]). It means that for each such μ we have

$$|\lambda| + l(\lambda) \geq |\mu| + l(\mu),$$

and for this reason also $\hat{\mathfrak{h}}_\lambda$ is a polynomial in k of degree at most $|\lambda| + l(\lambda)$.

For any partition μ such that $|\mu| = k + 1 - 2g$ we have $k = p_1(\mu) + 2g - 1$, where p_1 is a power symmetric function, hence there exists a symmetric function f_λ of degree $|\lambda| + l(\lambda)$ which does not depend on k such that $\hat{\mathfrak{h}}_\lambda = f_\lambda(\mu)$. Proposition 3.2.4 finishes the proof. □

3.3 Divisibility of polynomials

3.3.1 Implications of divisibility

At this step we proved Proposition 3.1.2. In order to prove Theorem 3.1.1 we would like to show that for each integer $g \geq 1$, functions $\tilde{f}_g$ and $\tilde{h}_g$ are divisible by the symmetric function

$$(p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!},$$

where p_1 denotes the power symmetric function. By word divisible we mean that there exist symmetric functions f_g and h_μ such that

$$\tilde{f}_g = (p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!} f_g$$

and

$$\tilde{h}_g = (p_1 + 2g - 1)\frac{(p_1 + 2g)(p_1 + 2g - 1)(p_1 + 2g - 2)}{3!} h_g.$$

Observe that for fixed $g \geq 1$ and for any partition μ there exists number k such that $|\mu| = k + 1 - 2g$ and then

$$\tilde{f}_g(\mu) = k\binom{k+1}{3} f_g(\mu),$$

$$\tilde{h}_g(\mu) = k\binom{k+1}{3} h_g(\mu)(\mu).$$

The idea of showing divisibility of symmetric function by symmetric function of degree 1 is similar to the case of showing divisibility of some polynomial by some monomial. The main idea is dividing with a remainder and using a fact that if some integer is divisible by infinite number of primes then it has to be equal to zero. The remainning of this section is a formalisation of this idea.

The first difficulty is that we have to deal with polynomials in several variables. Hence, in order to show some generalisation of the "dividing with remainder" technique, we need the following technical lemma:

Lemma 3.3.1. *Let m be a fixed integer and f be a polynomial in variables $x_k, \dots, x_l$ with the property that $f(\mu_k, \dots, \mu_l) = 0$ for all integers $\mu_k, \dots, \mu_l \geq 1$ which fulfill the following equations:*

$$\sum_{k \leq j \leq l} \mu_j > m, \tag{3.11}$$

$$\mu_i > \mu_{i+1} + \cdots + \mu_l \tag{3.12}$$

for all values of i for which it makes sense. Then $f = 0$.

The proof of this lemma can be found in Section 3.4.

The next lemma is key for this section: it allows to translate information about arithmetic properties of Kerov polynomials into information about the polynomials governing the coefficients.

Lemma 3.3.2. *Let f, respectively h, be a symmetric function of degree at most d with rational coefficients, $g \geq 1$ be an integer; we define*

$$L_k = \sum_{|\mu| = k+1-2g} \big(l(\mu) + 2g - 2\big)!\ f(\mu)\ \mathcal{R}_\mu,$$

respectively,

$$L'_k = \sum_{|\mu| = k+1-2g} \big(2g - 1\big)^{l(\mu)}\ h(\mu)\ \mathcal{Q}_\mu$$

and view it as a polynomial in $R_2, R_3, \dots$.

Assume that an integer Δ has a property that all coefficients of $L_{p+\Delta}$ (respectively, all coefficients of $L'_{p+\Delta}$) are integers divisible by p for an infinite number of prime numbers p. Then there exists a symmetric function $\tilde{f}$ (respectively, $\tilde{h}$) with rational coefficients of degree at most $d-1$ such that

$$L_k = (k - \Delta) \sum_{|\mu| = k+1-2g} \big(l(\mu) + 2g - 2\big)!\ \tilde{f}(\mu)\ \mathcal{R}_\mu,$$

respectively,

$$L'_k = (k - \Delta) \sum_{|\mu| = k+1-2g} \big(2g - 1\big)^{l(\mu)}\ \tilde{h}(\mu)\ \mathcal{Q}_\mu.$$

Proof. For simplicity assume that the coefficients of f (respectively, h) are integer numbers; if this is not the case we multiply L_k and f (respectively, L'_k and h) by some common multiple of the denominators.

Let $\mu = (\mu_1, \mu_2, \dots)$ be a sequence of indeterminates. We use the notation $|\mu| = \mu_1 + \mu_2 + \cdots$ and define variable $z = |\mu| + 2g - 1 - \Delta$. The family of indeterminates μ can be alternatively parametrized by $z, \mu_2, \mu_3, \dots$; we just use the substitution $\mu_1 = z + \Delta + 1 - 2g - \mu_2 - \mu_3 - \cdots$. Now we can consider $f, h \in \Lambda[z]$ as polynomials in one variable z with coefficients in the ring Λ of symmetric functions in variables $\mu_2, \mu_3, \dots$ and we can divide f and h by z with a remainder. Hence

$$f(\mu) = (|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu) + r(\mu_2, \mu_3, \dots),$$
$$h(\mu) = (|\mu| + 2g - 1 - \Delta)\tilde{h}(\mu) + s(\mu_2, \mu_3, \dots)$$

for some $\tilde{f}, \tilde{h} \in \Lambda[z]$ and for some $r, s \in \Lambda$. Below we will show that $r = s = 0$. This would imply that

$$f(\mu) = (|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu),$$
$$h(\mu) = (|\mu| + 2g - 1 - \Delta)\tilde{h}(\mu),$$

where by substitution $z = |\mu| + 2g - 1 - \Delta$ we view $\tilde{f}, \tilde{h} \in \Lambda[\mu_1]$ as polynomials in one variable μ_1 with coefficients in Λ. For any permutation π of the set of positive integers which moves only finitely many elements we have

$$(|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu_1, \mu_2, \dots) = f(\mu_1, \mu_2, \dots) =$$
$$f(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots) = (|\mu| + 2g - 1 - \Delta)\tilde{f}(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots)$$

hence from the cancellation property

$$\tilde{f}(\mu_1, \mu_2, \dots) = \tilde{f}(\mu_{\pi(1)}, \mu_{\pi(2)}, \dots)$$

is a symmetric function. In an analogous way we show that $\tilde{h}$ is a symmetric function. The lemma follows now immediately.

It remains now to show that $r = s = 0$. From the following on let $\mu_2 > \cdots > \mu_l$ be fixed integers bigger than 1 which fulfill Equations (3.12) and (3.11) with $m = \Delta - 2g$. Define $\mu_1 = p + \Delta + 1 - 2g - \mu_2 - \mu_3 - \cdots - \mu_l$, where p is a prime number. Notice that $\mu_1 - 1 < p$, because we required that $\Delta + 1 - 2g - \mu_2 - \mu_3 - \cdots - \mu_l < 1$. We consider the integral vector $\mu = (\mu_1, \dots, \mu_l)$.

If p is large enough, the parts of μ are all distinct and it follows that

$$[R_{\mu_1} R_{\mu_2} \cdots R_{\mu_l}] L_{p+\Delta} = (l + 2g - 2)!\, f(\mu)\, (\mu_1 - 1)(\mu_2 - 1) \cdots (\mu_l - 1).$$

Also, if prime number p is big enough then it does not divide $(l + 2g - 2)!\ (\mu_1 - 1)(\mu_2 -$

$1)\cdots(\mu_l - 1)$. It follows that for infinitely many prime numbers p the number

$$f(\mu) = p\tilde{f}(\mu) + r(\mu_2, \dots, \mu_l)$$

is divisible by p. We proved in this way that $r(\mu_2, \dots, \mu_l, 0, \dots)$ is an integer which is divisible by an infinite number of primes hence $r(\mu_2, \dots, \mu_l, 0, \dots) = 0$. Finaly Lemma 3.3.1 shows that $r = 0$.

If p is big enough then condition (3.12) holds true for all $1 \leq i \leq l - 1$ therefore every partition resulting from μ by gluing together some of its parts cannot be obtained by gluing the parts of μ in some other way (in fact, this property is the main reason of introducing Equation (3.12)). From (3.1) it follows that

$$[R_{\mu_1} R_{\mu_2} \cdots] L'_{p+\Delta} = \sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu) - 1\big)!\, h(\nu),$$

where $\nu \geq \mu$ means that partition ν can be obtained from partition μ by gluing some parts of μ. We also know that for infinitely many prime numbers p the following number

$$\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu) - 1\big)!\, h(\nu) =$$

$$p \left[\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu) - 1\big)!\, \tilde{h}(\nu) \right] + \sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu) - 1\big)!\, s(\nu')$$

is divisible by p, where for $\nu = (\nu_1, \nu_2, \dots)$ we denote $\nu' = (\nu_2, \nu_3, \dots)$. Notice that the set of values of ν' which contribute to the right hand side does not depend on the choice of p, because only ν_1 depends on the choice of p. Thus we proved that for infinitely many prime numbers p the second summand on the right hand side does not depend on the choice of p and is a fixed integer divisible by all these prime numbers, hence

$$\sum_{\nu \geq \mu} (2g-1)^{l(\nu)} \big(l(\nu) - 1\big)!\, s(\nu') = 0. \tag{3.13}$$

We will use induction over k to show that $s(x_2, \dots, x_k, 0, \dots)$ is equal to the zero polynomial for any $k > 1$, which proves that $s = 0$. Indeed, assume, that $s(x_2, \dots, x_k, 0, \dots)$ is equal to the zero polynomial for $k < l$. From the induction hypothesis it follows that all summands on the left-hand side of (3.13) vanish, except for $\nu = \mu$, which shows that $s(\mu') = 0$. We use Lemma 3.3.1 to show that $s = 0$ as claimed. □

3.3.2 DIVISIBILITY

In order to prove Theorem 3.1.1 we would like to apply Lemma 3.3.2 to Kerov polynomials. For this, we need some interesting arithmetic properties of coefficients of Kerov polynomials. The next lemma, which was formulated as a conjecture by Światosław Gal [Gal08], shows some properties of these kind.

Lemma 3.3.3. *If p is an odd prime number then*

(a) $\frac{\mathrm{Ch}_p - R_{p+1} + 2R_2}{p}$,

(b) $\frac{\mathrm{Ch}_{p-1} - R_p}{p}$,

(c) $\frac{\mathrm{Ch}_{p+1} - R_{p+2} + R_3}{p}$

are polynomials in free cumulants $R_2, R_3, \dots$ with nonnegative integer coefficients.

We will prove this Lemma in Section 3.4, because the proof is very technical. Finally, we can prove the main result.

3.3.3 PROOF OF THE MAIN RESULT

Proof of Theorem 3.1.1. We know by Proposition 3.1.2 that for any integer $g \geq 1$ there exist inhomogeneous symmetric functions $\tilde{f}_g$ and $\tilde{h}_g$, having maximal degree $4g$, such that

$$\begin{aligned} L_k := L'_k := k\ K_{k,k+1-2g} &= \sum_{|\mu|=k+1-2g} \big(l(\mu) + 2g - 2\big)!\ \tilde{f}_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}_g(\mu)\mathcal{Q}_\mu. \end{aligned}$$

By applying Lemma 3.3.2 for $\Delta = 0$ we obtain that

$$\begin{aligned} k\ K_{k,k+1-2g} &= k \sum_{|\mu|=k+1-2g} \big(l(\mu) + 2g - 2\big)!\ \tilde{f}'_g(\mu)\mathcal{R}_\mu \\ &= k \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}'_g, \tilde{h}'_g$ are symmetric functions of degree at most $4g-1$.

Let

$$\begin{aligned} L_k := L'_k := K_{k,k+1-2g} &= \sum_{|\mu|=k+1-2g} \big(l(\mu) + 2g - 2\big)!\ \tilde{f}'_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'_g(\mu)\mathcal{Q}_\mu. \end{aligned}$$

Lemma 3.3.3(a) shows that Lemma 3.3.2 can be applied for $\Delta = 0$, thus

$$\begin{aligned} K_{k,k+1-2g} &= k \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}''_g(\mu)\mathcal{R}_\mu \\ &= k \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}''_g, \tilde{h}''_g$ are symmetric functions of degree at most $4g-2$.

Let

$$\begin{aligned} L_k := L'_k := \frac{AK_{k,k+1-2g}}{k} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!A\ \tilde{f}''_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}A\ \tilde{h}''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where A is the common multiple of the denominators of coefficients of $\tilde{h}''_g$ and $\tilde{f}''_g$. We know that $L_{p-1} = L'_{p-1}$ has integer coefficients as polynomial in $R_2, R_3, \dots$ and we know, thanks to Lemma 3.3.3(b), that for infinitely many prime numbers p the coefficients of $(p-1)L_{p-1} = (p-1)L'_{p-1}$ are divisible by p, hence coefficients of $L_{p-1} = L'_{p-1}$ are also divisible by p, because $p-1$ and p are coprime. Then we can apply Lemma 3.3.2 for $\Delta = -1$ and we obtain that there exist symmetric functions $\tilde{f}'''_g, \tilde{h}'''_g$ of degree at most $4g-3$ such that

$$\begin{aligned} K_{k,k+1-2g} &= k\ (k+1) \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\ \tilde{f}'''_g(\mu)\mathcal{R}_\mu \\ &= k\ (k+1) \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\ \tilde{h}'''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where $\tilde{f}''_g, \tilde{h}''_g$ are symmetric functions of degree at most $4g-3$.

Similarly as before, thanks to Lemma 3.3.3(c) and thanks to the fact that for prime number $p > 2$ the numbers p and $p+1$ are coprime and the numbers $p+2$ and p are coprime, we can apply Lemma 3.3.2 for $\Delta = 1$ for

$$\begin{aligned} L_k := L'_k := \frac{BK_{k,k+1-2g}}{k(k+1)} &= \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!B\ \tilde{f}'''_g(\mu)\mathcal{R}_\mu \\ &= \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}B\ \tilde{h}'''_g(\mu)\mathcal{Q}_\mu, \end{aligned}$$

where B is the common multiple of the denominators of coefficients of $\tilde{h}'''_g$ and $\tilde{f}'''_g$ and we obtain that there exist symmetric functions $\tilde{f}_g'''', \tilde{h}_g''''$ of degree at most $4g-4$ such that

$$\begin{aligned} K_{k,k+1-2g} &= (k-1)\,k\,(k+1) \sum_{|\mu|=k+1-2g} \big(l(\mu)+2g-2\big)!\; \tilde{f}_g''''(\mu)\mathcal{R}_\mu \\ &= (k-1)\,k\,(k+1) \sum_{|\mu|=k+1-2g} (2g-1)^{l(\mu)}\; \tilde{h}_g''''(\mu)\mathcal{Q}_\mu, \end{aligned}$$

which finishes the proof. □

3.4 Technical lemmas

In this Section we prove all technical lemmas we used in this article.

3.4.1 Identities on symmetric functions

Lemma 3.4.1. *The following abstract equalities hold:*

$$\sum_{\mu\cup\rho=\nu} \mathfrak{h}_\lambda(\mu) \frac{(l(\mu)+2g-2)!}{l(\mu)!} u_\mu u_\rho = \frac{\mathfrak{h}_\lambda(\nu)}{l(\lambda)+2g-1} \frac{(l(\nu)+2g-2)!}{l(\nu)!} u_\nu; \tag{3.14}$$

$$\sum_{\mu\cup\rho=\nu} \mathfrak{h}_\lambda(\mu)(2g-1)^{l(\mu)} z_\mu^{-1} z_\rho^{-1} = \left(\frac{2g-1}{2g}\right)^{l(\lambda)} \mathfrak{h}_\lambda(\nu)(2g)^{l(\nu)} z_\nu^{-1}. \tag{3.15}$$

Proof. From the definition of the monomial symmetric function we know that $\mathfrak{h}_\lambda(\mu) = 0$ for all partitions μ such that $l(\mu) < l(\lambda)$. We use an identity that for every integer n such that $l(\lambda) \le n \le l(\nu)$ we have

$$\binom{l(\nu)-l(\lambda)}{n-l(\lambda)} \mathfrak{h}_\lambda(\nu) = \sum_{\substack{\mu\cup\rho=\nu,\\ l(\mu)=n}} \mathfrak{h}_\lambda(\mu) \left(\prod_{i\ge1} \frac{m_i(\nu)!}{m_i(\mu)!\, m_i(\rho)!}\right). \tag{3.16}$$

Indeed,

$$\begin{aligned} \sum_{\substack{\mu\cup\rho=\nu,\\ l(\mu)=n}} \mathfrak{h}_\lambda(\mu) \left(\prod_{i\ge1} \frac{m_i(\nu)!}{m_i(\mu)!\, m_i(\rho)!}\right) &= \sum_{\substack{\mu\subset\nu,\\ l(\mu)=n}} \mathfrak{h}_\lambda(\mu) = \\ \sum_{\substack{\mu'\subset\mu\subset\nu,\\ l(\mu)=n, l(\mu')=l(\lambda)}} \mathfrak{h}_\lambda(\mu') &= \sum_{\substack{\mu'\subset\nu,\\ l(\mu')=l(\lambda)}} \sum_{\substack{\mu'\subset\mu\subset\nu,\\ l(\mu)=n}} \mathfrak{h}_\lambda(\mu') = \\ \binom{l(\nu)-l(\lambda)}{n-l(\lambda)} \sum_{\substack{\mu'\subset\nu,\\ l(\mu')=l(\lambda)}} \mathfrak{h}_\lambda(\mu') &= \binom{l(\nu)-l(\lambda)}{n-l(\lambda)} \mathfrak{h}_\lambda(\nu), \end{aligned}$$

where $\sum_{\substack{\mu\subset\nu,\\ l(\mu)=n}}$ means that $\mu=(\nu_{\sigma(1)},\ldots,\nu_{\sigma(n)})$ for some $\sigma\in\mathfrak{S}_{l(\nu)}$ such that $\sigma(i)<\sigma(i+1)$ for all $i\in\{1,\ldots,n-1\}$ and we are summing over all such permutations σ. Now, we can write the left hand side of (3.14) in the following way:

$$\sum_{l(\lambda)\le n\le l(\nu)}\frac{(n+2g-2)!}{n!}\left(\sum_{\substack{\mu\cup\rho=\nu,\\ l(\mu)=n}}\mathfrak{h}_\lambda(\mu)\prod_{i\ge1}\frac{m_i(\nu)!}{m_i(\mu)\,!m_i(\rho)!}\right)\times$$
$$u_\nu\frac{n!\,(l(\nu)-n)!}{l(\nu)!}=$$
$$\sum_{l(\lambda)\le n\le l(\nu)}\frac{(n+2g-2)!}{n!}\frac{(l(\nu)-l(\lambda))!}{(l(\nu)-n)!(n-l(\lambda))!}\mathfrak{h}_\lambda(\nu)\times$$
$$u_\nu\frac{n!\,(l(\nu)-n)!}{l(\nu)!}=$$
$$\frac{(l(\nu)-l(\lambda))!}{l(\nu)!}\sum_{l(\lambda)\le n\le l(\nu)}\frac{(n+2g-2)!}{(n-l(\lambda))!}\mathfrak{h}_\lambda(\nu)u_\nu$$

and using the equality

$$\sum_{0\le i\le b}\binom{a+i}{i}=\binom{a+b+1}{b}$$

we have

$$\frac{(l(\nu)-l(\lambda))!}{l(\nu)!}\sum_{l(\lambda)\le n\le l(\nu)}\frac{(n+2g-2)!}{(n-l(\lambda))!}=$$
$$\frac{(l(\nu)-l(\lambda))!(l(\lambda)+2g-2)!}{l(\nu)!}\sum_{0\le n\le l(\nu)-l(\lambda)}\frac{(n+l(\lambda)+2g-2)!}{n!(l(\lambda)+2g-2)!}=$$
$$\frac{(l(\nu)-l(\lambda))!(l(\lambda)+2g-2)!}{l(\nu)!}\sum_{0\le n\le l(\nu)-l(\lambda)}\binom{l(\lambda)+2g-2+n}{n}=$$
$$\frac{(l(\nu)-l(\lambda))!\,(l(\lambda)+2g-2)!\,(l(\nu)+2g-1)!}{l(\nu)!\,(l(\nu)-l(\lambda))!\,(l(\lambda)+2g-1)!}=\frac{(l(\nu)+2g-1)!}{l(\nu)!(l(\lambda)+2g-1)}$$

which finishes the proof of (3.14).

Using (3.16) we can write the left hand side of (3.15) in the following form:

$$\sum_{l(\lambda)\leq n\leq l(\nu)} (2g-1)^n \left(\sum_{\substack{\mu\cup\rho=\nu,\\ l(\mu)=n}} \mathfrak{h}_\lambda(\mu) \prod_{i\geq 1} \frac{m_i(\nu)!}{m_i(\mu)\,!m_i(\rho)!} \right) z_\nu^{-1} =$$

$$\left(\sum_{l(\lambda)\leq n\leq l(\nu)} (2g-1)^n \binom{l(\nu)-l(\lambda)}{n-l(\lambda)} \right) \mathfrak{h}_\lambda(\nu) z_\nu^{-1} =$$

$$(2g-1)^{l(\lambda)} \left(\sum_{0\leq n\leq l(\nu)-l(\lambda)} \binom{l(\nu)-l(\lambda)}{n} (2g-1)^n \right) \mathfrak{h}_\lambda(\nu) z_\nu^{-1} =$$

$$\left(\frac{2g-1}{2g}\right)^{l(\lambda)} \mathfrak{h}_\lambda(\nu)(2g)^{l(\nu)} z_\nu^{-1},$$

where the last equality holds because of the binomial identity:

$$\sum_{0\leq n\leq m} \binom{m}{n} a^n = (a+1)^m,$$

which finishes the proof. □

3.4.2 Proof of Lemma 3.3.1

Proof of Lemma 3.3.1. Let l be fixed; we will use backward induction over k. For $k = l$ we know that $f(\mu_k) = 0$ for infinitely many choices of μ_k. In other words, polynomial f has infinitely many zeros hence $f = 0$, as claimed.

Let us assume that the inductive assertion holds for some $k \leq l$. We can write

$$f(x_{k-1},\dots,x_l) = \sum_{0\leq i\leq N} x_{k-1}^i \, f_i(x_k,\dots,x_l)$$

for some N. Let us fix integers $\mu_k,\dots,\mu_l$ bigger than 1 which satisfy (3.12) and (3.11). Then we can find infinitely many integer numbers μ_{k-1} for which the vector $(\mu_{k-1},\dots,\mu_l)$ satisfies both (3.12) and (3.11); for each such a number we have $f(\mu_{k-1},\dots,\mu_l) = 0$ therefore the polynomial $x_{k-1} \mapsto f(x_{k-1},\mu_k,\dots,\mu_l)$ has infinitely many zeros hence it is the zero polynomial and $f_i(\mu_k,\dots,\mu_l) = 0$. This shows that the inductive assertion can be applied to the polynomial $f_i(x_k,\dots,x_l)$ and therefore $f_i(x_k,\dots,x_l) = 0$. This finishes the proof. □

3.4.3 Arithmetic properties of Kerov polynomials

Auxiliary results

We present two theorems we need to prove Lemma 3.3.3.

Theorem 3.4.2 (Dołęga, Féray, Śniady [DFŚ10]). *Let $k \geq 1$ and let $s_2, s_3, \dots$ be a sequence of non-negative integers with only finitely many non-zero elements. The coefficient of $R_2^{s_2} R_3^{s_3} \cdots$ in the Kerov polynomial K_k is equal to the number of triples (σ_1, σ_2, q) with the following properties:*

(a) σ_1, σ_2 is a factorization of the cycle; in other words $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ are such that $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$;

(b) the number of cycles of σ_2 is equal to the number of factors in the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_2)| = s_2 + s_3 + \cdots$;

(c) the total number of cycles of σ_1 and σ_2 is equal to the degree of the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$;

(d) $q : C(\sigma_2) \to \{2, 3, \dots\}$ is a coloring of the cycles of σ_2 with a property that each color $i \in \{2, 3, \dots\}$ is used exactly s_i times (informally, we can think that q is a map which to cycles of $C(\sigma_2)$ associates the factors in the product $R_2^{s_2} R_3^{s_3} \cdots$);

(e) for every set $A \subset C(\sigma_2)$ which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq C(\sigma_2)$) there are more than $\sum_{i \in A} \big(q(i) - 1\big)$ cycles of σ_1 which intersect $\bigcup A$.

We say that a partition Π of the set $[k] = \{1, \dots, k\}$ is a *pushing partition* if any pair of neighboring elements of $[k]$ with respect to the cyclic order (i.e. i and $i+1$ are a pair of neighboring elements for any $1 \leq i \leq k-1$ as well as 1 and k) does not belong to the same block of Π.

The cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on the set of all partitions (respectively, the set of pushing partitions) of the set $[k]$ as follows: for a partition Π of $[k]$ and $i \in \mathbb{Z}/k\mathbb{Z}$ we define $i + \Pi$ as the partition of $[k]$ with a property that a, b belong to the same block of Π if and only if a', b' belong to the same block of $i + \Pi$ for all $a, b, a', b' \in [k]$ such that $a + i \equiv a' \pmod{k}$, $b + i \equiv b' \pmod{k}$.

For any pushing partition Π it is possible (see [Śni06a]) to define the normalized character Ch_Π. It has a property that $\mathrm{Ch}_\Pi = \mathrm{Ch}_\pi$, where the right-hand side should be understood as in (2.1) for some $\pi \in \mathfrak{S}_l$, $l \geq 1$. So defined partition-indexed character has the following properties:

Theorem 3.4.3 (Proposition 4.4, Claim 3.1, Proposition 3.2 in [Śni06a]).

- *The map $\Pi \mapsto \mathrm{Ch}_\Pi$ is constant on the orbits of the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on the set of pushing partitions of $[k]$.*

- *For any integer $k \geq 2$*

$$R_k = \sum_{\Pi} I_\Pi \ \mathrm{Ch}_\Pi, \tag{3.17}$$

where the sum runs over pushing partitions of $[k]$ and $I_\Pi \in \mathbb{Z}$, called free index, is constant on the orbits of the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on the set of pushing partitions of $[k]$.

- *For the minimal partition $\Pi = \big\{\{1\},\dots,\{k\}\big\}$ the corresponding character is given by*

$$\mathrm{Ch}_{\{\{1\},\dots,\{k\}\}} = \mathrm{Ch}_{k-1}.$$

PROOF OF LEMMA 3.3.3

Proof of Lemma 3.3.3. In the following we shall prove that the coefficients are integer numbers. Their nonnegativity would follow from Theorem 3.4.2.

In order to prove that the coefficients of $\frac{\mathrm{Ch}_p - R_{p+1} + 2R_2}{p}$ are integer we consider the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ on the set of triples (σ_1, σ_2, q) which contribute to Theorem 3.4.2 defined by conjugation

$$\psi(i)(\sigma_1, \sigma_2, q) = \big(c^i \sigma_1 c^{-i}, c^i \sigma_2 c^{-i}, q'\big),$$

where $c = (1, 2, \dots, k)$ is the cycle and $q'(a) = q(c^{-i} a c^i)$ for $a \in C(\sigma_2)$. All orbits of this action consist of p elements except for the fixpoints of this action which are of the form $\sigma_1 = c^a$, $\sigma_2 = c^{1-a}$. These fixpoints contribute to the monomial R_{p+1} (with multiplicity 1) and to the monomial R_2 (with multiplicity $p-2$). This finishes the proof of the integrality of coefficients of (a).

We apply Theorem 3.4.3 in the case when $k = p$ is a prime number. The right-hand side of (3.17) is constant on each orbit of the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$. Each orbit of this action consists of p elements, except for the fixpoints. The only pushing partition of $[p]$ which is invariant under the action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ is the minimal partition $\big\{\{1\},\dots,\{p\}\big\}$. In this way we proved that

$$R_p = \mathrm{Ch}_{p-1} + p\,\Big(\text{linear combination of the characters } \mathrm{Ch}_\pi \\ \text{for } \pi \in \mathfrak{S}_l,\ l \geq 1 \text{ with integer coefficients}\Big).$$

Thanks to Kerov polynomials, each Ch_π can be written as a polynomial in free cumulants with integer coefficients. This shows part (b).

In the following we shall use the notations and results presented in the paper of Biane [Bia03]. In order to prove part (c) we consider the formal power series

$$H(z) = z - \sum_{j \geq 1} B_{j+1} z^{-j}$$

where B_j are Boolean cumulants. Biane showed that

$$(-p-1)\operatorname{Ch}_{p+1} = [z^{-1}]H(z)H(z-1)\cdots H(z-p) \tag{3.18}$$

and

$$(-p-1)R_{p+2} = [z^{-1}]H(z)^{p+1}.$$

We know from [Bia03] that B_j is a polynomial in free cumulants $R_2, R_3, \dots$ with integer coefficients as well as Ch_{p+1} is a polynomial in Boolean cumulants $B_2, B_3, \dots$ with integer coefficients; hence it suffices to show that $(-p-1)(\operatorname{Ch}_{p+1} - R_{p+2} + R_3)$ is a polynomial in Boolean cumulants with all coefficients divisible by p. It is equivalent to show that $\operatorname{Ch}_{p+1} - R_{p+2} + R_3 = 0$ under additional assumption that all coefficients of the power series are taken from a field of characteristic p, hence all formulas are considered in a field of characteristic p from now.

From (3.18) it follows that

$$[B_3]\operatorname{Ch}_{p+1} = \frac{1}{p+1}\sum_{0\le z\le p} \frac{1}{2}\frac{d^2}{dz^2}\big[z(z-1)\cdots(z-p)\big].$$

From Fermat's little theorem (see for example [GKP88]) it follows that in the field of characteristic p

$$z(z-1)\cdots(z-p) = z(z^p - z) = z^{p+1} - z^2$$

hence

$$[B_3]\operatorname{Ch}_{p+1} = \frac{1}{p+1}\sum_{0\le z\le p}(-1) = -1. \tag{3.19}$$

We define $B_0 = -1$ and $B_1 = 0$; then using binomial formula we have

$$H(z-i) = \sum_{j\ge -1}\sum_{k\ge 0}(-1)^{k+1}\binom{-j}{k} i^k B_{j+1} z^{-(j+k)},$$

hence

$$-\frac{1}{p+1}H(z)H(z-1)\cdots H(z-p) = \\ -\frac{1}{p+1}\sum_{k\in A}\sum_{j\in B}(-1)^{|k|_1+p+1} \\ \left(\prod_{0\le i\le p}\binom{-j_i}{k_i} i^{k_i} B_{j_i+1}\right) z^{-|j|_0-|k|_1}, \tag{3.20}$$

where $A, B \subset \mathbb{Z}^{p+1}$ such that

$$A = \{(k_0, k_1, \dots, k_p) : k_i \geq 0 \text{ for } 0 \leq i \leq p\},$$

$$B = \{(j_0, j_1, \dots, j_p) : j_i \geq -1 \text{ for } 0 \leq i \leq p\}$$

and for $k = (k_0, k_1, \dots, k_p)$, $i \in \{0, 1\}$ the sum $k_i + k_{i+1} + \cdots + k_p$ is denoted by $|k|_i$. For any $a \in \mathbb{Z}/p\mathbb{Z}$ such that $a \neq 0$ the map $x \mapsto ax$ is a bijection of the multiset $(0, 1, \dots, p) \subset \mathbb{Z}/p\mathbb{Z}$ (notice that $0 = p$ appears twice in this multiset) therefore the left-hand side of (3.20) is equal to

$$-\frac{1}{p+1} H(z)H(z-a)\cdots H(z-pa) = \\ -\frac{1}{p+1} \sum_{k \in A} \sum_{j \in B} (-1)^{|k|_1 + p + 1} a^{|k|_1} \\ \left(\prod_{0 \leq i \leq p} \binom{-j_i}{k_i} i^{k_i} B_{j_i+1} \right) z^{-|j|_0 - |k|_1}. \quad (3.21)$$

The coefficient of z^{-1} in (3.21) can be viewed as a polynomial in a; we shall denote it by $P(a)$. In the following we will study its coefficients of highest degrees. We are interested only in the summands for which $|j|_0 + |k|_1 = 1$; since $|j|_0 \geq -p-1$ therefore $|k|_1 \leq p+2$ and the degree of $P(a)$ is at most $p+2$.

However, $|k|_1 = p+2$ would correspond to the case $j = (-1, 1, \dots, -1)$ which is equivalent to setting $B_2 = B_3 = \cdots = 0$; therefore $[a^{p+2}]P(a) = 0$.

For $|k|_1 = p+1$ there is no summand for which $j_0, \dots, j_p \neq 0$ hence $[a^{p+1}]P(a) = 0$.

For $|k|_1 = p$ every summand which contributes is of the following form: one of the numbers $j_0, \dots, j_p$ is equal to 1 and all the others are equal to -1. This shows that $[a^p]P(a)$ viewed as a polynomial in $B_2, B_3, \dots$ contains only one monomial, namely a multiple of B_2; also $[B_2]P(a)$ viewed as a polynomial in a contains only one monomial namely a multiple of a^p. Therefore $[a^p]P(a)$ is a multiple of B_2 and the value of the coefficient of B_2 fulfills:

$$[B_2][a^p]P(a) = [B_2]P(1) = [B_2] \operatorname{Ch}_{p+1}.$$

Since p is odd, the expansion of Ch_{p+1} into Boolean cumulants contains only summands which are of odd degree [Bia03]; it follows that $[a^p]P(a) = 0$.

In an analogous way we prove that $[a^{p-1}]P(a)$ is a multiple of B_3 and

$$[B_3][a^{p-1}]P(a) = [B_3] \operatorname{Ch}_{p+1} = -1$$

from (3.19).

In this way we proved that $P(a)$ is a polynomial of degree $p-1$ which takes the same value for all $a \in \{1, \dots, p-1\}$. Polynomial

$$\widetilde{P}(a) = -B_3 a^{p-1} + P(0)$$

has the same properties. It follows that $P - \widetilde{P}$ has degree at most $p-2$ which takes the same value for all $a \in \{1, \dots, p-1\}$ hence it must be equal to the constant. It follows that $\widetilde{P} = P$.

Therefore

$$\mathrm{Ch}_{p+1} = P(1) = -B_3 + R_{p+2}.$$

Observation that $B_3 = R_3$ finishes the proof for the third expression. □

It is interesting that for the first two expressions we managed to find combinatorial proofs while for the last expression there seems to be no natural candidate for a combinatorial approach.

4

On Kerov polynomials for Jack characters

Abstract

We consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that some quantities describing the structure of Kerov polynomials are polynomials with prescribed degree in the Jack parameter α. As a consequence we prove some conjectures of Lassalle.

Our result has several interesting consequences in various directions. Firstly, we give a new proof of the fact that the coefficients of Jack polynomials expanded in the monomial or power-sum basis depend polynomially in α. Secondly, a small part of Matching Jack conjecture of Goulden and Jackson is proved. Finally, the last and the main consequence is Law of Large Numbers and Central Limit Theorem for random Young diagrams under Jack measure, which is a one-parameter deformation of Plancherel measure. This result is a generalization of celebrated Vershik-Kerov's limit shape and Kerov's Central Limit Theorem and is proved using multivariate Stein's method.

4.1 Introduction

4.1.1 Jack polynomials

In a seminal paper [Jac71], H. Jack introduced a family of symmetric functions $J_\lambda^{(\alpha)}$ depending on an additional parameter α. These functions are now called *Jack polynomials*. For some special values of α, they coincide with some established families of symmetric functions. Namely, up to multiplicative constants, for $\alpha = 1$ Jack polynomials coincide with Schur polynomials, for $\alpha = 2$ they coincide with zonal polynomials, for $\alpha = \frac{1}{2}$ they coincide with symplectic zonal polynomials, for $\alpha = 0$ we recover the elementary symmetric functions and finally their highest degree component in α are the monomial symmetric functions. Moreover, some other specializations appear in different contexts: the case $\alpha = 1/k$, where k is an integer, has been considered by Kadell in relation with generalizations of Selberg's integral [Kad97]. In addition, Jack polynomials for $\alpha = -(k+1)/(r+1)$ verify some interesting annihilation conditions [FJMM02] and this property makes them useful in some statistical physics models.

Over the time it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials (Section (VI,10) of I.G. Macdonald's book [Mac95] gives a few results of this kind), therefore Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions.

4.1.2 A weak form of Macdonald-Stanley ex-conjecture

One of the most surprising features of Jack polynomials is that they have several equivalent classical definitions but none of them makes obvious the fact that the coefficients of their expansion on the augmented monomial basis are polynomials in α with non-negative integer coefficients (this does not follow either from Stanley's combinatorial interpretation [Sta89]). This conjecture of Stanley and Macdonald [Mac95, Equation (10.26?)] has been proved after a while by Knop and Sahi [KS97].

In fact, even the weaker fact that these coefficients are polynomials in α with rational coefficients is quite hard (by construction, they are only rational functions). It has been established by Lapointe and Vinet [LV95] shortly before Knop and Sahi's paper.

One of the results of this chapter is a new proof of this weaker form of Stanley-Macdonald conjecture.

Theorem 4.1.1 (Lapointe and Vinet [LV95])**.** *The coefficients of the expansion of Jack polynomials in the monomial basis are polynomials in α.*

This theorem is proved in Section 4.2.7. We believe that this new proof is interesting in itself, because it relies on a very different approach to Jack polynomials.

4.1.3 Dual approach

Despite the result of Knop and Sahi, there seems still to remain some not well understood combinatorics in this topic. We recall (see Subection 2.3.2) that *Jack character* are equal (up to some simple normalization constants) to the coefficient $[p_\mu]J_\lambda^{(\alpha)}$ in the expansion of the Jack polynomial $J_\lambda^{(\alpha)}$ in the basis of power-sum symmetric functions. The idea of dual approach is to consider Jack character as a function of λ and not as a function of μ as usual. In more concrete words, we would like to express the Jack character as a sum of some quantities depending on λ over some combinatorial set depending on μ (in Stanley's or Knop-Sahi's results, it is roughly the opposite).

Inspired by the case $\alpha = 1$ (which corresponds to the usual characters of the symmetric groups), Lassalle [Las09] suggested to express Jack characters in terms of, so called, free cumulants of the transition measure of the Young diagram λ (we recall that Lassalle used different normalization; we compare his normalization to our in Subsection 2.3.2). This expression, called *Kerov polynomials for Jack characters*, involves rational functions in α, which are conjecturally polynomials with non-negative coefficients in α and $\beta = 1 - \alpha$ (we refer to this as Lassalle's conjecture). This suggests the existence of a combinatorial interpretation. A result of this type holds true in the case $\alpha = 1$, see [DFŚ10].

In this chapter, we prove a part of Lassalle's conjecture, that is the polynomiality in α (but not the non-negativity) of the coefficients.

Theorem 4.1.2. *The coefficients of Kerov polynomials for Jack characters are polynomials in α with rational coefficients.*

This theorem is proved in Section 4.2.6. Our method even gives (several) bounds for the degree of the coefficients. Theorem 4.1.1 is a consequence of one of these bounds.

Remark 4.1.3. One more time, we recall that Theorem 4.1.2 is formulated using Lassalle's normalization which is different than our.

4.1.4 Structure constants

Kerov polynomials turn out to be a suitable tool to study the multiplication table of Jack characters. In particular, we are able to show that the structure constants in this multiplication table are polynomials in the variable $\gamma = \frac{1-\alpha}{\sqrt{\alpha}}$ with a prescribed degree. This may seem anecdotal, but has consequences on several objects, introduced independently in the literature.

- We give a new proof of a recent result of Aker and Can [AC12], who studied the structure constants in the Hecke algebra of the pair (S_{2n}, H_n) (see Section 4.3.4 for definition and details).
- I. P. Goulden and D. M. Jackson have defined in [GJ96a] an interpolation between these structure constants and the ones of the center of the symmetric group algebra. By con-

struction, these quantities are rational functions in α but they conjectured that they are in fact polynomials with non-negative integer coefficients having some combinatorial interpretation [GJ96a, Section 4]. Here, we prove that they are *polynomials* in α. Unfortunately, we are not able to prove neither the integrality nor the positivity of the coefficients.

- We are also able to prove two conjectures of S. Matsumoto [Mat10, Section 9], arising in the context of matrix integrals (Section 4.3.7).

4.1.5 Asymptotics of Jack deformation of Plancherel measure

Our bounds for coefficients of Kerov polynomials for Jack characters imply in particular that some coefficients (corresponding to the leading term for some gradation) are independent on α. In Section 4.5, we use this simple remark to describe asymptotically the shape of random Young diagrams whose distribution is a deformation of Plancherel measure. Moreover, using Multivariate Stein's Method [RR09], our bounds for coefficients of Kerov polynomials for Jack characters and for structure constants, we are able to show that the fluctuations of the random Young diagrams around this limit shape are Gaussian. This result, presented in Sections 4.6, 4.7, is a generalization of celebrated Kerov's Central Limit Theorem [Ker93a, IO02].

4.1.6 Outline of this chapter

This chapter is organized as follows. Section 2.3 gives all necessary definitions and background; in particular we recall the notions of free cumulants and Kerov polynomials. In Section 4.2 we prove Theorem 4.1.2 with bounds on the degree of the polynomials. Then we consider the multiplication table of Jack characters and related problems in Section 4.3. The last four sections are devoted to the computation of some coefficients of Kerov polynomials for Jack characters and, as a consequence, to the study of the aforementioned large random Young diagrams.

4.2 Polynomiality

4.2.1 Notations

As in the previous sections, most of our objects are indexed by partitions of integers. Therefore it will be useful to use some short notations for small modifications (adding or removing a box or a part) of partitions. We denote by $\mu \cup (r)$ (resp. $\mu \setminus (r)$) the partition obtained from μ by adding (resp. deleting) one part equal to r. We denote by $\mu_{\downarrow r} = \mu \setminus (r) \cup (r-1)$ the partition obtained by removing one box in a row of size r. The reader might wonder what $\mu \setminus (r)$ and $\mu_{\downarrow r}$ mean if μ does not have a part equal to r: we will not use these notations in this context. Finally, if o is an outer corner of λ, we denote by $\lambda^{(o)}$ the diagram obtained from λ by adding a box at place o.

4.2.2 How to compute Jack character polynomials?

M. Lassalle [Las09] gave an algorithm for computing $K_\mu^{(\alpha)}$ by induction over μ. In this section we present a slightly simpler version of this algorithm which allows to compute $L_\mu^{(\alpha)}$ instead of $K_\mu^{(\alpha)}$.

One of the base ingredients is the following formula, which corresponds to [Las09, Proposition 8.3].

Proposition 4.2.1. *Let $k \geq 2$, λ be a Young diagram and $o = (x, y)$ an outer corner of λ.*

$$M_k^{(\alpha)}(\lambda^{(o)}) - M_k^{(\alpha)}(\lambda) = \sum_{\substack{r \geq 1, s,t \geq 0,\\ 2r+s+t \leq k}} z_o^{k-2r-s-t} \binom{k-t-1}{2r+s-1} \binom{r+s-1}{s} \left(\frac{-\beta}{\sqrt{\alpha}}\right)^s M_t^{(\alpha)}(\lambda),$$

where $\beta = 1 - \alpha$ and $z_o = \sqrt{\alpha}x - \sqrt{\alpha}^{-1} y$ is the anisotropic content of the corner corresponding to o in the stretched diagram $T_{\sqrt{\alpha}, \sqrt{\alpha}^{-1}}(\lambda)$.

Proof. As mentioned above, this is exactly [Las09, Proposition 8.3]. To help the reader see that, we compare our notations to Lassalle's ones (we use boldface to refer to his notations):

$$M_k^{(\alpha)}(\lambda^{(o)}) = \alpha^{k/2} \boldsymbol{M_k(\lambda^{(i)})};$$
$$M_t^{(\alpha)}(\lambda) = \alpha^{t/2} \boldsymbol{M_t(\lambda)};$$
$$z_o = \sqrt{\alpha} \cdot \boldsymbol{x_i}.$$

□

Note that the quantity $\gamma := \frac{\beta}{\sqrt{\alpha}}$ seems to play a particular role in the above formula.

For any partition ρ we define $M_\rho^{(\alpha)}(\lambda^{(o)}) := \prod_i M_{\rho_i}^{(\alpha)}(\lambda^{(o)})$ by multiplicativity. The above proposition implies immediately the following corollary:

Corollary 4.2.2. *For any partition ρ, any diagram λ and any outer corner o of λ,*

$$M_\rho^{(\alpha)}(\lambda^{(o)}) = M_\rho^{(\alpha)}(\lambda) + \sum_{\substack{g,h \geq 0,\\ \pi \vdash h}} b_{g,\pi}^\rho(\gamma)\, z_o^g\, M_\pi^{(\alpha)}(\lambda),$$

where $b_{g,\pi}^\rho(\gamma)$ is a polynomial in γ.

Proof. The case when ρ consists of only one part is a direct consequence of Proposition 4.2.1 (one even has an explicit expression for $b_{g,\pi}^\rho(\gamma)$ in this case). The general case follows by multiplication. □

This corollary is an analogue of Equation (8.1) in [Las09].

Let μ be a partition. By definition of $L_\mu^{(\alpha)}$, there exist some numbers a_ρ^μ (depending on α)

such that, for any Young diagram λ,

$$\mathrm{Ch}^{(\alpha)}_{\mu}(\lambda) = \sum_{\rho} a^{\mu}_{\rho}\, M^{(\alpha)}_{\rho}(\lambda).$$

Using Corollary 4.2.2 we can compute

$$\mathrm{Ch}^{(\alpha)}_{\mu}(\lambda^{(o)}) = \sum_{\rho} a^{\mu}_{\rho}\; M^{(\alpha)}_{\rho}(\lambda^{(o)}) = \mathrm{Ch}^{(\alpha)}_{\mu}(\lambda) + \sum_{\rho} a^{\mu}_{\rho} \left(\sum_{\substack{g,h\geq 0,\\ \pi\vdash h}} b^{\rho}_{g,\pi}(\gamma)\; z^{g}_{o}\; M^{(\alpha)}_{\pi}(\lambda) \right). \tag{4.1}$$

The second ingredient of Lassalle's algorithm is some linear identity between the values of Jack character evaluated on different diagrams. We denote by $c_o(\lambda)$ the probability of the corner o in the transition measure $\mu_{T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)}$, so that

$$M^{(\alpha)}_{k}(\lambda) = \sum_{o} c_o(\lambda) z^{k}_{o}.$$

Then we have [Las09, Equation (3.6)] the following proposition.

Proposition 4.2.3. *For any (regular generalized, see Section 2.5) Young diagram λ and any partition μ*

$$\sum_{o\in\mathbb{O}_{\lambda}} c_o(\lambda)\,\mathrm{Ch}^{(\alpha)}_{\mu}(\lambda^{(o)}) = m_1(\mu)\,\mathrm{Ch}^{(\alpha)}_{\mu\setminus 1}(\lambda) + \mathrm{Ch}^{(\alpha)}_{\mu}(\lambda),$$
$$\sum_{o\in\mathbb{O}_{\lambda}} c_o(\lambda) z_o\,\mathrm{Ch}^{(\alpha)}_{\mu}(\lambda^{(o)}) = \sum_{r\geq 2} r m_r(\mu)\,\mathrm{Ch}^{(\alpha)}_{\mu_{\downarrow r}}.$$

Proof. It is an exercise to adapt Equations (3.6) of [Las09] to our notations. Note that our formulation (unlike Lassalle's) works also if μ has some parts equal to 1 and is more pleasant for the second equation. □

Plugging Equation (4.1) into this proposition we obtain the following equalitites between functions on the set of (generalized) Young diagrams:

$$\sum_{\rho} a^{\mu}_{\rho} \left(\sum_{\substack{g,h\geq 0,\\ \pi\vdash h}} b^{\rho}_{g,\pi}(\gamma) M^{(\alpha)}_{\pi\cup(g)} \right) = m_1(\mu)\,\mathrm{Ch}^{(\alpha)}_{\mu\setminus 1}, \tag{A}$$

$$\sum_{\rho} a^{\mu}_{\rho} \left(\sum_{\substack{g,h\geq 0,\\ \pi\vdash h}} b^{\rho}_{g,\pi}(\gamma) M^{(\alpha)}_{\pi\cup(g+1)} \right) = \sum_{r\geq 2} r\cdot m_r(\mu)\,\mathrm{Ch}^{(\alpha)}_{\mu_{\downarrow r}}. \tag{B}$$

Fix some partitions μ and suppose that for all partitions ν of size smaller than $|\mu|$, we know the polynomial $L_\nu^{(\alpha)}$. Then we can identify the coefficient of a given monomial $M_\tau^{(\alpha)}$ in the above equations. This gives us two linear equations which will be denoted by (A_τ) and (B_τ), where the variables are the coefficients a_ρ^μ.

This leads to a *finite* system of linear equations (indeed, $a_\rho^\mu = 0$ as soon as $|\rho| \geq |\mu| + \ell(\mu)$ [Las09, Proposition 9.2 (ii)]). As explained by M. Lassalle, the system obtained that way has a unique solution (we shall see another explanation of that in the next paragraph) and thus, one can compute the coefficients a_ρ^μ by induction over $|\mu|$.

4.2.3 A TRIANGULAR SUBSYSTEM

In the previous section we explained how to determine the coefficients a_ρ^μ (where ρ runs over partitions without any part equal to 1) of $L_\mu^{(\alpha)}$ as the solution of an overdetermined linear system of equations. Recall that this is a copy of Lassalle's method of computing the coefficients of $K_\mu^{(\alpha)}$. In this section we shall extract a triangular subsystem. Note that this idea would not work with $K_\mu^{(\alpha)}$.

We will need an order on all partitions: let us define $<_1$ as follows:

$$\rho <_1 \rho' \iff \begin{cases} |\rho| < |\rho'|; \\ |\rho| = |\rho'| \text{ and } \ell(\rho) > \ell(\rho'); \\ |\rho| = |\rho'|,\ \ell(\rho) = \ell(\rho') \text{ and } \min(\rho) > \min(\rho'). \end{cases}$$

We say that an equation involves a variable if its coefficient is non-zero.

Lemma 4.2.4. *Let ρ be a partition and $q = \min(\rho)$ its smallest part.*

- *If $q = 2$, set $\tau = \rho \setminus (2)$. Then Equation (A_τ) involves the variable a_ρ^μ and involves some of the variables $a_{\rho'}^\mu$ for $\rho' >_1 \rho$ (and no other variables $a_{\rho'}^\mu$).*
- *If $q > 2$, set $\tau = \rho_{\downarrow q}$. Then Equation (B_τ) involves the variable a_ρ^μ and some of the variables $a_{\rho'}^\mu$ for $\rho' >_1 \rho$ (and no other variables $a_{\rho'}^\mu$).*

Proof. Using Proposition 4.2.1, we have the following equality:

$$M_\rho^{(\alpha)}(\lambda^{(o)}) = M_\rho^{(\alpha)}(\lambda) + \sum_{i \leq \ell(\rho)} M_{\rho\setminus\rho_i}^{(\alpha)} \left(\sum_{\substack{t\geq 0,\\ g=\rho_i-2-t}} (\rho_i - t - 1) M_t^{(\alpha)}(\lambda) z_o^g \right) + \sum_{\substack{\pi,g\\ |\pi|+g<|\rho|-2}} b_{g,\pi}^\rho(\gamma) M_\pi^{(\alpha)}(\lambda) z_o^g.$$

Indeed, it is true for $\rho = (k)$ and follows directly for any ρ by multiplication. Please notice that in the right-hand side, for all terms, one has:

$$|\pi| + g \leq |\rho| - 2$$

with equality only if $\pi \cup (g)$ (resp. $\pi \cup (g+1)$) is obtained from ρ by choosing a part, removing 2 (resp. removing 1) to this part and splitting it in two parts (which may be empty).

Fix a partition τ. Let us look which variables appear in the coefficient of $M_\tau^{(\alpha)}$ in the left-hand side of Equation (A_τ). In other terms, we want to determine for which ρ', the partition τ can appear as $\pi \cup (g)$ in the equation above. The first condition is that $|\tau| = |\pi| + g \leq |\rho'| - 2$, *i.e.* $|\rho'| \geq |\tau| + 2$. Moreover, if $|\rho'| = |\tau| + 2$, then τ must be obtained from ρ' by removing 2 from some part and splitting it into two. In particular, τ is at least as long as ρ', unless both new parts are empty. This can happen only if the split part of ρ' was 2, that is if $\rho' = \tau \cup (2)$. Denote $\rho = \tau \cup (2)$, we have proved that (A_τ) can involve $a^\mu_{\rho'}$ only if $\rho' >_1 \rho$. It is easy to check that the coefficient of a^μ_ρ is $m_2(\rho)$, which finishes the proof of the first point.

The proof of the second point is quite similar. Fix a partition ρ, denote q its smallest part and $\tau = \rho_{\downarrow q}$. The variables $a^\mu_{\rho'}$ can appear in the equation (B_τ) only if $|\tau| = |\pi| + g + 1 \leq |\rho'| - 1$, i.e. $|\rho'| \geq |\tau| + 1$. Moreover, if there is equality, τ must be obtained from ρ' by removing 1 from some part and splitting it into two. One of the two new parts is always non-empty, thus τ is at least as long as ρ'. If they have the same length, it means that τ is obtained from ρ' by shortening a part. If this part is equal to q, then $\rho' = \rho$. Otherwise ρ' contains a part $q - 1$ and thus $\rho' >_1 \rho$ (they have same size and same length). Finally, we have proved that (B_τ) can involve $a^\mu_{\rho'}$ only if $\rho' >_1 \rho$. Once again, the coefficient of a^μ_ρ in (B_τ) is easy to compute: it is $(q-1)m_q(\rho)$. □

The first interesting consequence is the following.

Corollary 4.2.5. *The coefficient a^μ_ρ is a polynomial in γ with rational coefficients. The same is true for the coefficients of Kerov's polynomials $K_\mu^{(\alpha)}$.*

Proof. We proceed by induction over $|\mu|$. The quantities a^μ_ρ are the solution of a triangular linear system, whose right-hand side is a vector of $a^{\mu'}_\tau$ with $|\mu'| < |\mu|$. By induction hypothesis, the right-hand sides belong to $\mathbb{Q}[\gamma]$. The coefficients $b^\rho_{g,\pi}(\gamma)$ of the system also belong to $\mathbb{Q}[\gamma]$. Moreover, the diagonal coefficients of the system (given in the proof above) are invertible in $\mathbb{Q}[\gamma]$, hence the solution is also in $\mathbb{Q}[\gamma]$.

For the second statement, it is enough to say that each $M_k^{(\alpha)}$ is a polynomial in the $R_k^{(\alpha)}$'s with integer coefficients. □

4.2.4 THE FIRST BOUND ON THE DEGREE

Recall that $(M_k^{(\alpha)})_{k\geq 2}$ is an algebraic basis of the ring $\Lambda_\star^{(\alpha)}$ of α-polynomial functions on Young diagrams. Hence, we can define a gradation on $\Lambda_\star^{(\alpha)}$ by choosing arbitrarily the degree of each of the generators $M_k^{(\alpha)}$. In this section, we will use the most natural one:

$$\deg_1(M_k^{(\alpha)}) = k \qquad \text{for } k \geq 2.$$

Our goal is to obtain a bound on the degree of the polynomial $a_\rho^\mu \in \mathbb{Q}[\gamma]$. We begin by the following lemma concerning the polynomials $b_{g,\pi}^\rho(\gamma)$.

Lemma 4.2.6. *Let ρ and π be two partitions and $g \geq 0$ be an integer. One has*

$$\deg(b_{g,\pi}^\rho(\gamma)) \leq \deg_1(M_\rho^{(\alpha)}) - \deg_1(M_{\pi\cup(g)}^{(\alpha)}) - 2.$$

Moreover, if the right-hand side is an even (resp. odd) number, then $b_{g,\pi}^\rho(\gamma)$ is an even (resp. odd) polynomial.

Proof. By Proposition 4.2.1, $M_k^{(\alpha)}(\lambda^{(o)})$ can be written as a linear combination of terms of the form $b(\gamma)\ M_\pi^{(\alpha)}\ z_o^g$. We define the degree of such a term to be the quantity $\deg(b) + |\pi| + g$. This degree is multiplicative. Then,

$$M_k^{(\alpha)}(\lambda^{(o)}) = M_k^{(\alpha)}(\lambda) + \text{ terms of degree smaller or equal to } k-2.$$

By multiplying this kind of expressions we obtain that

$$M_\rho^{(\alpha)}(\lambda^{(o)}) = M_\rho^{(\alpha)}(\lambda) + \text{ terms of degree smaller or equal to } |\rho|-2,$$

which corresponds to our bound on the degree. The parity also follows immediately from the one-part case by multiplication. □

This yields the following result.

Proposition 4.2.7. *The coefficient a_ρ^μ of $M_\rho^{(\alpha)}$ in Jack character polynomial $L_\mu^{(\alpha)}$ is a polynomial in γ of degree smaller or equal to $|\mu|+\ell(\mu)-|\rho|$. Moreover, it has the same parity as the integer $|\mu|+\ell(\mu)-|\rho|$.*

The same is true for $K_\mu^{(\alpha)}$.

Proof. We proceed by induction over (μ, ρ). The base case $\mu = (1)$ is trivial as $L_{(1)}^{(\alpha)} = M_2^{(\alpha)}$. Fix two partitions μ and ρ. We assume that our result holds for any pair (μ', ρ') with $|\mu'| < |\mu|$ or $|\mu'| = |\mu|$ and $\rho' >_1 \rho$.

It may seem strange to assume that the result holds for $\rho' >_1 \rho$. We are indeed doing some kind of *descending induction*. This is possible because, for a given μ, the number of partitions

ρ we shall consider is finite: indeed, $a^\mu_\rho = 0$ as soon as $|\rho| \geq |\mu| + \ell(\mu)$ [Las09, Proposition 9.2 (ii)]. The same remark holds for most proofs in this section.

Let us first consider the case when $\rho = \tau \cup (2)$ contains a part equal to 2. By Lemma 4.2.4, Equation (A_τ) can be written as:

$$m_2(\rho) \cdot a^\mu_\rho = m_1(\mu) a^{\mu\setminus 1}_\tau - \sum_{\substack{\pi, g, \\ \pi \cup (g) = \tau}} \sum_{\rho' >_1 \rho} b^{\rho'}_{g,\pi}(\gamma) a^\mu_{\rho'}.$$

The first term on the right-hand-side is by convention equal to 0 if μ does not contain any part equal to 1. If μ contains a part equal to 1, as $|\mu \setminus 1|$ is smaller than $|\mu|$, by induction hypothesis $a^{\mu\setminus 1}_\tau$ is a polynomial of degree

$$|\mu \setminus 1| + \ell(\mu \setminus 1) - |\tau| = |\mu| - 1 + \ell(\mu) - 1 - (|\rho| - 2) = |\mu| + \ell(\mu) - |\rho|.$$

As $\rho' >_1 \rho$, we can also apply the induction hypothesis to each summand of the second term: $a^\mu_{\rho'}$ is polynomial of degree at most $|\mu| + \ell(\mu) - |\rho'|$. But using Lemma 4.2.6, $b^{\rho'}_{g,\pi}(\gamma)$ has degree at most $|\rho'| - |\pi \cup (g)| - 2$. Hence the degree of the product is bounded by

$$|\mu| + \ell(\mu) - (|\pi \cup (g)| + 2) = |\mu| + \ell(\mu) - |\rho|.$$

The last equality comes from the fact that $\pi \cup (g) = \tau = \rho \setminus 2$.

The proof of the case when the smallest part of ρ is $q > 2$ is similar. We use Equation (B_τ) for $\tau = \rho_{\downarrow q}$, which takes the form:

$$(q-1) m_q(\rho) \cdot a^\mu_\rho = \sum_{r \geq 2} r \cdot m_r(\mu) a^{\mu_{\downarrow r}}_\tau - \sum_{\substack{\pi, g, \\ \pi \cup (g+1) = \tau}} \sum_{\rho' >_1 \rho} b^{\rho'}_{g,\pi}(\gamma) a^\mu_{\rho'}.$$

Note that $|\mu_{\downarrow r}| < |\mu|$, therefore by induction hypothesis $a^{\mu_{\downarrow r}}_\tau$ is a polynomial in γ of degree at most

$$|\mu_{\downarrow r}| + \ell(\mu_{\downarrow r}) - |\tau| = |\mu| - 1 + \ell(\mu) - (|\rho| - 1) = |\mu| + \ell(\mu) - |\rho|.$$

For the second summand, the argument is the same as before, except that here the equality $|\pi \cup (g)| + 2 = |\rho|$ comes from the fact that $|\tau| = |\rho| - 1$ and $|\tau| = |\pi \cup (g+1)| = |\pi \cup (g)| + 1$.

The parity is obtained the same way. □

4.2.5 A second bound on degrees

For some purposes the bound on the degree of a^μ_ρ given by Proposition 4.2.7 is not strong enough. In this section we give another bound which is related to another gradation of $\Lambda^{(\alpha)}_\star$ defined by:

$$\deg_2(M^{(\alpha)}_k) = k - 2 \qquad \text{for } k \geq 2.$$

One has the following analogue of Lemma 4.2.6:

Lemma 4.2.8. *Let ρ and π be two partitions and $g \geq 0$ an integer. Then*

$$\deg(b^{\rho}_{g,\pi}(\gamma)) \leq \deg_2(M^{(\alpha)}_{\rho}) - \deg_2(M^{(\alpha)}_{\pi\cup(g)}),$$
$$\deg(b^{\rho}_{g,\pi}(\gamma)) \leq \deg_2(M^{(\alpha)}_{\rho}) - \deg_2(M^{(\alpha)}_{\pi\cup(g+1)}) - 1.$$

Proof. The proof is similar to the one of Lemma 4.2.6. We define the *pre-degree* of an expression of the form $b(\gamma)\ M^{(\alpha)}_{\pi}\ z^g_o$ to be $\deg(b) + \deg_2(M^{(\alpha)}_{\pi}) + g$. By Proposition 4.2.1, the pre-degree of $M^{(\alpha)}_k(\lambda^{(o)})$ is equal to $k-2$. Note that this pre-degree is multiplicative. Then $M^{(\alpha)}_{\rho}(\lambda^{(o)})$ has pre-degree $|\rho| - 2\ell(\rho) = \deg_2(M^{(\alpha)}_{\rho})$. The lemma follows because of the following inequalities: for $g \geq 0$,

$$\deg_2(M^{(\alpha)}_{\pi\cup(g)}) \leq \deg_2(M^{(\alpha)}_{\pi}) + g;$$
$$\deg_2(M^{(\alpha)}_{\pi\cup(g+1)}) \leq \deg_2(M^{(\alpha)}_{\pi}) + g - 1.$$

Note that in the first inequality the difference between the right hand side and the left hand side is equal to 2, unless $g = 0$; in that case the right hand side is equal to the left one. In the second inequality, the case $g = 0$ is obvious as $M^{(\alpha)}_{\pi\cup(1)} = 0$ and hence its degree is $-\infty$ by convention. In all other cases, we have an equality. $\square$

We deduce from this lemma a new bound on the degree of a^{μ}_{ρ}.

Proposition 4.2.9. *The coefficient a^{μ}_{ρ} of $M^{(\alpha)}_{\rho}$ in Jack character polynomial $L^{(\alpha)}_{\mu}$ is a polynomial in γ of degree smaller or equal to $|\mu| - \ell(\mu) - (|\rho| - 2\ell(\rho))$.*

The same is true for $K^{(\alpha)}_{\mu}$.

Proof. It is a straightforward exercise to adapt the proof of Proposition 4.2.7. We use Lemma 4.2.8 instead of Lemma 4.2.6 and $|\rho|$ has to be replaced by $|\rho| - 2\ell(\rho)$. $\square$

Note that this result is neither weaker nor stronger than Proposition 4.2.7. But it is convenient for the comparison with Lassalle's normalization that we shall do in the next section.

4.2.6 Comparison with Lassalle's normalizations

As in Section 4.2.2, we use boldface font for quantities defined in Lassalle's paper [Las09]. In particular, if μ does not contain a part equal to 1 then

$$\boldsymbol{\vartheta}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = z_{\mu}\theta^{\lambda}_{\mu,1^{|\lambda|-|\mu|}}(\alpha)$$

so that

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \boldsymbol{\vartheta_\mu^\lambda(\alpha)}.$$

Besides,

$$\boldsymbol{R_k(\lambda)} = \alpha^{-k/2} R_k^{(\alpha)}(\lambda)$$

and

$$\boldsymbol{\vartheta_\mu^\lambda(\alpha)} = \boldsymbol{K_\mu(R_2, R_3}, \cdots).$$

Finally, the coefficient $\boldsymbol{c_\rho^\mu}$ of $\boldsymbol{R_\rho}$ in $\boldsymbol{K_\mu}$ with Lassalle's normalization is related to the coefficient d_ρ^μ of $R_\rho^{(\alpha)}$ in $K_\mu^{(\alpha)}$ with our conventions by:

$$\boldsymbol{c_\rho^\mu} = \alpha^{\frac{|\mu|-\ell(\mu)}{2}+\frac{|\rho|}{2}} d_\rho^\mu.$$

But we have shown that d_ρ^μ is a polynomial in $\gamma = \frac{\beta}{\sqrt{\alpha}}$ of degree smaller than $|\mu| - \ell(\mu) - (|\rho| - 2\ell(\rho))$. Thus $\boldsymbol{c_\rho^\mu}$ is a polynomial in $\sqrt{\alpha}$ divisible by $\alpha^{|\rho|-\ell(\rho)}$. The parity of d_ρ^μ implies that $\boldsymbol{c_\rho^\mu}$ is in fact a polynomial in α, which finishes the proof of Theorem 4.1.2.

M. Lassalle had only proved in his article that these quantities were rational functions in α. He conjectured that they are polynomials with integer coefficients [Las09, Conjecture 1.1]. Our result is weaker than this conjecture as we are not able to prove the integrity of the coefficients. However, we also proved that the polynomials are divisible by $\alpha^{|\rho|-\ell(\rho)}$, fact which fits with Lassalle's data [Las09, Section 1], but was not mentioned by him.

4.2.7 A weak form of Macdonald-Stanley ex-conjecture

In this section we prove that $\theta_\mu(\lambda)$ is a polynomial in α. To deduce this from the results above, one has to see how $M_k^{(\alpha)}(\lambda)$ depends on α.

Lemma 4.2.10. *Let $k \geq 2$ be an integer and λ a partition. Then $\sqrt{\alpha}^{k-2} M_k^{(\alpha)}(\lambda)$ is a polynomial in α with integer coefficients.*

Proof. We use induction over $|\lambda|$ and k. Proposition 4.2.1 can be rewritten as

$$\sqrt{\alpha}^{k-2} M_k^{(\alpha)}(\lambda^{(o)}) - \sqrt{\alpha}^{k-2} M_k^{(\alpha)}(\lambda) = \sum_{\substack{r\geq 1, s,t\geq 0,\\ 2r+s+t\leq k}} \alpha^r (\sqrt{\alpha} z_o)^{k-2r-s-t} \binom{k-t-1}{2r+s-1}\binom{r+s-1}{s} (\alpha-1)^s \sqrt{\alpha}^{t-2} M_t^{(\alpha)}(\lambda).$$

Note that $\sqrt{\alpha} z_o = \alpha x - y$ is a polynomial in α with integer coefficients. Hence the induction is immediate. □

Now we write, for $\mu, \lambda \vdash n$,

$$z_\mu \theta_\mu(\lambda) = \alpha^{\frac{|\mu|-\ell(\mu)}{2}} \mathrm{Ch}^{(\alpha)}_\mu(\lambda) = \alpha^{\frac{|\mu|-\ell(\mu)}{2}} \sum_\rho a^\mu_\rho M^{(\alpha)}_\rho(\lambda)$$
$$= \sum_\rho \alpha^{\frac{|\mu|-\ell(\mu)-(|\rho|-2\ell(\rho))}{2}} a^\mu_\rho \left(\prod_{i \leq \ell(\rho)} \sqrt{\alpha}^{\rho_i - 2} M^{(\alpha)}_{\rho_i}(\lambda) \right).$$

The quantities $\alpha^{\frac{|\mu|-\ell(\mu)-(|\rho|-2\ell(\rho))}{2}} a^\mu_\rho$ and $\sqrt{\alpha}^{\rho_i-2} M^{(\alpha)}_{\rho_i}(\lambda)$ are polynomials in α (by Proposition 4.2.9 and Lemma 4.2.10), hence $\theta_\mu(\lambda)$ is a polynomial in α, which proves Theorem 4.1.1.

Remark 4.2.11. The remaining open part of the conjecture of Lassalle [Las09, Conjecture 1.1] is the claim that $\alpha^{\frac{|\mu|-\ell(\mu)-(|\rho|-2\ell(\rho))}{2}} a^\mu_\rho$ has integer coefficients (as a polynomial in α). Hence, using the argument above, it would imply that $\theta_\mu(\lambda)$ belongs to $\mathbb{Z}[\alpha]$, which corresponds exactly to Lapointe and Vinet's result [LV95, Theorem 2].

4.2.8 Yet another gradation and bound on degrees

The gradation introduced in Section 4.2.5 is suitable for some purposes (as we have seen in the previous section), but it has the unpleasant aspect that all homogeneous spaces have an infinite dimension. In particular, Proposition 4.2.9 does not give any information on the maximal power of $M^{(\alpha)}_2$ which can appear in $L^{(\alpha)}_\mu$. In this section we propose a way to avoid this difficulty. It is a little bit technical but will be quite useful in the next section.

We define a new algebraic basis of $\Lambda^{(\alpha)}_\star$ by:

$$M'_2 = M^{(\alpha)}_2,$$
$$M'_k = M^{(\alpha)}_k - (-\gamma)^{k-2} M^{(\alpha)}_2 \qquad \text{for } k \geq 3.$$

Obviously, there exists a polynomial L'_μ such that

$$\mathrm{Ch}^{(\alpha)}_\mu = L'_\mu(M'_2, M'_3, \dots).$$

For example, one has:

$$L'_{(2,2)} = (M'_3)^2 + 6(M'_2)^2 - 4M'_4 - 10\gamma M'_3 - 2M'_2.$$

We denote by $(a')^\mu_\rho$ the coefficient of M'_ρ in L'_μ. Then, one has the following result.

Proposition 4.2.12. *The coefficient $(a')^\mu_\rho$ is a polynomial in γ of degree at most $|\mu| - \ell(\mu) + m_1(\mu) - (|\rho| - 2\ell(\rho) + m_2(\rho))$.*

Remark 4.2.13. The analogous result is not true for a^μ_ρ, as it can be seen on the case $\mu = (2,2)$.

The algorithm to compute the coefficient $(a')^\mu_\rho$ is the same as for a^μ_ρ and the proof of the bound on degrees is similar to those of Propositions 4.2.7 and 4.2.9. We will give some details.

First, one can rewrite Proposition 4.2.1 in terms of the quantities M'_k:

$$\begin{aligned} M'_k(\lambda^{(o)}) - M'_k(\lambda) &= M^{(\alpha)}_k(\lambda^{(o)}) - M^{(\alpha)}_k(\lambda) - (-\gamma)^{k-2} \\ &= \sum_{\substack{r\geq 1, s,t\geq 0,\\ 2r+s+t\leq k \\ \boldsymbol{(r,s,t)\neq(1,k-2,0)}}} z_o^{k-2r-s-t}\binom{k-t-1}{2r+s-1}\binom{r+s-1}{s}. \\ &\qquad (-\gamma)^s(M'_t(\lambda)+(-\gamma)^{t-2}M'_2(\lambda)). \quad (4.2) \end{aligned}$$

Please note that the term $(-\gamma)^{k-2}$ corresponding to $(r,s,t) = (1, k-2, 0)$ does not belong to the sum any more. By multiplication, there exist some polynomials $(b')^\rho_{g,\pi}(\gamma)$ such that

$$M'_\rho(\lambda^{(o)}) = M'_\rho(\lambda) + \sum_{g,\pi} (b')^\rho_{g,\pi}(\gamma)\ z_o^g\ M'_\pi(\lambda).$$

Using Equation (4.1) and Proposition 4.2.3, we obtain the following equalities:

$$\sum_\rho (a')^\mu_\rho \left(\sum_{\substack{g,h\geq 0,\\ \pi\vdash h}} (b')^\rho_{g,\pi}(\gamma) M'_\pi M^{(\alpha)}_g \right) = m_1(\mu)\, \mathrm{Ch}^{(\alpha)}_{\mu\setminus 1}, \quad (A')$$

$$\sum_\rho (a')^\mu_\rho \left(\sum_{\substack{g,h\geq 0,\\ \pi\vdash h}} (b')^\rho_{g,\pi}(\gamma) M'_\pi M^{(\alpha)}_{g+1} \right) = \sum_{r\geq 2} r\cdot m_r(\mu)\, \mathrm{Ch}^{(\alpha)}_{\mu_{\downarrow r}}. \quad (B')$$

Plugging $M^{(\alpha)}_g = M'_g + (-\gamma)^{g-2} M'_2$ in these equations and identifying the coefficient of M'_τ on both sides, we obtain the following system:

$$\sum_\rho (a')^\mu_\rho \left(\sum_{\substack{g,\pi,\\ \pi\cup(g)=\tau}} (b')^\rho_{g,\pi}(\gamma) + \sum_{\substack{g>2,\pi,\\ \pi\cup(2)=\tau}} (-\gamma)^{g-2}(b')^\rho_{g,\pi}(\gamma) \right) = m_1(\mu)(a')^{\mu\setminus 1}_\tau, \quad (A'_\tau)$$

$$\begin{aligned} \sum_\rho (a')^\mu_\rho \left(\sum_{\substack{g,\pi,\\ \pi\cup(g+1)=\tau}} (b')^\rho_{g,\pi}(\gamma) + \sum_{\substack{g\geq 2,\pi,\\ \pi\cup(2)=\tau}} (-\gamma)^{g-1}(b')^\rho_{g,\pi}(\gamma) \right) \\ = \sum_{r\geq 2} r\cdot m_r(\mu)(a')^{\mu_{\downarrow r}}_\tau, \quad (B'_\tau) \end{aligned}$$

It is easy to check that Lemma 4.2.4 still holds for this system.

The next step is to give a bound on the degree of $(b')^{\rho}_{g,\pi}(\gamma)$.

Lemma 4.2.14.

$$\deg(b^{\rho}_{g,\pi}(\gamma)) \leq \deg_3(M'_\rho) - \deg_3(M'_\pi) - \max(g, 1),$$

where $\deg_3(M'_k) = \min(k-2, 1)$.

Proof. Let us call *pre-degree* of an expression of the form $b(\gamma)\ M^{(\alpha)}_{\pi}\ z_o^g$ the quantity $\deg(b) + \deg_3(M'_\pi) + g$. It is multiplicative. Clearly, $M'_k(\lambda^{(o)})$ has pre-degree $\min(k-2, 1)$ (see Equation (4.2)), thus $M'_\rho(\lambda^{(o)})$ has pre-degree $\deg_3(M'_\rho)$, which finishes the proof of the case $g \geq 1$. For $g = 0$, one has to look at the term which does not involve z_o. It is easy to check on Equation (4.2) (here, it is crucial to use M' and not M) that

$$M'_k(\lambda^{(o)})|_{z_o=0} = M'_k(\lambda)|_{z_o=0} + (\text{terms of pre-degree } k-3)\,.$$

Hence by multiplication,

$$M'_\rho(\lambda^{(o)})|_{z_o=0} = M'_\rho(\lambda)|_{z_o=0} + \big(\text{terms of pre-degree } \deg_3(M'_\rho) - 1\big)\,,$$

which ends the proof of the lemma. □

We have now all the tools to prove Proposition 4.2.12 by induction. As usual, we first consider the case where $\rho = \tau \cup (2)$ has the smallest part equal to 2. Then Equation (A_τ) can be written as:

$$m_2(\rho)\cdot(a')^{\mu}_{\rho} = m_1(\mu)(a')^{\mu\setminus 1}_{\tau} - \sum_{\substack{\pi,g,\\ \pi\cup(g)=\tau}} \sum_{\rho'>_1\rho} (b')^{\rho'}_{g,\pi}(\gamma)(a')^{\mu}_{\rho'}$$
$$- \sum_{\substack{g>2,\pi,\\ \pi\cup(2)=\tau}} \sum_{\rho'>_1\rho} (-\gamma)^{g-2}(b')^{\rho}_{g,\pi}(\gamma)(a')^{\mu}_{\rho'}.$$

With arguments similar to the ones used previously, the first two terms are polynomials in γ of degree at most $|\mu| - \ell(\mu) + m_1(\mu) - \deg_3(M'_\rho)$. Let us focus on the last summand. By induction hypothesis $(a')^{\mu}_{\rho'}$ is a polynomial of degree $|\mu| - \ell(\mu) + m_1(\mu) - \deg_3(M'_{\rho'})$. By Lemma 4.2.14, $(b')^{\rho}_{g,\pi}(\gamma)$ has degree $\deg_3(M'_\rho) - \deg_3(M'_\pi) - g$. Hence the product of these two terms with $(-\gamma)^{g-2}$ has degree at most

$$|\mu| - \ell(\mu) + m_1(\mu) - (\deg_3(M'_\pi) - 2) = |\mu| - \ell(\mu) + m_1(\mu) - \deg_3(M'_\rho).$$

The equality comes from the fact that $\rho = \tau \cup (2) = \pi \cup (2,2)$.

Finally one obtains that $(a')^{\mu}_{\rho}$ has degree at most $|\mu| - \ell(\mu) + m_1(\mu) - \deg_3(M'_\rho)$.

The case when ρ has no parts equal to 2 is similar. $\square$

4.2.9 Gradations and characters

In the previous sections we have defined three different gradations. The elements of our favorite basis $(\mathrm{Ch}_\mu^{(\alpha)})$ are not homogeneous, but have the following nice property: if we denote

$$V_i^d = \{x \in \Lambda_\star^{(\alpha)} : \deg_i(x) \leq d\}$$

then each V_i^d is spanned linearly by the functions $\mathrm{Ch}_\mu^{(\alpha)}$ that it contains.

4.3 Structure constants of Jack characters

4.3.1 Structure constants are polynomials in γ

Let μ and ν be two partitions. The product $\mathrm{Ch}_\mu^{(\alpha)} \cdot \mathrm{Ch}_\nu^{(\alpha)}$ is an α-symmetric function and, hence, can be written in the basis $(\mathrm{Ch}_\pi^{(\alpha)})$. Explicitly, there exist some real numbers $g_{\mu,\nu;\pi}^{(\alpha)}$, depending on α such that

$$\mathrm{Ch}_\mu^{(\alpha)} \cdot \mathrm{Ch}_\nu^{(\alpha)} = \sum_{\substack{\pi \text{ partition} \\ \text{of any size}}} g_{\mu,\nu;\pi}^{(\alpha)} \, \mathrm{Ch}_\pi^{(\alpha)} .$$

These numbers, called *structure constants* of the basis $(\mathrm{Ch}_\pi^{(\alpha)})$, describe the multiplicative structure of the algebra. Besides, we will see in the next sections that, for $\alpha = 1, 2$ (and conjecturally for general α), they have some combinatorial interpretations.

The results of the previous section have the following consequence.

Theorem 4.3.1. *Fix three partitions μ, ν and π. The structure constant $g_{\mu,\nu;\pi}^{(\alpha)}$ is a polynomial in γ with rational coefficients and of degree (at most)*

$$\min_{i=1,2,3} \deg_i(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_i(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_i(\mathrm{Ch}_\pi^{(\alpha)}).$$

Moreover, if $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$ *is even (respectively, odd), it is an even (respectively, odd) polynomial.*

Proof. Let us consider the bound involving $\deg_1$ (the others are similar). We know (Proposition 4.2.7) that

$$\mathrm{Ch}_\mu^{(\alpha)} = \sum_\rho a_\rho^\mu M_\rho^{(\alpha)},$$

where each a_ρ^μ is a polynomial in γ of degree $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) - \deg_1(M_\rho^{(\alpha)})$. Hence, we have

$$\mathrm{Ch}_\mu^{(\alpha)} \cdot \mathrm{Ch}_\nu^{(\alpha)} = \sum_\rho b_\rho^{\mu,\nu} M_\rho^{(\alpha)},$$

where each $b_\rho^{\mu,\nu}$ is a polynomial in γ of degree $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_1(M_\rho^{(\alpha)})$. In particular $\mathrm{Ch}_\mu^{(\alpha)} \cdot \mathrm{Ch}_\nu^{(\alpha)}$ has degree at most $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)})$ and hence, thanks to the remark of Section 4.2.9, $g_{\mu,\nu;\pi}^{(\alpha)} = 0$ whenever

$$\deg_i(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_i(\mathrm{Ch}_\nu^{(\alpha)}) < \deg_i(\mathrm{Ch}_\pi^{(\alpha)}).$$

The structure constants are obtained by solving the linear system:

$$\sum_\tau a_\rho^\tau g_{\mu,\nu;\tau}^{(\alpha)} = b_\rho^{\mu,\nu}, \tag{S}$$

where μ and ν are fixed and ρ runs over all partitions without parts equal to 1; the variables are $g_{\mu,\nu;\tau}^{(\alpha)}$.

We will prove our statement by induction over

$$\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_1(\mathrm{Ch}_\pi^{(\alpha)}).$$

If this quantity is equal to -1, the coefficient $g_{\mu,\nu;\pi}^{(\alpha)}$ is equal to 0 and the statement is true. Note that a_ρ^τ vanishes as soon as $\deg_1(\mathrm{Ch}_\tau^{(\alpha)}) < \deg_1(M_\rho^{(\alpha)})$. We fix a partition π and we suppose that for all partitions τ bigger than π (in the sense that $\deg_1(\mathrm{Ch}_\tau^{(\alpha)}) > \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$), the degree of $g_{\mu,\nu;\pi}^{(\alpha)}$ is bounded from above by $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$. Then from (S) we extract a subsystem

$$\sum_{\substack{\tau,\\ \deg_1(\mathrm{Ch}_\tau^{(\alpha)})=\deg_1(\mathrm{Ch}_\pi^{(\alpha)})}} a_\rho^\tau g_{\mu,\nu;\tau}^{(\alpha)} = b_\rho^{\mu,\nu} - \sum_{\substack{\tau,\\ \deg_1(\mathrm{Ch}_\tau^{(\alpha)})>\deg_1(\mathrm{Ch}_\pi^{(\alpha)})}} a_\rho^\tau g_{\mu,\nu;\tau}^{(\alpha)}, \tag{S'}$$

where ρ runs over partitions such that $\deg_1(M_\rho^{(\alpha)}) = \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$. The variables are $g_{\mu,\nu;\tau}^{(\alpha)}$ for τ with $\deg_1(\mathrm{Ch}_\tau^{(\alpha)}) = \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$. This system is invertible (because $(\mathrm{Ch}_\pi^{(\alpha)})$ is a basis of $\Lambda_\star^{(\alpha)}$) and the coefficients are rational numbers (by Proposition 4.2.7). Besides, all terms on the right-hand side are polynomials in γ of degree at most $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) + \deg_1(\mathrm{Ch}_\nu^{(\alpha)}) - \deg_1(\mathrm{Ch}_\pi^{(\alpha)})$ which finishes the proof.

The proof of the parity follows in the same way. □

4.3.2 Projection on functions on Young diagrams of size n

Recall that $\Lambda_\star^{(\alpha)}$ is a subalgebra of the algebra of functions on all Young diagrams. The latter has a natural projection map φ_n onto $\mathcal{F}(\mathbb{Y}_n, \mathbb{Q})$, the algebra of functions on Young diagrams of size n. Note that, as Jack symmetric functions J_λ form a basis of the symmetric function ring, the functions $(\theta_\mu^{(\alpha)})_{\mu\vdash n}$ form a basis of $\mathcal{F}(\mathbb{Y}_n, \mathbb{Q})$ (see [Fér10a, Proposition 4.1]).

We consider the structure constants $c_{\mu,\nu;\pi}^{(\alpha)}$ (depending on α) of $\mathcal{F}(\mathbb{Y}_n, \mathbb{Q})$ with basis $(\theta_\mu^{(\alpha)})_{\mu\vdash n}$, that is the numbers defined by:

$$\theta_\mu^{(\alpha)}(\lambda) \cdot \theta_\nu^{(\alpha)}(\lambda) = \sum_{\pi \vdash n} c_{\mu,\nu;\pi}^{(\alpha)} \theta_\pi^{(\alpha)}(\lambda) \qquad \text{for } \lambda \vdash n. \tag{4.3}$$

It is important to keep in mind that the c's are indexed by triples of partitions of *the same size*, while the g's are indexed by any triple of partitions.

It turns out that the quantities $c_{\mu,\nu;\pi}^{(\alpha)}$ can be expressed in terms of the quantities $g_{\mu,\nu;\tau}^{(\alpha)}$. To explain that, for any partition μ, let us denote $\tilde{\mu}$ the partition obtained by erasing all parts equal to 1. Fix two partitions μ and ν of the same integer n; then

$$\mathrm{Ch}_{\tilde{\mu}}^{(\alpha)} \cdot \mathrm{Ch}_{\tilde{\nu}}^{(\alpha)} = \sum_\tau g_{\tilde{\mu},\tilde{\nu};\tau}^{(\alpha)} \, \mathrm{Ch}_\tau^{(\alpha)} .$$

But using the definition of $\mathrm{Ch}^{(\alpha)}$, this implies that, for all $\lambda \vdash n$, one has:

$$\alpha^{-\frac{|\mu|-\ell(\mu)}{2}} z_{\tilde{\mu}} \, \theta_\mu^{(\alpha)}(\lambda) \cdot \alpha^{-\frac{|\nu|-\ell(\nu)}{2}} z_{\tilde{\nu}} \, \theta_\nu^{(\alpha)}(\lambda) \\ = \sum_{\substack{\tau, \\ |\tau| \le n}} g_{\tilde{\mu},\tilde{\nu};\tau}^{(\alpha)} \, \alpha^{\frac{|\tau|-\ell(\tau)}{2}} z_\tau \binom{n-|\tau|+m_1(\tau)}{m_1(\tau)} \theta_{\tau 1^{n-|\tau|}}^{(\alpha)}(\lambda).$$

Every partition τ with $|\tau| \le n$ can be written uniquely as $\tilde{\pi}1^i$ where π is a partition of n and $i \le m_1(\pi)$. Denoting

$$d(\mu,\nu;\pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi)),$$

one has

$$\theta_\mu^{(\alpha)}(\lambda) \cdot \theta_\nu^{(\alpha)}(\lambda) = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{\pi \vdash n} \left(\sum_{0 \le i \le m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i}^{(\alpha)} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n-|\tilde{\pi}|}{i} \right) \theta_\pi^{(\alpha)}(\lambda).$$

As this is true for all partitions λ of n, one can use Equation (4.3) to obtain:

$$c_{\mu,\nu;\pi}^{(\alpha)} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \le i \le m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i}^{(\alpha)} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n-|\tilde{\pi}|}{i}. \tag{4.4}$$

In particular, one has the following result:

Proposition 4.3.2. *Let μ, ν and π be three partitions without parts equal to* 1. *Then,* $\alpha^{-d(\mu,\nu;\pi)/2} c_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}^{(\alpha)}$ *is a polynomial in n and γ with rational coefficients, of total degree at most $d(\mu,\nu;\pi)$.*

Moreover, seen as a polynomial in γ, it has the same parity as $d(\mu,\nu;\pi)$.

Corollary 4.3.3. *The quantity $c^{(\alpha)}_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}$ is a polynomial in n and α. Moreover, it has degree at most $d(\mu,\nu;\pi)$ in n and at most $d(\mu,\nu;\pi)$ in α (the total degree may be bigger).*

4.3.3 Case $\alpha = 1$: symmetric group algebra

In the case $\alpha = 1$, the structure constants considered in the previous section are linked with the symmetric group algebras. Let $\mathfrak{S}_n$ denote the symmetric group of size n, *i.e.* the group of permutations of the set $[n] := \{1,\dots,n\}$. Recall that the cycle-type of a permutation $\sigma \in \mathfrak{S}_n$ is the integer partition $\mu \vdash n$ obtained by sorting the lengths of the cycles of σ. We consider the group algebra $\mathbb{Q}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$ over the rational field $\mathbb{Q}$. Its center $Z(\mathbb{Q}[\mathfrak{S}_n])$ is spanned linearly by the conjugacy classes, that is the elements

$$\mathcal{C}_\mu = \sum_{\substack{\sigma\in\mathfrak{S}_n,\\ \text{cycle-type}(\sigma)=\mu}} \sigma.$$

By a classical result of Frobenius (see [Fro00] and [Mac95, (I,7.8)]), for any $\lambda \vdash n$,

$$\frac{\text{Tr}\,\rho^\lambda(\mathcal{C}_\mu)}{\text{dimension of }\rho^\lambda} = \theta^{(1)}_\mu(\lambda).$$

In other words: $\theta^{(1)}_\mu$ is the image of $\mathcal{C}_\mu$ by the abstract Fourier transform, which is an algebra morphism. Hence, the structure constants of the algebra $Z(\mathbb{Q}[\mathfrak{S}_n])$ with the basis $(\mathcal{C}_\mu)_{\mu\vdash n}$ coincide with $c^{(1)}_{\mu,\nu;\pi}$.

These structure constants have been widely studied in the last fifty years in the frameworks of algebra and, because they count some families of graphs drawn on orientable surfaces, combinatorics. The most famous result in this topic is due to Farahat and Higman [FH59, Theorem 2.2]: the quantity $c^{(1)}_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}$ is a polynomial in n. Note that this is a consequence of Proposition 4.3.2.

Remark 4.3.4. The quantities $g^{(1)}_{\mu,\nu;\pi}$ also have a direct combinatorial interpretation in terms of partial permutations, see [IK99].

4.3.4 Case $\alpha = 2$: Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

An analogous interpretation of the structure constants is available in the case $\alpha = 2$. We explain it here, following the development given in [GJ96b].

We can view the elements of the symmetric group $\mathfrak{S}_{2n}$ as permutations of the following set: $\{1,\bar{1},\dots,n,\bar{n}\}$. A subgroup, denoted H_n and called *hyperoctahedral group*, is formed by permutations σ such that

$$\overline{\sigma(i)} = \sigma(\bar{i}) \qquad \text{for } i \in \{1,\dots,n\},$$

where by convention $\overline{\overline{j}} = j$. We consider the subalgebra $\mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n]$ of $\mathbb{Q}[\mathfrak{S}_{2n}]$ of the elements which are invariant by multiplication on the left or on the right by any element of H_n; in other words

$$x \in \mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n] \stackrel{\text{def}}{\iff} hxh' = x \quad \text{for all } h, h' \in H_n.$$

A non-trivial result is that this algebra is commutative.

The equivalence classes for the relation $x \sim hxh'$ (for $x \in \mathfrak{S}_{2n}$ and $h, h' \in H_n$) are called double-coset. They are naturally indexed by partitions of n, see [Mac95, (VII,2)]. We denote by $\mathcal{C}_\mu^{(2)} \in \mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n]$ the sum of all elements in the double coset corresponding to μ. The family $(\mathcal{C}_\mu^{(2)})_{\mu \vdash n}$ is a basis of $\mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n]$.

One can show (see [GJ96b, Equation (3) and (5)]) that there exist some orthogonal idempotents E_λ such that:

$$\mathcal{C}_\mu^{(2)} = 2^n n! \sum_{\lambda \vdash n} \theta_\mu^{(2)}(\lambda)\, E_\lambda.$$

Hence, one has

$$\begin{aligned}\mathcal{C}_\mu^{(2)} \cdot \mathcal{C}_\nu^{(2)} &= (2^n n!)^2 \sum_{\lambda \vdash n} \theta_\mu^{(2)}(\lambda)\, \theta_\nu^{(2)}(\lambda)\, E_\lambda \\ &= (2^n n!)^2 \sum_{\pi \vdash n} \sum_{\lambda \vdash n} c_{\mu,\nu;\pi}^{(2)}\, \theta_\pi^{(2)}(\lambda)\, E_\lambda = (2^n n!) \sum_{\pi \vdash n} c_{\mu,\nu;\pi}^{(2)}\, \mathcal{C}_\pi^{(2)}.\end{aligned}$$

Hence, the structure constants $h_{\mu,\nu;\pi}$ of the algebra $\mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n]$ for the basis $(\mathcal{C}_\mu^{(2)})_{\mu \vdash n}$ are, up to a factor $2^n n!$, the same as the ones of algebra $\mathcal{F}(\mathbb{Y}_n, \mathbb{Q})$ with the basis $(\theta_\mu^{(2)})_{\mu \vdash n}$.

In particular, Proposition 4.3.2 implies the following result.

Proposition 4.3.5. *Let μ, ν and π be partitions without parts equal to 1. The renormalized structure constant of the algebra $\mathbb{Q}[H_n \backslash \mathfrak{S}_{2n}/H_n]$*

$$\frac{h_{\mu 1^{n-|\mu|}, \nu 1^{n-|\nu|}; \pi 1^{n-|\pi|}}}{n!\; 2^n\; \sqrt{2}^{d(\mu,\nu,\pi)}}$$

is a polynomial in n of degree at most $d(\mu,\nu,\pi)$. Moreover, it has the same dominant coefficient as $c^{(1)}_{\mu 1^{n-|\mu|}, \nu 1^{n-|\nu|}; \pi 1^{n-|\pi|}}$. In particular,

- *When $|\mu| - \ell(\mu) + |\nu| - \ell(\nu) = |\pi| - \ell(\pi)$, one has*

$$\frac{h_{\mu 1^{n-|\mu|}, \nu 1^{n-|\nu|}; \pi 1^{n-|\pi|}}}{n!\; 2^n} = c^{(1)}_{\mu 1^{n-|\mu|}, \nu 1^{n-|\nu|}; \pi 1^{n-|\pi|}}$$

and this quantity is independent of n.

- *When* $|\mu| - \ell(\mu) + |\nu| - \ell(\nu) = |\pi| - \ell(\pi) - 1$,

$$\frac{h_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}}{n!\, 2^n}$$

is independent of n.

Proof. The claim that the renormalized structure constant mentioned above is a polynomial and the bound on its degree follow from Proposition 4.3.2 specialized to $\alpha = 2$. The dominant coefficient is a polynomial in α of degree 0, so it is the same for $\alpha \in \{1, 2\}$. The second item comes from the existence of the sign on $\mathfrak{S}_n$, which implies that

$$c^{(1)}_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}} = 0$$

whenever $d(\mu, \nu; \pi)$ is odd. □

A part of this result (the polynomiality and the first item) has been recently proved combinatorially by Aker and Can in their paper [AC12].

Goulden and Jackson [GJ96b] described the coefficients $h_{\mu,\nu;\pi}$ combinatorialy. Let $\mathcal{F}_\mathcal{S}$ be the set of all (perfect) matchings on a set $\mathcal{S}$. For $F_1, \dots, F_k \in \mathcal{F}_\mathcal{S}$, let $G(F_1, \dots, F_k)$ be the multigraph with vertex-set $\mathcal{S}$ whose edges are formed by the pairs in $F_1, \dots, F_k$. The components of $G(F_1, F_2)$ are even cycles. Let the list of their lengths in weakly decreasing order be $(2\theta_1, 2\theta_2, \dots) = 2\theta$, and define Λ by $\Lambda(F_1, F_2) = \theta$. Let $\mathcal{F}_n$ denote the set of all matchings on the set $\{1, 2 \dots, 2n\}$.

Lemma 4.3.6 ([GJ96b, Lemma 2.2.]). *Let* F_1, F_2 *be two fixed matchings in* $\mathcal{F}_n$ *such that* $\Lambda(F_1, F_2) = \pi$, *where* $\rho \vdash n$. *Then, for any* $\mu, \nu \vdash n$ *we have*

$$h_{\mu,\nu;\pi} = 2^n n! |\{F_3 \in \mathcal{F}_n : \Lambda(F_1, F_3) = \mu, \Lambda(F_2, F_3) = \nu\}|.$$

4.3.5 Back to general α-case: technical lemma

Here, we are going to prove some technical lemma about connection coeffcients which will be very helpful in the next sections.

Lemma 4.3.7. *We have the following identities:*

1. $g^{(\alpha)}_{\mu,\nu;\rho} = \delta_{\rho,\mu\cup\nu}$ *for* $|\rho| + \ell(\rho) \geq |\mu| + \ell(\mu) + |\nu| + \ell(\nu)$;
2. $g^{(\alpha)}_{\mu,\nu;1^k} = 0$ *for* $\tilde{\mu} \neq \tilde{\nu}$ *and* $2k \geq |\mu| + \ell(\mu) + |\nu| + \ell(\nu) - 2$;
3. $g^{(\alpha)}_{(k),(l);1^{\lceil (k+l)/2 \rceil}} = \delta_{k,l} k$.

Proof. First, we notice that

$$g^{(\alpha)}_{\mu,\nu;\rho} = (-1)^{|\mu|+|\nu|+|\rho|-\ell(\mu)-\ell(\nu)-\ell(\rho)} g^{(1/\alpha)}_{\mu,\nu;\rho},$$

since

$$\mathrm{Ch}^{(\alpha)}_{\mu}(\lambda) = (-1)^{|\mu|-\ell(\mu)} \mathrm{Ch}^{(1/\alpha)}_{\mu}(\lambda').$$

By Theorem 4.3.1 we know that

$$\deg(g^{(\alpha)}_{\mu,\nu;\rho}) \leq |\mu| + \ell(\mu) + |\nu| + \ell(\nu) - (|\rho| + \ell(\rho)),$$

hence the case 1 is obvious, since this is true for $\alpha = 1$ [IO02, Proposition 4.9.].

In order to prove case 2, we notice that $\deg(g^{(\alpha)}_{\mu,\nu;1^k}) \leq 2$ and we know that $g^{(1)}_{\mu,\nu;1^k} = 0$ in that case. Hence, it is enough to show that $g^{(2)}_{\mu,\nu;1^k} = 0$. It is clear, from Lemma 4.3.6 that $h_{\mu,\nu;1^n} = 0$ for any pair of different partitions $\mu, \nu \vdash n$, hence also $c^{(2)}_{\mu,\nu;1^n} = 0$ whenever $\mu \neq \nu$. It means that for any sufficiently big n, thanks to (4.4), we have the following equation:

$$0 = \sum_{0 \leq i \leq k+1} g^{(2)}_{\tilde{\mu},\tilde{\nu};(1^i)} \cdot i! \cdot \binom{n}{i},$$

hence

$$g^{(2)}_{\tilde{\mu},\tilde{\nu};(1^i)} = 0.$$

Since for any $\lambda \vdash n$ one has

$$\mathrm{Ch}^{(\alpha)}_{\mu} = (n - |\tilde{\mu}|)_{m_1(\mu)} \mathrm{Ch}^{(\alpha)}_{\tilde{\mu}}$$

we have that $g^{(2)}_{\mu,\nu;1^k} = 0$ which finishes the proof of the case 2.

Finaly, if $k \neq l$, the last case follows from the case 2. Let $k = l$. For the proof of the last case we use Equation (4.4) again. W have that

$$c^{(2)}_{(k1^n),(k1^n);(1^{k+n})} = \frac{2^{k-1}}{(kn!)^2} \sum_{0 \leq i \leq k} g^{(2)}_{(k),(k);(1^i)} (i!)^2 \binom{k+n}{i}.$$

By Lemma 4.3.6 one has that

$$h_{(k1^n),(k1^n);(1^{k+n})} = 2^{k+n}(k+n)! \binom{k+n}{k} 2^{k-1}(k-1)!,$$

hence

$$c^{(2)}_{(k1^n),(k1^n);(1^{k+n})} = \binom{k+n}{k} 2^{k-1}(k-1)!.$$

Indeed, if F_1, F_2 are two fixed matchings such that $\Lambda(F_1, F_2) = (1^{k+n})$, then $F_1 = F_2$. There are k elements of F_1 which are in the same connected component of $G(F_1, F_3)$ and them can be chosen on $\binom{k+n}{k}$ ways. Then, these elements can be connected by F_3 in one connected component on $(2k-2)(2k-4)\cdots 2 = 2^{k-1}(k-1)!$ ways. It gives us

$$\binom{k+n}{k} 2^{k-1}(k-1)! = \frac{2^{k-1}}{(kn!)^2} \sum_{0 \le i \le k} g^{(2)}_{(k),(k);(1^i)} (i!)^2 \binom{k+n}{i}$$

and since both sides of the equation are polynomials in n, the equation $g^{(2)}_{(k),(k);(1^k)} = k$ follows. An equality $g^{(1)}_{(k),(k);(1^k)} = k$ finishes the proof of the case 3. □

4.3.6 Back to general α-case: matching-Jack conjecture

In this paragraph, we use the convention that the boldface quantities refer to the notations of Goulden and Jackson [GJ96a].

In the case when α is general it is not known whether the quantities $c_{\mu,\nu;\pi}$ can be interpreted as the structure coefficients of some combinatorial algebra. However, in this section we will show that they correspond to the quantity $\boldsymbol{c}^{\boldsymbol{\pi}}_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{b})$ studied previously by I. P. Goulden and D. M. Jackson [GJ96a], which has conjecturally a combinatorial meaning in terms of matchings (see the matching-Jack conjecture [GJ96a, Section 4]). To establish this result we will need to use the α scalar product on the symmetric functions, for which Jack polynomials and power sums are orthogonal basis [Mac95, (VI,10)].

Proposition 4.3.8. *Let μ, ν and π be three partitions of the same integer n. Then*

$$c_{\mu,\nu;\pi} = \frac{n!}{z_\pi} \alpha^{\ell(\pi)} \sum_{\lambda \vdash n} \frac{\theta_\pi(\lambda)\, \theta_\mu(\lambda)\, \theta_\nu(\lambda)}{\langle J_\lambda, J_\lambda \rangle}.$$

Proof. Let partitions $\mu \vdash n$ and $\nu \vdash n$ be fixed. We consider the following symmetric function:

$$F := \sum_{\lambda \vdash n} \frac{\theta_\mu(\lambda)\, \theta_\nu(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda.$$

By definition of $c_{\mu,\nu;\pi}$, one has:

$$F = \sum_{\lambda \vdash n} \sum_{\pi \vdash n} c_{\mu,\nu;\pi} \left(\frac{\theta_\pi(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda \right). \tag{4.5}$$

But $\theta_\pi(\lambda)$ is defined by

$$J_\lambda = \sum_{\pi \vdash n} \theta_\pi(\lambda)\, p_\pi.$$

As p_π is an orthogonal basis, this implies

$$\theta_\pi(\lambda) = \frac{\langle J_\lambda, p_\pi \rangle}{\langle p_\pi, p_\pi \rangle}.$$

But J_λ is also an orthogonal basis, hence:

$$p_\pi = \sum_\lambda \frac{\langle J_\lambda, p_\pi \rangle}{\langle J_\lambda, J_\lambda \rangle} J_\lambda = \langle p_\pi, p_\pi \rangle \sum_\lambda \frac{\theta_\pi(\lambda)}{\langle J_\lambda, J_\lambda \rangle} J_\lambda. \tag{4.6}$$

Plugging this into (4.5), one has:

$$F = \sum_{\pi \vdash n} c_{\mu,\nu;\pi} \frac{p_\pi}{\langle p_\pi, p_\pi \rangle}$$

and thus,

$$c_{\mu,\nu;\pi} = \langle F, p_\pi \rangle = \sum_{\lambda \vdash n} \frac{\theta_\mu(\lambda)\, \theta_\nu(\lambda)}{\langle J_\lambda, J_\lambda \rangle} \langle J_\lambda, p_\pi \rangle$$

$$= \sum_{\lambda \vdash n} \frac{\theta_\mu(\lambda)\, \theta_\nu(\lambda)}{\langle J_\lambda, J_\lambda \rangle} \langle p_\pi, p_\pi \rangle \theta_\pi(\lambda).$$

As $\langle p_\pi, p_\pi \rangle = n!/z_\pi \cdot \alpha^{\ell(\pi)}$, we obtain the claimed formula. □

Comparing the proposition with the definition of the connection series $\boldsymbol{c^\pi_{\mu,\nu}(b)}$ [GJ96a, equations (1) and (5)], we get that

$$c_{\mu,\nu;\pi} = \boldsymbol{c^\pi_{\mu,\nu}(b)}. \tag{4.7}$$

I. P. Goulden and D. M. Jackson had conjectured that they were polynomials with non-negative integer coefficients in $b = \alpha - 1$. Corollary 4.3.3 implies the following weaker statement, which was not known yet.

Proposition 4.3.9. *The connection series* $\boldsymbol{c^\pi_{\mu,\nu}(b)}$ *introduced in [GJ96a] is a polynomial in* b *of degree at most* $d(\mu, \nu; \pi)$.

4.3.7 Symmetric functions of contents

In this section we consider a closely related problem considered by S. Matsumoto in [Mat10, Section 8] in connection with matrix integrals. The above results allow us to prove two conjectures stated in his paper.

For a box $\square = (i, j)$ of a Young diagram λ (i is the row-index, j the column index and $j \le \lambda_i$), we define its *(α-)content* as $c(\square) = \sqrt{\alpha}(j-1) - \sqrt{\alpha}^{-1}(i-1)$. The *alphabet of the content* of λ is the multiset $\mathcal{C}_\lambda = \{c(\square) : \square \in \lambda\}$.

Matsumoto [Mat10, Equation (8.9)] (beware that in his paper the normalization is different than ours) showed the following remarkable result: for any partition λ

$$e_k(\mathcal{C}_\lambda) = \sum_{\substack{\mu:\\ |\mu|-\ell(\mu)=k,\\ m_1(\mu)=0}} \frac{\mathrm{Ch}^{(\alpha)}_\mu(\lambda)}{z_\mu}. \tag{4.8}$$

In particular, $\lambda \mapsto e_k(\mathcal{C}_\lambda)$ is a shifted symmetric function. Therefore for any symmetric function F, the map $\lambda \mapsto F(\mathcal{C}_\lambda)$ is also a shifted symmetric function and one may wonder how it can be expressed in the Ch basis. Explicitly, we are interested in the coefficients $a_\mu(F)$ defined by:

$$F(\mathcal{C}_\lambda) = \sum_{\mu \text{ partition}} a_\mu(F)\, \mathrm{Ch}^{(\alpha)}_\mu(\lambda). \tag{4.9}$$

Using the results of Section 4.3.1, one has the following result:

Proposition 4.3.10. *Let F be a symmetric function and μ be a partition. The coefficient $a_\mu(F)$ is a polynomial in γ of degree at most $\deg(F) - (|\mu| - \ell(\mu) + m_1(\mu))$.*

Proof. By (4.8), the proposition is true for $F = e_k$ for any $k \geq 1$. Besides, if it is true for two symmetric functions F_1 and F_2, it is clearly true for any linear combination of them. By Theorem 4.3.1, it is also true for $F_1 \cdot F_2$. Since the elementary symmetric functions form a basis of symmetric functions, it follows that the proposition is true for any symmetric function F. □

From now on, we use the convention that the boldface quantities refer to the notation of Matsumoto. The coefficients $a_\mu(F)$ are closely related to the quantities $\boldsymbol{\mathcal{A}^{(\alpha)}_\mu(F, n)}$ introduced by S. Matsumoto [Mat10]. Indeed, one has the following lemma (which extends [Mat10, Lemma 8.5]):

Lemma 4.3.11. *Let μ be a partition. For $n \geq |\mu| + \ell(\mu)$, we denote π the partition $\mu + (1^{n-|\mu|})$ of n obtained from μ by adding 1 to every part and adding new parts equal to 1. Then, for any symmetric function F one has:*

$$\boldsymbol{\mathcal{A}^{(\alpha)}_\mu(F, n)} = \alpha^{\frac{\deg(F) - (|\pi| - \ell(\pi))}{2}} \left[\sum_{i \leq m_1(\pi)} a_{\tilde{\pi}1^i}(F) z_{\tilde{\pi}} i! \binom{n - |\tilde{\pi}|}{i} \right],$$

where $\boldsymbol{\mathcal{A}^{(\alpha)}_\mu(F, n)}$ is the quantity defined in [Mat10, Section 8.3].

Proof. If we fix the integer n, one may rewrite Equation (4.9) using the definition of Ch:

$$F(\mathcal{C}_\lambda) = \sum_{\substack{\nu,\\ |\nu|\leq n}} a_\nu(F)\alpha^{\frac{|\nu|-\ell(\nu)}{2}} z_\nu \binom{n-|\nu|+m_1(\nu)}{m_1(\nu)} \theta_{\nu 1^{n-|\nu|}}(\lambda)$$

$$= \sum_{\pi\vdash n} \theta_\pi(\lambda) \left[\alpha^{\frac{|\pi|-\ell(\pi)}{2}} \sum_{i\leq m_1(\pi)} a_{\tilde{\pi}1^i}(F) z_{\tilde{\pi}} i! \binom{n-|\tilde{\pi}|}{i} \right].$$

The notations are the same as in Section 4.3.2. The second equality comes from the fact that each partition ν of size at most n writes uniquely as $\tilde{\pi}1^i$ where π is a partition of n and i a non-negative integer smaller or equal to $m_1(\pi)$. We denote A_π the expression in the bracket in the equation above.

As in the proof of Proposition 4.3.8 we shall use the Jack deformation of Hall scalar product on symmetric function.

$$\sum_{\lambda\vdash n} F(\mathcal{C}_\lambda)\frac{J_\lambda}{\langle J_\lambda, J_\lambda\rangle} = \sum_{\lambda,\pi\vdash n} A_\pi \theta_\pi(\lambda)\frac{J_\lambda}{\langle J_\lambda, J_\lambda\rangle} = \sum_{\pi\vdash n} A_\pi \frac{p_\pi}{\langle p_\pi, p_\pi\rangle}.$$

The last equality corresponds to (4.6). We deduce that

$$A_\pi = \left\langle \sum_{\lambda\vdash n} F(\mathcal{C}_\lambda)\frac{J_\lambda}{\langle J_\lambda, J_\lambda\rangle}, p_\pi \right\rangle = \sum_{\lambda\vdash n} F(\mathcal{C}_\lambda)\frac{\theta_\pi(\lambda)\cdot\langle p_\pi, p_\pi\rangle}{\langle J_\lambda, J_\lambda\rangle}.$$

This formula is close to the definition of $\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)}$ in [Mat10, paragraph 8.3]. More precisely,

$$\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)} = \alpha^{\frac{\deg(F)}{2}} A_\pi.$$

The only difficulty is the difference of notation. To help the reader, we provide the following dictionary. First recall that $\langle p_\pi, p_\pi\rangle = z_\pi\ \alpha^{\ell(\pi)}$. Then our partition π corresponds to $\boldsymbol{\mu} + (\boldsymbol{1^{n-|\mu|}})$. In particular, one has $|\boldsymbol{\mu}| = |\pi| - \ell(\pi)$ and $\boldsymbol{z_{\mu+(1^{n-|\mu|})}} = z_\pi$. Besides, $F(\mathcal{C}_\lambda)$ in this chapter corresponds to $\alpha^{\deg(F)2}\ F(A_\lambda^\alpha)$ in [Mat10]. Finally, the probability $\mathbb{P}_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{\lambda})$ is simply given by $\frac{n!\alpha^n}{\langle J_\lambda, J_\lambda\rangle}$. □

Proposition 4.3.10, when translated into Matsumoto's notation by Lemma 4.3.11, has several interesting consequences.

- If $\deg(F) = |\mu|$, the only term of the sum which can be non-zero corresponds to $i = 0$ (by Proposition 4.3.10). Moreover, it does not depend on α. Besides, the exponent of α in the formula is equal to zero. Finally, $\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)}$ does not depend neither on α nor on n, which proves [Mat10, Conjecture 9.2].

- If $\deg(F) = |\mu| + 1$, there are only two terms (corresponding to $i = 0, 1$) which can be non zero in the sum. Besides, the coefficient $a_{\tilde{\pi}1}$ does not depend on α because of Proposition 4.3.10. But it is easy to prove that it is equal to 0 in the case $\alpha = 1$ (it comes from the combinatorial interpretation of $\boldsymbol{\mathcal{A}_\mu^{(1)}(F,n)}$, see [Mat10, Example 9.2]). Hence, $a_{\tilde{\pi}1} = 0$ and only the term corresponding to $i = 0$ is non-zero. In particular, one can see that $\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)}$ does not depend on n, which proves [Mat10, Conjecture 9.3].

- In the general case, non-zero terms of the sum are indexed by values of i smaller or equal to $\deg(F)-|\mu|$ (by Proposition 4.3.10). Hence $\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)}$ is a polynomial in n of degree at most $\deg(F) - |\mu|$. This result is not stronger that the bound of S. Matsumoto on the degree of $\boldsymbol{\mathcal{A}_\mu^{(\alpha)}(F,n)}$ [Mat10, Theorem 8.8]. Nevertheless, it is better in some cases (as illustrated by the proofs of the conjectures above). Besides, we also give information on the dependence on α.

4.4 Structure of Kerov polynomials

4.4.1 Linear terms in Kerov polynomials

In this short section, we compute the top degree part of the coefficients of linear terms in Kerov polynomials. This proves a conjecture of Lassalle [Las09, page 31].

Theorem 4.4.1. *For any integers $k > 0$ and $k-1 \geq i \geq 0$, we have*

$$[R_{k+1-i}^{(\alpha)}]K_k^{(\alpha)} = \begin{bmatrix} k \\ k-i \end{bmatrix} \gamma^i + \textit{lower degree terms},$$

where $\begin{bmatrix} k \\ k-i \end{bmatrix}$ denotes the positive Stirling number of the first kind.

Proof. It is enough to prove that

$$a_{k+1-i}^k = \begin{bmatrix} k \\ k-i \end{bmatrix} \gamma^i + \text{lower degree terms}$$

for any positive integers $k > 0$ and $k-1 \geq i \geq 0$. We will do it by induction over k. For $k = 1$ we have that $K_1^{(\alpha)} = M_2^{(\alpha)} = R_2^{(\alpha)}$ and the inductive assertion holds in this case.

Putting $\mu = k$ in Equation (B) we have that

$$\sum_\rho a_\rho^k \left(\sum_{\substack{g,h \geq 0, \\ \pi \vdash h}} b_{g,\pi}^\rho(\gamma) M_{\pi \cup (g+1)}^{(\alpha)} \right) = k L_{k-1}^{(\alpha)},$$

hence

$$\sum_{\rho} a_{\rho}^{k} b_{k-1-i,0}^{\rho}(\gamma) = k a_{k-i}^{k-1}$$

for any integer $0 \leq i \leq k-1$. We have that $b_{k-1-i,0}^{\rho} = 0$ for $|\rho| < k+1-i$ by Lemma 4.2.6. Moreover, by Proposition 4.2.7 and by Lemma 4.2.8, we have that

$$\deg(a_{\rho}^{k} b_{k-1-i,0}^{\rho}(\gamma)) \leq k+1-|\rho|+|\rho|-2\ell(\rho)-(k-3-i) = i-2(\ell(\rho)-1),$$

hence by inductive assertion we have that

$$\sum_{k+1 \geq r \geq k+1-i} a_{r}^{k} b_{k-1-i,0}^{r}(\gamma) = k \begin{bmatrix} k-1 \\ k-1-i \end{bmatrix} \gamma^{i} + \text{lower degree terms} \tag{4.10}$$

for any integer $0 \leq i \leq k-1$. By Proposition 4.2.1 we know that

$$b_{k-1-i,0}^{r}(\gamma) = \binom{r-1}{r-(k-i)} (-\gamma)^{r-(k-i+1)}.$$

Putting it into Equation (4.10) we obtain that in order to finish the proof it is enough to prove the following identity:

$$\sum_{0 \leq j \leq i} \binom{k-i+j}{j+1} (-1)^{j} \begin{bmatrix} k \\ k-i+j \end{bmatrix} = k \begin{bmatrix} k-1 \\ k-1-i \end{bmatrix}$$

for any integer $0 \leq i \leq k-1$.

We will prove it by double induction over k and i. Let us assume that the inductive assertion is true for all $k < K$ and for $k = K$ and all $i < I < K-1$. The following equations hold:

$$\sum_{0\le j\le I}\binom{K-I+j}{j+1}(-1)^j\left[{K \atop K-I+j}\right]=$$
$$\sum_{0\le j\le I-1}\binom{(K-1)-(I-1)+j}{j+1}(-1)^j\left[{K \atop (K-1)-(I-1)+j}\right]+$$
$$\binom{K}{I+1}(-1)^I=$$
$$\sum_{0\le j\le I-1}\binom{(K-1)-(I-1)+j}{j+1}(-1)^j\times$$
$$\left((K-1)\left[{K-1 \atop (K-1)-(I-1)+j}\right]+\left[{K-1 \atop K-1-I+j}\right]\right)+\binom{K}{I+1}(-1)^I=$$
$$(K-1)^2\left[{K-2 \atop K-1-I}\right]+\sum_{0\le j\le I}\binom{K-I+j}{j+1}(-1)^j\left[{K-1 \atop K-1-I+j}\right]=$$
$$(K-1)^2\left[{K-2 \atop K-1-I}\right]+\sum_{0\le j\le I}\binom{K-1-I+j}{j+1}(-1)^j\left[{K-1 \atop K-1-I+j}\right]+$$
$$\sum_{0\le j\le I}\binom{K-1-I+j}{j}(-1)^j\left[{K-1 \atop K-1-I+j}\right]=$$
$$(K-1)^2\left[{K-2 \atop K-1-I}\right]+(K-1)\left[{K-2 \atop K-2-I}\right]+\left[{K-1 \atop K-1-I}\right]+$$
$$\sum_{0\le j\le I-1}\binom{(K-1)-(I-1)+j}{j+1}(-1)^j\left[{K-1 \atop (K-1)-(I-1)+j}\right]=$$
$$(K-1)\left((K-2)\left[{K-2 \atop K-1-I}\right]+\left[{K-2 \atop K-2-I}\right]\right)+\left[{K-1 \atop K-1-I}\right]=$$
$$K\left[{K-1 \atop K-1-I}\right] \quad (4.11)$$

The second equality is a conseqence of a well-known recurrsion:

$$\left[{N \atop M}\right]=(N-1)\left[{N-1 \atop M}\right]+\left[{N-1 \atop M-1}\right].$$

The fourth equality is a conseqence of a well-known recurrsion:

$$\binom{N}{M}=\binom{N-1}{M}+\binom{N-1}{M-1}.$$

The third equality follows by an inductive assertion for a pair $K-1, I-1$, The fifth equality follows by an inductive assertion for a pair $K-1, I$ and the sixth equality follows again by an inductive assertion for a pair $K-1, I-1$.

$\square$

4.4.2 High degree terms of Kerov polynomials for the first degree

The highest degree term of $K_\mu^{(\alpha)}$ for $\deg_1$ is easy to compute. Indeed, thanks to Section 4.2.4, one knows that $K_\mu^{(\alpha)}$ has at most degree $|\mu|+\ell(\mu)$ (this has also been proved by Lassalle [Las09, Proposition 9.2 (ii)]). Moreover, its component of degree $|\mu|+\ell(\mu)$ does not depend on α. As this dominant term is known in the case $\alpha = 1$ (see for example [Śni06a, Theorem 4.9]), one obtains the following result (which extends [Las09, Theorem 10.2]):

$$K_\mu^{(\alpha)} = \prod_{i \leq \ell(\mu)} R_{\mu_i+1}^{(\alpha)} + \text{smaller degree terms.}$$

In this section, we explore two directions:

- we give explicit formulas for more terms, confirming Lassalle's conjectural data;
- we use this result on the dominant term to describe the asymptotic shape under some deformation of the Plancherel measure on Young diagrams.

Explicit formulas for smaller degrees

Let $\mathfrak{h}_\pi(\mu)$ denote the monomial symmetric function indexed by π evaluated in variables μ_1,μ_2, ... For example,

$$\mathfrak{h}_{1^2}(\mu) = \sum_{i<j} \mu_i \mu_j.$$

We also introduce the notation $\tilde{R}_i = (i-1)R_i^{(\alpha)}$ and $\tilde{R}_\mu = \prod_i \frac{\tilde{R}_i^{m_i(\mu)}}{m_i(\mu)!}$.

Theorem 4.4.2. *For $k \geq 1$, one has*

$$K_k^{(\alpha)} = R_{k+1}^{(\alpha)} + \gamma \frac{k}{2} \sum_{|\mu|=k} (\ell(\mu)-1)! \tilde{R}_\mu + \sum_{|\mu|=k-1} \left(\frac{1}{4}\binom{k+1}{3} + \gamma^2 k \frac{3\mathfrak{h}_2(\mu) + 4\mathfrak{h}_{1^2}(\mu) + 2\mathfrak{h}_1(\mu)}{24} \right) \ell(\mu)! \tilde{R}_\mu + \text{terms of lower degree with respect to } \deg_1. \quad (4.12)$$

Proof. Let us write:

$$K_k^{(\alpha)} = \sum_\mu c_\mu R_\mu^{(\alpha)}.$$

By Proposition 4.2.7, c_μ is a polynomial in γ of degree at most $k+1-|\mu|$, hence c_μ is a polynomial in γ of degree at most 2 for $|\mu| \geq k-1$. Moreover, we know explicitly how to express $K_k^{(\alpha)}$ in terms of free cumulants for $\alpha \in \{\frac{1}{2}, 1, 2\}$ (which corresponds to $\gamma \in \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$). The case $\alpha = 1$ has been solved separately in papers [GR07, Śni06a], while the cases $\alpha = 1/2$ and 2 follows from the combinatorial interpretation given in [FŚ11b] and the explicit computation done in [CJ11]. □

Remark 4.4.3. One can notice, that the explicit formulas for c_μ with $|\mu| \geq k$ were also proved by Lassalle [Las09, Theorems 10.2 and 10.3]. Moreover, our calculations for c_μ with $|\mu| = k-1$ are consistent with Lassalle's computer experiments [Las09, p. 2257], which provide a new evidence to Conjecture 11.2 of Lassalle [Las09].

Characters for more complicated partitions

In the previous section we have focused on character values on a single cycle. Let us consider now the case of more complicated conjugacy classes, that is the functions $\mathrm{Ch}^{(\alpha)}_{k_1,\dots,k_l}$.

In fact, it turns out to be more convenient to consider *cumulant of character values* $\kappa^{\mathrm{id}}(\mathrm{Ch}^{(\alpha)}_{k_1}, \dots, \mathrm{Ch}^{(\alpha)}_{k_l})$. Precise definition of these quantities can be found in [RŚ08], for the purpose of this article it is enough to know that their relation to the characters $\mathrm{Ch}^{(\alpha)}_{k_1,\dots,k_l}$ is analogous to the relation between classical cumulants of random variables and their moments. As pointed out in [Śni06c], in the special case $\alpha = 1$, the above quantities $\kappa^{\mathrm{id}}(\mathrm{Ch}^{(1)}_r, \mathrm{Ch}^{(1)}_s, \dots)$ are very useful in the study of fluctuations of random Young diagrams; in fact this is also the case for any parameter α. Let $\tilde{K}_\mu$ denote the polynomial, which expresses the cumulant $\kappa^{\mathrm{id}}(\mathrm{Ch}^{(\alpha)}_{\mu_1}, \dots, \mathrm{Ch}^{(\alpha)}_{\mu_{\ell(\mu)}})$ in terms of free cumulants. One can found the following upper bound for degree of polynomial $\tilde{K}_\mu$:

Theorem 4.4.4. $\deg_1(\tilde{K}_\mu) \leq \max(|\mu| + 2 - \ell(\mu), |\mu| + \ell(\mu) - 3)$.

Proof. Let us write:

$$\tilde{K}_\mu = \sum_\rho c_{\mu,\rho} R^{(\alpha)}_\rho.$$

We know that $c_{\mu,\rho}$ are polynomials in γ of degree at most $|\mu| + \ell(\mu) - |\rho|$, hence for $|\rho| \geq |\mu| + \ell(\mu) - 2$ they are polynomials in γ of degree at most 2. Moreover, strictly from the definition of $\tilde{K}_\mu$ and from the combinatorial interpretations given in [DFŚ10, FŚ11b], we have that $\deg_1(\tilde{K}^\alpha_\mu) = |\mu| + 2 - \ell(\mu)$ for $\alpha \in \{\frac{1}{2}, 1, 2\}$. In other words, $c_{\mu,\rho}(\gamma) = 0$ for $\gamma \in \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}$ and for $|\rho| > |\mu| + 2 - \ell(\mu)$. This implies that $c_{\mu,\rho} \equiv 0$ for $|\rho| > \max(|\mu| + 2 - \ell(\mu), |\mu| + \ell(\mu) - 3)$ which ends the proof. □

In our opinion this is not the optimal bound for the degree. We conjecture that the bound established for $\alpha \in \{1, 1/2, 2\}$ still holds for a general parameter α:

Conjecture 4.4.5. $\deg_1(\tilde{K}_\mu) = |\mu| + 2 - \ell(\mu)$.

We consider the following deformation of the Plancherel measure

$$\mathbb{P}_n^{(\alpha)}(\lambda) = \frac{\alpha^n n!}{j_\lambda^{(\alpha)}},$$

where $j_\lambda^{(\alpha)}$ is the following deformation of the square of the hook products:

$$j_\lambda^{(\alpha)} = \prod_{\square \in \lambda} \Big(\alpha a(\square) + \ell(\square) + 1\Big)\Big(\alpha a(\square) + \ell(\square) + \alpha\Big).$$

Here, $a(\square) := \lambda_j - i$ and $\ell(\square) := \lambda'_i - j$ are respectively the arm and leg length of the box $\square = (i, j)$ drawn in the Franch convention (the same definition as in [Mac95, Chapter I]). The probability measure $\mathbb{P}_n^{(\alpha)}$ on Young diagrams of size n has appeared recently in several research papers [Ful04, Mat08, Ols10, Mat10] and is presented as an important area of research in Okounkov's survey on random partitions [Oko03, Section 3.3]. When $\alpha = 1$, it specializes to the well-known Plancherel measure for the symmetric groups.

The following property, which corresponds to the case $\pi = (1^n)$ in [Mat10, Equation (8.4)], characterizes the Jack measure:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\theta_\mu^{(\alpha)}(\lambda)) = \delta_{\mu,1^n},$$

where λ is a random Young diagram with n boxes distributed according to $\mathbb{P}_n^{(\alpha)}$.

Using the definition of $\mathrm{Ch}_\mu^{(\alpha)}$ we have:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\mathrm{Ch}_\mu^{(\alpha)}) = \begin{cases} n(n-1)\cdots(n-k+1) & \text{if } \mu = 1^k \text{ for some } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathrm{Ch}_\mu^{(\alpha)}$ is a linear basis of $\Lambda_\star^{(\alpha)}$, it implies the following lemma (which is an analogue of [Ols10, Theorem 5.5] with another gradation).

Lemma 4.5.1. *Let F be an α-polynomial function. Then $\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(F)$ is a polynomial in n of degree at most* $\deg_1(F)/2$.

Proof. It is enough to verify this lemma on the basis $\mathrm{Ch}_\mu^{(\alpha)}$. But in this case $\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(F)$ is explicit (see the formula above) and the lemma is obvious (recall that $\deg_1(\mathrm{Ch}_\mu^{(\alpha)}) = |\mu| + \ell(\mu)$, see Section 4.2.4). □

Let $(\lambda_n)_{n\geq 1}$ be a sequence of random partitions, where λ_n has distribution $\mathbb{P}_n^{(\alpha)}$. In the case $\alpha = 1$, it has been proved in 1977 separately by Logan and Shepp [LS77] and Kerov and

Vershik [KV77] that, in probability,

$$\left\|\omega(D_{1/\sqrt{n}}(\lambda)) - \Omega\right\| = 0,$$

where $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|$, and Ω is the limit shape given explicitly as follows:

$$\Omega(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$$

Recall that $\omega(\lambda)$ is by definition the function whose graphical representation is the border of λ, rotated by $45°$ (see Section 2.1.4) and stretched by $\sqrt{2}$.

In the case α is general, we have the following weak convergence result:

Proposition 4.5.2. *For any 1-polynomial function $F \in \Lambda_\star^{(1)}$, when n tends to ∞, one has*

$$F\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda_n)\big) \xrightarrow{\mathbb{P}_n^{(\alpha)}} F(\Omega),$$

where $\xrightarrow{\mathbb{P}_n^{(\alpha)}}$ means convergence in probability.

Proof. As $(R_k^{(1)})_{k\geq 2}$ is an algebraic basis of $\Lambda_\star^{(1)}$, it is enough to prove the proposition for any $R_k^{(1)}$.

Let μ be partition. As mentioned in the previous sections (as a direct consequence of Theorem 4.1.2), one has:

$$\prod_{i\leq\ell(\mu)} R_{\mu_i+1}^{(\alpha)} = \mathrm{Ch}_\mu^{(\alpha)} + \text{terms of degree at most}$$
$$|\mu| + \ell(\mu) - 1 \text{ with respect to } \deg_1. \quad (4.13)$$

Together with Lemma 4.5.1 and the formula for $\mathbb{E}(\mathrm{Ch}_\mu^{(\alpha)})$, this implies:

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\Big(\prod_{i\leq\ell(\mu)} R_{\mu_i+1}^{(\alpha)}\Big) = \begin{cases} n(n-1)\cdots(n-k+1) + O(n^{k-1}) & \text{if } \mu = 1^k \text{ for some } k; \\ o(n^{\frac{|\mu|+\ell(\mu)}{2}}) & \text{otherwise.} \end{cases}$$

In particular

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda_n))) = \frac{1}{n^{k/2}}\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)}) = \delta_{k,2} + O\left(\frac{1}{\sqrt{n}}\right),$$
$$\mathrm{Var}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda_n))) = \frac{1}{n^k}\left(\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}\big((R_k^{(\alpha)})^2\big) - \mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(R_k^{(\alpha)})^2\right) = O\left(\frac{1}{n}\right).$$

Thus, for each k, $R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda_n))$ converges in probability towards $\delta_{k,2}$. But, by definition

$$R_k^{(\alpha)}(D_{1/\sqrt{n}}(\lambda_n)) = R_k^{(1)}\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda_n)\big)$$

and $(\delta_{k,2})_{k\geq 2}$ is the sequence of free cumulants of the continuous diagram Ω (see [Bia01, Section 3.1]), *i.e.*

$$\delta_{k,2} = R_k^{(1)}(\Omega).$$

□

The last thing we need to prove the uniform convergence of the function associated to the diagram (see Section 2.1.4) is the following technical lemma, proved by Fulman [Ful04, Lemma 6.6]:

Lemma 4.5.3. *Suppose that* $\alpha > 0$. *Then*

1.
$$\mathbb{P}_n^{(\alpha)}\left(\lambda_1 \geq 2e\sqrt{\frac{n}{\alpha}}\right) \leq \alpha n^2 4^{-e\sqrt{\frac{n}{\alpha}}},$$

2.
$$\mathbb{P}_n^{(\alpha)}(\lambda_1' \geq 2e\sqrt{n\alpha}) \leq \frac{n^2}{\alpha}4^{-e\sqrt{n\alpha}}.$$

In particular
$$\lim_{n\to\infty} \mathbb{P}_n^{(\alpha)}\left(\left[-\frac{\lambda_1'}{\sqrt{n}}; \frac{\lambda_1}{\sqrt{n}}\right] \subseteq [-2e, 2e]\right) = 1.$$

It gives us the following theorem:

Theorem 4.5.4. *For each* n*, let* λ_n *be a random Young diagram of size* n *distributed with Jack measure. Then*

$$\left\|\omega\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda_n)\big) - \Omega\right\| = 0$$

in probability.

Proof. It follows from Proposition 4.5.2 and Lemma 4.5.3 by the same argument as the one given in [IO02, Theorem 5.5]. □

The idea of using polynomial functions to study the asymptotic shape of Young diagrams has been developped by Kerov (see [IO02]). In the case $\alpha = 1$, he gave more precise result that what we have here: he proved that for any polynomial function F, the quantity $F(\lambda_n)$ has Gaussian fluctuations, which is refered as Kerov's Central Limit Theorem. In next section we are going to show that the fluctuation phenomenon holds in the general α case.

4.6 Jack measure: Central limit theorem for Jack characters

4.6.1 Multivariate Stein's method

In this section we are going to prove a Central Limit Theorem for Jack characters. To be more precise we are going to prove the following theorem.

Theorem 4.6.1. *Choose a sequence $(\Xi_k)_{k=2,3,\dots}$ of independent standard Gaussian random variables. As $n \to \infty$, we have:*

$$\left(\frac{\mathrm{Ch}_k^{(\alpha)}}{\sqrt{k} n^{k/2}}\right)_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots},$$

where $\mathrm{Ch}_k^{(\alpha)}$ are random variables distributed by Jack measure.

Our main tool will be a multivariate analogon of Stein's theorem [RR09, Theorem 2.1]. For any discrete random variables W, W^* with values in $\mathbb{R}^d$, we say that the pair (W, W^*) is *exchangeable* if for any $w_1, w_2 \in \mathbb{R}^d$ one has $\mathbb{P}(W = w_1, W^* = w_2) = \mathbb{P}(W = w_2, W^* = w_1)$. Let $\mathbb{E}^W(\cdot)$ denote for the conditional expected value given W. The theorem of Reinert and Röllin is the following:

Theorem 4.6.2. *[RR09, Theorem 2.1] Let (W, W^*) be an exchangeable pair of $\mathbb{Q}^d$-valued random variables such that $\mathbb{E}(W) = 0$ and $\mathbb{E}(WW^t) = \Sigma$, where $\Sigma \in M_{d\times d}(\mathbb{R})$ is symmetric and positive definite matrix. Suppose that $\mathbb{E}^W(W^* - W) = -\Lambda W$, where $\Lambda \in M_{d\times d}(\mathbb{R})$ is invertible. Then, if Z has d-dimensional standard normal distribution, we have for every three times differentiable function $h : \mathbb{R}^d \to \mathbb{R}$,*

$$\left|\mathbb{E}h(W) - \mathbb{E}h(\Sigma^{1/2}Z)\right| \leq \frac{|h|_2}{4}A + \frac{|h|_3}{12}B, \tag{4.14}$$

where, with $\lambda^{(i)} := \sum_{1\leq m\leq d} |(\Lambda^{-1})_{m,i}|$ and with $|h_n| = \sup_{i_1,\dots,i_n} \left\|\frac{\partial^n}{\partial x_{i_1}\cdots\partial x_{i_n}} h\right\|$,

$$A = \sum_{1\leq i,j\leq d} \lambda^{(i)} \sqrt{\mathrm{Var}\, \mathbb{E}^W (W_i^* - W_i)(W_j^* - W_j)},$$

$$B = \sum_{1\leq i,j,k\leq d} \lambda^{(i)} \mathbb{E}|(W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)|.$$

Firstly, we recall briefly from the work of Fulman [Ful04], how for any given random variable W defined on the set of Young diagrams one can associate a random variable W^* defined on the set of Young diagrams such that the pair (W, W^*) is exchangeable with respect to Jack measure.

4.6.2 EXCHANGEABLE PAIR WITH RESPECT TO JACK-PLANCHEREL MEASURE

Let $(X, \mathbb{P})$ be a finite probability space, and let W be a random variable on X. Let M be Markov chain on X, which is reversible with respect to $\mathbb{P}$. Let $x \in X$ and let x^* be obtained from x by one step in the chain M. Then the pair $(W, W^*) := (W(x), W(x^*))$ is exchangeable with respect to $\mathbb{P}$. Of course, if $W_1, \ldots, W_k$ are random variables, then the pair of random vectors $(\tilde{W}_k, \tilde{W}_k^*)$ is exchangeable, where $\tilde{W}_k = (W_1, \ldots, W_k)$ and $\tilde{W}_k^* = (W_1^*, \ldots, W_k^*)$. Now, we are going to recall the reversible Markov chain with respect to Jack measure, constructed by Fulman [Ful04].

Let $\tau \vdash n-1$ and $\lambda \vdash n$, and let $C_{\lambda/\tau}$ ($R_{\lambda/\tau}$ respectively) be the union of columns (rows respectively) of λ that intersect λ/τ. We define

$$\phi^{(\alpha)}(\lambda/\tau) = \prod_{\square \in C_{\lambda/\tau} \setminus R_{\lambda/\tau}} \frac{(\alpha a_\lambda(\square) + \ell_\lambda(\square) + 1)(\alpha a_\tau(\square) + \ell_\tau(\square) + \alpha)}{(\alpha a_\lambda(\square) + \ell_\lambda(\square) + \alpha)(\alpha a_\tau(\square) + \ell_\tau(\square) + 1)}.$$

Let

$$c_\lambda^{(\alpha)} = \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + 1).$$

and

$$(c_\lambda')^{(\alpha)} = \prod_{\square \in \lambda} (\alpha a(\square) + \ell(\square) + \alpha).$$

We recall that $j_\lambda^{(\alpha)} = c_\lambda^{(\alpha)} (c_\lambda')^{(\alpha)}$. For $\lambda, \rho \vdash n$ we define two functions:

$$M^{(\alpha)}(\lambda, \rho) = \frac{(c_\lambda')^{(\alpha)}}{n\alpha c_\rho^{(\alpha)}} \sum_{\tau \vdash n-1} \frac{\phi^{(\alpha)}(\lambda/\tau)\phi^{(\alpha)}(\rho/\tau) c_\tau^{(\alpha)}}{(c_\tau')^{(\alpha)}} \tag{4.15}$$

and

$$L^{(\alpha)}(\lambda, \rho) = \frac{1}{\alpha^n n! j_\lambda^{(\alpha)}} \sum_{\mu \vdash n} (z_\mu)^2 \alpha^{2\ell(\mu)} \theta_\mu^{(\alpha)}(\lambda) \theta_\mu^{(\alpha)}(\rho) \theta_\mu^{(\alpha)}((n-1, 1)). \tag{4.16}$$

As it was explained by Fulman [Ful04], both $M^{(\alpha)}$ and $L^{(\alpha)}$ are defined to be a deformation of a certain Markov chain which is reversible with respect to Plancherel measure. Roughly speaking, this Markov chain remove one box from a given Young diagram with certain probability and add another box with some probability to obtain a new Young diagram of the same size as the one from which we started. Fulman [Ful04] proved the following theorem:

Theorem 4.6.3. *[Ful04, Section 4]*

1. *If* $\rho \neq \lambda$ *then*

$$L^{(\alpha)}(\lambda, \rho) = \frac{\alpha(n-1)+1}{\alpha(n-1)} M^{(\alpha)}(\lambda, \rho);$$

2. *Let $\lambda \vdash n$. Then*

$$\sum_{\rho \vdash n} L^{(\alpha)}(\lambda, \rho) = \sum_{\rho \vdash n} M^{(\alpha)}(\lambda, \rho) = 1;$$

3. *$L^{(\alpha)}$ (hence $M^{(\alpha)}$ as well) is reversible with respect to Jack measure.*

For more details about this construction we refer to Fulman [Ful04]. Finally, for a random vector W defined on the set of Young diagrams with n boxes we define a pair (W, W^*) of the random vectors by a Markov chain given by $M^{(\alpha)}$ and we define a pair (W, W') of the random vectors by a function $L^{(\alpha)}$.

4.6.3 Necessary conditions

In this section we are going to prove, that for any $d \in \mathbb{N}$, the pair $(\tilde{W}_d, \tilde{W}_d^*)$ of random vectors, where

$$\tilde{W}_d = (W_2, \dots, W_{d+1})$$

with

$$W_k = n^{-k/2} \sqrt{k}^{-1} \operatorname{Ch}_k^{(\alpha)},$$

satisfies conditions of Theorem 4.6.2.

Lemma 4.6.4. *1. [Mac95, Page 382]*

$$\sum_{\rho \vdash n} \frac{\theta_\mu^{(\alpha)}(\rho)\theta_\nu^{(\alpha)}(\rho)}{j_\rho^{(\alpha)}} = \frac{\delta_{\mu,\nu}}{\delta_{\mu,\nu} z_\mu \alpha^{\ell(\mu)}};$$

2. [Sta89, Page 107]

$$\theta_\mu^{(\alpha)}((n-1,1)) = \frac{\alpha^{n-\ell(\mu)} n!}{z_\mu} \frac{(\alpha(n-1)+1)m_1(\mu) - n}{\alpha n(n-1)}.$$

Proposition 4.6.5. *Let $d \in \mathbb{N}$ and let $2 \le k \le d+1$. Then*

$$(\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_k^*) = \left(1 - \frac{k}{n}\right) W_k.$$

and

$$(\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_k') = \left(1 - \frac{k(\alpha(n-1)+1)}{\alpha n(n-1)}\right) W_k$$

Proof. By [Ful04, Proposition 6.2.] we have that $\theta_\mu^{(\alpha)}(\lambda)$ is an eigenvector of $M^{(\alpha)}$ with eigenvalue

$$1 + \frac{\alpha(n-1)}{\alpha(n-1)+1} \left(\frac{z_\mu}{\alpha^{n-\ell(\mu)} n!} \theta_\mu^{(\alpha)}((n-1,1)) - 1 \right)$$

and an eigenvector of $L^{(\alpha)}$ with eigenvalue

$$\frac{z_\mu}{\alpha^{n-\ell(\mu)}n!}\theta_\mu^{(\alpha)}((n-1,1)).$$

Applying the case (2) of Lemma 4.6.4 we see that this eigenvalue is equal to:

$$\frac{(\alpha(n-1)+1)m_1(\mu)}{n(\alpha(n-1)+1)} = \frac{m_1(\mu)}{n}$$

in the first case and

$$\frac{(\alpha(n-1)+1)m_1(\mu)-n}{\alpha n(n-1)}$$

in the second case. Setting $\mu = (k, 1^{n-k})$, the eigenvalue is equal to $1 - \frac{k}{n}$ in the first case and $1 - \frac{k(\alpha(n-1)+1)}{\alpha n(n-1)}$ in the second case. Since for $\lambda \vdash n$ one has that $W_k(\lambda)$ is equal to $\theta^{(\alpha)}_{(k,1^{n-k})}(\lambda)$ up to a multiplication by a constant, it finishes the proof. □

Corollary 4.6.6. *Let $d \in \mathbb{N}$. Then*

$$(\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(\tilde{W}_d^* - \tilde{W}_d) = -\Lambda \tilde{W}_d,$$

with $\Lambda_{i,j} = \delta_{i,j}\frac{i+1}{n}$. In particular

$$\lambda^{(i)} = \frac{n}{i+1}.$$

Proof. It is a straightforward consequence of Proposition 4.6.5. □

Proposition 4.6.7. *Let $d \in \mathbb{N}$ and let*

$$(\mathbb{E}_n^{(\alpha)})(\tilde{W}_d\tilde{W}_d^t) = \Sigma.$$

Then there exists a constant $A_{d,\alpha}$ which depends only on d and α such that for any $n \geq A_{d,\alpha}$ the matrix Σ is positive definite. Moreover,

$$\Sigma^{1/2} = \operatorname{Id} + O(n^{-1/2})\Sigma',$$

where $\Sigma' \in M_{d\times d}(\mathbb{R})$ depends only on α.

Proof. Strictly from the definition we have that

$$\begin{aligned}\Sigma_{i,j} = \mathbb{E}_n^{(\alpha)}&\left(\frac{1}{\sqrt{i+1}\sqrt{j+1}n^{(i+j+2)/2}}\operatorname{Ch}_{i+1}^{(\alpha)}\operatorname{Ch}_{j+1}^{(\alpha)}\right)\\ &= \frac{1}{\sqrt{i+1}\sqrt{j+1}n^{(i+j+2)/2}}\sum_\lambda g^{(\alpha)}_{(i+1),(j+1);\lambda}\mathbb{E}_n^{(\alpha)}(\operatorname{Ch}_\lambda^{(\alpha)}). \quad (4.17)\end{aligned}$$

Since

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\mathrm{Ch}_\mu^{(\alpha)}) = \begin{cases} (n)_k & \text{if } \mu = 1^k \text{ for some } k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$\Sigma_{i,j} = \frac{1}{\sqrt{i+1}\sqrt{j+1}n^{(i+j+2)/2}} \sum_\lambda g^{(\alpha)}_{(i+1),(j+1);1^l}(n)_l,$$

and by the cases (1) and (3) of Lemma 4.3.7 we have, that

$$\Sigma_{i,j} = \delta_{i,j} + O(n^{-1}).$$

It means that

$$\Sigma = \mathrm{Id} + O(n^{-1})\Sigma'',$$

where $\Sigma'' \in M_{d\times d}(\mathbb{R})$ is a symmetric matrix, which depends only on α. It finishes the proof. □

4.6.4 Error term

Since we checked that the pair $(\tilde{W}_d, \tilde{W}_d^*)$ of the random vectors satisfies the assumptions of Theorem 4.6.2, we would like to estimate an error term in that theorem. Since we are going to prove that the random vector $\tilde{W}_d$ is asymptotically Gaussian, we need to show that the terms A and B from Theorem 4.6.2 vanish as $n \to \infty$. This section is devoted to making these calculations.

Lemma 4.6.8. *The following inequality holds:*

$$\mathrm{Var}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i - W_i^*)(W_j - W_j^*)\right) \le \frac{1}{ijn^{i+j+2}} \times \left(\sum_{\mu_1,\mu_2,\mu} H^{(i,j)}_{\mu_1,\mu_2;\mu}\mathbb{E}_n^{(\alpha)}(\mathrm{Ch}_\mu^{(\alpha)}(\lambda)) - (i+j)^2\left(\sum_\mu g^{(\alpha)}_{(i),(j);\mu}\mathbb{E}_n^{(\alpha)}(\mathrm{Ch}_\mu^{(\alpha)})\right)^2\right), \quad (4.18)$$

where

$$H^{(i,j)}_{\mu_1,\mu_2;\mu} := (i+j-|\mu_1|+m_1(\mu_1))(i+j-|\mu_2|+m_1(\mu_2))g^{(\alpha)}_{(i),(j);\mu_1}g^{(\alpha)}_{(i),(j);\mu_2}g^{(\alpha)}_{\mu_1,\mu_2;\mu}.$$

Proof. Following Fulman [Ful04, Proof of Proposition 6.4.], from Jensen's inequality for conditional expectations, the fact that $\tilde{W}_d$ is determined by λ implies that

$$\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i - W_i^*)(W_j - W_j^*)\right)^2 \le \mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\lambda}(W_i - W_i^*)(W_j - W_j^*)\right)^2.$$

We have, by Theorem 4.6.3, that

$$(\mathbb{E}_n^{(\alpha)})^\lambda (W_i^* - W_i)(W_j^* - W_j) = \frac{\alpha(n-1)}{\alpha(n-1)+1} (\mathbb{E}_n^{(\alpha)})^\lambda (W_i' - W_i)(W_j' - W_j)$$
$$= \frac{\alpha(n-1)}{\alpha(n-1)+1} \left((\mathbb{E}_n^{(\alpha)})^\lambda (W_i' W_j') - (\mathbb{E}_n^{(\alpha)})^\lambda (W_i') W_j - (\mathbb{E}_n^{(\alpha)})^\lambda (W_j') W_i) + W_i W_j \right)$$
$$= \frac{\alpha(n-1)}{\alpha(n-1)+1} \left((\mathbb{E}_n^{(\alpha)})^\lambda (W_i' W_j') + \left(\frac{(i+j)(\alpha(n-1)+1)}{\alpha n(n-1)} - 1 \right) W_i W_j \right), \quad (4.19)$$

where the last inequality follows from Proposition 4.6.5. Strictly from the definition of $L^{(\alpha)}$, one has

$$(\mathbb{E}_n^{(\alpha)})^\lambda (W_i' W_j') = \sum_{\mu \vdash n} \theta_\mu^\lambda(\alpha) \theta_\mu^{(n-1,1)}(\alpha) \frac{(z_\mu)^2 \alpha^{2\ell(\mu)}}{\alpha^n n!}$$
$$\times \frac{1}{\sqrt{ij} n^{(i+j/2)}} \sum_\lambda g_{(i),(j);\lambda}^{(\alpha)} \sum_{\rho \vdash n} \frac{\mathrm{Ch}_\lambda^{(\alpha)}(\rho) \theta_\mu^\rho(\alpha)}{j_\rho(\alpha)}. \quad (4.20)$$

By the definition (2.3.2) of $\mathrm{Ch}_\lambda^{(\alpha)}(\rho)$ and by the case (1) of Lemma 4.6.4, above equation became

$$(\mathbb{E}_n^{(\alpha)})^\lambda (W_i' W_j') = \frac{1}{\sqrt{ij} n^{(i+j/2)}} \sum_\mu g_{(i),(j);\mu}^{(\alpha)} \mathrm{Ch}_\mu^{(\alpha)}(\lambda) \theta_{\mu \cup 1^{n-|\mu|}}^{(n-1,1)}(\alpha) \frac{z_{\mu \cup 1^{n-|\mu|}} \alpha^{\ell(\mu \cup 1^{n-|\mu|})}}{\alpha^n n!}.$$

Applying the case (2) of Lemma 4.6.4 to the above equation we obtain

$$(\mathbb{E}_n^{(\alpha)})^\lambda (W_i' W_j') = \frac{1}{\sqrt{ij} n^{(i+j/2)}} \sum_\mu g_{(i),(j);\mu}^{(\alpha)} \mathrm{Ch}_\mu^{(\alpha)}(\lambda)$$
$$\times \frac{(\alpha(n-1)+1)(n-|\mu|+m_1(\mu)) - n}{\alpha n(n-1)}. \quad (4.21)$$

Moreover, strictly from the definition, one has

$$W_i W_j(\lambda) = \frac{1}{\sqrt{ij} n^{(i+j/2)}} \sum_\mu g_{(i),(j);\mu}^{(\alpha)} \mathrm{Ch}_\mu^{(\alpha)}(\lambda).$$

One can substitute two above equations to the equation (4.19) and simplifes it to obtain

$$(\mathbb{E}_n^{(\alpha)})^\lambda (W_i^* - W_i)(W_j^* - W_j) = \frac{1}{\sqrt{ij} n^{(i+j/2)}} \sum_\mu \frac{i+j-|\mu|+m_1(\mu)}{n} g_{(i),(j);\mu}^{(\alpha)} \mathrm{Ch}_\mu^{(\alpha)}(\lambda). \quad (4.22)$$

It gives us, that

$$\begin{aligned}
&\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\lambda}(W_i^*-W_i)(W_j^*-W_j)\right)^2 \\
&\qquad= \frac{1}{ijn^{i+j+2}}\mathbb{E}_n^{(\alpha)}\left(\sum_{\mu}(i+j-|\mu|+m_1(\mu))g^{(\alpha)}_{(i),(j);\mu}\operatorname{Ch}^{(\alpha)}_{\mu}(\lambda)\right)^2 \\
&\qquad\qquad= \frac{1}{ijn^{i+j+2}}\sum_{\mu_1,\mu_2,\mu} H^{(i,j)}_{\mu_1,\mu_2;\mu}\mathbb{E}_n^{(\alpha)}(\operatorname{Ch}^{(\alpha)}_{\mu}(\lambda)), \qquad (4.23)
\end{aligned}$$

where

$$H^{(i,j)}_{\mu_1,\mu_2;\mu} := (i+j-|\mu_1|+m_1(\mu_1))(i+j-|\mu_2|+m_1(\mu_2))g^{(\alpha)}_{(i),(j);\mu_1}g^{(\alpha)}_{(i),(j);\mu_2}g^{(\alpha)}_{\mu_1,\mu_2;\mu}.$$

Moreover

$$\begin{aligned}
&\left(\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i^*-W_i)(W_j^*-W_j)\right)\right)^2 \\
&= \left(2\mathbb{E}_n^{(\alpha)}(W_iW_j)-\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i^*)\mathbb{E}_n^{(\alpha)}(W_j)\right)-\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_j^*)\mathbb{E}_n^{(\alpha)}(W_i)\right)\right)^2 \\
&\qquad= \left(\frac{i+j}{n}\mathbb{E}_n^{(\alpha)}(W_iW_j)\right)^2 = \frac{(i+j)^2}{ijn^{i+j+2}}\sum_{\mu} g^{(\alpha)}_{(i),(j);\mu}\mathbb{E}_n^{(\alpha)}(\operatorname{Ch}^{(\alpha)}_{\mu}), \qquad (4.24)
\end{aligned}$$

where the first equality comes from the fact that W_i has the same distribution as W_i^*, and the second equality follows from Proposition 4.6.5. We finish the proof by the following inequality:

$$\begin{aligned}
&\operatorname{Var}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i-W_i^*)(W_j-W_j^*)\right) \\
&= \mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i^*-W_i)(W_j^*-W_j)\right)^2 - \left(\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i^*-W_i)(W_j^*-W_j)\right)\right)^2 \\
&\le \mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\lambda}(W_i^*-W_i)(W_j^*-W_j)\right)^2 - \left(\mathbb{E}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i^*-W_i)(W_j^*-W_j)\right)\right)^2. \qquad (4.25)
\end{aligned}$$

□

Proposition 4.6.9. *Let $d \in \mathbb{N}$ and let*

$$A = \sum_{2\le i,j\le d+1}\frac{n}{i}\sqrt{\operatorname{Var}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i-W_i^*)(W_j-W_j^*)\right)}.$$

Then $A = O(n^{-1/2})$.

Proof. By Lemma 4.6.8 we need to estimate the following sum:

$$\left(\sum_{\mu_1,\mu_2,\mu} H^{(i,j)}_{\mu_1,\mu_2;\mu}\mathbb{E}^{(\alpha)}_n(\mathrm{Ch}^{(\alpha)}_\mu(\lambda)) - (i+j)^2\left(\sum_{\mu} g^{(\alpha)}_{(i),(j);\mu}\mathbb{E}^{(\alpha)}_n(\mathrm{Ch}^{(\alpha)}_\mu)\right)^2\right)^{1/2}.$$

Since

$$\mathbb{E}_{\mathbb{P}^{(\alpha)}_n}(\mathrm{Ch}^{(\alpha)}_\mu) = \begin{cases} (n)_k & \text{if } \mu = 1^k \text{ for some } k \le n,\\ 0 & \text{otherwise,}\end{cases}$$

we, in fact, have to count the following:

$$\left(\sum_{\mu_1,\mu_2,l} H^{(i,j)}_{\mu_1,\mu_2;(1^l)}(n)_l - (i+j)^2\sum_{l,k} g^{(\alpha)}_{(i),(j);(1^l)}g^{(\alpha)}_{(i),(j);(1^k)}(n)_l(n)_k\right)^{1/2}.$$

By the cases (1) and (3) of Lemma 4.3.7 we have that

$$(i+j)^2\sum_{l,k} g^{(\alpha)}_{(i),(j);(1^l)}g^{(\alpha)}_{(i),(j);(1^k)}(n)_l(n)_k = \delta_{i,j}ij(i+j)^2n^{i+j} + O(n^{i+j-1}).$$

For

$$|\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) > 2(i+j+2)$$

we have by the case (1) of Lemma 4.3.7 that $g^{(\alpha)}_{(i),(j);\mu_1}$ or $g^{(\alpha)}_{(i),(j);\mu_2}$ vanishes, hence $H^{(i,j)}_{\mu_1,\mu_2;(1^l)} = 0$. For

$$|\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) = 2(i+j+2)$$

we have by the case (1) of Lemma 4.3.7 that $g^{(\alpha)}_{(i),(j);\mu_1}$ or $g^{(\alpha)}_{(i),(j);\mu_2}$ vanishes unless $\mu_1 = \mu_2 = (i,j)$. But then

$$(i+j-|\mu_1| + m_1(\mu_1))(i+j-|\mu_2| + m_1(\mu_2)) = 0,$$

hence $H^{(i,j)}_{\mu_1,\mu_2;(1^l)} = 0$. In particular $H^{(i,j)}_{\mu_1,\mu_2;(1^l)} = 0$ for $l \ge i+j+2$. Moreover, for $l < i+j$, we have that

$$H^{(i,j)}_{\mu_1,\mu_2;(1^l)}(n)_l = O(n^{i+j-1}).$$

It means, that there are two cases left for analysis:

- $l = i+j+1$. By Lemma 4.3.7 (1), $H^{(i,j)}_{\mu_1,\mu_2,(1^{i+j+1})}$ is equal to zero, unless

  $$2(i+j+2) > |\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) \ge 2(i+j+1).$$

  In that case, assuming $|\mu_i| + \ell(\mu_k) \le i+j+2$ for $k \in \{1,2\}$, we have that $\tilde{\mu_1} \neq \tilde{\mu_2}$ (and by Lemma 4.3.7 (2) $g^{(\alpha)}_{\mu_1,\mu_2;(1^{i+j+1})} = 0$) or $\mu_k = \mu_{1-k} \cup 1$ and $|\mu_k| + \ell(\mu_k) = i+j+2$

for some $k \in \{1,2\}$. But then, by Lemma 4.3.7 (1) $g^{(\alpha)}_{(i),(j);\mu_k} = 0$. Concluding,

$$H^{(i,j)}_{\mu_1,\mu_2,(1^{i+j+1})} = 0.$$

- $l = i+j$. By Lemma 4.3.7 (1), $H^{(i,j)}_{\mu_1,\mu_2,(1^{i+j})}$ is equal to zero, unless

$$2(i+j+2) > |\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) \geq 2(i+j).$$

When

$$|\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) = 2(i+j)+3,$$

then $g^{(\alpha)}_{(i),(j);\mu_1} g^{(\alpha)}_{(i),(j);\mu_2} = 0$ since $g^{(\alpha)}_{(i),(j);\mu} = 0$ for $|\mu| + \ell(\mu) = i+j+1$ (which follows from Theorem 4.4.4) and for $|\mu| + \ell(\mu) > i+j+2$ (which follows from Lemma 4.3.7 (1)), hence $H^{(i,j)}_{\mu_1,\mu_2;(1^{i+j})} = 0$. For

$$2(i+j)+1 \leq |\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) \leq 2(i+j+1),$$

assuming $|\mu_k| + \ell(\mu_k) \leq i+j+2$ for $k \in \{1,2\}$, we have that $\tilde{\mu_1} \neq \tilde{\mu_2}$ (and by Lemma 4.3.7 (2) $g^{(\alpha)}_{\mu_1,\mu_2;(1^{i+j})} = 0$) or $\mu_k = \mu_{1-k} \cup 1$ and $|\mu_k| + \ell(\mu_k) = i+j+2$ for some $k \in \{1,2\}$. But then, by Lemma 4.3.7 (1) $g^{(\alpha)}_{(i),(j);\mu_k} = 0$, hence, in both cases, $H^{(i,j)}_{\mu_1,\mu_2;(1^{i+j})} = 0$. Finaly, when

$$|\mu_1| + \ell(\mu_1) + |\mu_2| + \ell(\mu_2) = 2(i+j)$$

then, by Lemma 4.3.7 (1) we have that $g^{(\alpha)}_{\mu_1,\mu_2;(1^{i+j})} = 0$ unless $\mu_1 \cup \mu_2 = 1^{i+j}$. In that case, there exists $m \geq \lceil (i+j)/2 \rceil$ such that $\mu_k = (1^m)$ for $k \in \{1,2\}$. Then, by Lemma 4.3.7 (1) one has that $g^{(\alpha)}_{(i),(j);\mu_1} g^{(\alpha)}_{(i),(j);\mu_2} = 0$ unless $i = j$ and $\mu_1 = \mu_2 = (1^i)$ and in that case, by Lemma 4.3.7 (1) (3) one has that $H_{(1^i),(1^j);(1^{i+j})} = (i+j)^2 ij$. Concluding,

$$H_{\mu_1,\mu_2,(1^{i+j})} = \delta_{\mu_1,(1^i)} \delta_{\mu_2,(1^j)} \delta_{i,j} (i+j)^2 ij.$$

It means that

$$\sum_{\mu_1,\mu_2,l} H^{(i,j)}_{\mu_1,\mu_2;(1^l)} (n)_l = \delta_{i,j} (i+j)^2 ij n^{i+j} + O(n^{i+j-1}).$$

Summing up, it gives

$$\left(\sum_{\mu_1,\mu_2,l} H^{(i,j)}_{\mu_1,\mu_2;(1^l)} (n)_l - (i+j)^2 \sum_{l,k} g^{(\alpha)}_{(i),(j);(1^l)} g^{(\alpha)}_{(i),(j);(1^k)} (n)_l (n)_k \right)^{1/2} = O(n^{(i+j-1)/2}), \quad (4.26)$$

which implies that

$$A = \sum_{2 \le i,j \le d+1} \frac{1}{i\sqrt{ij}n^{(i+j)/2}} O(n^{(i+j-1)/2}) = O(n^{-1/2}),$$

which finishes the proof. $\square$

Lemma 4.6.10. *For any $k \ge 2$, and $\lambda \vdash n$ there exists $B_{\alpha,k} \in \mathbb{R}$, which depends only on k and α such that*

$$|M_k^{(\alpha)}(\lambda)| \le B_{\alpha,k} \max(\lambda_1, \lambda_1')^k.$$

Proof. This is an immediate consequence of the fact that $M_k^{(\alpha)}(\lambda) = h_k(\mathbb{O}_\lambda - \mathbb{I}_\lambda)$. $\square$

Proposition 4.6.11. *Let $d \in \mathbb{N}$ and let*

$$B = \sum_{2 \le i,j,k \le d+1} \frac{n}{i} \mathbb{E}_n^{(\alpha)} |(W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)|.$$

Then $B = O(n^{-1/2})$.

Proof. From the definition of $M^{(\alpha)}$ it is clear that λ^* is obtained from λ by removing a box from the diagram of λ and reattaching it somewhere. It means that $\lambda = \mu^{(o_1)}$ and $\lambda^* = \mu^{(o_2)}$ for some $\mu \vdash n-1$. It implies that

$$|W_k^* - W_k| \le \sqrt{k}^{-1} n^{-k/2} \left| \mathrm{Ch}_k^{(\alpha)}(\mu^{(o_1)}) - \mathrm{Ch}_k^{(\alpha)}(\mu^{(o_2)}) \right|.$$

By equation (4.1), the right hand side of the above inequality is equal to

$$\sqrt{k}^{-1} n^{-k/2} \left| \sum_{\rho} a_\rho^{(k)} \left(\sum_{\substack{g,h \ge 0, \\ \pi \vdash h}} b_{g,\pi}^{\rho}(\gamma)\, M_\pi^{(\alpha)}(\mu)\, \left(z_{o_1}^g - z_{o_2}^g\right) \right) \right|,$$

where $|\pi| \le |\rho| - g - 2$. By Proposition 4.2.7 we know, that $a_\rho^{(k)} = 0$ for $|\rho| > k+1$, hence $a_\rho^{(k)} b_{g,\pi}^{\rho}(\gamma) = 0$ for $|\pi| > k - g - 1$. It gives us, thanks to Lemma 4.6.10, that there exists some $C_{\alpha,k} \in \mathbb{R}$ which depends only on k and α such that

$$|W_k^* - W_k| \le n^{-k/2} C_{\alpha,k} \max(\lambda_1, \lambda_1')^{k-1}.$$

If $\lambda_1 \le 2e\sqrt{\frac{n}{\alpha}}$ and $\lambda_1' \le 2e\sqrt{n\alpha}$, then

$$\left|(W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)\right| \le |(W_i^* - W_i)||(W_j^* - W_j)||(W_k^* - W_k)| = O(n^{-3/2}) \quad (4.27)$$

and $B = O(n^{-1/2})$. If not, then

$$\left|(W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)\right| \leq |(W_i^* - W_i)||(W_j^* - W_j)||(W_k^* - W_k)| = O(n^{(i+j+k)/2-3}) \quad (4.28)$$

and $B = O(n^{(i+j+k)/2-4})$, but by Lemma 4.5.3, the desired result follows, since the probability that it occurs is small enough. It finishes the proof. □

4.6.5 Proof of the central limit theorem

Now, since we checked, that all necessary conditions required by Theorem 4.6.2 are satisfied, we are ready to prove our main result from this section:

Proof of Theorem 4.6.1. In order to show that

$$\left(\frac{\mathrm{Ch}_k^{(\alpha)}}{\sqrt{k}n^{k/2}}\right)_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots}$$

as $n \to \infty$, it is enough to show, that for all $d \in \mathbb{N}$ and for any smooth function h on $\mathbb{R}^d$, with all derivatives bounded, one has:

$$\left|\mathbb{E}_n^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}_n^{(\alpha)} h(\tilde{\Xi}_d)\right| \to 0$$

as $n \to \infty$, where $\tilde{W}_d = (W_2, \dots, W_{d+1})$ and $\tilde{\Xi}_d = (\Xi_2, \dots, \Xi_{d+1})$, where $W_k = \frac{\mathrm{Ch}_k^{(\alpha)}}{\sqrt{k}n^{k/2}}$ for $2 \leq k \leq d+1$. Let $(\mathbb{E}_n^{(\alpha)})(\tilde{W}_d \tilde{W}_d^t) = \Sigma$. By Proposition 4.6.7 , we know, that

$$\lim_{n\to\infty} \left|\mathbb{E}_n^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}_n^{(\alpha)} h(\tilde{\Xi}_d)\right| = \lim_{n\to\infty} \left|\mathbb{E}_n^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}_n^{(\alpha)} h(\Sigma\tilde{\Xi}_d)\right|.$$

By Corollary 4.6.6 and by Proposition 4.6.7 we know, that the pair $(\tilde{W}_d, \tilde{W}_d^*)$ satisfies all necessary conditions of Theorem 4.6.2. By Theorem 4.6.2, Corollary 4.6.6 and by Proposition 4.6.7 we have that

$$\left|\mathbb{E}_n^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}_n^{(\alpha)} h(\Sigma\tilde{\Xi}_d)\right| \leq |h|_2 \frac{A}{4} + |h|_3 \frac{B}{12},$$

where

$$A = \sum_{2\leq i,j\leq d+1} \frac{n}{i} \sqrt{\mathrm{Var}_n^{(\alpha)}\left((\mathbb{E}_n^{(\alpha)})^{\tilde{W}_d}(W_i - W_i^*)(W_j - W_j^*)\right)}$$

and

$$B = \sum_{2\leq i,j,k\leq d+1} \frac{n}{i} \mathbb{E}_n^{(\alpha)} |(W_i^* - W_i)(W_j^* - W_j)(W_k^* - W_k)|.$$

Propositions 4.6.9, 4.6.11 imply that

$$\left|\mathbb{E}_n^{(\alpha)} h(\tilde{W}_d) - \mathbb{E}_n^{(\alpha)} h(\Sigma\tilde{\Xi}_d)\right| = O(n^{-1/2}),$$

which finishes the proof. □

4.7 Jack measure: Central limit theorems for Young diagrams and transition measures

In this section we are going to prove two more kinds of Central Limit Theorem for Jack measure, which generalize Central Limit Theorems of Kerov [IO02]. Before we state our result, we need some preparations. Firstly, since the central limit theorems are described in terms of modified Chebyshev polynomials, we need to define them. We define

$$t_k(x) = 2T_K(x/2) = \sum_{0\le j\le \lfloor k/2\rfloor} (-1)^j \frac{k}{k-j}\binom{k-j}{j} x^{k-2j}$$

and

$$u_k(x) = U_k(x/2) = \sum_{0\le j\le \lfloor k/2\rfloor} (-1)^j \binom{k-j}{j} x^{k-2j},$$

where T_k and U_k are the Chebyshev polynomials of the first and of the second kind respectively. They can be alternatively defined by the following equations:

$$t_k(2\cos(\theta)) = 2t_k(\cos(k\theta))$$

and

$$u_k(2\cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)}.$$

Moreover, they form a family of orthonormal polynomials with respect to the measures $\frac{\sqrt{4-x^2}}{2\pi}dx$ and $\frac{1}{2\pi\sqrt{4-x^2}}dx$, respectively, i. e. :

$$\int_{-2}^{2} t_k(x)t_l(x)\frac{1}{2\pi\sqrt{4-x^2}}dx = \delta_{k,l}$$

and

$$\int_{-2}^{2} u_k(x)u_l(x)\frac{\sqrt{4-x^2}}{2\pi}dx = \delta_{k,l}.$$

The measure $\frac{\sqrt{4-x^2}}{2\pi}dx$ supported on the interval $[-2,2]$ is called the *semi-circular distribution* and will be denoted by the μ_{S-C}. It was shown by Kerov [Ker93a] that the transition measure (see Subsection 2.4.2) of the continual Young diagram Ω is exactly the semi-circular distribution. We recall that the Theorem 4.5.4 shows that the limit shape of a scaled Young diagram

$\omega\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda_n)$ is given by Ω, where λ_n is a random Young diagram with n boxes. Hence, in order to study fluctuations of the random Young diagrams around the limit shape, we introduce the function defined on the set of Young diagrams with n boxes:

$$\Delta_n^{(\alpha)}(x)(\lambda) := \sqrt{n}\frac{\omega\big(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda)\big)(x) - \Omega(x)}{2}.$$

Similarly, we define the function defined on the set of Young diagrams with n boxes with real measures values, which describes the difference between the transition measure of the rescaled Young diagram and the limiting semi-circular measure:

$$\widehat{\Delta}_n^{(\alpha)}(\lambda) := \sqrt{n}\left(\mu_{\left(T_{\sqrt{\alpha/n},1/\sqrt{n\alpha}}(\lambda)\right)} - \mu_{S-C}\right).$$

Now, we are ready to formulate the central limit theorem for the Jack measure:

Theorem 4.7.1. *Choose a sequence $(\Xi_k)_{k=2,3,\ldots}$ of independent standard Gaussian random variables. As $n \to \infty$, we have:*

1. *Central limit theorem for Young diagrams:*

$$\left(u_{k,n}^{(\alpha)}\right)_{k=1,2,\ldots} \xrightarrow{d} \left(\frac{\Xi_k}{\sqrt{k}}\right)_{k=2,3,\ldots},$$

where $u_{k,n}^{(\alpha)} = \int_{\mathbb{R}} u_k(x)\Delta_n^{(\alpha)}(x)\, dx$;

2. *Central limit theorem for transition measures:*

$$\left(t_{k,n}^{(\alpha)}\right)_{k=3,4,\ldots} \xrightarrow{d} \left(\sqrt{k}\Xi_k\right)_{k=2,3,\ldots},$$

where $t_{k,n}^{(\alpha)} = \int_{\mathbb{R}} t_k(x)\widehat{\Delta}_n^{(\alpha)}(dx)$.

Remark 4.7.2. Notice that this corollary is a continous generalization of Kerov's central limit theorems for Plancherel measure [Ker93a, IO02] which coincide with $\alpha = 1$ case.

The above theorem will be proved using following characterization of Gaussian processes.

4.7.1 Characterization of Gaussian process

The main tools and the definitions of the different cumulants used here come from [Śni06b]. Because of the representation theory character of Śniady's work, our result is stated in the different frame, however the ideas of the proof are the same as him. In particular, we shall use joint cumulants of random variables $k(X_1, \ldots, X_r)$, which can be defined by induction on r:

$$\mathbb{E}[X_1 X_2 \cdots X_r] = \sum_{\substack{\pi \text{ partition} \\ \text{of } \{1,2\ldots,r\}}} k(X_{i\in\pi_1})k(X_{i\in\pi_2})\cdots k(X_{i\in\pi_l}).$$

A centered Gaussian vector $(Y_1, \dots, Y_k)$ is characterized by the fact that all cumulants $k(Y_{i_1}, \dots, Y_{i_r})$ are equal to 0 for $r > 2$. Moreover, the values for $r = 2$ give then the covariance matrix. We will show a stronger result than the convergence towards 0 for $x_1, \dots, x_r \in \Lambda_\star^{(\alpha)}$ and for $r \geq 3$, namely:

Lemma 4.7.3. *For any $x_1, x_2, \dots, x_r \in \Lambda_\star^{(\alpha)}$, one has:*

$$k(x_1, \dots, x_r) = O(\sqrt{n}^{\deg_1(x_1)+\dots+\deg_1(x_r)-(r+1)+2\delta_{r,1}+\delta_{r,2}}).$$

Let us show, how to prove Theorem 4.7.1, using Lemma 4.7.3:

Proof of Theorem 4.7.1. From [IO02][Proof of Theorem 7.1] we have, that

$$u_{k,n}^{(\alpha)}(\lambda) = \frac{1}{n^{(k+1)/2}} \frac{1}{\sqrt{k+1}} \left(\mathrm{Ch}_{(k+1)}^{(1)} \left(T_{\sqrt{\alpha},1/\sqrt{\alpha}}(\lambda) \right) + F_{k+1} \left(T_{\sqrt{\alpha},1/\sqrt{\alpha}}(\lambda) \right) \right),$$

where F_{k+1} is an 1-polynomial function on the set of Young diagrams such that $\deg_1(F_k) = k+1$. Since

$$\mathrm{Ch}_{(k+1)}^{(1)} \left(T_{\sqrt{\alpha},1/\sqrt{\alpha}}(\lambda) \right) = \mathrm{Ch}_{(k+1)}^{(\alpha)}(\lambda) + G_{k+1}^{(\alpha)}(\lambda),$$

where $G_{k+1}^{(\alpha)}$ is an α-polynomial function such that $\deg_1(G_{k+1}^{(\alpha)}) = k+1$, we conclude that

$$u_{k,n}^{(\alpha)}(\lambda) = \frac{1}{n^{(k+1)/2}} \frac{1}{\sqrt{k+1}} \left(\mathrm{Ch}_{(k+1)}^{(\alpha)}(\lambda) + H_{k+1}^{(\alpha)}(\lambda) \right),$$

where $H_{k+1}^{(\alpha)}$ is an α-polynomial function with $\deg_1(H_{k+1}^{(\alpha)}) = k+1$. By Lemma 4.7.3 we see that, when $n \to \infty$, the asymptotics of the mixed moments of the random variables $u_{1,n}^{(\alpha)}, u_{2,n}^{(\alpha)}, \dots$ is the same as that for random variables $\frac{1}{n^{(k+1)/2}} \frac{1}{\sqrt{k+1}} \mathrm{Ch}_{(k+1)}^{(\alpha)}$ with $k = 1, 2 \dots$. Thanks to Theorem 4.6.1 we conclude the part 1. Proof of the part 2 follows, again, from [IO02] and it is almost the same as before. Thanks to the [IO02][Proof of Theorem 8.8] we have, that

$$t_{k,n}^{(\alpha)}(\lambda) = \frac{1}{n^{(k-1)/2}} \left(\mathrm{Ch}_{(k-1)}^{(1)} \left(T_{\sqrt{\alpha},1/\sqrt{\alpha}} \right) + F'_{k-1} \left(T_{\sqrt{\alpha},1/\sqrt{\alpha}} \right) \right),$$

where F'_{k-1} is an 1-polynomial function on the set of Young diagrams with $\deg_1(F'_{k-1}) = k-1$. Then, using the same argument as in the part 1, we conclude the proof of the part 2. □

The remaining thing is to prove Lemma 4.7.3. In order to do that, we need to show, that it is enough to calculate cumulants only for an algebraic basis of $Pola$, in other words, we will show the following:

Lemma 4.7.4. *Let $x_1, x_2, \dots, \in \Lambda_\star^{(\alpha)}$ form an algebraic basis of $\Lambda_\star^{(\alpha)}$. If for any $r \geq 1$ and for any $i_1, \dots, i_r \geq 1$ one has:*

$$k(x_{i_1}, \dots, x_{i_r}) = O(\sqrt{n}^{\deg_1(x_{i_1})+\dots+\deg_1(x_{i_r})-(r+1)+2\delta_{r,1}+\delta_{r,2}}),$$

then for any $r \geq 1$ and for any $x_1, \dots, x_r \in \Lambda_\star^{(\alpha)}$, one has

$$k(x_1, \dots, x_r) = O(\sqrt{n}^{\deg_1(x_1)+\cdots+\deg_1(x_r)-(r+1)+2\delta_{r,1}+\delta_{r,2}}).$$

Proof. Let $i_1 < i_2 < \cdots < i_{r+1}$ be integers and let $X_{i_1+1}, X_{i_1+2}, \dots, X_{i_{r+1}}$ be a family of random variables. It was shown by Śniady [Śni06b, Theorem 4.4., Lemma 4.6.] that

$$k(\prod_{i_1+1 \leq j \leq i_2} X_j, \dots, \prod_{i_r+1 \leq j \leq i_{r+1}} X_j) = \sum_\pi \prod_i k(X_{\pi_{i,1}}, \dots, X_{\pi_{i,m(i)}}), \tag{4.29}$$

where for partition $\pi = (\pi_1, \dots, \pi_l)$, one has $\pi_i = \{\pi_{i,1}, \dots, \pi_{i,m(i)}\}$, and where the sum runs over some special partitions π of the set $\{i_1+1, i_1+2, \dots, i_{r+1}\}$ with the property that

$$\sum_i (|\pi_i| - 1) \geq r - 1.$$

It means, that

$$\sum_i \left(2\delta_{|\pi_i|,1} + \delta_{|\pi_i|,1} - (|\pi_i| + 1)\right) \leq \sum_i \left(2(\delta_{|\pi_i|,1} - 1) + \delta_{|\pi_i|,1}\right) - (r+1).$$

To finish the proof, it is enough to notice that

$$\sum_i \left(2(\delta_{|\pi_i|,1} - 1) + \delta_{|\pi_i|,1}\right) \leq 0 \leq 2\delta_{r,1} + \delta_{r,2}.$$

□

Lemma 4.7.5. *For any $r \geq 1$ and for any $i_1, \dots, i_r \geq 1$ one has:*

$$k(\mathrm{Ch}_{i_1}^{(\alpha)}, \dots, \mathrm{Ch}_{i_r}^{(\alpha)}) = O(\sqrt{n}^{i_1+\cdots+i_r-1+2\delta_{r,1}+\delta_{r,2}}).$$

Proof. We know, that for any $r \geq 1$ and for any $i_1, \dots, i_r \geq 1$ the quantity

$$k(\mathrm{Ch}_{i_1}^{(\alpha)}, \dots, \mathrm{Ch}_{i_r}^{(\alpha)})$$

is a polynomial in n. By Theorem 4.6.1 and by the characterization of Gaussian process, we know that for $r \geq 2$ and $i_1, \dots, i_r \geq 2$, we have that

$$\sqrt{n}^{-(i_1+\cdots+i_r)} k(\mathrm{Ch}_{i_1}^{(\alpha)}, \dots, \mathrm{Ch}_{i_r}^{(\alpha)}) \to \sqrt{i_1 \cdots i_r}\,\delta_{2,r},$$

as $n \to \infty$, which implies that

$$k(\mathrm{Ch}_{i_1}^{(\alpha)}, \dots, \mathrm{Ch}_{i_r}^{(\alpha)}) = O(\sqrt{n}^{i_1+\cdots+i_r-1+2\delta_{r,1}+\delta_{r,2}}).$$

Moreover, since $\mathrm{Ch}_1^{(\alpha)} \equiv n$ on the set of the Young diagrams of size n, then

$$k(\mathrm{Ch}_1^{(\alpha)}, \mathrm{Ch}_{i_1}^{(\alpha)}, \dots, \mathrm{Ch}_{i_r}^{(\alpha)}) = 0$$

for $r \geq 1$ and $i_1, \dots, i_r \geq 1$. Since

$$k(\mathrm{Ch}_i^{(\alpha)}) = \mathbb{E}_n^{(\alpha)}(\mathrm{Ch}_i^{(\alpha)}) = n\delta_{1,i} = O(\sqrt{n}^{i+1}),$$

we conclude the proof. □

Proof of Lemma 4.7.3. The family $\mathrm{Ch}_1^{(\alpha)}, \mathrm{Ch}_2^{(\alpha)}, \dots$ forms an algebraic basis of the algebra $\Lambda_\star^{(\alpha)}$. Moreover, for any $r \geq 1$ and for any $i_1, \dots, i_r \geq 1$ one has:

$$\deg_1(\mathrm{Ch}_{i_1}^{(\alpha)}) + \cdots + \deg_1(\mathrm{Ch}_{i_r}^{(\alpha)}) - (r+1) + 2\delta_{r,1} + \delta_{r,2} = i_1 + \cdots + i_r - 1 + 2\delta_{r,1} + \delta_{r,2}.$$

Lemmas 4.7.4 and 4.7.5 finish the proof. □

5

Jack polynomials and orientability generating series of maps

Abstract

We study Jack characters, which are the coefficients of the power-sum expansion of Jack symmetric functions with a suitable normalization. These quantities have been introduced by Lassalle who formulated some challenging conjectures about them. We conjecture existence of a weight on maps (i.e., graphs drawn on surfaces), allowing to express Jack characters as weighted sums of some simple functions indexed by maps. We provide a candidate for this weight which gives a positive answer to our conjecture in some, but unfortunately not all, cases. This candidate weight measures somehow the non-orientability of a given map.

5.1 Introduction

The main goal of this chapter is to *understand the combinatorial structure of Jack characters* $\mathrm{Ch}_\mu^{(\alpha)}$. In the following we will give more details on this problem.

5.1.1 The main conjecture

Based on the formulas (2.11), (2.12), (2.13), on some theoretical results of this chapter and some computer exploration, we formulate the following conjecture:

Main Conjecture 5.1.1. *To each map M, one can associate some weight $\mathrm{wt}_M(\gamma)$ such that*

- $\mathrm{wt}_M(\gamma)$ *is a polynomial with non-negative rational coefficients in γ of degree (at most)*

$$d(M) := 2(\textit{number of connected components of } M) - \chi(M),$$

where

$$\chi(M) := |V(M)| - |\mathcal{E}(M)| + |F(M)|$$

is the Euler characteristic of M. Moreover, the polynomial $\mathrm{wt}_M(\gamma)$ is an even (respectively, odd) polynomial if and only if the Euler characteristic $\chi(M)$ is an even number (respectively, an odd number).

- *for every λ and μ, the following formula holds*

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_M \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(G)|} \left(\sqrt{\alpha}\right)^{|V_\circ(G)|} \mathrm{wt}_M\left(\frac{1-\alpha}{\sqrt{\alpha}}\right) N_M(\lambda),$$

where the sum runs over all non-oriented maps M with the face-type specified by μ.

Throughout this chapter, we shall denote $\gamma := (1-\alpha)/\sqrt{\alpha}$.

We know that $\mathrm{Ch}_\mu^{(\alpha)}$ can be written as a linear combination of functions N_G over some bipartite graphs G: it is a consequence of the fact that $\mathrm{Ch}_\mu^{(\alpha)}$ is an α-shifted symmetric function, see [Las08b, Proposition 2]. However, since functions N_G, seen as functions on Young diagrams, are not linearly independent [Fér09, Proposition 2.2.1], this expansion is not unique; therefore our conjecture should be understood as a claim about the existence of a *particularly nice* expansion of $\mathrm{Ch}_\mu^{(\alpha)}$ in terms of functions N_G.

5.1.2 A concrete version of the conjecture

As we have seen above, the case $\alpha = 1$ (Eq. (2.11)) corresponds to summation over *oriented maps*, while the cases $\alpha = 2$ and $\alpha = \frac{1}{2}$ correspond to summation over *non-oriented maps* (Eqs. (2.12), (2.13)) with some simple coefficients which depend only on general features of the map, such as the number of the vertices. Thus one can expect that the coefficient $\mathrm{wt}_M(\gamma)$ should be interpreted as a kind of *measure of non-orientability of a given map M*.

This notion of *measure of non-orientability* is not very well defined. For example, one could require that for $\alpha = 1$ the corresponding coefficient $\mathrm{wt}_M(0)$ is equal to 1 if M is orientable and zero otherwise; and that for $\alpha \in \left\{\frac{1}{2}, 2\right\}$ the coefficient $\mathrm{wt}_M(\pm 1/\sqrt{2})$ takes some fixed value on all (orientable and non-orientable) maps.

In Section 5.2.5 we will define some quantity mon_M which indeed — although in some perverse sense — measures non-orientability of a given map M. Roughly speaking, it is defined as follows: we remove edges of the map M one after another (in a random order). For each edge which is to be removed we check the *type* of this edge (for example, an edge may be *twisted* if, in some sense, it is a part of *Möbius band*). We multiply the factors corresponding to the types of all edges. The quantity mon_M is defined as the mean value of this product.

This is a rather strange definition. For example, one could complain that this is a weak measure of non-orientability of a map; in particular for $\alpha = 1$ the corresponding weight mon_M does not vanish on non-orientable maps. Nevertheless, this weight mon_M *often* (but not always!) gives a positive answer to our Main conjecture. We state it precisely as the following conjecture.

Conjecture 5.1.2. *For arbitrary Young diagram λ, partition μ and $\alpha > 0$, the value of Jack character is given by*

$$\mathrm{Ch}_{\mu}^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_{M} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_{\bullet}(G)|} \left(\sqrt{\alpha}\right)^{|V_{\circ}(G)|} \mathrm{mon}_M \, N_M(\lambda), \tag{5.1.1}$$

where the summation runs over all non-oriented maps with face-type μ.

With extensive computer calculations we were able to check that Conjecture 5.1.2, in general, not true; the simplest counterexample is $\mu = (9)$ (see Section 5.6 for more details). Nevertheless, as we shall present in the following, it seems that this conjecture predicts *some* properties of Jack characters surprisingly well. We hope that investigation of Conjecture 5.1.2 might shed some light on the problem and eventually lead to the correct formulation of the solution of Main Conjecture 5.1.1.

For example, the conjecture holds true for the following special cases: $\lambda = (n)$ which consists of a single part for $1 \leq n \leq 8$, furthermore for $\lambda = (2, 2)$, and $\lambda = (3, 2)$ (the proofs are computer-assisted). Corollary 5.3.3 shows that the conjecture holds also for any of these partitions augmented by an arbitrary number of parts equal to 1. In a forthcoming paper [CJŚ13] we will present a human-readable proof that Conjecture 5.1.2 is true for partitions $\mu = (n)$ consisting of a single part for $1 \leq n \leq 6$. In the following (Section 5.1.4, Section 5.1.5, Section 5.1.6) we will present some other special cases for which Conjecture 5.1.2 seems to be true.

5.1.3 Orientability generating series

We define the *orientability generating series* $\widehat{\mathrm{Ch}}_{\mu}^{(\alpha)}(\lambda)$ as the right hand-side of (5.1.1):

$$\widehat{\mathrm{Ch}}_{\mu}^{(\alpha)}(\lambda) := (-1)^{\ell(\mu)} \sum_{M} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_{\bullet}(M)|} \left(\sqrt{\alpha}\right)^{|V_{\circ}(M)|} \mathrm{mon}_M \; N_M(\lambda), \tag{5.1.2}$$

where the sum runs over all non-oriented maps M with the face-type μ. With this notation, Conjecture 5.1.2 may be equivalently reformulated as follows: for any partition μ the corresponding

Jack character and the orientability generating series are equal:

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \widehat{\mathrm{Ch}}_\mu^{(\alpha)}(\lambda).$$

5.1.4 Rectangular Young diagrams

Investigation of the normalized characters $\mathrm{Ch}_\mu(\lambda)$ in the case when $\lambda = p \times q$ is a *rectangular* Young diagram was initiated by Stanley [Sta04] who noticed that they have a particularly simple structure; in particular he showed that formula (2.11) holds true in this special case.

This line of research was continued by Lassalle [Las08b] who (apart from other results) studied Jack characters $\mathrm{Ch}_\mu^{(\alpha)}(p \times q)$ on rectangular Young diagrams. In particular, Lassalle found a recurrence relation [Las08b, formula (6.2)] fulfilled by such characters; this recurrence relates values of the characters on a fixed rectangular Young diagram $p \times q$, corresponding to various partitions μ. This recurrence relation is essential for the current chapter Conjecture 5.1.2 was formulated by a careful attempt of reverse-engineer the hidden hypothetical combinatorial structure behind Lassalle's recurrence.

In particular, our measure of non-orientability of maps mon_M was from the very beginning chosen in such a way that Main Conjecture 5.1.1 is true for an arbitrary rectangular Young diagram $\lambda = p \times q$. We will discuss these issues and prove Conjecture 5.1.2 for rectangular Young diagrams in Section 5.3.

Extensive computer exploration leads us to believe that Conjecture 5.1.2 might be true if the Young diagram λ is not far from being rectangular. We state it precisely as follows.

Conjecture 5.1.3. *Conjecture 5.1.2 is true if* $\lambda = (p_1, p_2) \times (q_1, q_2)$ *is a multirectangular Young diagram consisting of (at most) two rectangles.*

Our computer exploration supports this conjecture. It would imply explicit formulas for quadratic terms of Kerov polynomials for Jack characters (analogous formulas for the linear terms are known to hold true because Conjecture 5.1.1 holds true for rectangular Young diagrams); for details see [CJŚ13].

5.1.5 Top-twisted part

Conjecture 5.1.4. *The* top-twisted parts *of the Jack character and the orientability generating series are equal, i.e. for any partition* μ *and any Young diagram* λ

$$\lim_{s\to\infty} \frac{1}{s^{|\mu|+\ell(\mu)}} \mathrm{Ch}_\mu^{(s\alpha)}(s\lambda) = \lim_{s\to\infty} \frac{1}{s^{|\mu|+\ell(\mu)}} \widehat{\mathrm{Ch}}_\mu^{(s\alpha)}(s\lambda).$$

For details and the missing notation we refer to [CJŚ13]. Our computer exploration supports this conjecture.

5.1.6 Links with other problems

Other problems of combinatorics of Jack polynomials and non-orientability of maps

In this chapter we investigate the combinatorics of *Jack characters* related to maps. The study of analogous connections between *Jack polynomials* and maps is much older. In particular, Goulden and Jackson [GJ96a] formulated a conjecture (called b-*Conjecture*) which claims, roughly speaking, that the connection coefficients of Jack polynomials can be explained combinatorially as summation over certain maps with coefficients that should describe *non-orientability of a given map*. An extensive bibliography to this topic can be found in [LC09].

Although there is no direct link between our problem and the b-Conjecture (we are unable to show, for instance, that one implies the other), both problems seem quite close and we hope that any progress on one of them could give ideas to solve the other.

5.1.7 Outline of the chapter

In Section 5.2, we define the weight mon_M. In Section 5.3 and Section 5.4, we prove that Conjecture 5.1.2 holds respectively for rectangular Young diagrams and $\alpha = 2$. In Section 5.5, we explain the link between our Main Conjecture and some conjecture of Lassalle. Then finally, in Section 5.6, we present our numerical exploration and the counterexample.

5.2 The measure of non-orientability of maps

5.2.1 Removal of edges

Let P be a pairing of a set S and s_1, s_2 be two distinct elements of S. We define a pairing $P_{\{s_1,s_2\}}$ of the set $S \setminus \{s_1, s_2\}$ as follows:

- if $\{s_1, s_2\}$ is a pair of P, then $P_{\{s_1,s_2\}} := P \setminus \big\{\{s_1, s_2\}\big\}$.
- otherwise, consider the partners t_1 and t_2 of s_1 and s_2. Elements s_1, s_2, t_1, t_2 are distinct. We define
$$P_{\{s_1,s_2\}} := \Big(P \setminus \big\{\{s_1, t_1\}, \{s_2, t_2\}\big\}\Big) \cup \{t_1, t_2\}.$$
In other words, we remove the pairs containing s_1 or s_2 and we match together the unmatched pair of elements of $S \setminus \{s_1, s_2\}$.

Lemma 5.2.1. *Let P be a pairing of a set S and s_1, s_2, s_3, s_4 be four distinct elements of S. Then*
$$\big(P_{\{s_1,s_2\}}\big)_{\{s_3,s_4\}} = \big(P_{\{s_3,s_4\}}\big)_{\{s_1,s_2\}}.$$

Proof. Easy case by case analysis. □

Let $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ be a map and E be an edge of M. Then we define $M \setminus \{E\}$ (or $M \setminus E$) as the triplet $(\mathcal{B}_E, \mathcal{W}_E, \mathcal{E}_E)$. The lemma above states that, for any two edges E_1, E_2 in a map M, one has

$$\big(M \setminus \{E_1\}\big) \setminus \{E_2\} = \big(M \setminus \{E_2\}\big) \setminus \{E_1\},$$

which allows to define the map $M \setminus \{E_1, \cdots, E_i\}$ for an arbitrary subset $\{E_1, \cdots, E_i\} \subseteq \mathcal{E}(M)$.

Graphically, this corresponds to erasing the edge E in the map M. If one extremity (or both extremities) of this edge is a leaf (are leaves), we also remove it (them). Beware that removal of an edge might change the topology of the surface on which the map is drawn.

The following lemma is immediate to prove.

Lemma 5.2.2. *Let P and P' be two pairings of the same base set and let E be a pair in P. Suppose that the couple (P, P') has type μ and that E lies in a polygon of size r (in particular, r is a part of μ). Then, the type of the couple (P_E, P'_E) is obtained from μ by replacing a part equal to r by a part equal to $r - 1$.*

5.2.2 Effect of edge removal on faces

Let us look at the faces of the map $M \setminus \{E\}$, that is the polygons associated to $(\mathcal{B}_E, \mathcal{W}_E)$.

Straight edge removal

Suppose E is a straight edge of M. By definition it means that the two edge-sides s_1, s_2 of E belong to the same face F of M. Besides, we know that, if we fix an orientation of F there is an even number, let us say $2i$, of the edge-sides between s_1 and s_2 in F. With the other orientation, there would also be an even number, let us say $2j$, of the edge-sides between s_1 and s_2 in F. This means that the face F has size $2i + 2j + 2$ (we call *size* of a face the number of edges in the corresponding polygon; in particular, it is always an even number). When we remove the edge E, the face F is split into two faces F_1 and F_2 of respective sizes $2i$ and $2j$ (in the degenerate case where i or j is equal to 0, the corresponding face does not exist). This can be easily seen graphically, see Figure 5.2.1 (here $t_1^{\mathcal{B}}$ is the partner of s_1 in $\mathcal{B}$, etc.). The other faces are not modified.

Twisted edge removal

Suppose E is a twisted edge of M. By definition it means that the two edge-sides s_1, s_2 of E belong to the same face F of M and are in odd position. Let us denote $2r$ the size of the face

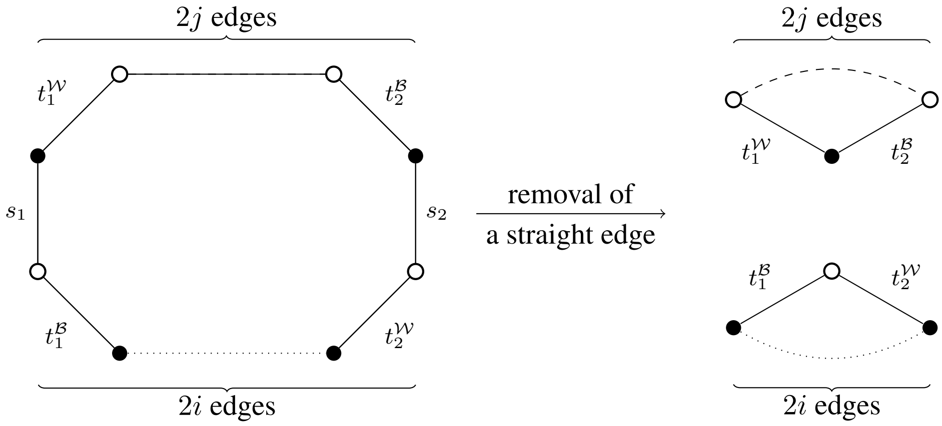


Figure 5.2.1: Result on faces of a straight edge removal.

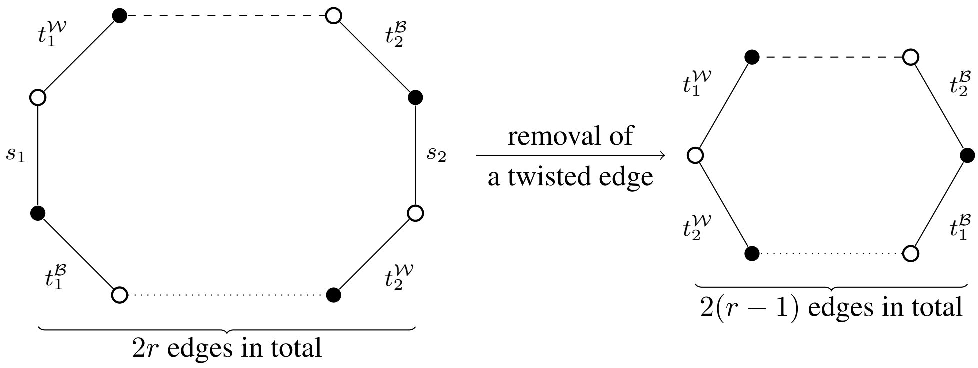


Figure 5.2.2: Result on faces of a twisted edge removal.

F. Then after removal of edge E, the face F is replaced by a face of size $2(r-1)$; see Figure 5.2.2. The other faces are not modified.

INTERFACE EDGE REMOVAL

Suppose E is an interface edge of M. By definition, it means that the two edge-sides s_1, s_2 of E belong to different faces F_1 and F_2 of M. Let us denote $2r$ and $2s$ the sizes of faces F_1 and F_2. Then after removal of edge E, faces F_1 and F_2 are replaced by a new face of size $2(r+s-1)$; see Figure 5.2.3. Other faces are not modified.

5.2.3 BRIDGES

From the case analysis from Section 5.2.2 one obtains a result on *bridges*.

Definition 5.2.3. An edge E of a map M is a *bridge* if

- either at least one of its extremity is a leaf;

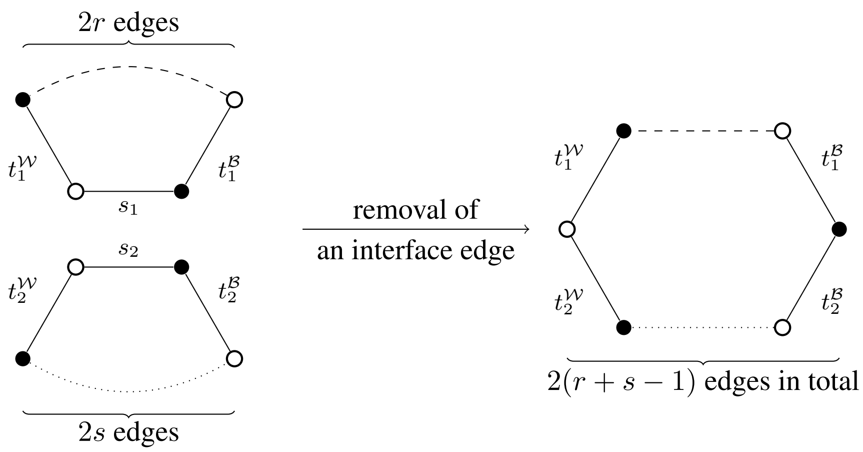


Figure 5.2.3: Result on faces of an interface edge removal.

- or its extremities lie in different connected components of $M \setminus \{E\}$.

If E has an extremity of degree 1 (that is, a *leaf*), this vertex is not a vertex of $M \setminus \{E\}$ anymore, so the second point does not make sense.

Lemma 5.2.4. *A bridge is always a straight edge.*

Proof. Consider a bridge E. If one extremity of E is a leaf, apply Lemma 2.8.9.

Otherwise, the extremities of E lie in different connected components of $M \setminus \{E\}$. But, the case analysis above shows that after removal of a twisted or interface edge, the extremities of this edge lie in the same face and hence in the same connected component of $M \setminus \{E\}$. So E must be straight. □

5.2.4 Weight associated to a map with a history

Let M be a map and let some linear order $\prec$ on the edges be given. This linear order will be called *history*.

Let $E \in E(M)$. We define

$$\mathrm{mon}_{M,E} = \begin{cases} 1 \text{ if } E \text{ is straight,} \\ \gamma \text{ if } E \text{ is twisted,} \\ \frac{1}{2} \text{ if } E \text{ is interface.} \end{cases} \tag{5.2.1}$$

Let $E_1, \ldots, E_n$ be the sequence of edges of M, listed according to the linear order $\prec$. We set $M_i = M \setminus \{E_1, \ldots, E_i\}$ and define

$$\mathrm{mon}_{M,\prec} := \prod_{0 \le i \le n-1} \mathrm{mon}_{M_i, E_{i+1}} \tag{5.2.2}$$

This quantity $\text{mon}_{M,\prec}$ can be interpreted as follows: from the map M we remove (one by one) all the edges, in the order specified by the history. For each edge which is about to be removed we consider its weight relative to the current map.

Please note that the type of a given edge (i.e., *straight* versus *twisted* versus *interface*) might change in the process of removing edges and the weight $\text{mon}_{M,\prec}$ usually depends on the choice of the history $\prec$.

5.2.5 MEASURE OF NON-ORIENTABILITY OF A MAP

Let M be a map with n edges. We define

$$\text{mon}_M =:= \frac{1}{n!} \sum_{\prec} \text{mon}_{M,\prec} .$$

This quantity can be interpreted as the mean value of the weight associated to the map M equipped with a randomly selected history (with all histories having equal probability). This is the central quantity for the current chapter, we call it *the measure of non-orientability* of the map M.

Example 5.2.5. We consider the map M depicted in Figure 5.2.4. For calculations involving removal of edges it is more convenient to represent this map as a *ribbon graph*, see Figure 5.2.5. For the history $E_A \prec E_B \prec E_C$ the corresponding weight is equal to $\text{mon}_{M,\prec} = 1 \cdot \frac{1}{2} \cdot 1$ while for the history $E_B \prec E_A \prec E_C$ the corresponding weight is equal to $\text{mon}_{M,\prec} = \gamma \cdot \gamma \cdot 1$. The other histories are analogous to these two cases; finally

$$\text{mon}_M = \frac{2 \times 1 \cdot \frac{1}{2} \cdot 1 + 4 \times \gamma \cdot \gamma \cdot 1}{6}.$$

Lemma 5.2.6. *Let M be a map. Then mon_M is a polynomial in variable γ of degree (at most)*

$$d(M) := 2(\textit{number of connected components of } M) - \chi(M),$$

where

$$\chi(M) := |V(M)| - |\mathcal{E}(M)| + |F(M)|$$

is the Euler characteristic of M.

The polynomial $\text{mon}_M(\gamma)$ is an even (respectively, odd) polynomial if and only if the Euler characteristic $\chi(M)$ is an even number (respectively, an odd number).

Proof. We claim that for an arbitrary map M and its edge E:

(A) $\text{mon}_{M,E}$ is a polynomial in γ of degree (at most) $d(M) - d(M \setminus E)$,

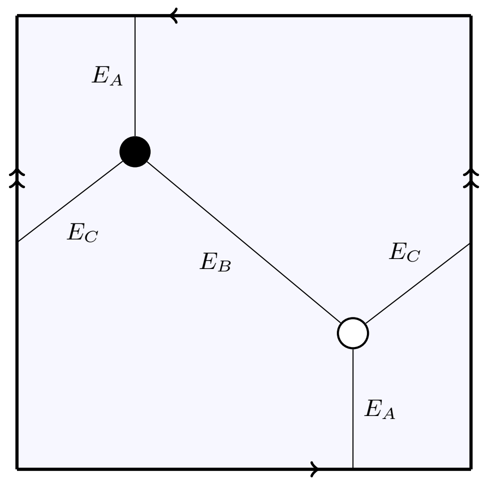


Figure 5.2.4: The map M considered in Example 5.2.5. This map is drawn on Klein bottle: the left-hand side of the square should be glued to the right-hand one (without a twist) and the top side should be glued to the bottom one (with a twist), as indicated by the arrows. For simplicity the labels of the edge-sides were removed and each edge carries only one label.

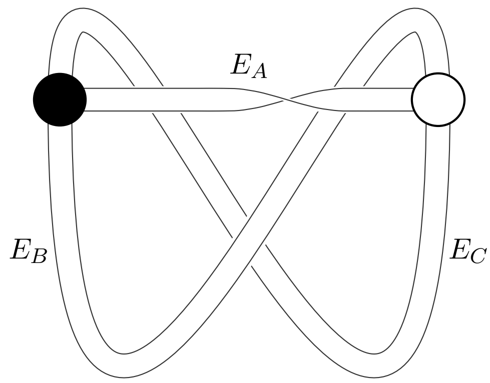


Figure 5.2.5: The map from Figure 5.2.4 drawn as a *ribbon graph*, i.e., each vertex is represented as a small disc, each edge is represented by a thin ribbon connecting two discs.

(B) $\text{mon}_{M,E}$ is an even (respectively, odd) polynomial if and only if $d(M) - d(M \setminus E)$ is an even number (respectively, an odd number).

Indeed, this statement follows by a careful investigation of each of the three cases considered in Section 2.8.3 (cases when at least one of the endpoints of E has degree 1 must be considered separately). We present the details in the following.

- Assume that both extremities of E are leaves. Then E is straight by Lemma 2.8.9, so $\text{mon}_{M,E} = 1$. But $M\backslash\{E\}$ has two vertices less, one edge less, one face less and one connected component less than M. So $d(M \setminus \{E\}) = d(M)$ and the claim holds in this case.

- Assume that exactly one extremity of E is a leaf. Then E is straight by Lemma 2.8.9, so $\text{mon}_{M,E} = 1$. But $M\backslash\{E\}$ has one vertex less, one edge less, and the same number of faces and connected components than M. So $d(M \setminus \{E\}) = d(M)$ and claim holds in this case.

- Assume that E is straight, but none of its extremities is a leaf. Recall that $\text{mon}_{M,E} = 1$ in this case. But $M\backslash\{E\}$ has one edge less and one face more (see Section 5.2.2) than M. The number of vertices is unchanged, while the number of connected components can be constant or increase by 1. In the first case, $d(M \setminus \{E\}) = d(M) - 2$ and in the second $d(M \setminus \{E\}) = d(M)$. In both cases our claim holds.

- Assume that E is twisted. In this case, $\text{mon}_{M,E} = \gamma$. According to Lemma 5.2.4, the numbers of connected components of M and $M \setminus \{E\}$ are the same. The number of faces is not changed either (see Section 5.2.2). The number of edges decreases by 1 and the number of vertices is constant. Thus $d(M \setminus \{E\}) = d(M) - 1$, and our claim holds in this case.

- Assume that E is an interface edge. In this case, $\text{mon}_{M,E} = \frac{1}{\cdot}$. According to Lemma 5.2.4, the numbers of connected components of M and $M \setminus \{E\}$ are the same. The number of faces decreases by 1 (see Section 5.2.2). The number of edges also decreases by 1, while the number of vertices remains the same. Thus $d(M \setminus \{E\}) = d(M)$, and our claim holds in this case.

We apply claims *(A)* and *(B)* to each factor on the right-hand side of (5.2.2); by a telescopic product it follows that the statement of the lemma holds true if the polynomial mon_M is replaced by $\text{mon}_{M,\prec}$, where $\prec$ is an arbitrary history. The latter result finishes the proof by taking the average over $\prec$. □

Remark 5.2.7. Notice that if M is a *connected* map on an *orientable* surface, then $d(M)$ has a natural interpretation as the Euler *genus* of the surface on which M is drawn. We shall see later that, if its underlying surface is orientable, then $\text{mon}_M(\gamma)$ does not depend on γ. Hence the degree of the polynomial $\operatorname{mon}_M(\gamma)$ satisfies the same bound as some invariants defined by Brown and Jackson [BJ07, Lemma 3.3] and La Croix [LC09, Theorem 4.4].

5.3 Support for the conjectures: rectangular Young diagrams and Lassalle's recurrence

The main result of the current section is the following.

Theorem 5.3.1. *For a rectangular Young diagram*

$$\lambda = p \times q = (\underbrace{q, \dots, q}_{p \text{ times}}),$$

where $p, q \in \mathbb{N}$ and for arbitrary partition μ and parameter $\alpha > 0$, Conjecture 5.1.2 holds true, i.e.,

$$\operatorname{Ch}_\mu^{(\alpha)}(\lambda) = \widehat{\operatorname{Ch}}_\mu^{(\alpha)}(\lambda). \tag{5.3.1}$$

The main idea of the proof is to use recurrence (5.3.2) found by Lassalle [Las08b, formula (6.2)] and to find a combinatorial interpretation of this recurrence.

5.3.1 Lassalle's recurrence

Following Lassalle [Las08b] we denote by $\mu \cup (s)$ the partition μ with extra part s added and by $\mu \setminus (s)$ the partition μ with one part s removed. We also denote

$$\mu \cup 1^l = \mu \cup \underbrace{(1) \cup \cdots \cup (1)}_{l \text{ times}}, \qquad \mu_{\downarrow(s)} = \mu \setminus (s) \cup (s-1),$$
$$\mu_{\uparrow(rs)} = \mu \setminus (r+s+1) \cup (r,s), \qquad \mu_{\downarrow(rs)} = \mu \setminus (r,s) \cup (r+s-1).$$

Consider a rectangular Young diagram $\lambda = p \times q$ and a partition μ such that $m_1(\mu) = 0$ (i.e., μ does not contain any part equal to 1). Then Lassalle's recurrence relation [Las08b, formula

(6.2)], after adapting to our normalizations takes the form:

$$\left(\frac{p}{\sqrt{\alpha}} - \sqrt{\alpha}q\right) \sum_r r\, m_r(\mu)\ \mathrm{Ch}^{(\alpha)}_{\mu\downarrow(r)}(\lambda) \\ + \sum_r r\, m_r(\mu) \sum_{i=1}^{r-2} \mathrm{Ch}^{(\alpha)}_{\mu\uparrow(i,r-i-1)}(\lambda) \\ - \gamma \sum_r r(r-1)\, m_r(\mu)\ \mathrm{Ch}^{(\alpha)}_{\mu\downarrow(r)}(\lambda) \\ + \sum_{r,s} rs\, m_r(\mu)\big(m_s(\mu) - \delta_{r,s}\big)\ \mathrm{Ch}^{(\alpha)}_{\mu\downarrow(rs)}(\lambda) \\ = -|\mu|\ \mathrm{Ch}^{(\alpha)}_{\mu}(\lambda). \quad (5.3.2)$$

A difficulty in this formula comes from the fact that it was proved only under assumption that $m_1(\mu) = 0$. In the following we will show how to overcome this issue.

5.3.2 Partitions with parts equal to 1

Jack characters corresponding to partition μ with some parts equal to 1 can be deduced from the case without parts equal to 1. Indeed, strictly from the definition of Jack characters (2.4), we have the following identity

$$\mathrm{Ch}^{(\alpha)}_{\mu\cup 1^l}(\lambda) = (|\lambda| - |\mu|)_l\ \mathrm{Ch}^{(\alpha)}_{\mu}(\lambda). \quad (5.3.3)$$

The following result shows that analogous property is fulfilled by the orientability generating series $\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}$ as well.

Lemma 5.3.2. *Let μ be a partition and λ be an arbitrary Young diagram. Then*

$$\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\cup 1^l}(\lambda) = (|\lambda| - |\mu|)_l\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\lambda). \quad (5.3.4)$$

Proof. It is enough to prove that for any partition μ the following holds:

$$\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\cup(1)}(\lambda) = (|\lambda| - |\mu|)\ \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\lambda). \quad (5.3.5)$$

For a partition μ let F_μ be the set of pairs $(M', \prec)$, where M' is a map with a face-type μ and with history $\prec$. More explicitly: we start by fixing two pairings $\mathcal{B}', \mathcal{W}'$ of the same set S so that $(\mathcal{B}', \mathcal{W}')$ has type μ. Then maps of face-type μ are triplet $(\mathcal{B}', \mathcal{W}', \mathcal{E}')$, where $\mathcal{E}'$ is a pairing of S'. In other words, each element of F_μ is a pair-partition of S, equipped with some linear order on the pairs.

Consider two elements b_1, b_2 that are *not* in S and denote $S = S' \sqcup \{b_1, b_2\}$. We also consider the pairings $\mathcal{B} = \mathcal{B}' \sqcup \{\{b_1, b_2\}\}$ and $\mathcal{W} = \mathcal{W}' \sqcup \{\{b_1, b_2\}\}$ of S. The couple $(\mathcal{B}, \mathcal{W})$

has type $\mu \cup (1)$. Hence maps of face type $\mu \cup (1)$ are triplet $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$, where $\mathcal{E}$ is a pairing of S.

If $(M, \prec) \in F_{\mu \cup (1)}$ is such that $\{b_1, b_2\}$ is a pair of $\mathcal{E}$, we say that $(M, \prec) \in F^0_{\mu \cup (1)}$; in the other case we say that $(M, \prec) \in F^1_{\mu \cup (1)}$. This gives a disjoint decomposition

$$F_{\mu \cup (1)} = F^0_{\mu \cup (1)} \sqcup F^1_{\mu \cup (1)}. \tag{5.3.6}$$

Let $\mathcal{H} : F_{\mu \cup (1)} \to F_\mu$ be a function $\mathcal{H} : (M, \prec) \mapsto (M', \prec')$ defined as follows:

- If $(M, \prec) \in F^0_{\mu \cup (1)}$, then $\mathcal{E}' := \mathcal{E} \setminus \big\{\{b_1, b_2\}\big\}$ is by definition the pairing $\mathcal{E}$ with the pair $\{b_1, b_2\}$ removed; as the linear order $\prec'$, we take the restriction of $\prec$ to $M' \subset M$.

  In this case M, viewed as a bipartite graph, is a disjoint sum of the bipartite graph M' and the bipartite graph consisting of two vertices connected by the edge $\{b_1, b_2\}$. The process of calculating $\operatorname{mon}_{M,\prec}$ is almost identical to the analogous process of calculating $\operatorname{mon}_{M',\prec'}$ except for the additional edge $\{b_1, b_2\}$ which is clearly a straight edge. Thus

$$\begin{aligned} N_M(\lambda) =& |\lambda| \; N_{M'}(\lambda), \\ |V_\circ(M)| =& |V_\circ(M')| + 1, \\ |V_\bullet(M)| =& |V_\bullet(M')| + 1, \\ \operatorname{mon}_{M,\prec} =& \operatorname{mon}_{M',\prec'} . \end{aligned}$$

- Let $(M, \prec) \in F^1_{\mu \cup (1)}$. The edge-sides b_1 and b_2 appear in two different edges $\{e_1, b_i\}, \{e_2, b_j\} \in \mathcal{E}$; we choose the indices i, j in such a way that $\{e_1, b_i\} \prec \{e_2, b_j\}$. Then

$$\mathcal{E}' := \big(\mathcal{E} \cup \big\{\{e_1, e_2\}\big\}\big) \setminus \big\{\{e_1, b_i\}, \{e_2, b_j\}\big\}.$$

  As the linear order $\prec'$ we take the unique linear order that coincides with $\prec$ on the intersection of their domains and such that for any pair $P \in M' \cap M$ we have that

$$P \prec' \{e_1, e_2\} \iff P \prec \{e_2, b_j\};$$

  in other words the order $\prec'$ is obtained from $\prec$ by substituting the pair $\{e_2, b_j\}$ by $\{e_1, e_2\}$.

  The map M is obtained from M' by replacing the edge $\{e_1, e_2\}$ by a pair of edges in such a way that a new face is created (this face corresponds to the bigon $B = \{b_1, b_2\}$ in $\mathcal{L}(\mathcal{B}, \mathcal{W})$). The process of calculating $\operatorname{mon}_{M,\prec}$ is almost identical to the analogous process of calculating $\operatorname{mon}_{M',\prec'}$ except for the edge $\{e_1, b_1\}$, which is the one of the two edges adjacent to the bigon B which is removed first. This edge is clearly an interface edge. Thus

$$\begin{aligned} N_M(\lambda) =& N_{M'}(\lambda), \\ |V_\circ(M)| =& |V_\circ(M')|, \\ |V_\bullet(M)| =& |V_\bullet(M'), \\ \operatorname{mon}_{M,\prec} =& \frac{1}{2} \operatorname{mon}_{M',\prec'}. \end{aligned}$$

The left-hand side of (5.3.5) is equal to

$$\begin{aligned} &\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\cup(1)}(\lambda) \\ &\quad = \sum_{(M,\prec)\in F_{\mu\cup(1)}} \frac{(-1)^{\ell(\mu)+1}}{(|\mu|+1)!} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M)|} \left(\sqrt{\alpha}\right)^{|V_\circ(M)|} \operatorname{mon}_{M,\prec}\ N_M(\lambda) \\ &\quad = \sum_{(M,\prec)\in F^0_{\mu\cup(1)}} \frac{(-1)^{\ell(\mu)}}{(|\mu|+1)!} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M')|} \left(\sqrt{\alpha}\right)^{|V_\circ(M')|} \operatorname{mon}_{M',\prec'} |\lambda|\ N_{M'}(\lambda) \\ &\qquad - \sum_{(M,\prec)\in F^1_{\mu\cup(1)}} \frac{(-1)^{\ell(\mu)}}{(|\mu|+1)!} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M')|} \left(\sqrt{\alpha}\right)^{|V_\circ(M')|} \operatorname{mon}_{M',\prec'} \frac{1}{2}\ N_{M'}(\lambda), \end{aligned}$$

where $\mathcal{H} : (M,\prec) \mapsto (M',\prec')$ and in the last equality we used the decomposition (5.3.6).

In the following we will show that for each $(M',\prec') \in F_\mu$ its preimage fulfills:

$$\begin{aligned} |\mathcal{H}^{-1}(M',\prec') \cap F^0_{\mu\cup(1)}| &= |\mu|+1, \\ |\mathcal{H}^{-1}(M',\prec') \cap F^1_{\mu\cup(1)}| &= 2(|\mu|+1)|\mu|. \end{aligned}$$

Indeed, for all $(M,\prec) \in \mathcal{H}^{-1}(M',\prec') \cap F^0_{\mu\cup(1)}$ the maps M are all the same: their edge pairing is given by $\mathcal{E} = \mathcal{E}' \cup \big\{\{b_1,b_2\}\big\}$. The order $\prec$ is obtained from the order $\prec'$ by adding the additional pair $\{b_1,b_2\}$ anywhere between pairs of $\mathcal{E}'$ and this can be done in $|\mu|+1$ ways.

Similarly, consider $(M,\prec) \in \mathcal{H}^{-1}(M',\prec') \cap F^1_{\mu\cup(1)}$ Then $\mathcal{E}$ is obtained from $\mathcal{E}'$ by removing some pair $\{e_1,e_2\}$ and adding pairs $\{e_1,b_i\}$ and $\{e_2,b_j\}$ for some choice of $\{i,j\} = \{1,2\}$. Since we can replace the roles of the edges e_1 and e_2, we have altogether 4 choices for doing this. The pair $\{e_1,e_2\} \in M'$ can be equivalently specified by saying that there are ℓ elements which are smaller than $\{e_1,e_2\}$ (with respect to $\prec'$). The linear order $\prec$ is obtained by substituting the pair $\{e_1,e_2\}$ by $\{e_2,b_j\}$ and by adding the pair $\{e_1,b_i\}$ in such a way that $\{e_1,b_i\} \prec \{e_2,b_j\}$; there are $\ell+1$ choices for this. Thus the total number of choices is equal to

$$\sum_{0\le\ell\le|\mu|-1} 4(\ell+1) = 2(|\mu|+1)|\mu|,$$

just as we claimed.

Concluding, it gives us

$$\begin{aligned}
&\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\cup(1)}(\lambda)\\
&= \frac{1}{|\mu|+1}\sum_{(M',\prec')\in F_\mu} \frac{(-1)^{\ell(\mu)}}{(|\mu|)!}\left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M')|}\left(\sqrt{\alpha}\right)^{|V_\circ(M')|}\\
&\qquad\qquad \times \mathrm{mon}_{M',\prec'}\ N_{M'}(\lambda)\left(|\lambda|(|\mu|+1)-2(|\mu|+1)|\mu|\frac{1}{2}\right)\\
&= (|\lambda|-|\mu|)\\
&\qquad \times \sum_{(M',\prec')\in F_\mu} \frac{(-1)^{\ell(\mu)}}{(|\mu|)!}\left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(M')|}\left(\sqrt{\alpha}\right)^{|V_\circ(M')|}\mathrm{mon}_{M',\prec'}\ N_{M'}(\lambda)\\
&\qquad\qquad = (|\lambda|-|\mu|)\ \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\lambda)
\end{aligned}$$

which finishes the proof. □

Corollary 5.3.3. *If Conjecture 5.1.2 is true for some partition μ and some Young diagram λ, it is also true for $\mu' := \mu \cup 1$ and λ.*

Proof. It is enough to use the recurrence relations (5.3.3) and (5.3.4) □

5.3.3 Recurrence relation for the orientability generating series

In this Section, we shall see that the orientability generating series $\widehat{\mathrm{Ch}}^{(\alpha)}(p\times q)$, evaluated on a rectangular Young diagram, fulfills a recurrence relation analogous to (5.3.2).

There is an important simplification when we restrict to rectangular Young diagram, thanks to the following lemma.

Lemma 5.3.4. *For a rectangular Young diagram $\lambda = p\times q$, the number of embeddings of a bipartite graph G in λ is given by the particularly simple formula*

$$N_G(\lambda) = p^{|V_\bullet(G)|}\, q^{|V_\circ(G)|}.$$

Proof. It is a particular case of [FŚ11b, Lemma 3.9]. □

We can now prove the following.

Proposition 5.3.5. *If $\lambda = p \times q$ is a rectangular Young diagram and μ is a partition such that $m_1(\mu) = 0$ then*

$$\underbrace{\left(\frac{p}{\sqrt{\alpha}} - \sqrt{\alpha} q\right) \sum_r r\, m_r(\mu)\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(r)}(\lambda)}_{\text{(removing a leaf)}}$$

$$+ \underbrace{\sum_r r\, m_r(\mu) \sum_{i=1}^{r-2} \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\uparrow(i,r-i-1)}(\lambda)}_{\text{(removing a straight edge)}}$$

$$- \underbrace{\gamma \sum_r r(r-1)\, m_r(\mu)\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(r)}(\lambda)}_{\text{(removing a twisted edge)}}$$

$$+ \underbrace{\sum_{r,s} rs\, m_r(\mu)\big(m_s(\mu) - \delta_{r,s}\big)\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(rs)}(\lambda)}_{\text{(removing an interface edge)}}$$

$$= -|\mu|\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\lambda). \quad (5.3.7)$$

The comments concerning individual summands on the left-hand side of this recurrence relation are connected to its proof; see below.

Proof. Using Lemma 5.3.4, the right-hand side of (5.3.7) rewrites as

$$-|\mu|\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\lambda) = \frac{(-1)^{\ell(\mu)-1}}{(|\mu|-1)!} \sum_{(M,\prec)} \left(-\frac{p}{\sqrt{\alpha}}\right)^{|V_\bullet(M)|} \big(q\sqrt{\alpha}\big)^{|V_\circ(M)|} \mathrm{mon}_{M,\prec}. \quad (5.3.8)$$

Recall that the summation in (5.3.8) should be interpreted as follows: we fix a couple $(\mathcal{B}, \mathcal{W})$ of pairings of type μ and we sum over all pairings $\mathcal{E}$ of the same ground set S ; we also sum over all linear order on $\mathcal{E}$. For such a map $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ and a linear order $\prec$, we denote by $E = \{s_1, s_2\}$ the first edge, according to the linear order $\prec$ and by $\prec'$ the restriction of the linear order $\prec$ to the edges of $\mathcal{E} \setminus \{E\}$. In the following we will use the notation

$$\mathrm{contribution}_{M,\prec}(\lambda) := \left(-\frac{p}{\sqrt{\alpha}}\right)^{|V_\bullet(M)|} \big(q\sqrt{\alpha}\big)^{|V_\circ(M)|} \mathrm{mon}_{M,\prec}$$

for the contribution of the pair $(M, \prec)$ to the right-hand side of (5.3.8).

The summation over $(M, \prec)$ can be seen alternatively as follows: we first choose the first edge E and then sum over couple $(\mathcal{E}', \prec')$ where $\mathcal{E}' = \mathcal{E} \setminus E$ is a pairing of $S \setminus E$ and $\prec'$ a linear order on $\mathcal{E}'$. Summation over $\mathcal{E}'$ can be interpreted as a summation over maps $M' =$

$(\mathcal{B}, \mathcal{W}, \mathcal{E}')$ of face-type corresponding to the type of the couple $(\mathcal{B}_E, \mathcal{W}_E)$. Note that the map M' corresponds to $M \setminus E$. We shall use this idea repetitively in the proof.

Clearly, (5.2.2) is equivalent to a recursive relationship

$$\mathrm{mon}_{M,\prec} = \mathrm{mon}_{M,E} \cdot \mathrm{mon}_{M\setminus E,\prec'} .$$

We will split our sum depending of the type (straight, twisted or interface) of edge E). Note that, as $\mathcal{B}$ and $\mathcal{W}$ are fixed, this type depends only on the pair E, not on the remaining pairs in $\mathcal{E}$. According to the classification from Section 2.8.3 there are the following possibilities:

The edge E is straight and both endpoints of E have degree 1. This is not possible since it would imply that one of the faces of M is a bigon thus $m_1(\mu) \geq 1$.

The edge E is straight (hence $\mathrm{mon}_{M,E} = 1$) and only the black (respectively, white) endpoint of E has degree 1 *(i.e., it is a leaf).* In other terms, the pair E belongs also to the pairing $\mathcal{B}$ (respectively, $\mathcal{W}$). We consider the map $M \setminus E$; recall that it has one black (respectively, white) vertex less than M (the leaf extremity has been removed with the edge). It follows that

$$\mathrm{contribution}_{M,\prec}(\lambda) = \frac{-p}{\sqrt{\alpha}} \mathrm{contribution}_{M\setminus E,\prec'}(\lambda); \tag{5.3.9}$$

respectively,

$$\mathrm{contribution}_{M,\prec}(\lambda) = q\sqrt{\alpha}\, \mathrm{contribution}_{M\setminus E,\prec'}(\lambda).$$

Fix the black vertex $B = \{b_1, b_2\} \in \mathcal{B}$ and let us consider the total contribution of couples $(\mathcal{E}, \prec)$ such that B is the black endpoint of E; in other words $E = B$. This means that $\mathcal{E} = \mathcal{E}' \sqcup B$, where $\mathcal{E}'$ is a pairing of $S \setminus B$. But $(\mathcal{B} \setminus B, \mathcal{W}_B)$ is a couple of pairing of $S \setminus B$ of type $\mu_{\downarrow(r)}$, where $2r$ is the number of edge-sides in the polygon of $\mathcal{L}(\mathcal{B}, \mathcal{W})$ containing B (see Lemma 5.2.2). Then summing over pairings $\mathcal{E}'$ corresponds to summation over maps M' of face-type $\mu_{\downarrow(r)}$. By definition, the map $M' = (\mathcal{B} \setminus B, \mathcal{W}_B, \mathcal{E}')$ is equal to $M \setminus E$ and its contribution is given by Equation (5.3.9) above.

Therefore, the total contribution to the right-hand side of (5.3.8) of couples $(\mathcal{E}, \prec)$ as above is given by

$$\frac{(-1)^{\ell(\mu)-1}}{|\mu|!} \sum_{(M',\prec')} \frac{-p}{\sqrt{\alpha}} \mathrm{contribution}_{M',\prec'}(\lambda) = \frac{p}{\sqrt{\alpha}} \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu_{\downarrow(r)}}(\lambda).$$

But this holds for a fixed pair $B \in \mathcal{B}$ that belongs to a polygon of size $2r$ of $\mathcal{L}(\mathcal{B}, \mathcal{W})$. Each of the $m_r(\mu)$ polygons of size $2r$ of $\mathcal{L}(\mathcal{B}, \mathcal{W})$ contains r pairs of $\mathcal{B}$, hence the total contribution of pairs $(M, \prec)$ such that the first edge E belongs to $\mathcal{B}$ (i.e., its black extremity is a leaf) is

$$\frac{p}{\sqrt{\alpha}} \sum_{r \geq 2} r m_r(\mu) \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu_{\downarrow(r)}}(\lambda).$$

Symmetrically, the total contribution of pairs $(M, \prec)$ such that the first edge E belongs to $\mathcal{W}$ is equal to

$$-q\,\sqrt{\alpha} \sum_{r \geq 2} r m_r(\mu) \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu_{\downarrow(r)}}(\lambda).$$

Finally, both cases together yield the first term of the induction relation (5.3.7).

The edge E is straight (hence $\mathrm{mon}_{M,E} = 1$) and no endpoint of E has degree 1. Then

$$\mathrm{contribution}_{M,\prec}(\lambda) = \mathrm{contribution}_{M \setminus E, \prec'}(\lambda).$$

By definition, $E = \{s_1, s_2\}$ being straight means that its both edge-sides s_1 and s_2 belong to the same polygon $F \in \mathcal{L}(\mathcal{B}, \mathcal{W})$. Besides there is an even number of edge-sides, let say $2i$ ($i > 0$), between s_1 and s_2 if we turn around F in one direction and also an even number of edge-sides, let say $2j$ ($j > 0$), if we turn around the face in the other direction.

Fix such a pair E of edge-sides. Then $(\mathcal{B}_E, \mathcal{W}_E)$ is a couple of pairings of $S \setminus E$ of type $\mu_{\uparrow(i,j)}$ (see Section 5.2.2). As before, summation over $(\mathcal{E}, \prec)$ such that E is the first edge is equivalent to summation over pairings $\mathcal{E}'$ of $S \setminus E$ and order $\prec'$. By definition, this corresponds to summation over $(M', \prec')$, where $M' = (\mathcal{B}_E, \mathcal{W}_E, \mathcal{E}') = M \setminus E$ runs over maps of face-type $\mu_{\uparrow(i,j)}$.

Therefore, for a fixed E, the total contribution of corresponding pairs $(M, \prec)$ is equal to

$$\sum_{(M', \prec')} \frac{(-1)^{\ell(\mu)-1}}{(k-1)!} \mathrm{contribution}_{M', \prec'}(\lambda) = \mathrm{Ch}^{(\alpha)}_{\mu_{\uparrow(i,j)}}(\lambda).$$

Let us count how many pairs E correspond to a given value of i and j. First, s_1 must be chosen in a face F, containing $2r$ edge-sides. There are $m_r(\mu)$ such faces and $2r$ edge-side in it, so there are $2rm_r(\mu)$ possible choices for s_1. Once s_1 is fixed, there are two possible choices for s_2 (only one if $i = j$): we fix arbitrarily a direction to turn around F and then s_2 must be the $i+1$-th or $j+1$-th edge-side after s_1 in this direction. As s_1 and s_2 play identical role and E is a non-ordered pair, the number of pairs E corresponding to a pair of values $\{i, j\}$ is $(2 - \delta_{i,j}) r m_r(\mu)$. Hence the total contribution of couples $(M, \prec)$ such that E is straight and no of its endpoints is a leaf is equal to

$$\sum_{\substack{r \geq 1 \\ \{i,j\}: i+j=r-1}} (2 - \delta_{i,j}) r m_r(\mu)\, \mathrm{Ch}^{(\alpha)}_{\mu_{\uparrow(i,j)}}(\lambda) = \sum_{\substack{r \geq 1 \\ i+j=r-1}} r m_r(\mu)\, \mathrm{Ch}^{(\alpha)}_{\mu_{\uparrow(i,j)}}(\lambda).$$

Clearly, it is equal to the second summand on the left-hand side of (5.3.7).

The edge E is twisted and thus $\mathrm{mon}_{M,E} = \gamma$. Then, no endpoint of E has degree 1, hence

$$\mathrm{contribution}_{M,\prec}(\lambda) = \gamma\, \mathrm{contribution}_{M\setminus E,\prec'}(\lambda).$$

One again, we fix a pair $E = \{s_1, s_2\}$ such that both edge sides s_1 and s_2 lie in a polygon F of $\mathcal{L}(\mathcal{B}, \mathcal{W})$ and are in an odd position. As above, if we fix the number $2r$ of edge-sides in F, there are $2rm_r(\mu)$ possible choices for s_1. Once s_1 is fixed, there are $r-1$ possible choices for s_2, which makes $r(r-1)m_r(\mu)$ choices for the pair $\{s_1, s_2\}$ (beware of the symmetry between s_1 and s_2).

Fix such an edge E. The couple $(\mathcal{B}_E, \mathcal{W}_E)$ of pairings of $S \setminus E$ has type $\mu \downarrow (r)$ (see Section 5.2.2). Hence summation over couples $(M, \prec)$ such that E is the first edge is equivalent to summation over maps $M \setminus E$ of face-type $\mu \downarrow (r)$.

Finally, the total contribution of couples $(M, \prec)$ with a twisted first edge is equal to

$$\sum_r r(r-1)\, m_r(\mu) \sum_{M',\prec'} \frac{(-1)^{\ell(\mu)-1}}{(|\mu|-1)!} \gamma\, \mathrm{contribution}_{M',\prec'}(\lambda) \\ = -\gamma \sum_r r(r-1)\, m_r(\mu)\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(r)}(\lambda),$$

where the summation on the left-hand side is over maps M' with face-type $\mu \downarrow (r)$. Clearly, it is equal to the third summand on the left-hand side of (5.3.7).

The edge E is interface and thus $\mathrm{mon}_{M,E} = \frac{1}{2}$. Then, no endpoint of E has degree 1, hence

$$\mathrm{contribution}_{M,\prec}(\lambda) = \frac{1}{2}\, \mathrm{contribution}_{M\setminus E,\prec'}(\lambda).$$

Fix a pair E of edge-sides $\{s_1, s_2\}$ lying in different polygons F_1 and F_2 of $\mathcal{L}(\mathcal{B}, \mathcal{W})$. Suppose F_1 contains $2r$ edge-sides, while F_2 has $2s$. Then $(\mathcal{B}_E, \mathcal{W}_E)$ has face-type $\mu \downarrow (rs)$ (see Section 5.2.2). Summation over couples $(M, \prec)$ such that E is the first edge is equivalent to summation over maps $M' = M \setminus E$ of face-type $\mu \downarrow (rs)$. Therefore, for a fixed pair E as above, the total contribution of couple $(M, \prec)$ with first edge E is

$$\sum_{M',\prec'} \frac{(-1)^{\ell(\mu)} - 1}{(|\mu|-1)!} \frac{1}{2}\, \mathrm{contribution}_{M',\prec'}(\lambda) = \frac{1}{2} \widehat{\mathrm{Ch}}^{(\alpha)}_{\pi\downarrow(rs)}$$

How many pair E correspond to a given pair $\{r, s\}$? First, one should choose s_1 in a polygon of size $2r$ or $2s$, let us say $2r$, of $\mathcal{L}(S_1, S_2)$. There is $2rm_r(\mu)$ choices for that. Then we choose s_2 in a polygon of size $2s$ of $\mathcal{L}(S_1, S_2)$ (beware that if $r = s$, this polygon has to be different from the first one): there are $2s(m_s - \delta_{r,s})$ choices for that. If $r = s$, s_1 and s_2 play analogous role (if $r \neq s$, we broke the symmetry by assuming that s_1 lies in a polygon of size $2r$), so one should divide by 2 to count pairs $\{s_1, s_2\}$, and not couples. Finally, we get that the total contribution of couples $(M, \prec)$ with an the first edge being interface is equal to

$$\sum_{\{r,s\}} \frac{4}{1+\delta_{r,s}} rs\, m_r(\mu)\big(m_s(\mu) - \delta_{r,s}\big)\, \frac{1}{2} \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(rs)}(\lambda)$$
$$= \sum_{r,s} rs\, m_r(\mu)\big(m_s(\mu) - \delta_{r,s}\big)\, \widehat{\mathrm{Ch}}^{(\alpha)}_{\mu\downarrow(rs)}(\lambda).$$

Clearly, it is equal to the fourth summand on the left-hand side of (5.3.7).

Bringing all contributions together, this establishes Proposition 5.3.5. □

5.3.4 Proof of Theorem 5.3.1

Proof of Theorem 5.3.1. We will use induction over $|\mu|$. For $|\mu| = 0$ there is only the empty partition $\mu = \emptyset$; clearly in this case $\mathrm{Ch}^{(\alpha)}_{\emptyset}(\lambda) = \widehat{\mathrm{Ch}}^{(\alpha)}_{\emptyset}(\lambda) = 1$ holds true. Since this is a bit pathological case (empty polygon, empty function, etc), in order to avoid difficulties with the start of the induction, we also consider separately the case $|\mu| = 1$ for which there is only one partition $\mu = (1)$; we easily get that that $\mathrm{Ch}^{(\alpha)}_{1}(\lambda) = \widehat{\mathrm{Ch}}^{(\alpha)}_{1}(\lambda) = |\lambda|$ indeed holds true.

Let us assume that the inductive assertion holds for all μ such that $|\mu| < n$ and let μ be a partition with $|\mu| = n$. In the case when $m_1(\mu) \geq 1$ we apply (5.3.3) and (5.3.4) and the inductive assertion implies that (5.3.1) holds true for μ as well.

In the case when $m_1(\mu) = 0$, we compare the left-hand side of (5.3.7) with the left-hand side of (5.3.2). From the inductive assertion it follows that they are equal; so must be their right-hand sides. □

5.4 Support for the conjectures: special values of α

5.4.1 Reformulation of results in [FŚ11b]

The purpose of this paragraph is to explain how Equations (2.12) and (2.13) can be obtained easily from the results of [FŚ11b], even if the presentation there is a little bit different. We use boldface characters for notation of [FŚ11b].

First note the difference of notation and normalization

$$\mathrm{Ch}^{(2)}_{\mu}(\lambda) = \left(\frac{1}{\sqrt{2}}\right)^{|\mu|-\ell(\mu)} \boldsymbol{\Sigma}^{(\mathbf{2})}_{\mu}.$$

Note also the roles of black and white vertices in the definition of N_G are inverted. Hence

[FŚ11b, Theorem 5.2] writes with the notation of this dissertation as

$$\mathrm{Ch}^{(2)}_\mu(\lambda) = \left(\frac{1}{\sqrt{2}}\right)^{|\mu|-\ell(\mu)} \frac{(-1)^{|\mu|}}{2^{\ell(\mu)}} \sum_M (-2)^{|V_\circ(M)|} N_M,$$

where the sum runs over maps of face-type μ. This is clearly the same as Equation (2.12).

Equation (2.13) is deduced directly from equation (2.12) using the duality relation (2.6) and the fact that $N_G(\lambda') = N_{G'}(\lambda)$, where G' is obtained from G by inverting the colors of the vertices.

5.4.2 Proving Conjecture 5.1.2 for $\alpha = \frac{1}{2}$ and $\alpha = 2$

Theorem 5.4.1. *Conjecture 5.1.2 is true for $\alpha \in \left\{\frac{1}{2}, 2\right\}$.*

Proof. Condition $\alpha \in \left\{\frac{1}{2}, 2\right\}$ is equivalent to $\gamma^2 = \frac{1}{2}$. With the same case analysis as in the proof of Lemma 5.2.6 one can check that in this case for any edge E of an arbitrary map M

$$\mathrm{mon}_{M,E} = \gamma^{\left[|F(M)|-|V(M)|)\right]-\left[|F(M\setminus E)|-|V(M\setminus E)|\right]+1}.$$

By a telescopic sum it follows that for an arbitrary history $\prec$:

$$\mathrm{mon}_{M,\prec} = \gamma^{|F(M)|-|V(M)|+|\mathcal{E}(M)|} = \gamma^{\ell(\mu)-|V(M)|+|\mu|},$$

where μ is the face-type of M; in particular this expression does not depend on the choice of the history, hence

$$\mathrm{mon}_M = \gamma^{\ell(\mu)-|V(M)|+|\mu|}.$$

One can thus easily check that $\widehat{\mathrm{Ch}}^{(\alpha)}_\mu$ coincides with (2.12), respectively (2.13). □

5.5 Link with a positivity conjectures of Lassalle

5.5.1 Statement of Lassalle's conjectures

Concerning first conjecture, we will be interested in the evaluation $\mathrm{Ch}^{(\alpha)}_\mu(\mathbf{p}\times\mathbf{q})$ of Jack characters on a Young diagram given by its multirectangular coordinates. Lassalle stated the following conjecture.

Conjecture 5.5.1 ([Las08b, Conjecture 1])**.** *Let μ be a partition such that $m_1(\mu) = 0$ and let $\beta := \alpha - 1$. Then $(-1)^{|\mu|}\vartheta^{\mathbf{p}\times\mathbf{q}}_{\mu\cup\mathbf{1}^{\mathbf{n}-|\mu|}}(\alpha)$ is a polynomial in $(\mathbf{p}, -\mathbf{q}, \beta)$ with non-negative integer coefficients.*

In the following we will prove (in Corollary 5.5.3) that our Main Conjecture implies a weaker version of this conjecture, namely that the coefficients are non-negative rational numbers.

5.5.2 Positivity in multirectangular coordinates

Our first step is the following statement.

Theorem 5.5.2. *Let us assume that Main Conjecture 5.1.1 holds true. Then* $(-1)^{|\mu|}\operatorname{Ch}_{\mu}^{(\alpha)}(\mathbf{p}\times\mathbf{q})$ *is a polynomial in variables* $(\mathbf{p}/\sqrt{\alpha}, -\sqrt{\alpha}\mathbf{q}, -\gamma)$ *with non-negative rational coefficients.*

Proof. For a Young diagram $\mathbf{p}\times\mathbf{q}$ given in multirectangular coordinates, the number of embeddings $N_M(\mathbf{p}\times\mathbf{q})$ takes a particularly simple form (see [FŚ11b, Lemma 3.9] where the notation is slightly different), for this reason (5.1.1) would imply that

$$\operatorname{Ch}_{\mu}^{(\alpha)}(\mathbf{p}\times\mathbf{q}) = (-1)^{\ell(\mu)}\sum_{M}\operatorname{wt}_M(\gamma)\times \left[\sum_{\varphi:V_{\bullet}(M)\to\mathbb{N}^{\star}}\prod_{l\in V_{\bullet}(M)}\left(\frac{-p_{\varphi(l)}}{\sqrt{\alpha}}\right)\cdot\prod_{l'\in V_{\circ}(M)}\left(\sqrt{\alpha}\, q_{\psi(l')}\right)\right],$$

where the first sum is over all bipartite maps M of the face-type μ and $\psi(l')$ is defined as the maximum of $\varphi(l)$ over all white neighbors l of the black vertex l.

Quantity $\operatorname{wt}_M(\gamma)$ is a polynomial in γ with non-negative rational coefficients of the same parity as the Euler characterstic $\chi(M)$, that is the same parity as $|\mu|+\ell(\mu)+|V(M)|$. Rewriting the equation above as

$$(-1)^{|\mu|}\operatorname{Ch}_{\mu}^{(\alpha)}(\mathbf{p}\times\mathbf{q}) = \sum_{M}\operatorname{wt}_M(-\gamma)\times \left[\sum_{\varphi:V_{\bullet}(M)\to\mathbb{N}^{\star}}\prod_{l\in V_{\bullet}(M)}\left(\frac{p_{\varphi(l)}}{\sqrt{\alpha}}\right)\cdot\prod_{l'\in V_{\circ}(M)}\left(-\sqrt{\alpha}\, q_{\psi(l')}\right)\right],$$

this finishes the proof. ☐

Corollary 5.5.3. *Let us assume that Main Conjecture 5.1.1 holds true. Let* μ *be an arbitrary partition. Then* $(-1)^{|\mu|}\vartheta^{\mathbf{p}\times\mathbf{q}}_{\mu\cup\mathbf{1}^{\mathbf{n}-|\mu|}}(\alpha)$ *is a polynomial in* $(\mathbf{p},-\mathbf{q},\beta)$ *with non-negative rational coefficients.*

Proof. Using (2.5) we obtain:

$$(-1)^{|\mu|}\vartheta^{\mathbf{p}\times\mathbf{q}}_{\mu\cup\mathbf{1}^{\mathbf{n}-|\mu|}}(\alpha) = \sum_{M}\sqrt{\alpha}^{2|V_{\circ}(M)|+|\mu|-\ell(\mu)-|V(M)|}\operatorname{wt}_M(-\gamma) \times\left[\sum_{\varphi:V_{\bullet}(M)\to\mathbb{N}^{\star}}\prod_{l\in V_{\bullet}(M)}p_{\varphi(l)}\cdot\prod_{l'\in V_{\circ}(M)}\left(-q_{\psi(l')}\right)\right].$$

Recall that $\mathrm{wt}_M(\gamma)$ is a polynomial in γ of degree at most

$$\chi(M) = 2(\text{number of connected components of } M) - \ell(\mu) + |\mu| - |V(M)|$$

and with the same parity as $\chi(M)$. The number of connected components of M is at most equal to the number of white vertices, and since $-\gamma = \frac{\beta}{\sqrt{\alpha}}$ and $\alpha = \beta + 1$ we have that

$$\sqrt{\alpha}^{2|V_\circ(M)|+|\mu|-\ell(\mu)-|V(M)|}\, \mathrm{wt}_M(-\gamma)$$

is a polynomial in β with non-negative rational coefficients. It finishes the proof. □

5.6 Computer exploration and the counterexample

5.6.1 Counterexample $\mu = (9)$

For $\mu = (9)$ a computer calculation shows that

$$\widehat{\mathrm{Ch}}^{(\alpha)}_{(9)}(\mathbf{p} \times \mathbf{q}) - \mathrm{Ch}^{(\alpha)}_{(9)}(\mathbf{p} \times \mathbf{q}) = \frac{41}{70}(2\gamma^2 - 1) \sum_{i<j<k} p_i p_j p_k (q_k - q_j)(q_i - q_j) q_k \quad (5.6.1)$$

which might be non-zero for multirectangular Young diagrams consisting of at least $\ell \geq 3$ rectangles. It is worth pointing out that this is *not* a counterexample for Conjectures 5.1.3 and 5.1.4.

The quantity $\widehat{\mathrm{Ch}}^{(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$ was computed using the very definition given in this article. Computing $\mathrm{Ch}^{(\alpha)}_{\mu}(\mathbf{p} \times \mathbf{q})$ is a bit harder (while shorter in practice): we used some data made available by Lassalle [Las], that express it in terms of free cumulants (Lassalle gave an algorithm to do this computation [Las09, Section 9], but as his data was made available, we did not implement it again). Then the free cumulant $R_k(\mathbf{p} \times \mathbf{q})$ can be computed as the highest degree component (as polynomials in $\mathbf{p}$ and $\mathbf{q}$) of $\mathrm{Ch}^{(\alpha)}_{(k-1)}(\mathbf{p} \times \mathbf{q})$ (see [Las09, Theorem 10.2]).

The calculation of (5.6.1) took a week of computer time. Finding this counter-example was only possible because the theoretical results in this chapter, in [CJŚ13], and some additional tricks allow to reduce the computational complexity. Analogous calculation for $\mu = (10)$ would be, in a moment, rather challenging.

References

[AC12] K. Aker and M. Can. Generators of the Hecke algebra of (S_{2n}, H_n). *Adv. Math.*, 231(5):2465–2483, 2012.

[BH08] B Andrei Bernevig and FDM Haldane. Model fractional quantum hall states and jack polynomials. *Physical review letters*, 100(24):246802, 2008.

[Bia07] Philippe Biane. On the formula of Goulden and Rattan for Kerov polynomials. *Sém. Lothar. Combin.*, 55:Art. B55d, 5 pp. (electronic), 2005/07.

[Bia98] Philippe Biane. Representations of symmetric groups and free probability. *Adv. Math.*, 138(1):126–181, 1998.

[Bia01] Philippe Biane. Approximate factorization and concentration for characters of symmetric groups. *Internat. Math. Res. Notices*, (4):179–192, 2001.

[Bia02] P. Biane. Free probability and combinatorics. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 765–774, Beijing, 2002. Higher Ed. Press.

[Bia03] Philippe Biane. Characters of symmetric groups and free cumulants. In *Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001)*, volume 1815 of *Lecture Notes in Math.*, pages 185–200. Springer, Berlin, 2003.

[BJ07] D. R. L. Brown and D. M. Jackson. A rooted map invariant, non-orientability and Jack symmetric functions. *J. Combin. Theory Ser. B*, 97(3):430–452, 2007.

[BOO00] Alexei Borodin, Andrei Okounkov, and Grigori Olshanski. Asymptotics of plancherel measures for symmetric groups. *Journal of the American Mathematical Society*, 13(3):481–515, 2000.

[CJ11] A. Czyżewska-Jankowska. Topological expansion of the coefficients of zonal polynomials in genus one. arXiv preprint 1108.3173v1, 2011.

[CJŚ13] Agnieszka Czyżewska-Jankowska and Piotr Śniady. Jack polynomials and the measure of non-orientability of maps. Part 2. Kerov polynomials and genus expansion. In preparation, 2013.

[DF12] Maciej Dołęga and Valentin Féray. On Kerov polynomials for Jack characters. Preprint arXiv:1201.1806, 2012.

[DFŚ10] Maciej Dołęga, Valentin Féray, and Piotr Śniady. Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations. *Adv. Math.*, 225(1):81–120, 2010.

[DFŚ13] Maciej Dołęga, Valentin Féray, and Piotr Śniady. Jack polynomials and the measure of non-orientability of maps. Part 1. Orientability generating series. Preprint arXiv:1301.6531, 2013.

[DS81] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete*, 57(2):159–179, 1981.

[DŚ12] Maciej Dołęga and Piotr Śniady. Asymptotics of characters of symmetric groups: Structure of Kerov character polynomials. *J. Comb. Theory Ser. A*, 119(6):1174–1193, 2012.

[Fér09] Valentin Féray. Combinatorial interpretation and positivity of Kerov's character polynomials. *J. Algebraic Combin.*, 29(4):473–507, 2009.

[Fér10a] V. Féray. On complete functions in Jucys-Murphy elements. To appear in *Annals of Combinatorics*, arXiv:1009.0144, 2010.

[Fér10b] V. Féray. Stanley's formula for characters of the symmetric group. *Annals of Combinatorics*, 13(4):453–461, 2010.

[FH59] H. K. Farahat and G. Higman. The centres of symmetric group rings. *Proc. Roy. Soc. London Ser. A*, 250:212–221, 1959.

[FJMM02] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. A differential ideal of symmetric polynomials spanned by Jack polynomials at $\beta = -(r-1)/(k+1)$. *Int. Math. Res. Not.*, (23):1223–1237, 2002.

[Fro00] G. Frobenius. Über die Charaktere der symmetrischen Gruppe. *Sitz. Konig. Preuss. Akad. Wissen*, 516(534):148–166, 1900.

[FŚ11a] Valentin Féray and Piotr Śniady. Asymptotics of characters of symmetric groups related to Stanley character formula. *Ann. of Math. (2)*, 173(2):887–906, 2011.

[FŚ11b] Valentin Féray and Piotr Śniady. Zonal polynomials via Stanley's coordinates and free cumulants. *J. Algebra*, 334:338–373, 2011.

[Ful04] J. Fulman. Stein's method, Jack measure, and the Metropolis algorithm. *Journal of Combinatorial Theory, Series A*, 108(2):275–296, 2004.

[Gal08] Światosław Gal. Private communication, 2008.

[GJ96a] I. P. Goulden and D. M. Jackson. Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. *Trans. Amer. Math. Soc.*, 348(3):873–892, 1996.

[GJ96b] I. P. Goulden and D. M. Jackson. Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials. *Canad. J. Math.*, 48(3):569–584, 1996.

[GKP88] Ronald L. Graham, Donald Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley Pub (Sd), 1988.

[GR05] Adriano Garsia and Jeffrey B. Remmel. Breakthroughs in the theory of Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 102(11):3891–3894 (electronic), 2005.

[GR07] I. P. Goulden and A. Rattan. An explicit form for Kerov's character polynomials. *Trans. Amer. Math. Soc.*, 359(8):3669–3685 (electronic), 2007.

[IK99] V. Ivanov and S. Kerov. The algebra of conjugacy classes in symmetric groups, and partial permutations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 256(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 3):95–120, 265, 1999.

[IO02] Vladimir Ivanov and Grigori Olshanski. Kerov's central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric functions 2001: surveys of developments and perspectives*, volume 74 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 93–151. Kluwer Acad. Publ., Dordrecht, 2002.

[Jac71] Henry Jack. A class of symmetric polynomials with a parameter. *Proc. Roy. Soc. Edinburgh Sect. A*, 69:1–18, 1970/1971.

[Kad97] Kevin W. J. Kadell. The Selberg-Jack symmetric functions. *Adv. Math.*, 130(1):33–102, 1997.

[Ker93a] S. Kerov. The asymptotics of interlacing sequences and the growth of continual Young diagrams. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 205(Differentsialnaya Geom. Gruppy Li i Mekh. 13):21–29, 179, 1993.

[Ker93b] S. Kerov. Transition probabilities of continual Young diagrams and the Markov moment problem. *Funct. Anal. Appl.*, 27(3):104–117, 1993.

[Ker00a] S. Kerov. Talk in Institute Henri Poincaré, Paris, January 2000.

[Ker00b] S. V. Kerov. Anisotropic young diagrams and jack symmetric functions. *Funct. Anal. Appl.*, 34:41–51, 2000.

[KO94] Serguei Kerov and Grigori Olshanski. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(2):121–126, 1994.

[KS96] F. Knop and S. Sahi. Difference equations and symmetric polynomials defined by their zeros. *Internat. Math. Res. Notices*, 1996(10):473–486, 1996.

[KS97] F. Knop and S. Sahi. A recursion and a combinatorial formula for Jack polynomials. *Inventiones Mathematicae*, 128(1):9–22, 1997.

[KSW96a] Vladimir A Kazakov, Matthias Staudacher, and Thomas Wynter. Almost flat planar diagrams. *Communications in mathematical physics*, 179(1):235–256, 1996.

[KSW96b] Vladimir A Kazakov, Matthias Staudacher, and Thomas Wynter. Character expansion methods for matrix models of dually weighted graphs. *Communications in mathematical physics*, 177(2):451–468, 1996.

[KV77] S.V. Kerov and A.M. Vershik. Asymptotics of the plancherel measure of the symmetric group and the limiting form of young tables. *Soviet Math. Dokl*, 18:527–531, 1977.

[Las] Michel Lassalle. Jack polynomials and free cumulants. http://igm.univ-mlv.fr/~lassalle/free.html. Accessed: 1/12/2012.

[Las03] A. Lascoux. *Symmetric functions and combinatorial operators on polynomials*, volume 99 of *Cbms Regional Conference Series in Mathematics*. Amer Mathematical Society, 2003.

[Las08a] Michel Lassalle. An explicit formula for the characters of the symmetric group. *Math. Ann.*, 340(2):383–405, 2008.

[Las08b] Michel Lassalle. A positivity conjecture for Jack polynomials. *Math. Res. Lett.*, 15(4):661–681, 2008.

[Las08c] Michel Lassalle. Two positivty conjectures for Kerov polynomials. *Adv. in Appl. Math.*, 41(3):407–422, 2008.

[Las09] Michel Lassalle. Jack polynomials and free cumulants. *Adv. Math.*, 222(6):2227–2269, 2009.

[LC09] Michael Andrew La Croix. *The combinatorics of the Jack parameter and the genus series for topological maps*. PhD thesis, University of Waterloo, 2009.

[LS77] B. F. Logan and L. A. Shepp. A variational problem for random Young tableaux. *Advances in Math.*, 26(2):206–222, 1977.

[LV95] Luc Lapointe and Luc Vinet. A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture. *Internat. Math. Res. Notices*, (9):419–424, 1995.

[LV97] Luc Lapointe and Luc Vinet. Rodrigues formulas for the Macdonald polynomials. *Adv. Math.*, 130(2):261–279, 1997.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mat08] S. Matsumoto. Jack deformations of Plancherel measures and traceless Gaussian random matrices. *Elec. Jour. Comb.*, 15(R149):1, 2008.

[Mat10] S. Matsumoto. Jucys–Murphy elements, orthogonal matrix integrals, and Jack measures. *The Ramanujan Journal*, 26(1):69–107, 2010.

[Oko00] Andrei Okounkov. Random matrices and random permutations. *Internat. Math. Res. Notices*, (20):1043–1095, 2000.

[Oko03] A. Okounkov. The uses of random partitions. In *Fourteenth International Congress on Mathematical Physics*, pages 379–403. Word Scientists, 2003.

[Ols10] G. Olshanski. Plancherel averages: Remarks on a paper by Stanley. *Elec. Jour. Comb.*, 17(1):R43, 2010.

[OO97] Andrei Okounkov and Grigori Olshanski. Shifted Jack polynomials, binomial formula, and applications. *Math. Res. Lett.*, 4(1):69–78, 1997.

[Roi96] Yuval Roichman. Upper bound on the characters of the symmetric groups. *Invent. Math.*, 125(3):451–485, 1996.

[RR09] Gesine Reinert and Adrian Röllin. Multivariate normal approximation with stein's method of exchangeable pairs under a general linearity condition. *The Annals of Probability*, 37(6):2150–2173, 2009.

[RŚ08] Amarpreet Rattan and Piotr Śniady. Upper bound on the characters of the symmetric groups for balanced young diagrams and a generalized frobenius formula. *Advances in Mathematics*, 218(3):673–695, 2008.

[Śni06a] Piotr Śniady. Asymptotics of characters of symmetric groups, genus expansion and free probability. *Discrete Math.*, 306(7):624–665, 2006.

[Śni06b] Piotr Śniady. Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Related Fields*, 136(2):263–297, 2006.

[Śni06c] Piotr Śniady. Generalized Cauchy identities, trees and multidimensional Brownian motions. I. Bijective proof of generalized Cauchy identities. *Electron. J. Combin.*, 13(1):Research Paper 62, 27 pp. (electronic), 2006.

[Spe94] R. Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Mathematische Annalen*, 298(1):611–628, 1994.

[Sta04] Richard P. Stanley. Irreducible symmetric group characters of rectangular shape. *Sém. Lothar. Combin.*, 50:Art. B50d, 11 pp. (electronic), 2003/04.

[Sta89] Richard P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.*, 77(1):76–115, 1989.

[Sta02] Richard P. Stanley. Kerov's character polynomial and irreducible symmetric group characters of rectangular shape. Transparencies from a conference in Québec City, June 2002.

[Sta06] Richard P. Stanley. A conjectured combinatorial interpretation of the normalized irreducible character values of the symmetric group. Preprint arXiv:math.CO/0606467, 2006.

[Voi86] Dan Voiculescu. Addition of certain noncommuting random variables. *J. Funct. Anal.*, 66(3):323–346, 1986.