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u-invariant of Formally Real and Nonreal Fields

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Abstract

We introduce a notion o u-invariant of a field K. It is defined as a maximum of dimensions of all anisotropic quadratic forms over K. For fields that are not formally real, we prove that u-invariant can not take values 3, 5, 7. As one of our main results, we prove Kneser's Theorem for nonreal fields. For formally real fields, we generalize the notion of u-invariant. Moreover, we extend Kneser's Theorem to the case of arbitrary fields.

Keywords: u-invariant, quadratic forms, Witt ring, Pfister forms.

1 Introduction

One of the notions of algebraic theory of quadratic forms, which have been intensively studied in recent times is the notion of u-invariant of a field. Almost before our eyes the initial hypothesis that u-invariant is a power of 2 was overthrown. Moreover, another hypothesis that u-invariant is an even number was also rejected.

Considering u-invariant of not formally real fields, Irving Kaplansky ventured in 1953 the conjecture that the u-invariant of a field is either infinity, or a power of 2. This conjecture, known as Kaplansky's Hypothesis, stood open for many years, until 1989, when Alexander S. Merkurjev disproved it by constructing a field of u-invariant 6.

Concerning parity of u-invariant, it was also proved that if F is a complete discretely valuated field with residue class field K of characteristic not 2, then $u(F) = 2 \cdot u(K)$, where $u(K)$ denotes u-invariant of a field K. In particular, if K is algebraically closed and $F_n = K((t_1))((t_2))\dots((t_n))$, then $u(F_n) = 2^n$. This observation, due to I. Kaplansky, proved the existence of nonreal fields of any 2-power u-invariant.

Still, as an open question the problem of existence of fields of u-invariant greater than or equal to 9 remained, until 2000, when Oleg T. Izhboldin constructed a field of u-invariant 9, proving, *ipso facto*, the existence of a field of odd u-invariant greater than 1. These events led to my interest in the notion of u-invariant, to which this paper is devoted.

The aim of the paper is to present the most important statements about u-invariant of formally real and nonreal fields. The paper is based on my Master Thesis, [3], defended in July 2009.

This paper is organized as follows. Section 2 contains necessary to understand the concept of the work notions that concern quadratic forms and basic properties of quadratic forms useful in later discussions. Section 3 splits into two subsections. The first subsection presents a definition of the notion of u-invariant of a nonreal field. We proved there theorems that present relationship between u-invariant of a field and the fundamental ideal of the Witt ring of a field. In particular, we present Kneser's Theorem, which was first verified in special cases by I. Kaplansky, and, later, proved in full by Adolf Kneser. Moreover, we prove also that u-invariant of a field can not take values 3, 5, 7. In the second subsection of Section 3 we discuss the notion of u-invariant of formally real fields. Namely, by appropriate theorems, we extend Kneser's Theorem to the case of arbitrary fields. Further generalizations of Kneser's Theorem can be found in [2].

2 Preliminaries

The purpose of this chapter is to present basic notions and properties of quadratic forms. We shall confine to those facts that will be useful in subsequent considerations apart from the proof part. Throughout this paper we will consider only fields of characteristic different from 2. The word *field* will be understood as *the field of characteristic different from* 2.

2.1 Bilinear spaces and quadratic forms

Suppose that K is a fixed field. We define an (n-ary) *quadratic form* as the homogeneous polynomial φ of degree 2 in n variables over K

$$\varphi(X_1, \ldots, X_n) = \sum_{1 \le i < j \le n} c_{ij} X_i X_j.$$

Taking $a_{ij} = a_{ji} = \frac{1}{2} c_{ij}$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$ and $a_{ii} = c_{ii}$ for $i = 1, \ldots, n$ the quadratic form φ can be written in a symmetrical way

$$\varphi(X_1, \ldots, X_n) = \sum_{i,j=1}^{n} a_{ij} X_i X_j.$$

By a *matrix of coefficients* of quadratic form φ we mean the matrix $M_\varphi = [a_{ij}]$. If M_φ is diagonal, the form φ is called a *diagonal form.* The form φ can be written also in the following way

$$\varphi(X_1, \ldots, X_n) = \varphi(\mathbb{X}) = \mathbb{X}^T M_\varphi \mathbb{X},$$

$$\text{where } \mathbb{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

is a column of variables. When the matrix M_φ is nonsingular, the form φ is called a *nonsingular* form. We define a *determinant of the form* φ as the determinant of its matrix M_φ and denote it by $\det \varphi$. Dimension of the form φ will be denoted by $\dim \varphi$.
The n-dimensional forms φ, ψ will be called *equivalent*, denoted by $\varphi \cong \psi$, when there exists an invertible matrix $C \in K_n^n$ such that

$$\varphi(\mathbb{X}) = \psi(C\mathbb{X}), \text{ that is}$$

$$M_\varphi = C^T M_\psi C.$$

It follows that if $\varphi \cong \psi$, then $\det\varphi = c^2 \det\psi$ for some $c \neq 0$.

There is a close relationship between quadratic forms and bilinear spaces over K. We will present it below. But first let us recall that a *bilinear space over the field* K is defined as a pair (V, β), where V is a finite dimensional linear space over K and $\beta\colon V \times V \to K$ is a symmetric bilinear functional. If $\{v_1, \ldots, v_n\}$ is a basis of V, then the matrix $A = [\beta(v_i, v_j)]$ is called the *matrix of the functional* β *relative to the basis* $\{v_1, \ldots, v_n\}$.

Two bilinear spaces (V_1, β_1), (V_2, β_2) will be called *isometric*, if there exists an isomorphism $f\colon V_1 \to V_2$ such that $\beta_1(u, v) = \beta_2(f(u), f(v))$ for every $u, v \in V_1$, and will be denoted by $(V_1, \beta_1) \cong (V_2, \beta_2)$ or $V_1 \cong V_2$.

Suppose that $\varphi \in K[X_1, \ldots, X_n]$ is an n-dimensional quadratic form over K and V is an n-dimensional linear space over K with a fixed basis $\{v_1, \ldots, v_n\}$. The form φ determines a functional $Q_\varphi\colon V \to K$ such that $Q_\varphi(x_1v_1 + \ldots + x_nv_n) = \varphi(x_1, \ldots, x_n)$. It is easy to verify that the mapping $\beta_\varphi\colon V \times V \to K$ defined as

$$\beta_\varphi(u, w) = \frac{1}{2}(Q_\varphi(u + w) - Q_\varphi(u) - Q_\varphi(w))$$

is a symmetric bilinear functional and the matrix M_φ is the matrix of β_φ relative to the basis $\{v_1, \ldots, v_n\}$. Bilinear functional β_φ is nonsingular if the matrix M_φ is nonsingular. We define the *bilinear space determined by the form* φ as the pair (V, β_φ), while the pair (V, Q_φ) is called the *quadratic space determined by the form* φ. By a *quadratic functional determined by the form* φ we mean the functional Q_φ.

It is easy to see that if we fix another space V' with the basis $\{w_1, \ldots, w_n\}$, or just another basis of the space V, then, received as above, bilinear space (V', β'_φ) would be isometric with the space (V, β_φ). It is also easy to check that if forms φ and ψ are equivalent, then they determine isometric bilinear spaces (V, β_φ), (V', β_ψ).

So, every n-dimensional quadratic form determines, uniquely up to isometry, an n-dimensional quadratic space (V, Q_φ). Therefore, in the later part the quadratic space (V, Q_φ) will be denoted by (V, φ).

It turns out that the reverse situation also occurs — every bilinear space (V, β) determines some quadratic form. Namely, suppose that (V, β) is a bilinear space over K, that is V is a finite dimensional linear space and $\beta\colon V \times V \to K$ is a symmetric bilinear functional. The bilinear functional β determines the mapping $Q_\beta\colon V \to K$ defined as $Q_\beta(v) = \beta(v, v)$. The functional Q_β is called the *quadratic functional determined by* β.

Suppose that $\{v_1, \ldots, v_n\}$ is a fixed basis of V and A is the matrix of a functional β in this basis. Then,

$$\varphi(X_1, \ldots, X_n) = [X_1, \ldots, X_n] \cdot A \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

determines a quadratic form over field K with the matrix of coefficients $M_\varphi = A$, where $\beta_\varphi = \beta$. Notice that $Q_\beta(x_1v_1 + \ldots + x_nv_n) = \varphi(x_1, \ldots, x_n)$. Moreover, if $(V', \beta') \cong (V, \beta)$, then quadratic form φ' determined by β' would be equivalent to the form φ.

In conclusion of these considerations, let us note that there exists one-one correspondence between quadratic forms and bilinear spaces, where equivalent forms correspond with isometric bilinear spaces.

This connection will be used in two ways in the sequel. First, having the quadratic form φ we will be using the geometric tools with reference to the quadratic space (V, φ) and the bilinear space (V, β_φ). And conversely, from considerations about the bilinear space (V, β) we will conclude some facts about the quadratic form φ determined by β.

Now suppose that φ is an n-dimensional quadratic form and V is an n-dimensional linear space with fixed basis $\{v_1, \ldots, v_n\}$. Let (V, β_φ) be the bilinear space determined by the form φ. As we know, every bilinear space V over the field K has an orthogonal basis, which means that the form φ over K is equivalent to some diagonal quadratic form. The form φ of matrix equal to $diag(a_1, \ldots, a_n)$ will be denoted by $\varphi = \langle a_1, \ldots, a_n \rangle$. That is $\langle a_1, \ldots, a_n \rangle = a_1 X_1^2 + \ldots + a_n X_n^2$. This observation shows that, up to equivalent forms, we can limit our considerations to diagonal forms.

Suppose that φ, ψ are the n and m-dimensional quadratic forms over K, respectively. Let M_φ, M_ψ be the matrices of coefficients of the forms φ and ψ. The *orthogonal sum of forms* φ and ψ is an $n+m$-dimensional quadratic form $\varphi \perp \psi$ with the matrix of coefficients equal to

$$M_{\varphi\perp\psi} = M_\varphi \perp M_\psi = \begin{bmatrix} M_\varphi & 0 \\ 0 & M_\psi \end{bmatrix}.$$

Note that $\langle a_1, \ldots, a_n \rangle \perp \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$.

The *tensor product of forms* φ and ψ is an $n \cdot m$-dimensional quadratic form $\varphi \otimes \psi$ with the matrix of coefficients equal to the Kronecker product of the matrices M_φ and M_ψ, that is

$$M_{\varphi\otimes\psi} = M_\varphi \otimes M_\psi.$$

It is easy to notice that

$$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle = \langle a_1 b_1, \ldots, a_1 b_m, a_2 b_1, \ldots, a_n b_m \rangle.$$

The product of forms $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_m \rangle$ will be denoted by $\varphi \cdot \psi = \langle a_1, \ldots, a_n \rangle \cdot \langle b_1, \ldots, b_m \rangle$. Note also that $\det(\varphi \perp \psi) = \det \varphi \cdot \det \psi$ and $\det(\varphi \otimes \psi) = \det \varphi^{\dim \psi} \cdot \det \psi^{\dim \varphi}$.

The above constructions have natural interpretations in terms of bilinear spaces. Namely, suppose that (V_1, β_φ), (V_2, β_ψ) are the bilinear spaces determined by the forms φ and ψ. Then, the orthogonal sum of the forms φ and ψ determines the bilinear space $(V, \beta_{\varphi\perp\psi})$ such that $V = V_1 \oplus V_2$ and $\beta_{\varphi\perp\psi}$ is the bilinear functional that satisfies

$$\beta_{\varphi\perp\psi}(v_1 \oplus v_2, w_1 \oplus w_2) = \beta_\varphi(v_1, w_1) + \beta_\psi(v_2, w_2).$$

Whereas, the tensor product of forms φ and ψ determines the bilinear space $(V, \beta_{\varphi\otimes\psi})$ such that $V = V_1 \otimes V_2$ and $\beta\varphi \otimes \psi$ is the only bilinear functional on V that satisfies

$$\beta_{\varphi\otimes\psi}(v_1 \otimes v_2, w_1 \otimes w_2) = \beta_\varphi(v_1, w_1) \cdot \beta_\psi(v_2, w_2).$$

Remark 2.1. Let K be an arbitrary field. By the symbol $\dot{K}$ we will denote the set $K \setminus \{0\}$, which is a multiplicative group of the field K. By the symbol $\dot{K}^2$ we will denote the set $\{x^2 \colon x \in \dot{K}\}$, which is a subgroup of $\dot{K}$.

By a group of square classes of the field K we mean the quotient ring $\dot{K}/\dot{K}^2$ and its rank denote by $q(K)$. The elements of $\dot{K}/\dot{K}^2$ are denoted by $a\dot{K}^2 \in \dot{K}/\dot{K}^2$, and called the square classes of the element a.
The field K will be called quadratically closed, when $\dot{K} = \dot{K}^2$, that is when $q(K) = 1$.

We shall also need the following theorem in later paragraphs of this paper. For a proof see [5].

Theorem 2.2 (Witt's Cancellation Theorem)**.** *Let K be a field and let $c, a_1, \ldots, a_n, b_1, \ldots, b_m \in K$, $c \neq 0$. Then*

$$\langle c, a_1, \ldots, a_n \rangle \cong \langle c, b_1, \ldots, b_m \rangle \Rightarrow \langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_m \rangle$$

Suppose that φ is a quadratic form over the field K. A form ψ is called a *subform* of φ, when there exists a form σ such that $\varphi \cong \psi \perp \sigma$. The form

$$\varphi(X_1, \ldots, X_n) = \sum_{i,j=1}^{n} a_{ij} X_i X_j$$

will be called *isotropic* over K, when there exist the elements $x_1, \ldots, x_n \in K$, not all equal to 0, such that $\varphi(x_1, \ldots, x_n) = 0$. The bilinear space (V, β_φ) determined by the form φ will be called *isotropic* when there exists $0 \neq v \in V$ such that $Q_\varphi(v) = \beta_\varphi(v, v) = 0$. Then, the vector v will be called *isotropic*, and the space (V, β_φ) a *isotropic space.*

The binary quadratic form φ will be called a *hyperbolic form*, when φ is equivalent to the form $\langle 1, -1 \rangle$. We define a *hyperbolic plane* as the bilinear space determined by the hyperbolic form and denote it by $\mathbb{H}$. The orthogonal sum of m hyperbolic planes $\underbrace{\mathbb{H} \perp \ldots \perp \mathbb{H}}_{m}$ will be called a *hyperbolic space* and denoted by $m\mathbb{H}$. By the above notions we have the following characterization of hyperbolic forms.

Theorem 2.3. *Let φ be a binary nonsingular quadratic form over a field K. The following statements are equivalent:*
(1) $\varphi \cong \langle 1, -1 \rangle$*;*
(2) φ *is isotropic;*
(3) $\det \varphi \in -\dot{K}^2$.

Now, let V be an arbitrary bilinear space over a field K. Witt's Decomposition Theorem states that V splits into an orthogonal sum

$$V = V_t \perp V_h \perp V_{an},$$

where V_t is a totally isotropic subspace, V_h is a hyperbolic or zero subspace, V_{an} is an anisotropic subspace, and the isometry types of V_g, V_h, V_{an} are all uniquely determined. If (V, β) is a nonsingular bilinear space, then $V = V_h \perp V_{an}$, where V_h is a hyperbolic (or zero) space and V_{an} is an anisotropic space. So, it is easy to notice that (V, β) is isotropic iff V contains as a subspace the hyperbolic plane $\mathbb{H}$. It is also easy to prove that if the form $\varphi = \langle a_1, \ldots, a_n \rangle$, for $a_1, \ldots, a_n \in \dot{K}$, is isotropic, then φ splits into $\varphi = \langle 1, -1 \rangle \perp \langle b_3, \ldots, b_n \rangle$ for some $b_3, \ldots, b_n \in \dot{K}$.

2.2 The Witt ring of the field K

In this section we briefly recall the construction of the Witt ring of the field K, without going into details. These details can be found in [5].

Nonsingular symmetric bilinear spaces U and V over the field K are called *similar*, denoted by $U \sim V$, if there exist hyperbolic spaces $m\mathbb{H}$ and $n\mathbb{H}$ such that

$$U \perp m\mathbb{H} \cong V \perp n\mathbb{H}.$$

It turns out that so defined relation of similarity is an equivalence relation. So, we can consider the equivalence classes of the relation $\sim$. By a *similarity class* of some fixed space U we mean the class of all spaces similar to U and denote it by $[U]$. If $U \cong A$ in some basis of the space U, then instead of $[U]$ we will write $[A]$. In particular, if $A = (a_1, \ldots, a_n)$, then we write $[a_1, \ldots, a_n]$.

Let $W(K)$ be the set of all similarity classes of the nonsingular symmetric bilinear spaces over the field K. In $W(K)$ can be naturally defined the operations of addition and multiplication of classes. Namely, for any nonsingular symmetric spaces U and V

$$[U] + [V] := [U \perp V],$$

$$[U] \cdot [V] := [U \otimes V].$$

So defined set $W(K)$ with operations of addition and multiplication defined as above is a commutative ring with unity. It is called the *Witt ring of the field K*. Let (V, β) be a nonsingular bilinear space over a field K. By Witt's Decomposition Theorem we have

$$V \cong m\mathbb{H} \perp V_{an},$$

where V_{an} is an anisotropic space. Then, of course, $[V] = [V_{an}]$. Note also that if V_{an} and V'_{an} are both anisotropic, then $V_{an} \sim V'_{an}$ iff $V_{an} \cong V'_{an}$. Therefore, every element of the Witt ring is represented, uniquely up to isometry, by exactly one anisotropic form.

Remark 2.4. Let φ be a nonsingular quadratic form and let (V, β_φ) be a bilinear space determined by φ. The Witt class $[V]$ of the space V will be also denoted by φ. Thus, the symbol $\varphi = \langle a_1, \ldots, a_n \rangle$, where $a_1, \ldots, a_n \in \dot{K}$, will mean, depending on the context, either the quadratic form $\varphi = a_1 X_1^2 + \ldots + a_n X_n^2$, or the quadratic functional φ defined on the space V, with fixed basis $\{v_1, \ldots, v_n\}$, as follows $\varphi(x_1 v_1 + \ldots + x_n v_n) = a_1 x_1^2 + \ldots + a_n x_n^2$, or the element of the Witt ring $W(K)$ represented by the bilinear space (V, β_φ).

Now consider the mapping $e \colon W(K) \to \mathbb{Z}/2\mathbb{Z}$ defined as follows

$$e([U]) = dim\ U + 2\mathbb{Z} = \begin{cases} 2\mathbb{Z} & \text{, if } dim\ U \equiv 0\ (mod\ 2) \\ 1 + 2\mathbb{Z} & \text{, if } dim\ U \equiv 1\ (mod\ 2) \end{cases}$$

It is easy to check that e is a ring epimorphism. So defined epimorphism is called the *dimension-index*. Consider the kernel of the dimension-index. Note that it consists of the Witt classes represented by even-dimensional spaces

$$\ker e = \{[U] \in W(K) \colon e([U]) = 0\} = \{[U] \in W(K) \colon dim\ U \equiv 0\ (mod\ 2)\}.$$

Moreover, $\ker e$ is the maximal ideal of the Witt ring additively generated by the set

$$\{\langle 1, a\rangle \colon a \in \dot{K}\}.$$

Namely, for an arbitrary fixed nonzero element $\langle a_1, \ldots, a_n\rangle \in W(K)$ we have the following representation

$$\langle a_1, \ldots, a_n\rangle = \langle 1, a_1\rangle + \ldots + \langle 1, a_n\rangle - n\langle 1\rangle.$$

If $\langle a_1, \ldots, a_n\rangle \in \ker e$, then $n = 2m$ is an even number, and therefore

$$\langle a_1, \ldots, a_n\rangle = \langle 1, a_1\rangle + \ldots + \langle 1, a_n\rangle - m\langle 1, 1\rangle.$$

The kernel of the epimorphism e is called a *fundamental ideal* of the Witt ring, and is denoted by $I(K)$.

2.3 Fundamental ideal and Pfister forms

In this section we shall present the notion of n-fold Pfister form and focus on the relationship between the fundamental ideal of the Witt ring and Pfister forms. There will be presented these properties, which will be useful in later considerations.

Among the symmetric quadratic forms the *Pfister forms* are especially important. By a 1-*fold Pfister form* we mean a binary form $\langle 1, a\rangle$, where $a \in \dot{K}$. For $n \geq 2$, by an *n-fold Pfister form* we mean the tensor product of n 1-fold Pfister forms

$$\langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle,\ a_1, \ldots, a_n \in \dot{K},$$

and denote it by

$$\langle\langle a_1, \ldots, a_n\rangle\rangle := \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle.$$

Note that if $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ is the n-fold Pfister form, then $\dim \varphi = 2^n$, and if $n \geq 2$, then $\det \varphi = \dot{K}^2$.

Let φ be a Pfister form over a field K. It is easy to show that if the form $\langle 1, -1\rangle$ is a subform of φ, then φ is hyperbolic. Generally, for an arbitrary anisotropic form ψ we have

$$\langle 1, -1\rangle \otimes \psi = \psi \perp - \psi \sim 0.$$

To prove that property, it suffices to show that if (V, β_ψ) is a nonsingular symmetric bilinear space determined by the form ψ, then $(V, \beta_\psi) \perp (V, -\beta_\psi)$ is a hyperbolic space. Moreover, we have the following theorem (for a proof see [3]).

Theorem 2.5. *Let φ be an arbitrary Pfister form over a field K. Then φ is isotropic if and only if it is hyperbolic.*

Consider now the fundamental ideal $I(K)$ of the Witt ring of the field K. As we noticed in an earlier paragraph, $I(K)$ is additively generated by similarity classes of all 1-fold Pfister forms. Let us recall that by a product of two ideals I and J of the ring P we mean the ideal $I \cdot J$ generated by the set of all products $x \cdot y$, where $x \in I$, $y \in J$. In the case when $I = J$ the product of ideals $I \cdot J$ takes the following form

$$I^2 = \{\sum xy \colon x, y \in I\},$$

where all the sums $\sum xy$ contain a finite number of components.

So, let $\langle 1, x\rangle, \langle 1, y\rangle \in I(K)$. The product of these classes $\langle 1, x\rangle\langle 1, y\rangle = \langle 1, x, y, xy\rangle$ lies in $I^2(K)$. Since the classes $\langle 1, x\rangle, \langle 1, y\rangle$ are arbitrary, so the square of fundamental ideal $I^2(K)$ is additively generated by the set

$$\{\langle 1, x\rangle\langle 1, y\rangle : a, b \in \dot{K}\}.$$

Looking at higher powers of the fundamental ideal, we can show that for any $n \geq 2$, $I^n(K)$ is additively generated by similarity classes of all n-fold Pfister forms. The proof of the following proposition can be found in [3].

Proposition 2.6. *For any $n \in \mathbb{N}$ the ideal $I^n(K)$ is additively generated by the similarity classes of all n-fold Pfister forms.*

Analyzing forms from $I^n(K)$ we will reach Arason–Pfister Theorem known as Hauptsatz.

Theorem 2.7 (Arason–Pfister Hauptsatz)**.** *Let φ be a nonzero anisotropic form over a field K. If $\varphi \in I^n(K)$, then $\dim\varphi \geq 2^n$.*

Corollary 2.8. *Let φ be a 2^n-dimensional quadratic form over a field K. Then $\varphi \in I^n(K)$ iff there exists an n-fold Pfister form ψ and $a \in \dot{K}$ such that*

$$\varphi \cong a \cdot \psi$$

Proofs of the above theorem and corollary are beyond the scope of this work, so we omit them. They can be found in [4].

2.4 The set of nonzero values

In this paragraph we shall define a set of nonzero values of a quadratic form and present its properties. All of the proofs can be found in [3] or [5].

Let $\varphi \in K[X_1, \ldots, X_n]$ be an n-dimensional quadratic form over a field K. The element $a \in \dot{K}$ is called the *element represented by the form* φ, when there exist $x_1, \ldots, x_n \in K$ such that $\varphi(x_1, \ldots, x_n) = a$.

The *set of nonzero values* of the form φ will be denoted by $D_K(\varphi)$. Directly from this definition results that if the forms φ and ψ are equivalent, then $D_K(\varphi) = D_K(\psi)$.

Let (V, β_φ) be the bilinear space determined by the form φ. Then

$$D_K(\varphi) = \{\beta_\varphi(v, v) : v \in V\} \setminus \{0\}$$

The set $D_K(\varphi)$ will be also denoted by $D_K(V)$. In particular, if $\varphi = \langle a_1, \ldots, a_n\rangle$, then

$$D_K(\varphi) = \{a_1x_1^2 + \ldots + a_nx_n^2 : x_1, \ldots, x_n \in K\} \setminus \{0\}.$$

In this situation the set $D_K(\varphi) = D_K(\langle a_1, \ldots, a_n\rangle)$ will be denoted by $D_K(a_1, \ldots, a_n)$. The form φ will be called a *universal form*, if $D_K(\varphi) = \dot{K}$.

Let us take the element $a \in D_K(\varphi)$. Then there exists $v \in V$ such that $Q_\varphi(v) = a$. For an arbitrary $c \in \dot{K}$ we have $ac^2 = Q_\varphi(cv) \in D_K(\varphi)$. Therefore, if $a \in D_K(\varphi)$, then $a\dot{K}^2 \subset D_K(\varphi)$. Thus $D_K(\varphi)$ is a sum of some square classes of K. Moreover, the set $D_K(1, a)$ is a subgroup of the group $\dot{K}$ for any $a \in \dot{K}$. In the case of hyperbolic form the set $D_K(1, -1) = \dot{K}$. Thus, any hyperbolic form is universal. If the form $\varphi \cong \psi \perp \sigma$, then $D_K(\psi) \subset D_K(\varphi)$.

We shall now quote some properties of the set of nonzero values.

Theorem 2.9 (Representation Criterion)**.** *Let φ be a quadratic form over a field K and let $a \in \dot{K}$. If $a \in D_K(\varphi)$, then there exist elements $a_2, \ldots, a_n \in K$ such that $\varphi \cong \langle a, a_2, \ldots, a_n \rangle$.*

Proposition 2.10. *Let K be an arbitrary field and let $a, b, c \in \dot{K}$. Then,*

(i) $c \in D_K(a,b) \iff (a,b) \cong (c, abc)$;

(ii) $1 \in D_K(a,b) \iff (a,b) \cong (1, ab)$;

(iii) If $a + b \neq 0$, then $(a,b) \cong (a+b, ab(a+b))$.

Proof of the above proposition can be found in [3]. Now, let $\varphi \cong (a_1, \ldots, a_n)$, for $a_1, \ldots, a_n \in \dot{K}$. As we know $D_K(\varphi) \subseteq \dot{K}$. Let $a \in \dot{K}$ and suppose that $a\varphi \cong \varphi$. Hence, for an arbitrary element $c \in \dot{K}$ such that $a = \frac{c}{a_1}$ we have:

$$\varphi \cong a\varphi \cong (c, aa_2, \ldots, aa_n).$$

And therefore, $c \in D_K(\varphi)$. In this way, the following proposition has been proven.

Proposition 2.11. *If φ is a quadratic form over a field K such that $a\varphi \cong \varphi$ for every $a \in \dot{K}$. Then φ is a universal form.*

We shall need also these two following facts (for proofs see [3]).

Proposition 2.12. *Any binary form over a field K is universal iff any 1-fold Pfister form over K is universal.*

Proposition 2.13. *If $1 \in D_K(a,b)$ for every $a, b \in \dot{K}$, then $\dot{K} = D_K(a,b)$ for every $a, b \in \dot{K}$.*

By the above statements, every nonsingular isotropic form over K is universal (see [3]). Combining the condition of representation any element of the field K with quadratic forms we get the following theorem.

Theorem 2.14. *Let φ be a quadratic form over a field K and let $d \in \dot{K}$. Then $d \in D_K(\varphi)$ if and only if a form $\varphi \perp \langle -d \rangle$ is an isotropic form.*

Hence, by the above theorem, it results the following corollary, which combines the notions of isotropic and universal quadratic forms.

Corollary 2.15. *For any $n \in \mathbb{N}$ the following statements are equivalent:*

(i) Every n-dimensional quadratic form over K is a universal form;

(ii) Every $n+1$-dimensional quadratic form over K is an isotropic form.

Now, let φ be an arbitrary symmetric form. For $a \in \dot{K}$ we will consider the form $a\varphi$. It is easy to notice, by the definition of the set of nonzero values, that $D_K(a\varphi) = a \cdot D_K(\varphi)$. We will be most interested in the case when $\varphi \cong a\varphi$. We define the set

$$G_K(\varphi) = \{a \in \dot{K} \colon \varphi \cong a\varphi\}$$

as a *set of similarity factors of the form φ*. It turns out that the set $G_K(\varphi)$ is a subgroup of the group $\dot{K}$ and $\dot{K}^2 \subseteq G_K(\varphi)$. Moreover, $G_K(\varphi) \subseteq D_K(\varphi)$ if and only if $1 \in D_K(\varphi)$. In the case when the form φ is a Pfister form, sets $D_K(\varphi)$ and $G_K(\varphi)$ are equal.

Theorem 2.16. *Let φ be an arbitrary Pfister form over a field K. Then*

$$D_K(\varphi) = G_K(\varphi)$$

The proof can be found in [5].

3 u-invariant of nonreal and formally real fields

In this section we will present the definition of u-invariant of a field, its essential features and the connection between the notions of u-invariant and fundamental ideal $I(K)$ of a Witt ring.

First, let us recall that by a *formally real field* we mean a field such that $-1 \in K$ is not a sum of squares of elements from K. Otherwise, if -1 is a sum of squares of elements from K, then K is called a *nonreal field*.

3.1 u-invariant of a nonreal field

Let K be a nonreal field, that is -1 is a sum of squares of elements from K. So, there exist elements $a_1, \ldots, a_n \in K$ such that

$$-1 = a_1^2 + \ldots + a_n^2.$$

The smallest number $n \in \mathbb{N}$ with this property is called a *Pfister index* of the field K and it is denoted by $s(K)$.

Let M be the set of dimensions of all anisotropic quadratic forms over K.

$$M = \{\dim \varphi \colon \varphi \text{ is an anisotropic form over } K\}.$$

If M is bounded from above, we define *u-invariant* of a field K as a maximum of M and denote it by $u(K)$. If such a maximum does not exist, we put $u(K) = \infty$. That is,

$$u(K) = \max \{\dim \varphi \colon \varphi \text{ is an anisotropic form over } K\},$$

if the maximum exists, and $u(K) = \infty$ otherwise.

Example 3.1. Consider $K = \mathbb{C}$. The form $\varphi = \langle 1 \rangle$ is anisotropic over $\mathbb{C}$, while every form of dimension more than or equal to 2 is isotropic. Thus $u(\mathbb{C}) = 1$.

Due to connection between universal and isotropic forms, we quote the following proposition, where as the minimum of empty set we take ∞.

Proposition 3.2. *Let K be a nonreal field. Then*
(1) $u(K) = \min \{n \in \mathbb{N} \colon$ *every form φ of dimension $> n$ is isotropic*$\}$
(2) $u(K) = \min \{n \in \mathbb{N} \colon$ *every form φ of dimension $\geq n$ is universal*$\}$

Proof. Results from corollary 2.15.

□

Having defined the notion of u-invariant let us characterise its features. What values can the u-invariant of a field K take? Let's start our considerations with the case when $u(K) = 1$.

Theorem 3.3. *$u(K) = 1$ iff K is quadratically closed.*

Proof. ($\Leftarrow$) Suppose that K is quadratically closed, that is $\dot{K} = \dot{K}^2$. Let $a \in \dot{K}$. Then $D_K(a) = a\dot{K}^2 = a\dot{K} = \dot{K}$. Thus any quadratic form of dimension more than or equal to 1 is universal, which shows that $u(K) = 1$.

($\Rightarrow$) By the assumption that form $\langle 1 \rangle$ is universal, we have $D_K(1) = \dot{K}$. On the other hand $D_K(1) = \dot{K}^2$, so $\dot{K} = \dot{K}^2$. □

Considering the possible successive values of u-invariant we will show that $u(K) \notin \{3, 5, 7\}$ for any field K. But first let us present statements expressing the relationship between u-invariant and the fundamental ideal of the Witt ring of K.

Theorem 3.4. *Let K be an arbitrary field and let $I(K)$ be the fundamental ideal of the Witt ring of K. Then $I^2(K) = 0$ iff every binary form over K is universal.*

Proof. ($\Rightarrow$) Suppose $I^2(K) = 0$. For any $a, b \in \dot{K}$ we have $\langle 1, -a, -b, ab \rangle \in I^2(K)$. Therefore $\langle 1, -a, -b, ab \rangle = 0$. Hence

$$(1, -a, -b, ab) \cong (1, -1, 1, -1) \cong (a, -a, b, -b)$$

By Witt's Cancellation Theorem, we get $(1, ab) \cong (a, b)$. Thus, from Proposition 2.10, we conclude that $1 \in D_K(a, b)$. Referring to the Corollary 2.13 we get $D_K(a, b) = \dot{K}$. Hence every binary quadratic form over K is universal.

($\Leftarrow$) From Corollary 2.6 it follows that $I^2(K)$ is additively generated by the 2-fold products $\langle 1, a \rangle \otimes \langle 1, b \rangle$. It suffices to show that $\langle 1, a \rangle \otimes \langle 1, b \rangle = 0$ for every $a, b \in \dot{K}$. By the assumption we have $D_K(a, b) = \dot{K}$. In particular $-1 \in D_K(a, b)$. Then by Theorem 2.10 $(a, b) \cong (-1, -ab)$. So, we obtain the following class equality $\langle a, b \rangle = \langle -1, -ab \rangle$. Hence we get

$$\langle 1, a \rangle \langle 1, b \rangle = \langle 1, a, b, ab \rangle = \langle a, b \rangle + \langle 1, ab \rangle = \langle -1, -ab \rangle + \langle 1, ab \rangle = 0.$$

So, since $a, b \in \dot{K}$ are arbitrary, we get $I^2(K) = 0$. □

Theorem 3.5. *Let K be an arbitrary field. Then*

$$u(K) \leq 2 \iff I^2(K) = 0.$$

Proof. Follows from the previous theorem. □

On the basis of Theorems 3.4 and 3.5, we have the following conclusion.

Corollary 3.6. *Let K be an arbitrary nonreal field. Then the following conditions are equivalent:*

(*i*) $u(K) \leq 2$;

(*ii*) *Every binary form over K is universal;*

(*iii*) *Every 2-fold Pfister form over K is a hyperbolic form;*

(*iv*) $I^2(K) = 0$.

From this corollary, we know that if the u-invariant of a field K is less than or equal to 2, then $I^2(K) = 0$. It turns out that this case can be generalized, however the proof of this generalization requires a strong statement – Hauptsatz (Theorem 2.7).

Theorem 3.7. *Let K be an arbitrary field. If $u(K) < 2^n$, then $I^n(K) = 0$. Moreover, if $u(K) \leq 2^n$, then every nonzero anisotropic form in $I^n(K)$ is a universal n-fold Pfister form.*

Proof. First, suppose that $u(K) < 2^n$. Let φ be any n-fold Pfister form over K. Since $\dim \varphi = 2^n > u(K)$, by the definition of u-invariant, φ is an isotropic form. By Theorem 2.5, φ is also a hyperbolic form. Since $I^n(K)$ is additively generated by classes of n-fold Pfister forms and the form φ is arbitrary, so we have $I^n(K) = 0$.

Now suppose that $u(K) \leq 2^n$. Consider any nonzero anisotropic form $\psi \in I^n(K)$. Then $\dim \psi \leq 2^n$. From Theorem 2.7 it follows that $\dim \psi \geq 2^n$. Summarizing: $\dim \psi = 2^n \geq u(K)$, which implies that ψ is a universal form.

Now we have to show that ψ is an n-fold Pfister form. By Corollary 2.8, the form ψ can be presented as follows

$$\psi \cong a \cdot \sigma,$$

where $a \in \dot{K}$ and σ is an n-fold Pfister form. Since $\dim \sigma = 2^n \geq u(K)$, the form σ is a universal form. Thus $a \in D_K(\sigma) = G_K(\sigma)$. So, we obtain the following equalities

$$\psi \cong a \cdot \sigma \cong \sigma.$$

Therefore ψ is an n-fold Pfister form. □

The next theorem concerns the notion of fundamental ideal $I(K)$ and the issue of parity of u-invariant.

Theorem 3.8. *Let K be an arbitrary field. If $I^3(K) = 0$ and $1 < u(K) < \infty$, then u-invariant of K is even.*

Proof. Assume $I^3(K) = 0$ and $1 < u(K) < \infty$. Let $\sigma \in I^2(K)$. First we will show that the form σ is universal. Indeed, for $\sigma \in I^2(K)$ i $a \in \dot{K}$ we have

$$\langle 1, -a \rangle \cdot \sigma \in I^3(K) = 0.$$

From the above equality we get $\sigma \cong a \cdot \sigma$. Proposition 2.11 shows, in fact, that σ is a universal form.

Suppose, reasoning by contradiction, that $u = u(K)$ is odd. Let φ be a u-dimensional anisotropic form and $a = \det \varphi$. Since, by the definition of u-invariant, φ is a universal form, and we have:

$$\varphi \cong \langle a \rangle \perp \sigma_1 \cong \langle -a \rangle \perp \sigma_2,$$

for suitable $(u-1)$-dimensional forms σ_1, σ_2. Dimensions of σ_1 and σ_2 are $u-1$, so these forms are even-dimensional, and hence $\sigma_1, \sigma_2 \in I(K)$. Furthermore

$$a = \det \varphi = a \cdot \det \sigma_1 = -a \cdot \det \sigma_2.$$

Hence $\det \sigma_1 = 1$ and $\det \sigma_2 = -1$. But, since $\dim \sigma_1 = u - 1 = \dim \sigma_2$, then either σ_1 is in $I^2(K)$, or σ_2 is in $I^2(K)$. Assume that $\sigma_1 \in I^2(K)$. By the first part of the proof it follows that σ_1 is universal. In that case the form σ_1 presents the element $-a$. From the Representation Criterion $\sigma_1 \cong \langle -a\rangle \perp \rho$, for some form ρ. So,

$$\varphi = \langle a\rangle \perp \langle -a\rangle \perp \rho,$$

which implies that φ is an isotropic form, a contradiction. □

On the basis of statements presented so far we can now prove that $u(K) \notin \{3, 5, 7\}$.

Theorem 3.9. *Let K be an arbitrary field. Then $u(K) \notin \{3, 5, 7\}$.*

Proof. Note that we don't have to consider each of the above three cases separately. Theorem 3.7 shows that if $u(K) < 8$, then $I^3(K) = 0$. And from the previous theorem $u(K)$ is even, which leads to contradiction. □

Remark 3.10. Considered in this paragraph fields are nonreal. So, let $s(K) = n$ be a Pfister index of K. That means there is a presentation

$$-1 = a_1^2 + \ldots + a_n^2$$

for some $a_1, \ldots, a_n \in \dot{K}$, where $n \in \mathbb{N}$ is the smallest number with this property. Hence, and by the definition of u-invariant, we get the following inequality

$$s(K) \leq u(K).$$

Now we will present the connection between u-invariant of K and the cardinality of the group of square classes of K. But first, let us prove the following auxiliary lemma.

Lemma 3.11 (Kneser)**.** *Let K be a nonreal field and let φ be an anisotropic d-dimensional quadratic form over K such that $D_K(\varphi) \neq \dot{K}$. Then for any $a \in \dot{K}$*

$$D_K(\varphi) \subsetneq D_K(\varphi \perp \langle a\rangle).$$

Moreover, φ represents at least d distinct square classes of K.

Proof. Let $a \in \dot{K}$. Suppose that $D_K(\varphi) = D_K(\varphi \perp \langle a\rangle)$. Hence $a \in D_K(\varphi)$. Suppose $-1 = e_1^2 + \ldots + e_s^2$, where $s = s(K)$. Inductively on i we will show that $a(e_1^2 + \ldots + e_i^2) \in D_K(\varphi)$ for every $i \in \{1, \ldots, n\}$.

(1) Since by the assumption $a \in D_K(\varphi)$, so for $i = 1$ we have $ae_1^2 \in D_K(\varphi)$.

(2) Suppose that $i \geq 2$ and $a(e_1^2 + \ldots + e_{i-1}^2) \in D_K(\varphi)$. Then

$$a(e_1^2 + \ldots + e_{i-1}^2) + ae_i^2 \in D_K(\varphi \perp \langle a\rangle) = D_K(\varphi).$$

So, we showed that $a(e_1^2 + \ldots + e_i^2) \in D_K(\varphi)$ for every $i \in \{1, \ldots, s\}$, therefore $-a = a(e_1^2 + \ldots + e_s^2) \in D_K(\varphi)$. By Theorem 2.14 we conclude that the form $\varphi \perp \langle a\rangle$ is isotropic, and therefore also universal. Hence

$$D_K(\varphi) = D_K(\varphi \perp \langle a\rangle) = \dot{K},$$

a contradiction.

Now, by induction on d, we will show that the form φ represents at least d distinct square classes. Let $\varphi = (a_1, \ldots, a_d)$.

(i) For $d = 1$ the form $\varphi = (a_1)$ represents the class $a_1\dot{K}^2$.

(ii) Suppose that $d \geq 2$ and the form $\varphi' = (a_1, \ldots, a_{d-1})$ represents at least $d-1$ distinct square classes. Then

$$\varphi = (a_1, \ldots, a_{d-1}, a_d) = \varphi' \perp \langle a_d \rangle$$

The already proven part shows that $D_K(\varphi') \subsetneq D_K(\varphi' \perp \langle a_d \rangle)$, therefore $a_d \notin D_K(\varphi')$. □

Now, let φ be a quadratic form over an arbitrary field K. Let us consider the set of nonzero values of φ. If $D_K(\varphi)$ is a sum of m square classes, then the number m is denoted by $V(\varphi)$. If $m = \infty$, we put $V(\varphi) = \infty$. That is,

$$V(\varphi) = card\ \{a\dot{K}^2 \colon a \in D_K(\varphi)\}.$$

Notice that if the form φ is universal, that is $D_K(\varphi) = \dot{K}$, then $V(\varphi) = q(K)$. Therefore, by Kneser's Lemma 3.11, for an arbitrary anisotropic form φ we have $\dim \varphi \leq V(\varphi)$.

Theorem 3.12 (Kneser)**.** *For any nonreal field K we have the following inequality*

$$u(K) \leq q(K).$$

Proof. Proof of Kneser's Theorem follows directly from Kneser's Lemma 3.11 and the definition of u-invariant. Namely, for an arbitrary anisotropic form φ we have the following inequality:

$$\dim \varphi \leq V(\varphi).$$

So, if φ ranges over all anisotropic forms over K, then:

$$\max\ (\dim \varphi) \leq q(K).$$

And therefore,

$$u(K) \leq q(K),$$

as desired. □

Kneser's Theorem and the earlier remark leads us to relationships between $u(K)$, $s(K)$ and $q(K)$. Namely, there are such inequalities

$$s(K) \leq u(K) \leq q(K),$$

from which we can draw the following corollary.

Corollary 3.13. *Witt ring $W(K)$ of K is finite if and only if $s(K)$ and $q(K)$ are both finite.*

Proof. ($\Rightarrow$) Assume $W(K)$ is finite. Let $a, b \in \dot{K}$ and $a\dot{K}^2 \neq b\dot{K}^2$. Note that quadratic forms (a) and (b) are not isometric, and therefore, because they have equal dimensions, present different elements of the Witt ring $\langle a \rangle \neq \langle b \rangle$.

Suppose that the group $\dot{K}/\dot{K}^2$ is infinite. Then the Witt ring $W(K)$ contains an infinite subset $\{\langle a \rangle \in W(K) \colon a \in \dot{K}\}$, which gives a contradiction.

Now suppose that the Pfister index $s(K)$ of K is infinite. Then, for every $n \in \mathbb{N}$, the element $-1 \notin D_K(n(1))$. Thus, the form $n(1)$ is anisotropic. Moreover, if $n < m$, then $n(1) \not\cong m(1)$. Indeed, if $n(1) \cong m(1)$, then the form $(m-n)(1)$ would be isotropic, which, as we have already noticed, is imposible. So $W(K)$ contains an infinite set of elements $\{n\langle 1\rangle : n \in \mathbb{N}\}$, a contradiction.

($\Leftarrow$) Let $\{a_1\dot{K}^2, \ldots, a_q\dot{K}^2\}$ be all of the distinct square classes. From the previous theorem $u = u(K) \leq q$. Thus every anisotropic form over K has dimension less than or equal to q.

Notice that for $k \in \mathbb{N}$, every k-dimensional form, uniquely up to isometry, appears as $(a_{i_1}, \ldots, a_{i_k})$, where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, q\}$. Thus,uniquely up to isometry, there are at most $\left(\frac{q+k-1}{k}\right)$ k-dimensional forms over K.

On the other hand, every nonzero element of the Witt ring is uniquely determined by the isomerty class of an anisotropic form. Therefore, every element of $W(K)$ is uniquely determined by some form of dimension less than or equal to u. Finally, the ring $W(K)$ contains at most

$$1 + \sum_{k=1}^{n} \left(\frac{q+k-1}{k}\right)$$

elements. □

Theorem 3.14. *Let K be a nonreal field such that $q(K) < \infty$. Then*

$$|W(K)| \geq u(K) \cdot q(K)$$

Proof. Assume that $q(K) < \infty$. By Theorem 3.12, we have $u(K) < \infty$. If $s(K) = \infty$, then $W(K)$ is an infinite ring and the thesis is obvious. Suppose that $s(K) < \infty$. Then $W(K)$ is finite.

Let $u = u(K)$ and φ be an anisotropic form over K of dimension $\dim \varphi = u$. Assume that ψ is a subform of φ of odd dimension. Then, of course, ψ is an anisotropic form over K. Moreover, for every $a \in \dot{K}$ we have

$$\det(\langle a\rangle\psi) = a^{\dim \psi} \cdot \det \psi = a \det \psi \dot{K}^2.$$

Thus, if $a\dot{K}^2 \neq b\dot{K}^2$, then $\det(\langle a\rangle\psi)\dot{K}^2 \neq \det(\langle b\rangle\psi)\dot{K}^2$. In this case, forms $\langle a\rangle\psi$ and $\langle b\rangle\psi$ are not isometric, and because they have equal dimensions and are anisotropic, they represent different elements in $W(K)$. So, for each odd number $k \leq u$ there are at least $q(K)$ elements in $W(K)$ represented by anisotropic k-dimensional forms. Suppose that there are m odd numbers less than or equal to u. Then, there are at least $m \cdot q(K)$ different elements in $W(K)$ represented by odd-dimensional forms.

On the other hand, we know that $W(K)/I(K) \cong \mathbb{Z}_2$. Therefore, the number of elements of $W(K)$ represented by even-dimensional forms is equal to the number of elements represented by odd-dimensional forms, and consequently

$$|W(K)| \geq 2 \cdot m \cdot q(K).$$

If u is even, then $m = \frac{u}{2}$, and hence

$$|W(K)| \geq u \cdot q(K).$$

If u is odd, then $m = \frac{1}{2}(u+1)$, and hence

$$|W(K)| \geq (u+1) \cdot q(K).$$

□

3.2 General u-invariant

Suppose that K is a formally real field. Then -1 is not the sum of squares of elements of K. So, for every $m \in \mathbb{Z}$, the form $m \cdot \langle 1 \rangle$ is anisotropic. And hence, $u(K) = \infty$. Therefore, the aim of this paragraph is to generalize the definition of u-invariant over the formally real fields.

First, let us present necessary definitions. Let $W(K)$ be a Witt ring of a field K. Consider an arbitrary form $\varphi \in W(K)$. The form φ is called a *torsion form*, if the following condition is fulfilled:

$$\exists_{n \in \mathbb{N}} \quad n\varphi = 0 \in W(K).$$

It is easy to notice that if φ and ψ are torsion forms, then forms $\varphi \perp \psi$ and $\varphi \otimes \psi$ are torsion as well. Then, a subgroup $W_t(K)$ of the Witt ring $W(K)$ equal to

$$W_t(K) = \{\varphi \in W(K) \colon \exists n \in \mathbb{N} \quad n\varphi = 0\}$$

is called a *torsion subgroup* of additive group of the Witt ring $W(K)$.

Let K be an arbitrary field. Let M be the set of dimensions of all anisotropic torsion forms over K,

$$M = \{\dim \varphi \colon \varphi \text{ is an anisotropic form, } \varphi \in W_t(K)\}.$$

If M is bounded from above, we define *general u-invariant* of a field K as a maximum of M and denote it by $u_g(K)$. If such a maximum does not exist, we put $u_g(K) = \infty$. That is

$$u_g(K) = \max\{\dim \varphi \colon \varphi \text{ is an anisotropic form, } \varphi \in W_t(K)\},$$

if the maximum exists, and $u_g(K) = \infty$ otherwise.

Note that if K is the nonreal field, the above definition of u-invariant coincides with the definition introduced in the earlier paragraph. This follows from the fact that any quadratic form over the nonreal field is a torsion form, and then $W_t(K) = W(K)$ and $u_g(K) = u(K)$. While, if K is the formally real field, then $W_t(K) \subseteq I(K)$. So, any torsion form is even-dimensional. By the above considerations, in the later part of this paper, the u-invariant of the formally real field will be denoted by $u(K)$. That also leads us to the following corollary.

Corollary 3.15. *Let K be a formally real field and let $u(K) < \infty$. Then $u(K)$ is even.*

Theorem 3.16. *Let γ be a binary form and let φ be a form of dimension more than or equal to 2. If the form $\gamma \cdot \varphi$ is isotropic, then the form φ contains a binary form β such that $\gamma \cdot \beta = 0 \in W(K)$.*

Proof. Let φ be a quadratic form such that $\dim \varphi \geq 2$ and let β be a binary subform of the form φ. Note that if the form φ is isotropic, as a subform β we can put $\beta = \langle 1, -1 \rangle$. So, let φ be an anisotropic form and let $\gamma = \langle s, t \rangle$, where $s, t \in \dot{K}$, be a binary form. Suppose that the form $\gamma \cdot \varphi$ is isotropic. By the assumption, the form

$$\gamma \cdot \varphi \cong \langle s \rangle \varphi \perp \langle t \rangle \varphi$$

is isotropic, so $sx + ty = 0$ for some $x, y \in D_K(\varphi)$. Let $\varphi \cong \langle x \rangle \perp \varphi_1$ for some form φ_1. Then, we get the following equality $y = xw^2 + z$, where $w \in \dot{K}$ and

$z \in D_K(\varphi_1) \cup \{0\}$. Let us consider the following two cases.

(1) If $z = 0$, then $y = xw^2$ for $w \in \dot{K}$. Therefore, x and y represent the same element of the group of square classes $\dot{K}/\dot{K}^2$. Hence, and by the equality $sx = -ty$, it follows that the form γ is hyperbolic. So, in this case, as β we can put any binary subform of the form φ.

(2) Suppose that $z \neq 0$. Then, the form φ_1 can be represented as $\varphi_1 \cong \langle z \rangle \perp \varphi_2$, for suitable form φ_2. Therefore,

$$\varphi \cong \langle x \rangle \perp \varphi_1 \cong \langle x, z \rangle \perp \varphi_2.$$

So, the form φ consists the binary subform $\beta \cong \langle x, z \rangle$. Note that, since $y = xw^2 + z$, then $y \in D_K(x, z)$. By Theorem 2.10 we have $\langle x, z \rangle \cong \langle y, xyz \rangle$. Moreover, the following sequence of isometries takes place:

$$\langle -t \rangle \cdot \beta \cong \langle -ty, -ty \cdot xz \rangle \cong \langle sx, sx \cdot xz \rangle \cong \langle sx, sz \rangle \cong \langle s \rangle \cdot \beta.$$

So, $\langle s \rangle \cdot \beta \cong \langle -t \rangle \cdot \beta$. And hence, $\langle s, t \rangle \cdot \beta = \gamma \cdot \beta = 0$. □

By the above theorem we get the following corollary.

Corollary 3.17. *Let γ be a binary form such that $\gamma \not\cong \langle 1, -1 \rangle$ and let φ be an arbitrary quadratic form. If $\gamma \cdot \varphi = 0 \in W(K)$, then $\dim \varphi = 2r$ for some r, and there exists an isometry*

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r,$$

where $\beta_1, \ldots, \beta_r$ are binary forms such that $\gamma \cdot \beta_i = 0$ for $i = 1, \ldots, r$.

Proof. Let γ be a binary form such that $\gamma \not\cong \langle 1, -1 \rangle$ and let φ be an arbitrary form. Suppose that $\gamma \cdot \varphi = 0$. By induction on $\dim \varphi$, we will show the thesis.

Notice that the case when $\dim \varphi = 1$ is impossible, because γ is not a hyperbolic plane. So, $\gamma \langle a \rangle \neq 0$ for every $a \in \dot{K}$.

In the case when $\dim \varphi = 2$ the thesis is obvious. So, assume that $\dim \varphi > 2$. Since $\gamma \cdot \varphi = 0 \in W(K)$, so the form $\gamma \cdot \varphi$ is isotropic. By the previous theorem, it follows that $\varphi = \beta_1 \perp \varphi_1$, where $\dim \beta_1 = 2$ and $\gamma \cdot \beta_1 = 0$. Therefore, we have $\gamma \cdot \varphi = \gamma \cdot \varphi_1$, where $\dim \varphi_1 = \dim \varphi - 2 < \dim \varphi$. By Induction Thesis, it follows that $\varphi_1 \cong \beta_2 \perp \ldots \perp \beta_r$, where $\beta_2, \ldots, \beta_r$ are binary forms such that $\gamma \cdot \beta_i = 0$ for $i = 2, \ldots, r$. Hence, we have:

$$\varphi = \beta_1 \perp \ldots \perp \beta_r.$$

So, by induction, we get the thesis. □

By Theorem 3.16 and the previous corollary, it follows the case when $\gamma \cong \langle 1, 1 \rangle$, which will be useful in the later considerations.

Proposition 3.18. *Let φ be a quadratic form over a field K. Then, there exist $r \geq 0$, binary forms $\beta_1, \ldots, \beta_r$ over K and a form φ_0 over K such that*

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r \perp \varphi_0,$$

where $2\beta_i = 0$ for $i = 1, \ldots, r$ and either $2\varphi_0$ is anisotropic, or else $\dim \varphi = 1$.

Proof. We will induct on the dimension m of the form φ. If $\dim \varphi = 1$, existence of the decomposition is obvious. Assume that $\dim \varphi \geq 2$. If the form 2φ is anisotropic, then we put $r = 0$ and $\varphi = \varphi_0$.

Now assume that $2\varphi = \langle 1, 1 \rangle \varphi$ is isotropic. Then, by Theorem 3.16, the form φ contains a binary subform β_1 such that $2\beta_1 = 0$. So, $\varphi = \beta_1 \perp \varphi_1$ for some form φ_1. Hence, $2\varphi = 2\beta_1 \perp 2\varphi_1$, and in the Witt ring of K we have $2\varphi = 2\varphi_1$. Naturally, $\dim \varphi_1 < \dim \varphi$. So, by Induction Thesis, the form φ_1 has a decomposition $\varphi_1 = \beta_2 \perp \ldots \perp \beta_r \perp \varphi_0$, where $2\beta_i = 0$ for $i = 1, \ldots, r$ and either the form $2\varphi_0$ is anisotropic, or else $\dim \varphi_0 = 1$. Then $\varphi = \beta_1 \perp \ldots \perp \beta_r \perp \varphi_0$, as desired. □

Now, let us recall that by a *pythagorean field* we mean a field K such that every element $a \in K$ is a sum of two squares of elements from K. It is easy to prove that if K is pythagorean field, then $u(K) \leq 1$ (see [2]).

Presented in the previous paragraph Kneser's theorem concerns only the nonreal case. Therefore, we shall generalize Theorem 3.12 to the formally real case. But first, let us prove the following theorem.

Theorem 3.19. *Let K be an arbitrary field such that $u = u(K) < \infty$ and let $\varphi \in W_t(K)$ be any u-dimensional anisotropic form. Then*

$$V(\varphi) \geq u.$$

Proof. Suppose that $u = u(K) < \infty$. Let φ be a u-dimensional anisotropic torsion form. If $u(K) = 1$, then K is a quadratically closed field and thesis is obvious. So, we may assume that $u = u(K) \geq 2$. Therefore, K is not a pythagorean field. Let

$$\varphi \cong \beta_1 \perp \ldots \perp \beta_r \perp \varphi_0$$

be a decomposition of the form φ such as in Proposition 3.18. We will show that $V(\varphi) \geq u$. The proof will be presented in the following four steps.

(1) As already noticed either $2\varphi_0$ is anisotropic, or else $\dim \varphi_0 = 1$. In both cases $\dim 2\varphi_0 \leq u$. And hence,

$$4r = 2 \cdot 2r = 2 \cdot (\dim \varphi - \dim \varphi_0) \geq 2u - u = u.$$

(2) We shall show that the sets $D(\beta_1), \ldots, D(\beta_r), D(\varphi_0)$ are mutually disjoint subsets of $D(\varphi)$. Indeed, reasoning by contradiction, suppose that there exists an element $x \in D(\beta_i) \cap D(\beta_j)$ for $i \neq j$. Let $\beta_i \cong \langle x, y \rangle$ for some $y \in \dot{K}$ and let $\beta_j \cong \langle x, z \rangle$ for some $z \in \dot{K}$. Moreover, we have the following isometry:

$$\beta_i \cong \langle -1 \rangle \beta_i \cong \langle -x, -y \rangle.$$

Considering the form $\beta_i \perp \beta_j$ we get

$$\beta_i \perp \beta_j \cong \langle -x, -y \rangle \perp \langle x, z \rangle \cong \langle 1, -1 \rangle \perp \langle -y, z \rangle.$$

So, the form $\beta_i \perp \beta_j$ is isotropic, a contradiction. Similarly, the sets $D(\beta_i)$ and $D(\varphi_0)$ are disjoint.

(3) Assume that $s(K) \leq 2$. We will show that $V(\varphi) \geq u$. Let $\varphi_0 \cong \langle a_{2r+1}, \ldots, a_u \rangle$. Notice that the elements $a_{2r+1}, \ldots, a_u$ represent different square classes. Indeed, if $a_i \dot{K}^2 = a_j \dot{K}^2$ for $i \neq j$, then

$$\beta_{r+1} = \langle a_i, a_j \rangle \cong 2\langle a_i \rangle.$$

Therefore $2\beta_{r+1} = 0$, which contradicts with the choice of r. And hence, it follows the inequality:

$$V(\varphi_0) \geq \dim \varphi_0 = u - 2r.$$

Now, we will show that $V(\beta_i) \geq 2$ for every $1 \leq i \leq r$. Let $\beta_i \cong \langle x, y\rangle$. If $x\dot{K}^2 \neq y\dot{K}^2$, then of course $V(\beta_i) \geq 2$. So, suppose that $x\dot{K}^2 = y\dot{K}^2$. In this case $\beta_i \cong \langle x, x\rangle$. Since K is not a pythagorean field, so there exist elements $a, b \in \dot{K}$ such that $a^2 + b^2$ is not a square. Hence, the set $D(\beta_i)$ contains at least x and $x(a^2 + b^2)$, where x and $x(a^2 + b^2)$ represent distinct square classes. Therefore, also in this case $V(\beta_i) \geq 2$. Hence, and by the step (2), we have the following sequence of inequalities:

$$V(\varphi) \geq \sum_{i=1}^{r} V(\beta_i) + V(\varphi_0) \geq 2r + (u - 2r) = u.$$

(4) Now assume that $s(K) > 2$. We shall show that $V(\beta_i) \geq 4$ for every $i = 1, \ldots, r$. Let $\beta_i \cong \langle x, y\rangle$. Since $4\langle 1\rangle \neq 0$ and $2\beta_i = 0$, so $x\dot{K}^2 \neq y\dot{K}^2$. Therefore, the set $D(\beta_i)$ contains elements $x, y, -x, -y$, which represent four distinct square classes. So, $V(\beta_i) \geq 4$.

Using the fact that $V(\beta_i) \geq 4$, we shall show that $V(\varphi) \geq u + 2$. For this purpose, let us consider the following two cases.

(i) $2r = u$

Then $\varphi \cong \beta_1 \perp \ldots \perp \beta_r$. Hence, and by the step (2), we get the following sequence of inequalities:

$$V(\varphi) \geq \sum_{i=1}^{r} V(\beta_i) \geq 4r = 2u \geq u + 2.$$

(ii) $2r < u$

As already noticed, K is not a pythagorean field. So, if $\dim \varphi_0 \geq 2$, then $V(\varphi_0) \geq 2$. By the steps (1) and (2) we get the inequality:

$$V(\varphi) \geq \sum_{i=1}^{r} V(\beta_i) + V(\varphi_0) \geq 4r + 2 \geq u + 2.$$

While, if $\dim \varphi_0 = 1$, then u is an odd number. So, by the step (1) it follows that $4r \geq u + 1$. Finally,

$$V(\varphi) \geq 4r + V(\varphi_0) \geq 4r + 1 \geq u + 2.$$

□

Now, by the above theorem, we may extend Kneser's Theorem 3.12 to the case of arbitrary fields.

Corollary 3.20. *Let K be an arbitrary field. Then $u(K) \leq q(K)$.*

Proof. We may assume that $q(K) < \infty$. Then, the Witt ring $W(K)$ is finitely generated abelian group. The Structure Theorem for Abelian Groups states that every finitely generated abelian group G is isomorphic to a direct sum

$$\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^n,$$

where $k \geq 0$, $n \geq 0$ and $n_1, \ldots, n_k$ are powers of prime numbers (see [1]). Furthermore, the values of n, $n_1, \ldots, n_k$ are, up to rearranging the indices, uniquely determined by G. Therefore, we conclude that the torsion subgroup $W_t(K)$ of the Witt ring $W(K)$ is isomorphic to a group $\mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, and hence also finite. So, the set of dimensions of all anisotropic torsion forms is finite, which implies that $u(K) < \infty$. The inequality $u(K) \leq q(K)$ follows from Theorem 3.19. □

Corollary 3.21. *Let K be an arbitrary field and let $s(K) > 2$ and $u(K) > 1$. Then,*

$$V(\varphi) \geq u + 2.$$

Proof. The proof follows from the step (4) of Theorem 3.19. □

4 Conclusion

This paper is based on my Master Thesis, [3], written by supervision of dr hab. Alfred Czogała. Apart from notions and theorems presented in the paper, we also introduce a definition of *system u-invariant* as a u-invariant of the system of n quadratic forms.

My Master Thesis concerns many examples of u-invariants of selected fields. Namely, we prove that u-invariant of a finite field is equal to 2. By Tsen–Lang Theorem, we prove that u-invariant of a field of transcendence degree 1 over an algebraically closed field is also equal to 2. Moreover, we discuss complete discretely valuated field F with residue class field K of characteristic different from 2, and prove that $u(F) = 2 \cdot u(K)$. As an example of that field we give the field of formal Lauren series $K((t))$ and show that $u(\mathbb{C}((t_1))((t_2))\ldots((t_n))) = 2^n$.

A significant part of my Master Thesis is devoted to u-invariant of finite extensions. We prove that if L is a finite extension over a field K, then $u(L) \leq \frac{1}{2}(n+1) \cdot u(K)$. Furthermore, we show that if K is an arbitrary nonreal field and $L = K(\sqrt{a})$, then $u(L) \leq \frac{3}{2} \cdot u(K)$. First, we prove this theorem by properties of system u-invariant, and then, it is also proven by construction of Scharlau's Transfer for finite extensions.

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