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A note on bisimulations of Kripke models

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A note on bisimulations of Kripke models

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Abstract

We introduce the notion of bisimulation. Our discussion focuses on two cases: the propositional intuitionistic logic case and the first-order case. In both cases we present the theorem which states that bisimulation implies logical equivalence. Further, we present our contribution to the research of the inverse theorem.

Keywords: Kripke model, bisimulation, logical equivalence.

1 Introduction

The notion of bisimulation has been introduced by David Park in 1981 to test whether two processes behave the same ([4]). Orginally discovered in Computer Science, bisimulation nowadays is employed in many fields. Today it is used in a number of areas of Computer Science such as functional languages, data types, databases, program analysis, to name but a few. Growing interests in this notion led to the discovery of bisimulation in Modal Logic and Set Theory (see [5]). Finally, the notion was introduced into first-order logic, and found a straightforward game-theoretical interpretation. These intensive studies of bisimulation rivet my attention to it.

The aim of the paper is to present the notion of bisimulation in the case of the propositional intuitionistic logic and the first-order case, and to reveal the relationship between bisimulation and the notion of logical equivalence.

This paper is organised as follows. Section 2 is devoted to the propositional intuitionistic logic case. It contains notions such as propositional formula, complexity of a formula, Kripke model as well as logical equivalence, and bisimulation. First, we present a theorem which states that layered bisimulation between worlds of two Kripke models implies their logical equivalence, up to certain formula's complexity. Then, we invert the theorem.

In Section 3 we consider the first-order case. After introducing necessary notions, we present the main theorem of Section 3. Considering formulas of a specified number of implications and quantifiers, we prove that bisimulation between worlds of first-order Kripke models implies their logical equivalence.

As we want to prove the inverse theorem, next, in Section 4, we present an outline of the proof and discuss partial results that will inspire further work.

2 Propositional Case and Layered Bisimulation

This chapter introduces basic notions such as Kripke model, bisimulation and logical equivalence. We reveal the relationship between those notions in the case of the propositional intuitionistic logic. All definitions of this chapter can be found in [6].

Consider a set $\mathcal{P} = \{p, q, r, \ldots\}$ of propositional variables. We define a set $\mathcal{L}(\mathcal{P})$ of *propositional formulas* as the smallest set such that

- $\mathcal{P} \subseteq \mathcal{L}(\mathcal{P}), \bot, \top \in \mathcal{L}(\mathcal{P})$
- if $\varphi, \psi \in \mathcal{L}(\mathcal{P})$, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi \in \mathcal{L}(\mathcal{P})$

The negation is defined by means of the implication and falsum as $\neg\varphi := \varphi \to \bot$.

Further, we define a mapping $i\colon \mathcal{L}(\mathcal{P}) \to \omega$ that describes formula's *complexity*

- $i(p) := i(\bot) := i(\top) := 0$
- $i(\varphi \wedge \psi) := i(\varphi \vee \psi) := \max(i(\varphi), i(\psi))$
- $i(\varphi \to \psi) := \max(i(\varphi), i(\psi)) + 1$

We will consider formulas of certain complexity. Thus, for $\alpha \in \omega$ we define a set $I_\alpha(\mathcal{P}) := \{\varphi \in \mathcal{L}(\mathcal{P})\colon i(\varphi) \leq \alpha\}$ of propositional formulas of complexity less than or equal to α.

A *Kripke model* is a structure $\mathcal{K} = \langle K, \leq, \mathcal{P}, \Vdash \rangle$, where K is a non-empty set of elements called *worlds*, $\leq$ is a partial order on K, $\mathcal{P}$ is a (possibly empty) set of propositional variables and $\Vdash$ is a *forcing* relation in $\mathcal{K} \times \mathcal{P}$ such that

$$(k \Vdash p \ \wedge\ k' \geq k) \ \Rightarrow\ k' \Vdash p.$$

We define a set $PV_{\mathcal{K}}(k) := \{p \in \mathcal{P}\colon k \Vdash_{\mathcal{K}} p\}$ of propositional variables forced at the world k and inductively, over the construction of a formula, extend the definition of $\Vdash$ onto the set of propositional formulas $\mathcal{L}(\mathcal{P})$. Thus, for any $k \in K$,

$$k \nVdash_{\mathcal{K}} \bot \text{ and } k \Vdash_{\mathcal{K}} \top$$

$$k \Vdash_{\mathcal{K}} \varphi \wedge \psi \iff k \Vdash_{\mathcal{K}} \varphi \text{ and } k \Vdash_{\mathcal{K}} \psi$$

$$k \Vdash_{\mathcal{K}} \varphi \vee \psi \iff k \Vdash_{\mathcal{K}} \varphi \text{ or } k \Vdash_{\mathcal{K}} \psi$$

$$k \Vdash_{\mathcal{K}} \varphi \to \psi \iff \forall_{k' \geq k}(k' \Vdash_{\mathcal{K}} \varphi \Rightarrow k' \Vdash_{\mathcal{K}} \psi)$$

Note that by the above definition

$$k \Vdash_{\mathcal{K}} \neg\varphi \iff \forall_{k' \geq k}\ k' \nVdash_{\mathcal{K}} \varphi$$

The Kripke model $\mathcal{K}$ is said to be *finite* if the set of worlds K is finite. We define the *theory* of a world k in a Kripke model $\mathcal{K}$ as the set of formulas forced at k, i.e. $Th_{\mathcal{K}}(k) := \{\varphi\colon k \Vdash_{\mathcal{K}} \varphi\}$. Similarly, if we consider formulas of complexity less than or equal to α, we define a theory $Th_{I_\alpha(\mathcal{P})}(k) := \{\varphi \in I_\alpha(\mathcal{P})\colon k \Vdash_{\mathcal{K}} \varphi\}$.

Now, consider Kripke models $\mathcal{K} = \langle K, \leq, \mathcal{P}, \Vdash \rangle$ and $\mathcal{M} = \langle M, \leq, \mathcal{P}, \Vdash \rangle$. For worlds $k \in K$, $m \in M$ we define a relation $\equiv_\alpha$ as follows

$$k \equiv_\alpha m \iff Th_{I_\alpha(\mathcal{P})}(k) = Th_{I_\alpha(\mathcal{P})}(m) \iff \forall_{\varphi \in I_\alpha(\mathcal{P})}(k \Vdash_{\mathcal{K}} \varphi \iff m \Vdash_{\mathcal{M}} \varphi)$$

We say that k and m are *α-equivalent* if and only if $k \equiv_\alpha m$.

A natural question that arises is characterising when two worlds $k \in K$ and $m \in M$ force the same set of formulas. A notion that describes α-equivalence between worlds is a notion of *layered bisimulation.* Let $\mathcal{K} = \langle K, \leq, \mathcal{P}, \Vdash \rangle$ and $\mathcal{M} = \langle M, \leq, \mathcal{P}, \Vdash \rangle$ be Kripke models. Let ω^∞ be a set $\omega \cup \{\infty\}$. A *layered bisimulation* (*l-bisimulation*) $\sim$ between $\mathcal{K}$ and $\mathcal{M}$ is a ternary relation on K, ω^∞ and M, satisfying the conditions specified below. We will consider $\sim$ also as an ω^∞-indexed set of binary relations between K and M writing $k \sim_\alpha m$ whenever the triple (k, α, m) is in that relation. The conditions are as follows:

(*i*) $k \sim_\alpha m \Rightarrow PV_{\mathcal{K}}(k) = PV_{\mathcal{M}}(m)$

(*ii*) $k \sim_{\alpha+1} m \Rightarrow$ for every $k' \geq k$ there is $m' \geq m$ such that $k' \sim_\alpha m'$

(*iii*) $k \sim_{\alpha+1} m \Rightarrow$ for every $m' \geq m$ there is $k' \geq k$ such that $k' \sim_\alpha m'$

Conditions (*ii*) and (*iii*) specified above are called the 'zig' and the 'zag' property of the l-bisimulation respectively.

The following theorem reveals the relationship between notions of l-bisimulation and logical equivalence, taking the formula's complexity into consideration. Its shorter proof can be found in [6].

Theorem 2.1. *Let $\sim$ be an l-bisimulation between Kripke models $\mathcal{K}$ and $\mathcal{M}$. Then, for $k \in K$ and $m \in M$*

$$k \sim_\alpha m \Rightarrow k \equiv_\alpha m.$$

Proof. Induction on α and the complexity of $\varphi \in I_\alpha(\mathcal{P})$. Let $\alpha = 0$ and suppose $k \sim_0 m$. Then, the set $I_0(\mathcal{P})$ is a set of implicationless formulas. By the assumption $PV_{\mathcal{K}}(k) = PV_{\mathcal{M}}(m)$, so if $\varphi = p$ is a propositional variable, then $(k \Vdash_{\mathcal{K}} p \Leftrightarrow m \Vdash_{\mathcal{M}} p)$. Furthermore, $(k \nVdash_{\mathcal{K}} \bot \Leftrightarrow m \nVdash_{\mathcal{M}} \bot)$ and $(k \Vdash_{\mathcal{K}} \top \Leftrightarrow m \Vdash_{\mathcal{M}} \top)$.

If $\varphi = \varphi_1 \vee \varphi_2$, then, by the definition of forcing and the induction hypothesis,

$$k \Vdash_{\mathcal{K}} \varphi \iff k \Vdash_{\mathcal{K}} \varphi_1 \vee \varphi_2 \iff k \Vdash_{\mathcal{K}} \varphi_1 \text{ or } k \Vdash_{\mathcal{K}} \varphi_2$$

$$\iff m \Vdash_{\mathcal{M}} \varphi_1 \text{ or } m \Vdash_{\mathcal{M}} \varphi_2 \iff m \Vdash_{\mathcal{M}} \varphi_1 \vee \varphi_2 \iff m \Vdash_{\mathcal{M}} \varphi.$$

The case of $\varphi = \varphi_1 \wedge \varphi_2$ is analogous.

Now, assume the implication holds for $\alpha > 0$ and suppose $k \sim_{\alpha+1} m$. First, let $\varphi = \psi_1 \to \psi_2$, where $\varphi \in I_{\alpha+1}(\mathcal{P})$. Suppose $k \nVdash_{\mathcal{K}} \psi_1 \to \psi_2$. Then, by the definition of forcing, for some $k' \geq k$ we have $k' \Vdash_{\mathcal{K}} \psi_1$ and $k' \nVdash_{\mathcal{K}} \psi_2$. Notice that $\psi_1, \psi_2 \in I_\alpha(\mathcal{P})$. Moreover, since $k \sim_{\alpha+1} m$, then, by the 'zig' property, there is $m' \geq m$ such that $k' \sim_\alpha m'$. Hence, by the induction hypothesis applied for α we get $m' \Vdash_{\mathcal{M}} \psi_1$ and $m' \nVdash_{\mathcal{M}} \psi_2$. Thus, $m \nVdash_{\mathcal{M}} \psi_1 \to \psi_2$. By analogy, using the 'zag' property, we prove the reverse implication.

Now, let $\varphi = \psi_1 \vee \psi_2$, where $\varphi \in I_{\alpha+1}(\mathcal{P})$. First, assume $i(\psi_1) > i(\psi_2)$ and consider two cases. If $i(\varphi) > i(\psi_1)$, then, by induction hypothesis,

$$k \Vdash_{\mathcal{K}} \varphi \iff k \Vdash_{\mathcal{K}} \psi_1 \vee \psi_2 \iff k \Vdash_{\mathcal{K}} \psi_1 \text{ or } k \Vdash_{\mathcal{K}} \psi_2$$

$$\iff m \Vdash_{\mathcal{M}} \psi_1 \text{ or } m \Vdash_{\mathcal{M}} \psi_2 \iff m \Vdash_{\mathcal{M}} \psi_1 \vee \psi_2 \iff m \Vdash_{\mathcal{M}} \varphi.$$

Now, assume $i(\varphi) = i(\psi_1)$. If $\rightarrow$ is not the main connective of ψ_1, we may assume that ψ_1 is a $\wedge$, $\vee$-combination of formulas of the form $\gamma \rightarrow \delta$ such that $i(\gamma \rightarrow \delta) = i(\varphi)$. Since $\wedge$ and $\vee$ do not change the formula's complexity, the only subformula of ψ_1 that we have to consider is $\gamma \rightarrow \delta$, where $i(\gamma), i(\delta) < i(\psi_1)$. Then, by the previous part of the proof, we have

$$k \Vdash_{\mathcal{K}} \gamma \rightarrow \delta \iff m \Vdash_{\mathcal{M}} \gamma \rightarrow \delta,$$

and

$$k \Vdash_{\mathcal{K}} \psi_1 \iff m \Vdash_{\mathcal{M}} \psi_1.$$

Hence,

$$k \Vdash \varphi \iff m \Vdash_{\mathcal{M}} \varphi.$$

As previously, the case of $\varphi = \psi_1 \wedge \psi_2$ is analogous. □

The query that arises is whether the reverse implication holds. It turns out that to prove it, we must confine to the class of finite models. A.Patterson proved that implication for all propositional formulas and unbounded bisimulation $\sim_\infty$ (see [2]), while we present a detailed proof for layered bisimulation $\sim_\alpha$, taking formulas' complexity into consideration.

Theorem 2.2. *Let k and m be worlds of finite Kripke models $\mathcal{K}$ and $\mathcal{M}$, respectively. Then*

$$k \equiv_\alpha m \Rightarrow k \sim_\alpha m.$$

Proof. Let k and m be worlds of finite Kripke models $\mathcal{K} = \langle K, \leq, \mathcal{P}, \Vdash \rangle$ and $\mathcal{M} = \langle M, \leq, \mathcal{P}, \Vdash \rangle$ respectively. By induction on α we will show that the logical equivalence relation $\equiv_\alpha$ is an l-bisimulation. Let $\alpha = 0$. Assume $k \equiv_0 m$. It means that

$$\forall_{\varphi \in I_0(\mathcal{P})} (k \Vdash_{\mathcal{K}} \varphi \Leftrightarrow m \Vdash_{\mathcal{M}} \varphi),$$

where $I_0(\mathcal{P})$ is a set of implicationless formulas. So, if $\varphi = p$ is a propositional variable, then $(k \Vdash_{\mathcal{K}} p \Leftrightarrow m \Vdash_{\mathcal{M}} p)$. That means $PV_{\mathcal{K}}(k) = PV_{\mathcal{M}}(m)$. Hence $k \sim_0 m$.

Now, assume the implication holds for $\alpha > 0$ and suppose $k \equiv_{\alpha+1} m$. Since $\equiv$ is symmetric, it suffices to show only the 'zig' property. Let $k' \geq k$ and let $\Theta_{k',\alpha} = Th_{I_\alpha(\mathcal{P})}(k') = \{\varphi \in I_\alpha(\mathcal{P})\colon k' \Vdash_{\mathcal{K}} \varphi\}$. We will show that there exists a world $m' \geq m$ such that $\Theta_{k',\alpha} = \Theta_{m',\alpha}$. We will proceed in the three following steps.

(i) First we will show that every sentence of $\Theta_{k',\alpha}$ is satisfied at some world accessible from the world m. Let $\varphi \in \Theta_{k',\alpha}$. If there were no world accessible from m where φ were satisfied, then the sentence $\neg\varphi$ would be satisfied at m, i.e. $m \Vdash_{\mathcal{M}} \neg\varphi$. Since $k' \Vdash_{\mathcal{K}} \varphi$, so $k \nVdash_{\mathcal{K}} \neg\varphi$. But note that the sentence $\neg\varphi \in I_{\alpha+1}(\mathcal{P})$, and by the assumption $k \equiv_{\alpha+1} m$, thus $m \nVdash_{\mathcal{M}} \neg\varphi$, a contradiction. Therefore, any sentence $\varphi \in \Theta_{k',\alpha}$ is satisfied at some world accessible from the world m.

(ii) Now we will present a construction that provides the existence of at least one world accessible from m that satisfies all sentences of $\Theta_{k',\alpha}$. Let M_1 be a set of worlds in $\mathcal{M}$ that do not satisfy all sentences from $\Theta_{k',\alpha}$. For each world $n \in M_1$, pick a sentence $\varphi_n \in \Theta_{k',\alpha}$ such that $n \nVdash_{\mathcal{M}} \varphi_n$. Since M_1 is finite, let γ be the conjunction $\bigwedge_{n\in M_1} \varphi_n$. Note that every world of $\mathcal{M}$ that satisfies γ satisfies also any sentence of $\Theta_{k',\alpha}$, i.e. $(w \Vdash_{\mathcal{M}} \gamma \Rightarrow w \Vdash_{\mathcal{M}} \Theta_{k',\alpha})$. Indeed, if $w \Vdash_{\mathcal{M}} \gamma$ and $w \nVdash_{\mathcal{M}} \Theta_{k',\alpha}$, then we could pick a sentence $\varphi_w \in \Theta_{k',\alpha}$ such that $w \nVdash_{\mathcal{M}} \varphi_w$ and, by the construction of γ, φ_w would occur in γ. Thus, we would get $w \nVdash_{\mathcal{M}} \gamma$, a contradiction. Moreover, since γ is a finite conjunction of some sentences from $\Theta_{k',\alpha}$, then $k' \Vdash_{\mathcal{K}} \gamma$. And hence, $k \nVdash_{\mathcal{K}} \neg\gamma$, where $\neg\gamma \in I_{\alpha+1}(\mathcal{P})$. By the assumption $k \equiv_{\alpha+1} m$, so $m \nVdash_{\mathcal{M}} \neg\gamma$. That means there exists $m' \geq m$ such that $m' \Vdash_{\mathcal{M}} \gamma$. Hence, there is a world accessible from m that satisfies all sentences of $\Theta_{k',\alpha}$.

(iii) To finish the proof, we shall show that there exists a world $m' \geq m$ that satisfies sentences only from $\Theta_{k',\alpha}$. Let M_2 be a set of worlds in $\mathcal{M}$ that satisfy a sentence not from $\Theta_{k',\alpha}$. For each world $n \in M_2$, pick a sentence $\psi_n \notin \Theta_{k',\alpha}$ such that $\psi_n \in I_\alpha(\mathcal{P})$ and $n \Vdash_{\mathcal{M}} \psi_n$. Since M_2 is finite, let δ be the disjunction $\bigvee_{n\in M_2} \psi_n$. First notice that every world of $\mathcal{M}$ that refutes δ satisfies sentences only from $\Theta_{k',\alpha}$. Indeed, if there were a world $w \in \mathcal{M}$ such that $w \nVdash_{\mathcal{M}} \delta$ and a sentence $\psi_w \notin \Theta_{k',\alpha}$ such that $w \Vdash_{\mathcal{M}} \psi_w$, then, by the construction of δ, ψ_w would occur in δ, and hence we would get $w \Vdash_{\mathcal{M}} \delta$, a contradiction. Furthermore, since δ is a finite disjunction of sentences not from $\Theta_{k',\alpha}$, then $k' \nVdash_{\mathcal{K}} \delta$, and, as we showed in part (ii), $k' \Vdash_{\mathcal{K}} \gamma$. Thus, $k \nVdash_{\mathcal{K}} \gamma \to \delta$. Note that $\gamma \to \delta \in I_{\alpha+1}(\mathcal{P})$, and by the assumption $k \equiv_{\alpha+1} m$, so $m \nVdash_{\mathcal{M}} \gamma \to \delta$. That means there exists a world $m' \geq m$ that satisfies γ and refutes δ. Thus, $\Theta_{k',\alpha} = \Theta_{m',\alpha}$ for some $m' \geq m$, and hence $k' \sim_\alpha m'$. □

3 First-order Case and Bounded Bisimulation

This section is devoted to the notion of bisimulation in the first-order case. A comprehensive overview of classical model theory topics can be found in [1]. For definitions appearing in this part see [7] and [3].

We consider the classical first-order language. Its (possibly infinite) signature L consists of constants, function and relation symbols. First-order formulas are built from atoms and symbols $\bot$, $\top$ by means of $\wedge$, $\vee$, $\to$, and quantifiers $\exists$, $\forall$. As a measure of formula's complexity, we define the *characteristic* of a formula $\varphi(\overline{x})$, $char(\varphi)$, as follows

- If φ is an atomic formula, then $char(\varphi) = ({}^{\to}0, {}^{\forall}0, {}^{\exists}0)$.

Suppose that formulas φ_1, φ_2 are given and $char(\varphi_i) = ({}^{\to}p_i, {}^{\forall}q_i, {}^{\exists}r_i)$ for $i = 1, 2$. Let $p = \max(p_1, p_2)$, $q = \max(q_1, q_2)$ and $r = \max(r_1, r_2)$.

- If $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$, then $char(\varphi) = ({}^{\to}p, {}^{\forall}q, {}^{\exists}r)$.
- If $\varphi = \varphi_1 \to \varphi_2$, then $char(\varphi) = ({}^{\to}p + 1, {}^{\forall}q, {}^{\exists}r)$.
- If $\varphi = \forall_x \varphi_1$, then $char(\varphi) = ({}^{\to}p_1, {}^{\forall}q_1 + 1, {}^{\exists}r_1)$.
- If $\varphi = \exists_x \varphi_1$, then $char(\varphi) = ({}^{\to}p_1, {}^{\forall}q_1, {}^{\exists}r_1 + 1)$.

We put $({}^{\to}p, {}^{\forall}q, {}^{\exists}r) \preceq ({}^{\to}p', {}^{\forall}q', {}^{\exists}r')$ whenever (p, q, r) precedes (p', q', r') with respect to the product order.

In this section by a Kripke model we mean a structure $\mathcal{K} = (K, \leq, \{A_k : k \in K\}, \Vdash)$ (for a general see [7]). To any world $k \in K$ there is assigned a first-order structure A_k. In order to preserve the monotonicity on a frame $(K, \leq)$, for any two worlds $k, k' \in K$ we require that A_k is a substructure of $A_{k'}$ whenever $k \leq k'$, i.e.

$$k \leq k' \Rightarrow A_k \subseteq A_{k'}.$$

The forcing relation $\Vdash_{\mathcal{K}}$ on $\mathcal{K}$ is defined inductively over the construction of a formula. For a world $k \in K$ and a sequence $\overline{a}$ of elements of the structure A_k we put

- $k \nVdash_{\mathcal{K}} \bot$ and $k \Vdash_{\mathcal{K}} \top$
- $k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff A_k \models \varphi[\overline{a}]$ for all atomic formulas $\varphi(\overline{x})$
- $k \Vdash_{\mathcal{K}} \varphi \wedge \psi[\overline{a}] \iff k \Vdash_{\mathcal{K}} \varphi[\overline{a}]$ and $k \Vdash_{\mathcal{K}} \psi[\overline{a}]$
- $k \Vdash_{\mathcal{K}} \varphi \vee \psi[\overline{a}] \iff k \Vdash_{\mathcal{K}} \varphi[\overline{a}]$ or $k \Vdash_{\mathcal{K}} \psi[\overline{a}]$
- $k \Vdash_{\mathcal{K}} \varphi \to \psi[\overline{a}] \iff \forall_{k' \geq k}(k' \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Rightarrow k' \Vdash_{\mathcal{K}} \psi[\overline{a}])$
- $k \Vdash_{\mathcal{K}} \exists_y \varphi[\overline{a}, y] \iff k \Vdash_{\mathcal{K}} \varphi[\overline{a}, b]$ for some element $b \in A_k$
- $k \Vdash_{\mathcal{K}} \forall_y \varphi[\overline{a}, y] \iff \forall_{k' \geq k} k' \Vdash_{\mathcal{K}} \varphi[\overline{a}, b]$ for all elements $b \in A_{k'}$

We say that model $\mathcal{K}$ forces the formula $\varphi(\overline{x})$ if it is forced at every world of K, i.e.

$$\mathcal{K} \Vdash \varphi \iff k \Vdash_{\mathcal{K}} \varphi \text{ for all } k \in K.$$

Let A and B be first-order structures, and let $\overline{a} = a_1, \ldots, a_n$ and $\overline{b} = b_1, \ldots, b_n$ be sequences of elements of A and B respectively. Then, we will use the symbol $(\overline{a}; \overline{b})$ to denote the finite mapping $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ between A and B. We define a *partial isomorphism between structures A and B* as a finite one-to-one mapping $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq A \times B$ such that

$$A \models \varphi[\overline{a}] \iff B \models \varphi[\overline{b}]$$

for every atomic formula $\varphi(\overline{x})$.

Now, consider formulas of characteristic less than or equal to $({}^{\to}0, {}^{\forall}0, {}^{\exists}1)$. Notice that in the definition of forcing connectives $\wedge$, $\vee$ and quantifier $\exists$ refer only to the current world. As we show below, forcing of such formulas at some world k depends only on forcing at k all atomic formulas. Namely, the following holds.

Proposition 3.1. *Let k and m be worlds of Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively, and let $\overline{a}$ and $\overline{b}$ be sequences of the elements of the structures A_k and B_m respectively. Then,*

$$k \Vdash_{\mathcal{K}} \tilde{\varphi}[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \tilde{\varphi}[\overline{b}] \quad \textit{for every formula } \tilde{\varphi}(\overline{x}) \tag{1}$$

$$\textit{with } char(\tilde{\varphi}) \leq ({}^{\to}0, {}^{\forall}0, {}^{\exists}1)$$

if and only if

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \varphi[\overline{b}] \quad \textit{for all atomic formulas } \varphi(\overline{x}). \tag{2}$$

Proof. Let k and m be worlds of Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively, and $\overline{a}$ and $\overline{b}$ be sequences of the elements of the structures A_k and B_m respectively.

(1) $\Rightarrow$ (2) This implication is clear since the set of all positive formulas contains the set of all atomic formulas.

(2) $\Rightarrow$ (1) Consider a set $\Phi := \{\tilde{\varphi}(\overline{x})\colon char(\tilde{\varphi}) \leq (^{\neg}0,^{\forall}0,^{\exists}1)\}$. Assume that

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all atomic formulas $\varphi(\overline{x})$. Inductively, over the construction of a formula from Φ, we will show that

$$k \Vdash_{\mathcal{K}} \tilde{\varphi}[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \tilde{\varphi}[\overline{b}]$$

for every formula $\tilde{\varphi}(\overline{x}) \in \Phi$. The case of constants is obvious. If $\tilde{\varphi} = \psi_1 \wedge \psi_2$, then, by induction hypothesis,

$$k \Vdash_{\mathcal{K}} \tilde{\varphi}[\overline{a}] \iff k \Vdash_{\mathcal{K}} \psi_1 \wedge \psi_2[\overline{a}] \iff k \Vdash_{\mathcal{K}} \psi_1[\overline{a}] \text{ and } k \Vdash_{\mathcal{K}} \psi_2[\overline{a}]$$

$$\iff m \Vdash_{\mathcal{M}} \psi_1[\overline{b}] \text{ and } m \Vdash_{\mathcal{M}} \psi_2[\overline{b}] \iff m \Vdash_{\mathcal{M}} \psi_1 \wedge \psi_2[\overline{b}] \iff m \Vdash_{\mathcal{M}} \tilde{\varphi}[\overline{b}]$$

If $\tilde{\varphi} = \psi_1 \vee \psi_2$, we proceed by analogy. If $\tilde{\varphi}(\overline{x}) = \exists_y \psi(\overline{x}, y)$, then, by induction hypothesis,

$$k \Vdash_{K} \exists_y \psi[\overline{a}, y] \iff k \Vdash_{\mathcal{K}} \psi[\overline{a}, a] \text{ for some } a \in A_k$$

$$\iff m \Vdash_{\mathcal{M}} \psi[\overline{b}, b] \text{ for some } b \in B_m \iff m \Vdash_{\mathcal{M}} \exists_y \psi[\overline{b}, y].$$

□

Notice that proposition 3.1 cannot be extended to cover all quantifier-free formulas. For example, if an atomic formula $\varphi(\overline{x})$ is not decidable, i.e.

$$\mathcal{K} \nVdash \varphi \vee \neg\varphi \text{ and } \mathcal{M} \nVdash \varphi \vee \neg\varphi,$$

then for some $k \in K$ and $m \in M$ it may happen

$$k \nVdash_{\mathcal{K}} \varphi \quad \text{and} \quad k' \Vdash_{\mathcal{K}} \varphi \text{ for some } k' \geq k,$$

$$m \nVdash_{\mathcal{M}} \varphi \quad \text{and} \quad m' \nVdash_{\mathcal{M}} \varphi \text{ for all } m' \geq m.$$

Now, we present the definition of bisimulation for first-order Kripke models (see [3]). Let $\mathcal{K} = (K, \leq, \{A_k\colon k \in K\}, \Vdash)$ and $\mathcal{M} = (M, \leq, \{B_m\colon m \in M\}, \Vdash)$ be Kripke models, let k and m be worlds of K and M respectively, let π range over mappings between worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$, and let $p, q, r \geq 0$. A *bounded bisimulation* between Kripke models $\mathcal{K}$ and $\mathcal{M}$ is a 6-ary relation that satisfies conditions specified below. We will write $\pi\colon k \sim_{p,q,r} m$ whenever π, k, p, q, r, m are in that relation.

(*i*) $\pi\colon k \sim_{0,0,0} m \Longrightarrow \pi$ is a partial isomorphism between A_k and B_m

(*ii*) $\pi\colon k \sim_{p+1,q,r} m \Longrightarrow \pi$ is a mapping between A_k and B_m and

($\rightarrow$ -zig) for every $k' \geq k$ there is $m' \geq m$ such that $\pi\colon k' \sim_{p,q,r} m'$

($\rightarrow$ -zag) for every $m' \geq m$ there is $k' \geq k$ such that $\pi\colon k' \sim_{p,q,r} m'$

(*iii*) $\pi\colon k \sim_{p,q+1,r} m \Longrightarrow \pi$ is a mapping between A_k and B_m and

(∀-zig) for every $k' \geq k$ and $a \in A_{k'}$ there are $m' \geq m$ and $b \in B_{m'}$ such that $\pi \cup (a,b)\colon k' \sim_{p,q,r} m'$

(∀-zag) for every $m' \geq m$ and $b \in B_{m'}$ there are $k' \geq k$ and $a \in A_{k'}$ such that $\pi \cup (a,b)\colon k' \sim_{p,q,r} m'$

(*iv*) $\pi\colon k \sim_{p,q,r+1} m \Longrightarrow \pi$ is a mapping between A_k and B_m and

(∃-zig) for every $a \in A_k$ there exists $b \in B_m$ such that $\pi \cup (a,b)\colon k \sim_{p,q,r} m$

(∃-zag) for every $b \in B_m$ there exists $a \in A_k$ such that $\pi \cup (a,b)\colon k \sim_{p,q,r} m$

Below we present a result concerning bisimulations (see [3]).

Theorem 3.2. *Let k and m be worlds of Kripke models $\mathcal{K}$ and $\mathcal{M}$, respectively. Assume $p,q,r \geq 0$ and $(\overline{a};\overline{b})$ is a mapping between A_k and B_m such that for some bisimulation $\sim$ we have $(\overline{a};\overline{b})\colon k \sim_{p,q,r} m$. Then*

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff m \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for every formula $\varphi(\overline{x})$ such that $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$.

Proof. Let k and m be worlds of Kripke models $\mathcal{K}$ and $\mathcal{M}$, respectively. To prove the theorem, we will proceed by induction on characteristic and complexity of a formula.

First, let φ be an atomic formula. Then, by the definition of partial isomorphism, the thesis holds. Hence, by Proposition 3.1, it holds for all formulas of characteristic $({}^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0)$.

Assume the theorem holds for $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r) > ({}^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0)$. Notice that the only cases we need to consider are the cases of implication $\rightarrow$ and quantifiers $\forall$ and $\exists$.

Let $char(\varphi) \leq ({}^{\rightarrow}p+1, {}^{\forall}q, {}^{\exists}r)$. Then, $\varphi(\overline{x})$ is of the form $\psi_1(\overline{x}) \rightarrow \psi_2(\overline{x})$, where $char(\psi_1), char(\psi_2) \leq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$. Let $(\overline{a};\overline{b})$ be a mapping between structures A_k and B_m such that $(\overline{a};\overline{b})\colon k \sim_{p+1,q,r} m$. Assume

$$k \Vdash_{\mathcal{K}} (\psi_1 \rightarrow \psi_2)[\overline{a}]. \tag{3}$$

Consider $m' \geq m$ such that

$$m' \Vdash_{\mathcal{M}} \psi_1[\overline{b}].$$

By the ($\rightarrow$ -zag) property, we can find a world $k' \geq k$ such that $(\overline{a};\overline{b})\colon k' \sim_{p,q,r} m'$. Thus, by the induction hypothesis,

$$k' \Vdash_{\mathcal{K}} \psi_1[\overline{a}],$$

and consequently, by (3), we get

$$k' \Vdash_{\mathcal{K}} \psi_2[\overline{a}].$$

Again, by the induction hypothesis,

$$m' \Vdash_{\mathcal{M}} \psi_2[\overline{b}].$$

It proves that

$$m \Vdash_{\mathcal{M}} (\psi_1 \rightarrow \psi_2)[\overline{b}].$$

Similarly, refering to the ($\to$ -zig) property, we prove the opposite implication.

Now, let $char(\varphi) \leq (^{\to}p, ^{\forall}q+1, ^{\exists}r)$. It means that $\varphi(\overline{x}) = \forall_y \psi(\overline{x}, y)$, where $char(\psi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r)$. Let $(\overline{a}; \overline{b})$ be a mapping between structures A_k and B_m such that $(\overline{a}; \overline{b})\colon k \sim_{p,q+1,r} m$. Assume that

$$k \Vdash_{\mathcal{K}} \forall_y \psi[\overline{a}, y]. \tag{4}$$

Let $m' \geq m$ be arbitrary, and choose an element $b' \in B_{m'}$. By the ($\forall$-zag) property, there are $k' \geq k$ and $a' \in A_{k'}$ such that $(\overline{a}; \overline{b}) \cup (a', b')\colon k' \sim_{p,q,r} m'$. By (4), in particular

$$k' \Vdash_{\mathcal{K}} \psi[\overline{a}, a']$$

By the induction hypothesis,

$$m' \Vdash_{\mathcal{M}} \psi[\overline{b}, b']$$

Since $m' \geq m$ and $b' \in B_{m'}$ were chosen arbitrarily, it proves that

$$m \Vdash_{\mathcal{M}} \forall_y \psi[\overline{b}, y].$$

Similarly, refering to the ($\forall$-zig) property, we prove the opposite implication.

Finally, let $char(\varphi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r+1)$, and let $(\overline{a}; \overline{b})$ be a mapping between structures A_k and B_m such that $(\overline{a}; \overline{b})\colon k \sim_{p,q,r+1} m$. So, $\varphi(\overline{x}) = \exists_y \psi(\overline{x}, y)$, where $char(\psi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r)$. Assume that

$$k \Vdash_{\mathcal{K}} \exists_y \psi[\overline{a}, y].$$

By the definition of forcing in $\mathcal{K}$, for some element $a \in A_k$ we have

$$k \Vdash_{\mathcal{K}} \psi[\overline{a}, a].$$

By the ($\exists$-zig) property, we can find an element $b \in B_m$ such that $(\overline{a}; \overline{b}) \cup (a, b)\colon k \sim_{p,q,r} m$. Thus, by the induction hypothesis, we get

$$m \Vdash_{\mathcal{M}} \psi[\overline{b}, b].$$

And hence,

$$m \Vdash_{\mathcal{M}} \exists_y \psi[\overline{b}, y].$$

Again, refering to the ($\exists$-zag) property, we prove the opposite implication. □

4 Partial Results and Conclusions

In order to find an analogue of Theorem 2.2 in the case of first-order Kripke models and bounded bisimulation, we restrict our assumptions. As previously, we confine our considerations to the case of finite Kripke models. Having analysed corresponding theorems of classical model theory, we have noticed that some additional assumptions on signature or first-order structures are needed. To start our investigation, we assume also that the signature L and first-order structures (worlds of the Kripke model $\mathcal{K}$) are finite. So, we say that model $\mathcal{K}$ is finite if and only if both the frame $(K, \leq)$ and first-order structures assigned to the worlds of K are finite. Moreover, the signature L is considered with no function symbols.

Since the complete solution of the problem has not been found yet, we present our partial results and discuss burdens that we have encountered.

Proposition 4.1. *Let k and m be worlds of finite Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively, and let $\overline{a}$ and $\overline{b}$ be sequences of the elements of the structures A_k and B_m respectively. Then, if*

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff m \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)$, then $(\overline{a}; \overline{b})\colon k \sim_{p,0,0} m$ for some bisimulation $\sim$.

Proof. It turns out that in the propositional case the proof may be carried out in a similar way that the proof of Theorem 2.2.

Let k and m be worlds of finite Kripke models $\mathcal{K}$ and $\mathcal{M}$ assigned to finite first-order structures A_k and B_m respectively. Let $\overline{a}$ and $\overline{b}$ be sequences of the elements of A_k and B_m respectively. For $p \geq 0$, we define a relation $\sim$ as follows

$$k \sim_{p,0,0} m :\iff (k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)$. By induction on $p \geq 0$ we want to prove that $\sim$ is a bisimulation. First, consider all formulas of characteristic $({}^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0)$. Let $\varphi(\overline{x})$ be an atomic formula. Then, we have

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff A_k \models \varphi[\overline{a}],$$

$$m \Vdash_{\mathcal{K}} \varphi[\overline{b}] \iff B_m \models \varphi[\overline{b}].$$

So, by the assumption, for all atomic formulas $\varphi(\overline{x})$ we obtain

$$A_k \models \varphi[\overline{a}] \iff B_m \models \varphi[\overline{b}].$$

That means that the mapping $(\overline{a}; \overline{b})$ is a partial isomorphism between structures A_k and B_m. Thus, $(\overline{a}; \overline{b})\colon k \sim_{0,0,0} m$.

Now assume that the result holds for some $p > 0$. Let us consider a formula $\tilde{\varphi}(\overline{x})$ such that $char(\tilde{\varphi}) \leq ({}^{\rightarrow}p+1, {}^{\forall}0, {}^{\exists}0)$, i.e. $\tilde{\varphi}(\overline{x})$ is of the form $\alpha(\overline{x}) \rightarrow \beta(\overline{x})$ with $char(\alpha), char(\beta) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)$, and assume that $k \Vdash_{\mathcal{K}} \tilde{\varphi}[\overline{a}] \iff m \Vdash_{\mathcal{M}} \tilde{\varphi}[\overline{b}]$. We verify the ($\rightarrow$ -zig) property. So, it suffices to show that for every $k' \geq k$ there exists $m' \geq m$ such that

$$k' \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff m' \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)$. Let $k' \geq k$. Consider a set $\Theta_{k'} = \{\varphi(\overline{x})\colon k' \Vdash_{\mathcal{K}} \varphi[\overline{a}], char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)\}$. We will show that there exists $m' \geq m$ such that $\Theta_{k'} = \Theta_{m'}$, where $\Theta_{m'} = \{\varphi(\overline{x})\colon m' \Vdash_{\mathcal{M}} \varphi[\overline{b}], char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}0, {}^{\exists}0)\}$. We will proceed in the three following steps.

(*i*) Let $\varphi(\overline{x}) \in \Theta_{k'}$. If there were no world accessible from m where $\varphi(\overline{x})$ were forced by $\overline{b}$, then $\neg\varphi(\overline{x})$ would be forced by $\overline{b}$ at the world m, i.e. $m \Vdash_{\mathcal{M}} \neg\varphi[\overline{b}]$. Since $k' \Vdash_{\mathcal{K}} \varphi[\overline{a}]$, so $k \nVdash_{\mathcal{K}} \neg\varphi[\overline{a}]$. But note that $char(\neg\varphi) \leq ({}^{\rightarrow}p+1, {}^{\forall}0, {}^{\exists}0)$, so by the assumption $m \nVdash_{\mathcal{M}} \neg\varphi[\overline{b}]$, a contradiction. Therefore, any formula $\varphi(\overline{x}) \in \Theta_{k'}$ is forced by $\overline{b}$ at some world accessible from the world m.

(*ii*) Consider a set M_1 of worlds in $\mathcal{M}$ that do not force all formulas from $\Theta_{k'}$. For each world $w \in M_1$, we pick a formula $\varphi_w(\overline{x}) \in \Theta_{k'}$ such that $w \nVdash_{\mathcal{M}} \varphi_w[\overline{b}]$. Notice that M_1 is finite. So, let $\gamma(\overline{x})$ be the conjunction $\bigwedge_{w \in M_1} \varphi_w(\overline{x})$.

Then, every world of $\mathcal{M}$ that forces $\gamma(\overline{x})$ forces also any formula of $\Theta_{k'}$, i.e. $(w \Vdash_{\mathcal{M}} \gamma[\overline{b}] \Rightarrow w \Vdash_{\mathcal{M}} \Theta_{k'})$. Indeed, if there were a world $w \in M$ such that $w \Vdash_{\mathcal{M}} \gamma[\overline{b}]$ and $w \nVdash_{\mathcal{M}} \Theta_{k'}$, then we could pick a formula $\varphi_w(\overline{x}) \in \Theta_{k'}$ with $w \nVdash_{\mathcal{M}} \varphi_w[\overline{b}]$ and, by the construction of $\gamma(\overline{x})$, φ_w would occur in γ. Thus, we would get $w \nVdash_{\mathcal{M}} \gamma[\overline{b}]$, a contradiction. Furthermore, since $\gamma(\overline{x}) \in \Theta_{k'}$, then $k' \Vdash_{\mathcal{K}} \gamma[\overline{a}]$. And so, $k \nVdash_{\mathcal{K}} \neg\gamma[\overline{a}]$. Note that $char(\neg\gamma) \leq (^{\rightarrow}p+1,^{\forall}0,^{\exists}0)$, thus, by the assumption, $m \nVdash_{\mathcal{M}} \neg\gamma[\overline{b}]$. That means that there exists $m' \geq m$ such that $m' \Vdash_{\mathcal{M}} \gamma[\overline{b}]$, and hence, m' forces all formulas of $\Theta_{k'}$.

(*iii*) Consider a set M_2 of worlds in $\mathcal{M}$ that force a formula not from $\Theta_{k'}$ of characteristic less that or equal to $(^{\rightarrow}p,^{\forall}0,^{\exists}0)$. As earlier, for each world $w \in M_2$, we pick a formula $\psi_w(\overline{x}) \notin \Theta_{k'}$ such that $w \Vdash_{\mathcal{M}} \psi_w[\overline{b}]$. We define a formula $\delta(\overline{x})$ as the finite $\bigvee_{w \in M_2} \psi_w(\overline{x})$. Then, every world of $\mathcal{M}$ that does not force δ forces formulas only from $\Theta_{k'}$. Indeed, if there were a world $w \in M$ such that $w \nVdash_{\mathcal{M}} \delta[\overline{b}]$ and a formula $\psi_w \notin \Theta_{k'}$ such that $w \Vdash_{\mathcal{M}} \psi_w[\overline{b}]$, then, by the construction of $\delta(\overline{x})$, ψ_w would occur in δ. So, we would get $w \Vdash_{\mathcal{M}} \delta[\overline{b}]$, a contradiction. Moreover, since $\delta(\overline{x}) \notin \Theta_{k'}$, then $k' \nVdash_{\mathcal{K}} \delta[\overline{a}]$, and, as we showed in part (*ii*), $k' \Vdash_{\mathcal{K}} \gamma[\overline{a}]$. Thus, $k \nVdash_{\mathcal{K}} (\gamma \rightarrow \delta)[\overline{a}]$, where $char(\gamma \rightarrow \delta) \leq (^{\rightarrow}p+1,^{\forall}0,^{\exists}0)$. And hence, by the assumption, $m \nVdash_{\mathcal{M}} (\gamma \rightarrow \delta)[\overline{b}]$. That means there exists a world m' accessible from m such that m' forces $\gamma(\overline{x})$ and does not force $\delta(\overline{x})$. By the construction of formulas $\gamma(\overline{x})$ and $\delta(\overline{x})$, we conclude that $\Theta_{k'} = \Theta_{m'}$.

And so, for every $k' \geq k$ there exists $m' \geq m$ such that

$$k' \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff m' \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p,^{\forall}0,^{\exists}0)$. By the induction hypothesis applied for $(^{\rightarrow}p,^{\forall}0,^{\exists}0)$ we get $(\overline{a};\overline{b})\colon k' \sim_{p,0,0} m'$. □

It turns out that Proposition 4.1 cannot be extended on all formulas $\varphi(\overline{x})$ of characteristic less than or equal to $(^{\rightarrow}p,^{\forall}q,^{\exists}r)$. Our reasoning fails even in the case of the existential quantifier. Indeed, suppose that for $p, q, r \geq 0$ the relation $\sim$ is defined as follows

$$k \sim_{p,q,r} m :\iff (k \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow m \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p,^{\forall}q,^{\exists}r)$. Consider a formula $\tilde{\varphi}(\overline{x})$ such that $char(\tilde{\varphi}) \leq (^{\rightarrow}p,^{\forall}q,^{\exists}r+1)$, i.e. $\tilde{\varphi}(\overline{x})$ is of the form $\exists_y \psi(\overline{x}, y)$ with $char(\psi) \leq (^{\rightarrow}p,^{\forall}q,^{\exists}r)$, and assume that

$$k \Vdash_{\mathcal{K}} \tilde{\varphi}[\overline{a}] \iff m \Vdash_{\mathcal{M}} \tilde{\varphi}[\overline{b}].$$

We verify the ($\exists$-zig) property. So, it suffices to show that for every $a \in A_k$ there exists $b \in B_m$ such that

$$k \Vdash_{\mathcal{K}} \varphi[\overline{a}, a] \iff m \Vdash_{\mathcal{M}} \varphi[\overline{b}, b]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p,^{\forall}q,^{\exists}r)$. For simlicity we will suppress the parameters $\overline{a}$ and $\overline{b}$.

Let $a \in A_k$. Suppose such an element $b \in B_m$ does not exist. Then, for every $b \in B_m$ there exists a formula $\varphi_b(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p,^{\forall}q,^{\exists}r)$ such that

$$(k \Vdash_{\mathcal{K}} \varphi_b[a] \text{ and } m \nVdash_{\mathcal{M}} \varphi_b[b]) \text{ or } (k \nVdash_{\mathcal{K}} \varphi_b[a] \text{ and } m \Vdash_{\mathcal{M}} \varphi_b[b]).$$

Notice that, since the structure B_m is finite, the set $\{\varphi_b(\overline{x})\colon b \in B_m\}$ is finite too. So, consider two following sets

$$\Theta_0 = \{\varphi_b(\overline{x})\colon k \Vdash_{\mathcal{K}} \varphi_b[a] \text{ and } m \nVdash_{\mathcal{M}} \varphi_b[b]\}$$

and

$$\Theta_1 = \{\varphi_b(\overline{x})\colon k \nVdash_{\mathcal{K}} \varphi_b[a] \text{ and } m \Vdash_{\mathcal{M}} \varphi_b[b]\}.$$

Since $k \Vdash_{\mathcal{K}} \bigwedge \Theta_0[a]$ and $k \nVdash_{\mathcal{K}} \bigvee \Theta_1[a]$, then

$$k \Vdash_{\mathcal{K}} (\bigvee \Theta_1 \to \bigwedge \Theta_0)[a]. \tag{5}$$

Note also that $m \nVdash_{\mathcal{M}} \bigwedge \Theta_0[b]$ and $m \Vdash_{\mathcal{M}} \bigvee \Theta_1[b]$ for all $b \in B_m$. Thus,

$$m \nVdash_{\mathcal{M}} (\bigvee \Theta_1 \to \bigwedge \Theta_0)[b]. \tag{6}$$

And hence, by (5) and (6),

$$k \Vdash_{\mathcal{K}} \exists_x(\bigvee \Theta_1(x) \to \bigwedge \Theta_0(x))$$

and

$$m \nVdash_{\mathcal{M}} \exists_x(\bigvee \Theta_1(x) \to \bigwedge \Theta_0(x)).$$

We got a formula $\exists_x(\bigvee \Theta_1(x) \to \bigwedge \Theta_0(x))$ of characteristic less than or equal to $({}^{\to}p+1, {}^{\forall}q, {}^{\exists}r+1)$, not $({}^{\to}p, {}^{\forall}q, {}^{\exists}r+1)$ as it was expected. Similarly, in the case of the universal quantifier, the above reasoning fails.

In conclusion, we presume that the proof of the above problem requires approaching another method or considering another measure of formulas' complexity. Nonetheless, it is not an immediate generalisation.

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