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Bisimulations of Finite Kripke Models

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Abstract

In our paper we consider the notion of bounded bisimulation for Kripke models for intuitionistic first-order theories. As it is already known, in this case, existence of bisimulation between given two Kripke models implies their logical equivalence. We present a new result which states that, under some additional conditions, for every two first-order Kripke models that are equivalent, there is a bisimulation between them. We also discuss the possible ways of generalising this fact.

Keywords: Kripke model, bounded bisimulation, logical equivalence.

1 Introduction

Having given two classical structures, one of the most important question is whether they validate the same formulas. When we discard the notion of structure isomorphism as too strong and too restrictive one, we have to look for another suitable condition for logical equivalence. In this paper we consider the case of Kripke semantics for intuitionistic first-order theories.

The notion of bisimulation was defined by Johan Van Benthem (1976) in the context of modal logic. Then, it was introduced by David Park (1981) into transition systems theory to test whether two processes behave the same ([4]). Nowadays the notion of bisimulation is employed in many areas of Computer Science such as functional languages, databases, to name but a two. Finally, bisimulation was introduced into first-order logic, and found a straightforward game-theoretical interpretation.

As we already know, bisimulation between worlds of two Kripke models implies their logical equivalence. The subject of research is, however, the reverse implication. When and under what conditions it holds? Theorem which states that, in the case of propositional intuitionistic logic, logical equivalence of worlds of two finite Kripke models implies bisimulation between them was proved by Albert Visser ([5]), and, subsequently, an interesting and illuminating construction of this proof was presented by Anna Paterson ([2]). The problem arises when we turn into the first-order case and have to deal with quantifiers $\forall$ and $\exists$. The question is, assuming that two classical first-order structures validate the same first-order formulas, what form the appropriate relation of bisimulation will have.

The aim of the paper is to reveal the relationship between notions of bounded bisimulation and logical equivalence. It is organised as follows. Section 2 provides an overview of basic definitions needed in further considerations. It contains notions such as first-order formula, characteristic of a formula, first-order Kripke model as well as logical equivalence, and bounded bisimulation.

In Section 3 we quote the well-known result concerning bounded bisimulation and logical equivalence. Then, we present the main theorem which states that, if we confine to the case of finite signature L with no function symbols, logical equivalence of worlds of two *strongly finite* Kripke models implies bisimulation between them.

Finally, Section 4 is an attempt to reduce the assumptions of the main theorem. We consider many possibilities so that the theorem could refer to a wider class of Kripke models.

2 Preliminaries

The aim of this section is to present the notion of bisimulation in the intuitionistic first-order case. For a comprehensive overview of classical model theory topics see [1]. Definitions appearing in this section can be found in [6] and [3].

Let us consider the classical first-order language. Its (possibly infinite) signature L consists of constants, function and relation symbols. First-order formulas are built from atoms and symbols $\bot$, $\top$ by means of $\wedge$, $\vee$, $\to$, and quantifiers $\exists$, $\forall$. As a measure of formula's complexity, we define the *characteristic* of a formula $\varphi(\overline{x})$, $char(\varphi)$, as follows

- If φ is an atomic formula, then $char(\varphi) = (^{\to}0, ^{\forall}0, ^{\exists}0)$.

Suppose that formulas φ_1, φ_2 are given and $char(\varphi_i) = (^{\to}p_i, ^{\forall}q_i, ^{\exists}r_i)$ for $i = 1, 2$. Let $p = \max(p_1, p_2)$, $q = \max(q_1, q_2)$ and $r = \max(r_1, r_2)$.

- If $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$, then $char(\varphi) = (^{\to}p, ^{\forall}q, ^{\exists}r)$.
- If $\varphi = \varphi_1 \to \varphi_2$, then $char(\varphi) = (^{\to}p + 1, ^{\forall}q, ^{\exists}r)$.
- If $\varphi = \forall_x \varphi_1$, then $char(\varphi) = (^{\to}p_1, ^{\forall}q_1 + 1, ^{\exists}r_1)$.
- If $\varphi = \exists_x \varphi_1$, then $char(\varphi) = (^{\to}p_1, ^{\forall}q_1, ^{\exists}r_1 + 1)$.

We put $(^{\to}p, ^{\forall}q, ^{\exists}r) \preceq (^{\to}p', ^{\forall}q', ^{\exists}r')$ whenever (p, q, r) precedes (p', q', r') with respect to the product order.

By a Kripke model for a first-order language L we mean a structure $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash)$ (for a general definition see [6]). To any world $\alpha \in K$ there is assigned a first-order structure K_α. In order to preserve the monotonicity on a frame $(K, \leq)$, for any two worlds $\alpha, \alpha' \in K$ we require that K_α is a substructure of $K_{\alpha'}$ whenever $\alpha \leq \alpha'$, i.e.

$$\alpha \leq \alpha' \Rightarrow K_\alpha \subseteq K_{\alpha'}.$$

The forcing relation $\Vdash_{\mathcal{K}}$ on $\mathcal{K}$ is defined inductively over the construction of a formula. Consider a world $\alpha \in K$ and a sequence $\overline{a} := a_1, \dots, a_n$ of elements of the structure K_α, we put

- $\alpha \nVdash_{\mathcal{K}} \bot$ and $\alpha \Vdash_{\mathcal{K}} \top$
- $\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff K_\alpha \models \varphi[\overline{a}]$ for all atomic formulas $\varphi(\overline{x})$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \wedge \psi)[\overline{a}] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}]$ and $\alpha \Vdash_{\mathcal{K}} \psi[\overline{a}]$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \vee \psi)[\overline{a}] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}]$ or $\alpha \Vdash_{\mathcal{K}} \psi[\overline{a}]$

- $\alpha \Vdash_{\mathcal{K}} (\varphi \to \psi)[\overline{a}] \iff \forall_{\alpha' \geq \alpha}(\alpha' \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Rightarrow \alpha' \Vdash_{\mathcal{K}} \psi[\overline{a}])$
- $\alpha \Vdash_{\mathcal{K}} \exists_y \varphi[\overline{a}, y] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}, b]$ for some element $b \in K_\alpha$
- $\alpha \Vdash_{\mathcal{K}} \forall_y \varphi[\overline{a}, y] \iff \forall_{\alpha' \geq \alpha} \alpha' \Vdash_{\mathcal{K}} \varphi[\overline{a}, b]$ for all elements $b \in K_{\alpha'}$

Notice that the forcing relation is persistent in the sense that

$$(\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \ \wedge \ \alpha' \geq \alpha) \ \Rightarrow \ \alpha' \Vdash_{\mathcal{K}} \varphi[\overline{a}]$$

for any formula $\varphi(\overline{x})$. We say that model $\mathcal{K}$ forces the formula $\varphi(\overline{x})$ if it is forced at every world of K, i.e.

$$\mathcal{K} \Vdash \varphi \iff \alpha \Vdash_{\mathcal{K}} \varphi \text{ for all } \alpha \in K.$$

Having given two Kripke models $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash)$ and $\mathcal{M} = (M, \leq, \{M_\beta \colon \beta \in M\}, \Vdash)$, the essential question is whether worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$ validate the same formulas. Thus, for worlds $\alpha \in K$, $\beta \in M$ and sequences $\overline{a}$ and $\overline{b}$ of elements of worlds K_α and M_β respectively, we define a relation $\equiv_{p,q,r}$ as follows

$$\alpha \equiv_{p,q,r} \beta :\iff (\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\to}p, {}^{\forall}q, {}^{\exists}r)$. We say that α and β are *p, q, r-equivalent* if and only if $\alpha \equiv_{p,q,r} \beta$. Approaching more model-theoretical notation, to denote the fact of p, q, r-equivalence between worlds α and β we will write $(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b})$.

Now, consider two first-order structures A and B. Let $\overline{a} = a_1, \ldots, a_n$ and $\overline{b} = b_1, \ldots, b_n$ be sequences of elements of A and B respectively. To denote the finite mapping $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ between A and B we will use the symbol $(\overline{a}; \overline{b})$. We define a *partial isomorphism between structures A and B* as a finite one-to-one mapping $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq A \times B$ such that

$$A \models \varphi[\overline{a}] \iff B \models \varphi[\overline{b}]$$

for every atomic formula $\varphi(\overline{x})$.

Finally, we present the definition of bisimulation for first-order Kripke models (for a more general case see [3]). Consider two Kripke models $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash)$ and $\mathcal{M} = (M, \leq, \{M_\beta \colon \beta \in M\}, \Vdash)$. Let α and β be worlds of K and M respectively, let π range over mappings between worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$, and let $p, q, r \geq 0$. A *bounded bisimulation* between Kripke models $\mathcal{K}$ and $\mathcal{M}$ is a 6-ary relation that satisfies conditions specified below. We will write $\pi \colon k \sim_{p,q,r} m$ whenever π, k, p, q, r, m are in that relation.

(i) $\pi \colon \alpha \sim_{0,0,0} \beta \Longrightarrow \pi$ is a partial isomorphism between K_α and M_β

(ii) $\pi \colon \alpha \sim_{p+1,q,r} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\to$ -zig) for every $\alpha' \geq \alpha$ there is $\beta' \geq \beta$ such that $\pi \colon \alpha' \sim_{p,q,r} \beta'$

($\to$ -zag) for every $\beta' \geq \beta$ there is $\alpha' \geq \alpha$ such that $\pi \colon \alpha' \sim_{p,q,r} \beta'$

(iii) $\pi \colon \alpha \sim_{p,q+1,r} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\forall$-zig) for every $\alpha' \geq \alpha$ and $a \in K_{\alpha'}$ there are $\beta' \geq \beta$ and $b \in M_{\beta'}$ such that $\pi \cup \{(a, b)\} \colon \alpha' \sim_{p,q,r} \beta'$

($\forall$-zag) for every $\beta' \geq \beta$ and $b \in M_{\beta'}$ there are $\alpha' \geq \alpha$ and $a \in K_{\alpha'}$ such that $\pi \cup \{(a,b)\} \colon \alpha' \sim_{p,q,r} \beta'$

(iv) $\pi \colon \alpha \sim_{p,q,r+1} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\exists$-zig) for every $a \in K_\alpha$ there exists $b \in M_\beta$ such that $\pi \cup \{(a,b)\} \colon \alpha \sim_{p,q,r} \beta$

($\exists$-zag) for every $b \in M_\beta$ there exists $a \in K_\alpha$ such that $\pi \cup \{(a,b)\} \colon \alpha \sim_{p,q,r} \beta$

3 Bounded Bisimulation and Logical Equivalence

This section reveals the relationship between notions of bounded bisimulation and logical equivalence.

First, we present the well-known result concerning those notions.

Theorem 3.1. *Let α and β be worlds of Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively. Assume $p, q, r \geq 0$ and $(\overline{a}; \overline{b})$ is a mapping between K_α and M_β such that for some bisimulation $\sim$ we have $(\overline{a}; \overline{b}) \colon \alpha \sim_{p,q,r} \beta$. Then*

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for every formula $\varphi(\overline{x})$ such that $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$.

Proof. For proof see ([3]). □

A natural question is whether the converse implication also holds. It turns out that we have to confine our considerations to much smaller class of Kripke models. Having analysed corresponding theorems of classical model theory, we have noticed that some additional assumptions on Kripke models, signature or first-order structures are needed. To start with, we say that model $\mathcal{K}$ is *strongly finite* if and only if both the frame $(K, \leq)$ and first-order structures assigned to the worlds of K are finite. Moreover, the finite signature L is considered with no function symbols.

Theorem 3.2. *Let α and β be worlds of strongly finite Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively, let $\overline{a}$ and $\overline{b}$ be sequences of the elements of the structures K_α and M_β respectively, and let $P, Q, R \geq 0$. If*

$$(\alpha, \overline{a}) \equiv_{P+1,Q,R} (\beta, \overline{b}),$$

then $(\overline{a}; \overline{b}) \colon \alpha \sim_{P,Q,R} \beta$ for some bisimulation $\sim$.

Proof. Let α and β be worlds of finite Kripke models $\mathcal{K}$ and $\mathcal{M}$ assigned to finite first-order structures K_α and M_β respectively. Let $\overline{a}$ and $\overline{b}$ be sequences of the elements of K_α and M_β respectively. Assume that $(\alpha, \overline{a}) \equiv_{P+1,Q,R} (\beta, \overline{b})$, i.e.

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}P+1, {}^{\forall}Q, {}^{\exists}R)$. For $0 \leq p \leq P$, $0 \leq q \leq Q$, $0 \leq r \leq R$ we define a relation $\sim$ as follows

$$(\overline{a}; \overline{b}) \colon \alpha \sim_{p,q,r} \beta :\iff (\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$. By induction on p, q, r we want to prove that $\sim$ is a bisimulation.

First, consider all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\to}0, ^{\forall}0, ^{\exists}0)$. Let $\varphi(\overline{x})$ be an atomic formula. Then, we have

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff K_{\alpha} \models \varphi[\overline{a}],$$

$$\beta \Vdash_{\mathcal{K}} \varphi[\overline{b}] \iff M_{\beta} \models \varphi[\overline{b}].$$

So, by the assumption, for all atomic formulas $\varphi(\overline{x})$ we obtain

$$K_{\alpha} \models \varphi[\overline{a}] \iff M_{\beta} \models \varphi[\overline{b}].$$

That means that the mapping $(\overline{a}; \overline{b})$ is a partial isomorphism between structures K_{α} and M_{β}. Thus, $(\overline{a}; \overline{b})\colon \alpha \sim_{0,0,0} \beta$.

Now assume that the result holds for some $p, q, r > 0$. We will proceed in the three following steps.

(i) First, for $p < P, q \leq Q, r \leq R$, assume that $(\overline{a}; \overline{b})\colon \alpha \sim_{p+1,q,r} \beta$, i.e

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\to}p+1, ^{\forall}q, ^{\exists}r)$. We verify the ($\to$ -zig) property. So, it suffices to show that for every $\alpha' \geq \alpha$ there exists $\beta' \geq \beta$ such that

$$\alpha' \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta' \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r)$.

Suppose there exists $\alpha' \geq \alpha$ such that for every $\beta' \geq \beta$ there exists a formula $\varphi_{\beta'}(\overline{x})$ with $char(\varphi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r)$ such that

$$(\alpha' \Vdash_{\mathcal{K}} \varphi_{\beta'}[\overline{a}] \text{ and } \beta' \nVdash_{\mathcal{M}} \varphi_{\beta'}[\overline{b}]) \text{ or } (\alpha' \nVdash_{\mathcal{K}} \varphi_{\beta'}[\overline{a}] \text{ and } \beta' \Vdash_{\mathcal{M}} \varphi_{\beta'}[\overline{b}]).$$

Note that, since $\beta' \geq \beta$ range over finite Kripke model $\mathcal{M}$, the set $\{\varphi_{\beta'}(\overline{x})\colon \beta' \geq \beta\}$ is finite too. Consider two following sets

$$\Theta_0 = \{\varphi_{\beta'}(\overline{x})\colon \alpha' \Vdash_{\mathcal{K}} \varphi_{\beta'}[\overline{a}] \text{ and } \beta' \nVdash_{\mathcal{M}} \varphi_{\beta'}[\overline{b}]\}$$

and

$$\Theta_1 = \{\varphi_{\beta'}(\overline{x})\colon \alpha' \nVdash_{\mathcal{K}} \varphi_{\beta'}[\overline{a}] \text{ and } \beta' \Vdash_{\mathcal{M}} \varphi_{\beta'}[\overline{b}]\}.$$

Notice that

$$\alpha' \Vdash_{\mathcal{K}} \bigwedge \Theta_0[\overline{a}] \text{ and } \alpha' \nVdash_{\mathcal{K}} \bigvee \Theta_1[\overline{a}] \tag{1}$$

and

$$\beta' \nVdash_{\mathcal{M}} \bigwedge \Theta_0[\overline{b}] \text{ and } \beta' \Vdash_{\mathcal{M}} \bigvee \Theta_1[\overline{b}] \tag{2}$$

for every $\beta' \geq \beta$. Hence, by (1), we get

$$\alpha \nVdash_{\mathcal{K}} (\bigwedge \Theta_0 \to \bigvee \Theta_1)[\overline{a}]$$

and, by (2), we get

$$\beta \Vdash_{\mathcal{M}} (\bigwedge \Theta_0 \to \bigvee \Theta_1)[\overline{b}].$$

But note that $char(\bigwedge \Theta_0 \to \bigvee \Theta_1) \leq (^{\to}p+1, ^{\forall}q, ^{\exists}r) \leq (^{\to}P+1, ^{\forall}Q, ^{\exists}R)$ which is contrary to the assumption.

(ii) Now, for $p \leq P, q < Q, r \leq R$ assume that $(\overline{a};\overline{b})\colon \alpha \sim_{p,q+1,r} \beta$, i.e

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p, ^{\forall}q+1, ^{\exists}r)$. We verify the ($\forall$-zig) property. It suffices to show that for every $\alpha' \geq \alpha$ and every element $a_{\alpha'} \in K_{\alpha'}$ there exist $\beta' \geq \beta$ and $b_{\beta'} \in M_{\beta'}$ such that

$$\alpha' \Vdash_{\mathcal{K}} \varphi[\overline{a}, a_{\alpha'}] \iff \beta' \Vdash_{\mathcal{M}} \varphi[\overline{b}, b_{\beta'}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$. For simplicity we will suppress the parameters $\overline{a}$ and $\overline{b}$.

Let us suppose there exist $\alpha' \geq \alpha$ and an element $a_{\alpha'} \in K_{\alpha'}$ such that for every $\beta' \geq \beta$ and $b_{\beta'} \in M_{\beta'}$ there exists a formula $\varphi_{b_{\beta'}}(x)$ with $char(\varphi) \leq (^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$ such that

$$(\alpha' \Vdash_{\mathcal{K}} \varphi_{b_{\beta'}}[a_{\alpha'}] \text{ and } \beta' \nVdash_{\mathcal{M}} \varphi_{b_{\beta'}}[b_{\beta'}]) \text{ or } (\alpha' \nVdash_{\mathcal{K}} \varphi_{b_{\beta'}}[a_{\alpha'}] \text{ and } \beta' \Vdash_{\mathcal{M}} \varphi_{b_{\beta'}}[b_{\beta'}]).$$

Notice that, since $\beta' \geq \beta$ and $b_{\beta'}$ range over finite Kripke model $\mathcal{M}$ and finite structure $M_{\beta'}$ respectively, the set $\{\varphi_{b_{\beta'}}(x)\colon \beta' \geq \beta, b_{\beta'} \in M_{\beta'}\}$ is finite too. So, consider two following sets

$$\Theta_0 = \{\varphi_{b_{\beta'}}(x)\colon \alpha' \Vdash_{\mathcal{K}} \varphi_{b_{\beta'}}[a_{\alpha'}] \text{ and } \beta' \nVdash_{\mathcal{M}} \varphi_{b_{\beta'}}[b_{\beta'}]\}$$

and

$$\Theta_1 = \{\varphi_{b_{\beta'}}(x)\colon \alpha' \nVdash_{\mathcal{K}} \varphi_{b_{\beta'}}[a_{\alpha'}] \text{ and } \beta' \Vdash_{\mathcal{M}} \varphi_{b_{\beta'}}[b_{\beta'}]\}.$$

Note that

$$\alpha' \Vdash_{\mathcal{K}} \bigwedge \Theta_0[a_{\alpha'}] \text{ and } \alpha' \nVdash_{\mathcal{K}} \bigvee \Theta_1[a_{\alpha'}] \tag{3}$$

and

$$\beta' \nVdash_{\mathcal{M}} \bigwedge \Theta_0[b_{\beta'}] \text{ and } \beta' \Vdash_{\mathcal{M}} \bigvee \Theta_1[b_{\beta'}] \tag{4}$$

for every $\beta' \geq \beta$ and $b_{\beta'} \in M_{\beta'}$. Hence, by (3), we get

$$\alpha \nVdash_{\mathcal{K}} \forall_y (\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(y),$$

and, by (4), we get

$$\beta \Vdash_{\mathcal{K}} \forall_y (\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(y).$$

But notice that $char(\forall_y(\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)) \leq (^{\rightarrow}p+1, ^{\forall}q+1, ^{\exists}r) \leq (^{\rightarrow}P+1, ^{\forall}Q, ^{\exists}R)$ which is contrary to the assumption.

(iii) To finish the proof, for $p \leq P, q \leq Q, r < R$ assume that $(\overline{a};\overline{b})\colon \alpha \sim_{p,q,r+1} \beta$, i.e

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p, ^{\forall}q, ^{\exists}r+1)$. We verify the ($\exists$-zig) property. So, we have to show that for every $a \in K_\alpha$ there exists $b \in M_\beta$ such that

$$\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}, a] \iff \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}, b]$$

for all formulas $\varphi(\overline{x})$ with $char(\varphi) \leq (^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$. Once again, for simplicity, we will suppress the parameters $\overline{a}$ and $\overline{b}$.

Let $a \in K_\alpha$. Suppose such an element $b \in M_\beta$ does not exist. Then, for every $b \in M_\beta$ there exists a formula $\varphi_b(x)$ with $char(\varphi) \leq (^{\to}p, ^{\forall}q, ^{\exists}r)$ such that

$$(\alpha \Vdash_{\mathcal{K}} \varphi_b[a] \text{ and } \beta \nVdash_{\mathcal{M}} \varphi_b[b]) \text{ or } (\alpha \nVdash_{\mathcal{K}} \varphi_b[a] \text{ and } \beta \Vdash_{\mathcal{M}} \varphi_b[b]).$$

Notice that, since the structure M_β is finite, the set $\{\varphi_b(x) \colon b \in M_\beta\}$ is finite too. So, consider two following sets

$$\Theta_0 = \{\varphi_b(x) \colon \alpha \Vdash_{\mathcal{K}} \varphi_b[a] \text{ and } \beta \nVdash_{\mathcal{M}} \varphi_b[b]\}$$

and

$$\Theta_1 = \{\varphi_b(x) \colon \alpha \nVdash_{\mathcal{K}} \varphi_b[a] \text{ and } \beta \Vdash_{\mathcal{M}} \varphi_b[b]\}.$$

Since $\alpha \Vdash_{\mathcal{K}} \bigwedge \Theta_0[a]$ and $\alpha \nVdash_{\mathcal{K}} \bigvee \Theta_1[a]$, then

$$\alpha \Vdash_{\mathcal{K}} (\bigvee \Theta_1 \to \bigwedge \Theta_0)[a]. \tag{5}$$

Note also that $\beta \nVdash_{\mathcal{M}} \bigwedge \Theta_0[b]$ and $\beta \Vdash_{\mathcal{M}} \bigvee \Theta_1[b]$ for all $b \in M_\beta$. Thus,

$$\beta \nVdash_{\mathcal{M}} (\bigvee \Theta_1 \to \bigwedge \Theta_0)[b]. \tag{6}$$

And hence, by (5) and (6),

$$\alpha \Vdash_{\mathcal{K}} \exists_x (\bigvee \Theta_1 \to \bigwedge \Theta_0)(x)$$

and

$$\beta \nVdash_{\mathcal{M}} \exists_x (\bigvee \Theta_1 \to \bigwedge \Theta_0)(x).$$

But note that $char(\exists_x(\bigvee \Theta_1 \to \bigwedge \Theta_0)) \leq (^{\to}p{+}1, ^{\forall}q, ^{\exists}r{+}1) \leq (^{\to}P{+}1, ^{\forall}Q, ^{\exists}R)$ which is contrary to the assumption. □

Under the same conditions, the above Theorem leads us to the following

Corollary 3.3. *Consider worlds α and β of strongly finite Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively. Let $\overline{a}$ and $\overline{b}$ be sequences of the elements of the structures K_α and M_β respectively, and let $p, q, r \geq 0$. Then,*

$$(\alpha, \overline{a}) \equiv_{p+1,q,r} (\beta, \overline{b}),$$

if and only if $(\overline{a}; \overline{b}) \colon \alpha \sim_{p,q,r} \beta$ for some bisimulation $\sim$.

Proof. It follows from Theorems 3.1 and 3.2. □

4 Concluding remarks and further work

In order to generalise Main Theorem, we must establish which assumptions are possible to remove, and which are necessary to remain the theorem true.

Having analysed corresponding theorems of classical model theory, we assumed that the frame $(K, \leq)$ is finite. Let us notice that the possible way to generalise our result is by considering *locally* finite frame $(K, \leq)$, i.e. a frame such that for every node $\alpha \in K$ there are finitely many worlds accessible from α.

If we consider the conditions of finite signature L with no function symbols and finite first-order structures assigned to nodes of the frame $(K, \leq)$, the natural way that we should take is to consider unnested formulas.

As we saw in the proof of Main Theorem, we did not obtain the symmetry between p, q, r-equivalence $\equiv_{p,q,r}$ and p, q, r-bisimulation $\sim_{p,q,r}$. In order to describe the theory of the world $\alpha \in K$ we had to determine satisfied and refuted formulas, and consider one additional implication. Maybe we could think of the following normal form for intuitionistic first-order formulas:

$$\Theta_0 := \{\varphi(\overline{x})\colon char(\varphi) \leq ({}^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0)\},$$
$$\Gamma_n := \{\forall_x(\varphi_1 \rightarrow \varphi_2)(\overline{y}, x)\colon \varphi_1, \varphi_2 \in \Theta_{n-1}\},$$
$$\Psi_n := \{\exists_x(\varphi_1 \rightarrow \varphi_2)(\overline{y}, x)\colon \varphi_1, \varphi_2 \in \Theta_{n-1}\}$$

and

$$\Theta_n := \Gamma_n \cup \Psi_n.$$

Another way that we could consider is to describe the class of Kripke models that Main Theorem refers to.

The above considerations constitute only inspiration and motivation for further work. Nonetheless, the generalisation of Main Theorem is not an immediate result.

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