Marcin Boryc

Uniwersytet M. Curie-Skłodowskiej w Lublinie

Some Classical Singular Stochastic Control Problems – a Survey

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MARCIN BORYC
Institute of Mathematics
Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1
20-031 Lublin Poland

Abstract

The goal of this paper is to present several singular stochastic control problems on the infinite time horizon and the most important results related to them. The regularity of the value function and the free boundary will be considered under various assumptions. We will also discuss the behaviour of optimally controlled state process, which in some cases is any reflected Brownian motion with possible initial jump.

1 Notation, assumptions and HJB equation.

Let $(W_t, t \geq 0)$ be a standard n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t, t \geq 0)$ be the filtration generated by W.

Consider the state process described by the stochastic integral equation

$$X_t = x + \int_0^t f(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \upsilon_t, \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is an initial position, $f : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{M}^{n\times n}$ are the drift and diffusion of the state process respectively and $\upsilon = (\upsilon_t, t \geq 0)$ is a process of control.

We assume, that $f, \sigma \in C^2(\mathbb{R}^n)$ are Lipschitz continuous and σ is nonvanishing, i.e. for all $x \in \mathbb{R}^n$ we have $\sigma(x) \neq 0$. Let υ be a left-continuous, adapted to $(\mathcal{F}_t, t \geq 0)$, $\mathbb{R}^n$-valued process such that (s.t.), for all $T \geq 0$, P almost surely (a.s.), the variation of $\upsilon_\cdot(\omega)$ on the interval $\langle 0, T\rangle$ is finite and $\upsilon_0 = 0$ a.s.. Let $\mathcal{Q}$ denote the set of all such processes of control, called the set of admissible controls.

For every $\upsilon \in \mathcal{Q}$, we write

$$\upsilon_t = \int_0^t N_s d\xi_s, \quad t \geq 0 \tag{2}$$

where $|N_t| = 1$ for every $t \geq 0$ a.s. and ξ is a nondecreasing and left-continuous process. In other words, $\xi_t(\omega)$ is the total variation of $\upsilon_\cdot(\omega)$ on the interval $\langle 0, T\rangle$ and $N_t(\omega)$ is the Radon-Nikodym derivative of the measure induced on $\langle 0, \infty)$ by $\upsilon(\omega)$ with respect to its total variation $\xi_\cdot(\omega)$. We can heuristically understand N_t as direction of the control at time t and ξ_t as the intensity of the control up to time t. We shall identify the control υ with the pair (N, ξ).

For a given control process $\upsilon \in \mathcal{Q}$ we define the corresponding cost function by

$$V_\upsilon(x) = E^x \int_0^\infty e^{-\alpha t}\Big(h(X_t)\, dt + d\xi_t\Big), \quad x \in \mathbb{R}^n \tag{3}$$

where $\alpha \geq 0$ is a discount factor and $h : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative, stricly convex function of the class C^2.

We can understand the function h as the cost of state process per unit time and $d\xi_t$ as the analogous cost of control. **In singular stochastic control problems the function of the cost of control is nonnegative homogeneous.**

The task is to minimize $V_\upsilon(x)$ in the class of all admissible controls, i.e., to find the so called value function

$$V(x) = \inf_{\upsilon \in \mathcal{Q}} V_\upsilon(x), \quad x \in \mathbb{R}^n. \tag{4}$$

If this minimum is attained for some $(N^*, \xi^*) \in \mathcal{Q}$, then we say that

$$\upsilon_t^* = \int_0^t N_s^* d\xi_s^*, \quad t \geq 0 \tag{5}$$

is an optimal policy (for the initial position x).

It is known that for stochastic control problems the following statement holds (see [5], Chapter VIII.5).

Theorem 1.1. *Dynamic Programming Principle of Bellman.*
For every stopping time $T \geq 0$ of the filtration $(\mathcal{F}_t)$ and every $x \in \mathbb{R}^n$

$$V(x) = \inf_{\upsilon \in \mathcal{Q}} \left\{ E^x \int_0^T e^{-\alpha t} \Big(h\left(X_t\right) dt + d\xi_t \Big) + E^x \Big(e^{-\alpha T} V\left(X_T\right) \Big) \right\}. \tag{6}$$

Moreover, if we assume that $f \equiv 0$, $\sigma = I_n$, where I_n denotes the $n \times n$ identity matrix, then the above infimum is attained for the optimal policy ([6], Lemma 2.10). Hence we can conclude ([6], Section 2.5) that the optimally controlled process $(X_t, t \geq 0)$ is a strong Markov process with respect to the filtration $(\mathcal{F}_t, t \geq 0)$.

Formal differentation of the Dynamic Programming Principle ([5], Chapters III and VIII) leads us to **Hamilton-Jacobi-Bellman (HJB) equation**:

$$\max \Big\{ \alpha\, V(x) - tr\, \big(a(x)\ D^2 V(x) \big) - f^T(x)\, DV(x) - h(x)\ ;\ |DV(x)| - 1 \Big\} = 0, \quad x \in \mathbb{R}^n, \tag{7}$$

where $a(x) = \frac{1}{2}\sigma(x)\sigma^T(x)$, $DV = \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right]$ and $D^2 V = \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]_{i,j=1,\ldots,n}$.

Definition 1.2. *We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is classical solution of the HJB equation (7) if $u \in C^2(\mathbb{R}^n)$ satisfies the equation (7) for each $x \in \mathbb{R}^n$.*

Next we adduce the verification theorem ([5], Th.VIII.4.1).

Theorem 1.3. *Verification.*
Let $u \in C^2\left(\mathbb{R}^n\right)$ be a classical solution of the HJB equation (7). Then for every $x \in \mathbb{R}^n$ we have $u(x) \leq V_\upsilon(x)$ for each control $\upsilon \in \mathcal{Q}$ s.t.

$$\liminf_{t \to \infty} E^x e^{-\alpha t} u\left(X_t\right) = 0. \tag{8}$$

Moreover, for a given $x \in \mathbb{R}^n$, we suppose that there exist a control process $(N, \xi) \in \mathcal{Q}$ s.t. $V_{(N,\xi)} < \infty$ and the corresponding state process X_t satisfies

$$\Big|Du\left(X_t\right)\Big| - 1 < 0, \text{ for almost every } t \geq 0, \ a.s., \tag{9}$$

$$\int_0^t \mathbb{I}_{\{N_s = -Du(X_s)\}} d\xi_s = \xi_t, \text{ for all } t \geq 0, \ a.s., \tag{10}$$

$$u(X_t) - u(X_{t+}) = \xi_{t+} - \xi_t, \text{ for all } t \geq 0, \ a.s., \tag{11}$$

$$\text{there exists } \lim_{t\to\infty} E^x e^{-\alpha t} u\left(X_t\right) = 0. \tag{12}$$

Then $u(x) = V_{(N,\xi)}(x) = V(x)$.

Generally, we cannot expect that the value function is a function of the class C^2, so it is possible that the value function is not a classical solution of the HJB equation. Therefore, in stochastic control problems, it is convenient to use the notions of a generalized solution and a viscocity solution.

DEFINITION 1.4. *We say that a function* $u : \mathbb{R}^n \to \mathbb{R}$ *is a generalized solution of the HJB equation* (7) *if* u *is twice differentiable at almost every point* $x \in \mathbb{R}^n$ *and in these points satisfies the equation* (7).

DEFINITION 1.5. *We say that a function* $u : \mathbb{R}^n \to \mathbb{R}$ *is of polynomial growth (or polynomially growing) if there exist* $C > 0$ *and* $m \in \mathbb{N}$ *s.t. for all* $x \in \mathbb{R}^n$ *we have*

$$|u(x)| \leq C\left(1 + |x|^m\right). \tag{13}$$

DEFINITION 1.6. *Viscocity solution.*

Let $u : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function of polynomial growth. We say that* u *is a viscocity subsolution (supersolution) of the HJB equation* (7) *if for each function of polynomial growth* $w \in C^2\left(\mathbb{R}^n\right)$ *and* $\overline{x} \in \mathbb{R}^n$ *which is a maximizer (minimizer) of* $u - w$ *on* $\mathbb{R}^n$ *with* $u(\overline{x}) = w(\overline{x})$ *we have*

$$\max\Big\{\alpha \ w(\overline{x}) - tr\left(a(\overline{x}) \ D^2 w(\overline{x})\right) - f^T(\overline{x}) \ Dw(\overline{x}) - h(\overline{x}) \ ; \ |Dw(\overline{x})| - 1\Big\} \leq \ (\geq) \ 0. \tag{14}$$

We say that u *is a viscocity solution of the HJB equation if it is both a viscocity subsolution and viscocity supersolution of the HJB equation.*

THEOREM 1.7. *Assume that* $V \in C\left(\mathbb{R}^n\right)$ *is a function of polynomial growth and the Dynamic Programming Principle* (6) *holds. Then* V *is a viscocity solution of the HJB equation* (7) ([5], Th.VIII.5.1).

In Sections 4 and 5, we will see that the value function is a generalized solution of the HJB equation, but a generalized solution is not unique. However, from the above theorem we know that the value function V is a viscocity solution of the HJB equation provided that V is continuous and polynomially growing.

Ending this section we recall the notions of the Hölder space and the Sobolev space.

Definition 1.8. *Let $O \subset \mathbb{R}^n$ be an open set. The α-Hölder seminorm of function $u : O \to \mathbb{R}$, $\alpha \in (0, 1\rangle$ is defined as*

$$\sup_{x,y \in O} \left\{ \frac{u(x) - u(y)}{|x-y|^\alpha} : x \neq y \right\}. \tag{15}$$

Definition 1.9. *Let $O \subset \mathbb{R}^n$ be an open set. The Hölder space $C^{k,\alpha}(O)$, $k \in \mathbb{N}$, $\alpha \in (0, 1\rangle$ consists of all bounded functions $u \in C^k(O)$ s.t. each partial derivative of degree less or equal k of function u is bounded and for each partial derivative of degree k of function u the α-Hölder seminorm is finite. We say that $u \in C^{k,\alpha}_{loc}(O)$ if $u \in C^{k,\alpha}(U)$ for each open bounded set $U \subset \overline{U} \subset O$.*

Definition 1.10. *Let $O \subset \mathbb{R}^n$ be an open set, $i, j \in \{1, \ldots, n\}$ and $u : O \to \mathbb{R}$. The function $w : O \to \mathbb{R}$ is generalized $\frac{\partial^2}{\partial x_i \partial x_j} u$ on O if for each test function $\varphi \in C^\infty_c(O)$ (i.e. $\varphi \in C^\infty(O)$ and has compact support) we have*

$$\int_O u(x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) dx = \int_O w(x)\varphi(x) dx. \tag{16}$$

This generalized partial derivative, if it exists, is unique up to a set of measure 0 (see [4], Chapter 5).

Definition 1.11. *Let $O \subset \mathbb{R}^n$ be an open set. We say that a function $u : O \to \mathbb{R}^n$ belongs to the Sobolev space $W^{2,\infty}(O)$, if $u \in C^1(O) \cap L^\infty(O)$ and there exist all generalized partial derivatives of degree 2 which are of class $L^\infty(O)$. We say that $u \in W^{2,\infty}_{loc}(O)$ if $u \in W^{2,\infty}(U)$ for each open bounded set $U \subset \overline{U} \subset O$.*

2 The one-dimensional case.

The section is devoted to one-dimensional stochastic control problems. First, we assume the following (see [7]):

$$f(x) = ax + b, \quad a, b \in \mathbb{R}, \tag{17}$$

$$\exists_{C>0}\ \forall_{x \in \mathbb{R}} \quad \left|\sigma'(x)\right| + \left|\left(\sigma^2(x)\right)''\right| \leq C, \tag{18}$$

$$\forall_{x \in \mathbb{R}} \quad \left|\sigma(x)\right| \neq 0, \tag{19}$$

$$\exists_{c,C>0}\ \forall_{x \in \mathbb{R}} \quad 0 < c \leq h''(x) \leq C, \tag{20}$$

$$\exists_{\overline{x} \in \mathbb{R}}\ \forall_{x \in \mathbb{R}} \quad (x - \overline{x})h'(x) \geq 0, \quad h'(\overline{x}) = 0, \tag{21}$$

$$\alpha > \frac{1}{2} \sup_{x \in \mathbb{R}} \left|\left(\sigma^2(x)\right)''\right| + 2a. \tag{22}$$

So, the drift of the state process f is a linear function and diffusion of the state process σ is a nonvanishing, C^2-function of linear growth. The function h is stricly convex C^2-function and has the minimum equals 0 at $\overline{x}$. For simplicity, we assume that $\overline{x} = 0$.

The corresponding HJB equation takes the form

$$\max\left\{\alpha V(x) - \frac{1}{2}\sigma^2(x)V''(x) - (ax+b)V'(x) - h(x)\ ;\ \left|V'(x)\right| - 1\right\} = 0, x \in \mathbb{R}. \quad (23)$$

Jin Ma has proved in [7] the following result.
THEOREM 2.1. *Consider the equation*

$$\alpha V(x) - \frac{1}{2}\sigma^2(x)V''(x) - (ax+b)V'(x) - h(x) = 0, x \in \mathbb{R}. \quad (24)$$

There exists a unique interval $\langle A, B\rangle \subset \mathbb{R}$ *on which there exists a unique, convex classical solution of* (24) *with boundary conditions*

$$V'(A) = -1, \quad V'(B) = 1, \quad (25)$$

$$V''(A) = V''(B) = 0. \quad (26)$$

Let $V_{A,B}(x)$ *denote this solution. Then the value function of our singular stochastic control problem is given by*

$$V(x) = \begin{cases} A - x + V_{A,B}(A), & x < A, \\ V_{A,B}(x), & x \in \langle A, B\rangle, \\ x - B + V_{A,B}(B), & x > B. \end{cases} \quad (27)$$

Moreover, the optimal control (for the initial position x*) is given by*

$$\upsilon_t = \begin{cases} A - x + \xi_t^A - \xi_t^B, & x < A, \\ \xi_t^A - \xi_t^B, & x \in \langle A, B\rangle, \\ B - x + \xi_t^A - \xi_t^B, & x > B, \end{cases} \quad (28)$$

where ξ_t^A, ξ_t^B *are adapted to* $(\mathcal{F}_t, t \geq 0)$*, continuous, nondecreasing processes, which are* 0 *at* $t = 0$ *s.t. for all* $t \geq 0$ *the optimally controlled process satisfies*

$$X_t = x + \int_0^t (aX_s + b)ds + \int_0^t \sigma(X_s)\, dW_s + \xi_t^A - \xi_t^B \ \in\ \langle A, B\rangle, \quad (29)$$

and

$$\xi_t^A = \int_0^t \mathbb{I}_{\{X_s = A\}} d\xi_s^A, \quad \xi_t^B = \int_0^t \mathbb{I}_{\{X_s = B\}} d\xi_s^B. \quad (30)$$

Hence, the value function satisfies the second order differential equation (24) on some interval $\langle A, B\rangle$ and is a linear function outside this interval. **The value function exists, is unique, convex and of class C^2.**

The optimal state process is a diffusion process reflected at the boundary of the interval $\langle A, B\rangle$, with immediate jump to A or B if the initial posision x is outside the interval.

We remark that in the literature (see [2],[3]) the conditions (25), (26) are called *the smooth pasting condition* and *the super contact condition* respectively.

Example 2.2. (See [5], Ex.VIII.4.2). Let $f(x) \equiv 0$, $\sigma(x) \equiv \sqrt{2}$, $h(x) = x^2$, $\alpha = 1$. Obviously, the assumptions (17) – (22) hold. The corresponding HJB equation takes the form

$$\max\left\{V(x) - V''(x) - x^2 \ ; \ \left|V'(x)\right| - 1\right\} = 0, \ x \in \mathbb{R}. \tag{31}$$

We want to contruct a convex, polynomially growing solution u of (31) and then use the verification theorem (Th. 1.3) to show that $u \equiv V$. Now we suppose that u is indeed a convex solution of (31). Set

$$A = \inf\left\{x \in \mathbb{R} : u'(x) > -1\right\}, \tag{32}$$

$$B = \sup\left\{x \in \mathbb{R} : u'(x) < 1\right\}. \tag{33}$$

So from (31) we have

$$u(x) - u''(x) = x^2, \ x \in \langle A, B\rangle, \tag{34}$$

$$u'(x) = 1, \quad x > B, \tag{35}$$

$$u'(x) = -1, \quad x < A. \tag{36}$$

The values of A and B are not a priori given for us, so we solve (34), (35) and (36) for every real numbers $A < B$ and then determine the values of A, B by using (31) again. First, we observe that the general solution of the homogeneous equation $u(x) - u''(x) = 0$ is

$$u_H(x) = C_1 e^x + C_2 e^{-x}, \quad C_1, C_2\text{-constant} \tag{37}$$

and a particular solution of the inhomogeneous equation $u(x) - u''(x) = x^2$ is for example $u_P(x) = x^2 + 2$. Hence every solution of (34) takes the form

$$u(x) = C_1 e^x + C_2 e^{-x} + x^2 + 2, \ x \in \langle A, B\rangle. \tag{38}$$

Now the solutions of (35) and (36) are respectively

$$u(x) = x + C_3, \quad x > B, \ C_3\text{-constant}, \tag{39}$$

$$u(x) = -x + C_4, \quad x < A, \ C_4\text{-constant}, \tag{40}$$

We want to determine the values of A, B, C_1, C_2, C_3, C_4 s.t. the function u be twice continuously differentiable. So we must solve the system of equation

$$\begin{cases} u(A) = u(A-) \\ u(B) = u(B+) \\ u'_+(A) = u'_-(A) \\ u'_+(B) = u'_-(B) \\ u''_+(A) = u''_-(A) \\ u''_+(B) = u''_-(B) \end{cases} \Longleftrightarrow \begin{cases} C_1 e^A + C_2 e^{-A} + A^2 + 2 = -A + C_4 \\ C_1 e^B + C_2 e^{-B} + B^2 + 2 = B + C_3 \\ C_1 e^A - C_2 e^{-A} + 2A = -1 \\ C_1 e^B - C_2 e^{-B} + 2B = 1 \\ C_1 e^A + C_2 e^{-A} + 2 = 0 \\ C_1 e^B + C_2 e^{-B} + 2 = 0. \end{cases} \tag{41}$$

An elementary computation (See [5], Ex.VIII.4.2) yields

$$u(x) = \begin{cases} x^2 + 2 + \frac{1-2B}{\sinh(B)}\cosh(x), \ x \in \langle A, B\rangle \\ B^2 + x - B, \quad x > B \\ A^2 - x + A, \quad x < A, \end{cases} \tag{42}$$

where $A = -B$ and B is unique positive solution of

$$\tanh(B) = b - \frac{1}{2}. \tag{43}$$

Next, for a given initial position x, we take a cotrol $(N, \xi) \in \mathcal{Q}$ satisfies for all $t \geq 0$

$$N_t = \begin{cases} -1 & , \ X_t > 0 \\ 1 & , \ X_t \leq 0 \end{cases} \tag{44}$$

$$\xi_t = (x - B)^+ + (A - x)^+ + \xi_t^A + \xi_t^B, \tag{45}$$

where ξ_t^A, ξ_t^B are given by (30). We see that the control (N, ξ) satisfies the assumptions of the verification theorem (9) – (12). Consequently, $u(x) = V(x)$ for each $x \in \mathbb{R}$.

The optimally controlled process X_t is the Brownian motion reflected at the boundary of $\langle A, B \rangle$ with initial jump to A or B if $x \notin \langle A, B \rangle$. Also ξ_t^A, ξ_t^B are the sums of local times of $X.$ at A and B respectively.

Ending this section we observe that in this example the conclusion of Theorem 2.1 obviously holds.

3 The two-dimensional case.

This section contains results proved by H. M. Soner and E. Shreve in [9]. They concern the control of a two-dimensional Brownian motion, when control can cause displacement in any direction. Recall that $C^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in (0, 1\rangle$ denote the Hölder space and $W^{2,\infty}$ - the Sobolev space. Consider a two-dimensional state process for which the drift function $f \equiv 0$ and the diffusion function $\sigma = \sqrt{2} I_2$, i.e.

$$X_t = x + \sqrt{2} W_t + \upsilon_t, \ t \geq 0. \tag{46}$$

Let the function h satisfy

$$h \in C^{2,1}_{loc}(\mathbb{R}^2), \tag{47}$$

$$\exists_{C,q>0} \ \forall_{x \in \mathbb{R}^2} \quad 0 \leq h(x) \leq C\left(1 + |x|^q\right), \tag{48}$$

$$\exists_{C>0} \ \forall_{x \in \mathbb{R}^2} \quad |Dh(x)| \leq C\left(1 + h(x)\right), \tag{49}$$

$$\exists_{c,C>0} \ \forall_{x,y \in \mathbb{R}^2} \quad c|y|^2 \leq D^2 h(x) y \cdot y \leq C|y|^2\left(1 + h(x)\right), \tag{50}$$

where $\cdot$ denote the standard inner product of vectors. Without loss of generality, we also assume that

$$\forall_{x \in \mathbb{R}^2} \quad 0 = h(0) \leq h(x). \tag{51}$$

The corresponding HJB equation (7) takes the form

$$\max\Big\{\alpha \ V(x) - \Delta V(x) - h(x) \ ; \ |DV(x)| - 1\Big\} = 0, \quad x \in \mathbb{R}^2, \tag{52}$$

where $\Delta V(x) = \frac{\partial^2}{\partial x_1^2} V(x) + \frac{\partial^2}{\partial x_2^2} V(x)$ is the Laplacian of V.

Theorem 3.1. *The HJB equation* (52) *has a unique nonnegative, convex, polynomially growing generalized solution* $V \in W^{2,\infty}_{loc}$.

Throughout the remainder of this section, V will denote the unique solution of (52) according to Theorem 3.1.

Now we define a set $\mathcal{C}$ which is called *the non-action region* for our singular stochastic control problem.

DEFINITION 3.2.

$$\mathcal{C} = \{x \in \mathbb{R} : |DV(x)| < 1\}. \tag{53}$$

The set $\partial\mathcal{C}$ is the so called *free boundary of the non-action region.*

REMARK 3.3. The analogous definition is used in a general n-dimensional singular stochastic control case. From the HJB equation (7) we obviously see

$$\forall_{x \in \mathcal{C}} \quad \alpha\, V(x) - tr\left(a(x)\ D^2V(x)\right) - f^T(x)\ DV(x) - h(x) = 0. \tag{54}$$

REMARK 3.4. In the one-dimensional case, considered in Section 2, the set $\mathcal{C}$ is an interval (A, B). The optimal control acts if and only if the state process $X_t \notin \mathcal{C}$. Hence the name *the non-action region.* Moreover, on the free boundary $\partial\mathcal{C} = \{A, B\}$, the value function V satisfies (25)-(26). In this context the conditions (25)-(26) are called *the principle of smooth fit* which is used to determine the free parameters that arose in the solution of a singular stochastic control problem (see Example 2.2).

An important question is whether *the principle of smooth fit* **holds in multidimensional singular control problems**. The discovery of a C^2 value function provides strong support for belief in a widely applicable *principle of smooth fit.*

Let us return to the two-dimensional problem with $f \equiv 0$ and $\sigma = \sqrt{2}I_2$. Inside the set $\mathcal{C}$ the function V satisfies a elliptic equation $V - \Delta V = h$ and therefore is smooth at least $C^{4,\alpha}$ for all $\alpha \in (0, 1)$, because we have assumed (47). The next two theorems concern the behaviour of D^2V outside $\mathcal{C}$.

THEOREM 3.5. *For every $\alpha \in (0, 1)$, $V \in C^{2,\alpha}(\bar{\mathcal{C}})$, i.e. D^2V restricted to $\mathcal{C}$ has an α-Hölder continuous extension to $\bar{\mathcal{C}}$.*

THEOREM 3.6. *There is a Lipschitz continuous version of D^2V on $\mathbb{R}^2$.*

In the proofs of the above theorems the following important lemmas have been used.

LEMMA 3.7. *Let $z \in \partial\mathcal{C}$ will be given. Then*

$$\lim_{\mathcal{C} \ni x \to z} D^2V(x) = \underbrace{(V(z) - h(z)) \begin{bmatrix} V_2^2(z) & -V_1(z)V_2(z) \\ -V_1(z)V_2(z) & V_1^2(z) \end{bmatrix}}_{A(z)}, \tag{55}$$

where V_i denotes the ith partial derivative of V.

LEMMA 3.8. *We have for almost every $x \in \mathbb{R}^2 \setminus \mathcal{C}$*

$$D^2V(x)\ DV(x) = 0\ ;\ ||D^2V(x)|| = \Delta V(x), \tag{56}$$

where $||\cdot||$ denotes the euclidean norm of a matrix.

REMARK 3.9. The characterization of $A(z)$ used in the proof of Lemma 3.7 makes critical use of the fact that our problem is posed in two dimensions. The two-dimensional nature of the problem also plays a fundamental role in Lemma 3.8.

The next theorem concerns regularity of the free boundary $\partial\mathcal{C}$.

THEOREM 3.10. *For all $\alpha \in (0,1)$ the free boundary $\partial\mathcal{C}$ is of class $C^{2,\alpha}$. Moreover, if in place of assumption (47), we assume that $h \in C^{k,\alpha}_{loc}$ for some $k \geq 3,\ \alpha \in (0,1)$, then the free boundary is of class $C^{k,\alpha}$.*

Hence, in the two-dimensional case, we have C^2-regularity both of the value function and of the free boundary. This regularity suffices to define a reflected Brownian motion in $\overline{\mathcal{C}}$ with reflection direction $-DV$ along $\partial\mathcal{C}$. It has been proved that the optimal policy is a solution of the so called *Skorokhod problem.*

DEFINITION 3.11. *Let $x \in \overline{\mathcal{C}}$ will be given. A control process $\upsilon = (N, \xi) \in \mathcal{Q}$ is called a solution to the Skorokhod problem for reflected Brownian motion in $\overline{\mathcal{C}}$ starting at x with reflection direction $-DV$ along $\partial\mathcal{C}$ provided that*
(a) *ξ is continuous,*
(b) *the process $X_t = x + \sqrt{2}W_t + \upsilon_t$ satisfies $X_t \in \overline{\mathcal{C}}$ for all $t \geq 0$, a.s.,*
(c) *for all $t \geq 0$*

$$\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial\mathcal{C},\ N_s = -DV(X_s)\}} d\xi_s. \tag{57}$$

There has been proved that for every $x \in \mathcal{C}$ the Skorokhod problem has a solution starting at x.

THEOREM 3.12. *Let $x \in \mathbb{R}^2$ be given. If $x \in \overline{\mathcal{C}}$, then the solution to the Skorokhod problem of Definition 3.11 is the optimal control for the singular stochastic control problem with initial position x. If $x \notin \overline{\mathcal{C}}$, then there exists a unique $\widetilde{x} \in \partial\mathcal{C}$, s.t. (N, ξ) is optimal for the control problem with initial position x, where*

$$N_t = \begin{cases} -DV(x) & , \ t = 0 \\ \widetilde{N}_t & , \ t > 0, \end{cases} \tag{58}$$

$$\xi_t = \begin{cases} 0 & , \ t = 0 \\ \widetilde{\xi}_t + |x - \widetilde{x}| & , \ t > 0 \end{cases} \tag{59}$$

and $(\widetilde{N}, \widetilde{\xi})$ denotes the solution to the Skorokhod problem starting at $\widetilde{x}$.

So the optimally controlled process is a reflected Brownian motion in $\overline{\mathcal{C}}$ with reflection direction $-DV$ along $\partial\mathcal{C}$ with initial jump on $\partial\mathcal{C}$ if the starting position $x \notin \overline{\mathcal{C}}$.

4 Multidimensional case.

We consider a singular stochastic control problem, where $n \geq 2$. First, we discuss, when we can obtain results similar to those presented in Section 3.

We assume the following:

$$\forall_{x \in \mathbb{R}^n} \quad f(x) = 0,\ \sigma(x) = I_n,\ h(x) = \lambda |x|^2, \tag{60}$$

where $\lambda > 0$. Then the corresponding HJB equation takes the form

$$\max\left\{-\frac{1}{2}\Delta V(x)+\alpha V(x)-\lambda|x|^2\ ;\ |DV(x)|-1\right\}=0,\ x\in\mathbb{R}^n. \tag{61}$$

We observe that the value of function h depends actually only on the norm of its argument. So we can expect that the value function V has the same property, i.e. there exists a function $\widetilde{V}:\langle 0,\infty)\to\langle 0,\infty)$, s.t. $V(x)=\widetilde{V}(r)$, where $r=|x|$. Then we can write the HJB equation in the equivalent form

$$\max\left\{-\frac{1}{2}\widetilde{V}''(r)-\frac{n-1}{2r}\widetilde{V}'(r)+\alpha\widetilde{V}(r)-\lambda r^2\ ;\ |\widetilde{V}'(r)|-1\right\}=0,\ r\geq 0. \tag{62}$$

THEOREM 4.1. *Let* (60) *hold. Then the value function* V *is a unique,* C^2*, convex solution of the HJB equation* (61) *and* $V(x)=\widetilde{V}(|x|)$ *for any function* $\widetilde{V}:\langle 0,\infty)\to\langle 0,\infty)$*. The non-action region* $\mathcal{C}$ *(see Def. 3.2 and Rem. 3.3) is an open ball centred at the origin, so the principle of smooth fit holds. Moreover the optimal control exists, is unique and is a solution of the Skorokhod problem for a Brownian motion with reflection at the boundary of the ball* $\mathcal{C}$ *in the direction* $-DV$ *along* $\partial\mathcal{C}$*, modulo an initial jump on the boundary.*

This theorem has been proved by M. H. A. Davis and M. Zervos in [1]. These autors have given a explicit formula of the value function in this case.

Next we assume that the drift function $f\equiv 0$, the diffusion function $\sigma=\sqrt{2}I_n$, i.e

$$X_t=x+\sqrt{2}W_t+v_t,\ t\geq 0. \tag{63}$$

Let there exists $c,C,q>0$, s.t. for all $x,y,x'\in\mathbb{R}^n$, $|x'|\leq 1$, $\lambda\in\langle 0,1\rangle$ we have

$$h\in C^{2,1}(\mathbb{R}^n), \tag{64}$$

$$0\leq h(x)\leq C(1+|x|^q), \tag{65}$$

$$|h(x)-h(x+x')|\leq C\left(1+h(x)+h(x+x')\right)^{1-\frac{1}{q}}|x'|, \tag{66}$$

$$h(x+\lambda x')+h(x+\lambda x')-2h(x)\leq C\lambda^2\left(1+h(x)\right)^r,\ r=\left(1-\frac{2}{q}\right)^+, \tag{67}$$

$$c|y|^2\leq D^2h(x)y\cdot y. \tag{68}$$

For convenience, we also assume that

$$\inf_{x\in\mathbb{R}^n}h(x)=h(0)=0. \tag{69}$$

These assumptions are similar to (46)-(51).

THEOREM 4.2. *Consider an n-dimensional singular stochastic control problem with the assumptions* (63)-(69)*. Then the value function* $V\in W^{2,\infty}_{loc}(\mathbb{R}^n)$ *and it is a generalized solution of the HJB equation* (7)*. The optimal control exists and it is unique* (see [8]).

But there is a problem "**What is it an optimal control?**". Some regularity of both the domain and the vector field determining the reflection direction is necessary to define the Brownian motion with reflection at the boundary of such a domain. **We do not have sufficient information about regularity of both the free boundary** $\partial\mathcal{C}$

and the value function V to define the reflected Brownian motion in $\mathcal{C}$ with reflection direction $-DV$ along $\partial\mathcal{C}$.

Ł. Kruk in [6] has proved that in any dimension under the assumptions (63)-(69) the optimal control is a solution to *the modified Skorokhod problem*. This approach, however, leaves the question of smooth fit of the free boundary in higher dimension unanswered.

Below we present main results obtained by Ł. Kruk in [6].

DEFINITION 4.3. *Let $x \in \overline{\mathcal{C}}$ will be given. A control process $\upsilon = (N, \xi) \in \mathcal{Q}$ is called a solution to the modified Skorokhod problem for reflected Brownian motion in $\overline{\mathcal{C}}$ starting at x with reflection direction $-DV$ along $\partial\mathcal{C}$ provided that*

(a) *for all $t \geq 0$, a.s., we have*

$$\xi_{t+} > \xi_t \quad \text{(i.e. the process } X \text{ has a jump at } t) \tag{70}$$

if and only if there exists an interval

$$I = \{a + sDV(X_t) : \; s \in \langle 0, c\rangle\} \subseteq \partial\mathcal{C}, \tag{71}$$

for some $a \in \mathbb{R}^n, c > 0$, s.t. $X_t \in I \setminus \{a\}$ and $DV(\widetilde{x})$ has the same direction as $DV(X_t)$ for every $\widetilde{x} \in I$. Also, if we assume that I is a maximal interval with such properties (i.e., there is no $\widetilde{I}$ enjoying the same properties and s.t. I is properly contained in $\widetilde{I}$), then

$$X_{t+} = a, \tag{72}$$

(b) *the process $X_t = x + \sqrt{2}W_t + \upsilon_t$ satisfies $X_t \in \overline{\mathcal{C}}$ for all $t \geq 0$, a.s.,*

(c) *for all $t \geq 0$*

$$\xi_t = \int_0^t \mathbb{I}_{\{X_s \in \partial\mathcal{C},\ N_s = -DV(X_s)\}} d\xi_s. \tag{73}$$

In particular, if intervals of the type described in (a) do not exist (e.g., DV is never tangential to $\partial\mathcal{C}$ or $\mathcal{C}$ is stricly convex), then ξ_t is continuous in $\langle 0, \infty)$ a.s. and above definition reduces to the definition of the solution of the Skorokhod problem (see Definition 3.11).

THEOREM 4.4. *For every $x \in \overline{\mathcal{C}}$ the optimal policy $\upsilon = (N, \xi) \in \mathcal{Q}$ is a solution to the modified Skorokhod problem for W_t, $\mathcal{C}$, $-DV$.*

REMARK 4.5. Let the starting point $x \notin \overline{\mathcal{C}}$. Then the optimal policy jumps immediately to some point $\widetilde{x} \in \partial\mathcal{C}$ and then follows the optimal policy starting at $\widetilde{x}$.

REMARK 4.6. It is clear that any process which solves the modified Skorokhod problem for W_t, $\mathcal{C}$, $-DV$ is an optimal policy for our problem (see Verification Theorem 1.2). Thus, uniqueness of the optimal policy (Theorem 4.2) implies uniqueness of a solution to the modified Skorokhod problem for W_t, $\mathcal{C}$, $-DV$.

Usually, some assumptions about regularity of the boundary of a region are necessary to prove existence and uniqueness of a solution to the Skorokhod problem. Here, all such assumptions are hidden in the very nature of the stochastic control problem and, if $n \geq 3$ we do not know what they are, i.e., how regular $\partial\mathcal{C}$ really is. **The conjecture is that for higher dimensions "smooth fit" holds also, $\partial\mathcal{C}$ is $C^{2,\alpha}$ for every $\alpha \in (0,1)$, and DV is notangential to $\partial\mathcal{C}$, so the optimal policy is a solution to the Skorokhod problem in the usual sense, as in the two-dimensional case.**

5 Other singular stochastic control problems.

Consider a singular stochastic control problem, in which a state process X satisfies

$$X_t = x + ft + \sigma W_t + \upsilon_t,\ t \geq 0, \tag{74}$$

where $x \in \mathbb{R}^n$ is initial position, $f \in \mathbb{R}^n, \sigma \in \mathbb{M}^{n\times n}$ are constants representing the drift and the diffusion of the state process respectively, $\{W_t, t \geq 0\}$ is a standard n-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t, t \geq 0)$ is the filtration generated by W and $\{\upsilon_t, t \geq 0\}$ is a control process.

We assume that $a = \frac{1}{2}\sigma\sigma^T$ is a positive definite matrix. The control is realized by $2n$ nondecreasing, left-continuous, adapted to $(\mathcal{F}_t, t \geq 0)$ real-valued processes $\upsilon_t^{i+},\ \upsilon_t^{i-}$, $i = 1, 2, \ldots, n$, s.t.

$$\upsilon_t = \left(\upsilon_t^1, \upsilon_t^2, \ldots, \upsilon_t^n\right),\ t \geq 0, \tag{75}$$

$$\upsilon_t^i = \upsilon_t^{i+} - \upsilon_t^{i-},\ t \geq 0,\ i = 1, 2, \ldots, n, \tag{76}$$

$$\upsilon_0^{i+} = \upsilon_0^{i-} = 0\ a.s.,\ i = 1, 2, \ldots, n. \tag{77}$$

Let $\mathcal{Q}$ denote the set of all such processes of control, called the set of admissible controls. With each admissible control $\upsilon \in \mathcal{Q}$ we associate a cost

$$V_\upsilon(x) = E^x\left\{\int_0^\infty e^{-\alpha t}h(X_t)dt + \sum_{i=1}^n a_i\Big[\int_0^\infty e^{-\alpha t}d\upsilon_t^{i+} + \int_0^\infty e^{-\alpha t}d\upsilon_t^{i-}\Big]\right\}, \quad x \in \mathbb{R}^n, \tag{78}$$

where $a_i,\ i = 1, 2, \ldots, n$ are positive constant and $\alpha > 0$ is discount factor and $h : \mathbb{R}^n \to \mathbb{R}$ is stricly convex, nonnegative function of polynomial growth which satisfies the assumptions (65)-(67).

In this case we count costs of control, in some sense, separately for each control direction. Observe that **the costs of control in any direction are proportional to the variation of the control. This is a characteristic property of singular stochastic control problems**.

The task is to minimize $V_\upsilon(x)$ in the class of all admissible controls, i.e., to find the value function

$$V(x) = \inf_{\upsilon\in\mathcal{Q}} V_\upsilon(x), \quad x \in \mathbb{R}^n. \tag{79}$$

If this minimum is attained for some $\upsilon \in \mathcal{Q}$, then we say that this υ is an optimal policy (for initial position x).

The corresponding HJB equation takes a form

$$\max_{i=1,\dots,n}\left\{\alpha\, V(x) - tr\left(a\ D^2V(x)\right) - f^T\ DV(x) - h(x)\ ;\ \left|\frac{\partial V(x)}{\partial x_i}\right| - a_i\right\} = 0,\ x \in \mathbb{R}^n. \quad (80)$$

REMARK 5.1. Also in this modified case, the theorems from Section 1 are true (see [5]).

J. L. Menaldi and M. I. Taksar have proved in [8] under above assumptions the following theorem.

THEOREM 5.2. *The value function* $V \in W^{2,\infty}_{loc}(\mathbb{R}^n)$ *and is a nonnegative, convex, polynomially growing generalized solution of the HJB equation* (80). *Moreover the optimal control exists and is unique.*

REMARK 5.3. Theorem 5.2 ensures us that the optimal policy exists, but we do not know, what is it. Define so called *the non-action region*:

$$\mathcal{C} = \left\{x \in \mathbb{R}^n : \alpha\ V(x) - tr\left(a\ D^2V(x)\right) - f^T\ DV(x) - h(x) = 0\right\}. \quad (81)$$

It can be shown that the optimal control υ increases only if the corresponding state process X belongs to the so called *the free boundary* $\partial\mathcal{C}$, that is

$$\int_0^\infty \mathbb{I}_{\{X_s \notin \partial\mathcal{C}\}} d\upsilon_s = 0. \quad (82)$$

The conjecture is that the optimal controlled process is a Brownian motion in $\mathcal{C}$ with oblique reflections on $\partial\mathcal{C}$ in the direction $-DV$. However, to construct such a process one needs smoothness of $\partial\mathcal{C}$. **It remains an open problem to show that the free boundary $\partial\mathcal{C}$ and the gradient DV are sufficiently smooth for a construction of the reflected Brownian motion.**

Now we consider a simplification of the above situation, when a control can acts only in positive directions, i.e.

$$\upsilon_t = \left(\upsilon_t^1, \upsilon_t^2, \dots, \upsilon_t^n\right),\ t \geq 0, \quad (83)$$

where $\upsilon_t^i,\ i = 1,\dots,n$ are nondecreasing, adapted to $(\mathcal{F}_t, t \geq 0)$, left-continuous, real-valued processes with $\upsilon_0^i = 0$ a.s.. We denote the set of such control by $\mathcal{Q}^+$. For that control υ and each initial position $x \in \mathbb{R}^n$, the corresponding cost function takes a form

$$V_\upsilon(x) = E^x\left\{\int_0^\infty e^{-\alpha t}h(X_t)dt + \sum_{i=1}^n a_i \int_0^\infty e^{-\alpha t} d\upsilon_t^i\right\}, \quad x \in \mathbb{R}^n, \quad (84)$$

and the HJB equation

$$\max_{i=1,\dots,n}\left\{\alpha\, V(x) - tr\left(a\ D^2V(x)\right) - f^T\ DV(x) - h(x)\ ;\ -\frac{\partial V(x)}{\partial x_i} - a_i\right\} = 0, \quad x \in \mathbb{R}^n, \quad (85)$$

where $V(x) = \inf_{\upsilon \in \mathcal{Q}^+} V_\upsilon(x)$ is of course the value function.

Let assume that $h \in C^3(\mathbb{R}^n)$, h is convex and there exist $c, C > 0$, $p \geq 1$, s.t. for all $x, y \in \mathbb{R}^n$

$$c|x^+|^p - C \leq h(x) \leq C(1 + h(x))^p, \quad x^+ = (x_1^+, x_2^+, \dots, x_n^+), \tag{86}$$

$$|h(x) - h(y)| \leq C(1 + |x|^{p-1} + |y|^{p-1})|x - y|, \tag{87}$$

$$0 \leq \frac{\partial^2 h(x)}{\partial z^2} \leq C(1 + |x|^q), \quad q = (p-2)^+ \tag{88}$$

for any second order directional derivative $\frac{\partial}{\partial z^2}$.

THEOREM 5.4. *Let* (83), (84), (86), (87), (88) *hold. Then the value function* $V \in W^{2,\infty}_{loc}(\mathbb{R}^n)$ *is a convex, nonnegative, polynomially growing generalized solution of the HJB equation* (85).

Next we consider regularity of the free boundary $\partial\mathcal{C}$ (see Rem. 5.3). We know that some regularity the value function and the free boundary is needed to define a reflected Brownian motion in $\mathcal{C}$ with reflection direction $-DV$ along $\partial\mathcal{C}$. We expect the optimally controlled process X to be the reflected Brownian motion. The results below have been obtained by S. A. Williams, P. L. Chow and J. L. Menaldi in [10].

THEOREM 5.5. *Let the assumptions of Theorem 5.4 hold. Then for each* $i = 1, \dots, n$ *there exists a real-valued function* $\psi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, s.t. *for all* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$-\frac{\partial V(x)}{\partial x_i} - a_i = 0, \quad x_i \leq \psi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \tag{89}$$

$$-\frac{\partial V(x)}{\partial x_i} - a_i < 0, \quad x_i > \psi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \tag{90}$$

DEFINITION 5.6. *For any* $i = 1, \dots, n$ *define*

$$\mathcal{C}_i = \left\{ x \in \mathbb{R}^n : \ -\frac{\partial V(x)}{\partial x_j} - a_j < 0 \quad \text{for all } i \neq j \right\}, \tag{91}$$

$$\mathcal{D}_i = \mathcal{C}_i \cap \{x \in \mathbb{R}^n : \ x_i = \psi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\}. \tag{92}$$

Observe that the non-action region is equal

$$\mathcal{C} = \bigcap_{i=1}^{n} \mathcal{C}_i \tag{93}$$

and we can write the free boundary in the form

$$\partial\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{D}_i \cup \mathcal{D}, \tag{94}$$

where $\mathcal{D}$ be so called *the set of corner points.*

For example, for $n = 3$, if $\psi_1(x_2, x_3) \equiv \psi_2(x_1, x_3) \equiv \psi_3(x_1, x_2) \equiv 0$, then the non-action region $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$ is the principal octant, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are the "positive" quarter planes and the corner points are points on the nonnegative coordinate axes. The autors of [10] warn that they do not know whether the corner points always have this simple type of structure (although they believe that to be true).

The next theorem concerns regularity of the free boundary away from corner points.

THEOREM 5.7. *Let the assumptions of Theorem 5.4 hold. Moreover we assume that for each* $i = 1, \ldots, n$

$$\frac{\partial h(x)}{\partial x_i} + \alpha a_i \quad \text{and} \quad grad\left(\frac{\partial h(x)}{\partial x_i}\right) \text{ never vanish simultaneously.} \tag{95}$$

Then for each $i = 1, \ldots, n$

$$\mathcal{D}_i \text{ is a } C^1 \text{ hypersurface,} \tag{96}$$

$$\forall_{\alpha \in (0,1)} \quad \mathcal{D}_i \text{ is a } C^{1,\alpha} \text{ hypersurface,} \tag{97}$$

$$\text{if } h \in C^{k,\alpha} \text{ for any integer } k \geq 2 \text{ and } \alpha \in (0,1), \text{ then } \mathcal{D}_i \text{ is a } C^{k,\alpha} \text{ hypersurface,} \tag{98}$$

$$\text{if } h \text{ is real analityc, then } \mathcal{D}_i \text{ is a real analityc hypersurface.} \tag{99}$$

Ending this section we solve a simple one-dimensional example, in which a control acts only in positive direction (compare [5], Ex.VIII.4.1). It can be shown that in this case the value funtion $V \in C^2$ is a classical solution of the HJB equation (85) and the optimal state process is a diffusion process reflected in any unbounded interval $\langle A, \infty)$.

EXAMPLE 5.8. Let $f = 0, \sigma = \sqrt{2}, h(x) = x^2, \alpha = 1$. The state process satisfies

$$X_t = x + \sqrt{2} W_t + v_t, \tag{100}$$

where $x \in \mathbb{R}^n$ is the initial position and $(v_t, t \geq 0)$ is a left-continuous, adapted to $(\mathcal{F}_t, t \geq 0)$ nondecreasing process with $v_0 = 0$ a.s.. The corresponding HJB equation takes the form

$$\max\left\{V(x) - V''(x) - x^2 \;;\; -V'(x) - 1\right\} = 0, x \in \mathbb{R}. \tag{101}$$

We want to construct a convex, polynomially growing, C^2 solution u of (101) and then use The Verification Theorem 1.3 to show that $u \equiv V$. Suppose that u is indeed a convex, polynomially growing, C^2 solution of (101). Set

$$A = \inf_{x \in \mathbb{R}} \{u'(x) > -1\}. \tag{102}$$

The convexity of u yields

$$u'(x) > -1, \quad x > A. \tag{103}$$

So, using (101), we conclude that

$$u(x) - u''(x) = x^2, \quad x > A, \tag{104}$$

$$u'(x) = -1, \quad x \le A. \tag{105}$$

The value of A is not a priori given to us, so we solve (104),(105) for every real number A and then determine the value of A by using (101) again. The solution of (104) (see Example 2.2) is

$$u(x) = C_1 e^x + C_2 e^{-x} + x^2 + 2, \quad x > A, \tag{106}$$

and the solution of (105) is

$$u(x) = -x + C_3, \quad x \le A, \tag{107}$$

where C_1, C_2, C_3 are unknown constants. Because we have assumed that u is polynomially growing, $C_1 = 0$. We want to determine C_2, C_3, A s.t. u is twice continuously differentiable. So we must solve a system of equations:

$$\begin{cases} u(A) = u(A-) \\ u'_+(A) = u'_-(A) \\ u''_+(A) = u''_-(A) \end{cases} \iff \begin{cases} C_2 e^{-A} + A^2 + 2 = -A + C_3 \\ -C_2 e^{-A} + 2A = -1 \\ C_2 e^{-A} + 2 = 0. \end{cases} \tag{108}$$

An elementary computation yields

$$\begin{cases} A = -\frac{3}{2} \\ C_2 = -2e^{-\frac{3}{2}} \\ C_3 = \frac{3}{4} \end{cases} \tag{109}$$

$$u(x) = \begin{cases} -2e^{-\frac{3}{2}-x} + x^2 + 2, & x > -\frac{3}{2} \\ \frac{3}{4} - x, & x \le -\frac{3}{2}. \end{cases} \tag{110}$$

Hence, u is a convex, quadratically growing solution of (101).

Now we take a control $\upsilon \in \mathcal{Q}^+$, s.t. the corresponding state process X satisfies

$$X_t = x + \sqrt{2} W_t + \upsilon_t > A \quad \text{for almost every } t \ge 0,\ a.s., \tag{111}$$

$$\int_0^t \mathbb{I}_{\{X_s = A\}} d\upsilon_s = \upsilon_t, \quad \text{for all } t \ge 0,\ a.s.. \tag{112}$$

The problem of finding $\upsilon \in \mathcal{Q}^+$ satisfying (111)-(112) is known as the Skorokhod problem and, in this case, its solution is unique. So we see that for this control the assumptions (9)-(12) of The Verification Theorem 1.3 hold. Hence, $u(x) = V(x)$ for each $x \in \mathbb{R}$.

The optimally controlled process X is a Brownian motion reflected at A with initial jump to A if $x < A$. The optimal control υ_t is the local time of the state process X at A on the interval $\langle 0, t \rangle$ plus the length of possible initial jump which equals $(A - x)^+$.

6 A finite fuel problem.

The last section is devoted to the so called *finite fuel problem*. Actually, this problem has been the earliest problem of singular stochastic control. It was considered in the context of modelling a spacecraft with finite fuel.

We take the same notation, definitions and assumptions as in Section 1. Let $z > 0$ will be given. We can interpret z as the amount of fuel before start and $(N_t, \xi_t, t \geq 0)$ as the push direction at time t and the total amount of fuel used by the spacecraft up to time t respectively.

For a given initial position $x \in \mathbb{R}^n$ and initial amount of fuel $z > 0$ we define the set of admissible controls by

$$\mathcal{Q}_z = \{ v \in \mathcal{Q} : \xi_t \leq z \text{ for all } t \geq 0 \} . \tag{113}$$

Then the value function is defined by

$$V(x,z) = \inf_{v \in \mathcal{Q}_z} V_v(x), \tag{114}$$

where $V_v(x)$ is given by (3). The dynamic programming principle takes the form

$$V(x,z) = \inf_{v \in \mathcal{Q}_z} \left\{ E^x \int_0^T e^{-\alpha t} \Big(h\left(X_t\right) dt + d\xi_t \Big) + E^x \Big(e^{-\alpha T} V\left(X_T, z - \xi_T\right) \Big) \right\}. \tag{115}$$

for every stopping time $T \geq 0$ of the filtration $(\mathcal{F}_t)$. Hence, the dynamic programming HJB equation is

$$\max \Big\{ \alpha \; V(x,z) - tr \left(a(x) \; D_x^2 V(x,z) \right) - f^T(x) \; D_x V(x,z) - h(x) \; ;$$
$$|D_x V(x,z)| - 1 + D_z V(x,z) \Big\} = 0, \quad x \in \mathbb{R}^n, \; z > 0, \tag{116}$$

where $D_x V = \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right]$, $D_z V = \frac{\partial V}{\partial z}$, $D_x^2 V = \left[\frac{\partial^2 V}{\partial x_i \partial x_j} \right]_{i,j=1,\ldots,n}$.

When h is a convex, polynomially growing, nonnegative C^2 function, f is affine and $N_t \equiv m$ for some $m \in \mathbb{R}^n$ (i.e. the control can act only in one give direction), then we have an elegant simple solution of the finite fuel problem (see [5] Section VIII.6).

Theorem 6.1. *Let the above assumptions hold. Let $V(x)$ be the value function of the unconstrained problem and $H(x) = V(x,0)$ for all $x \in \mathbb{R}^n$ (so H is the cost function of the uncontrolled state process). Then*

$$V(x,z) = V(x) - V(x + zm) + H(x + zm), \; x \in \mathbb{R}^n, z \geq 0. \tag{117}$$

Moreover until $\xi_t < z$, then the optimal control for the finite fuel problem is equal to that of unconstrained problem.

Example 6.2. Let $z > 0$ will be given and consider the one-dimensional singular stochastic control problem as in Example 5.8. We assume also that $\xi_t \leq z$ for all $t \geq 0$. We recall that the optimal controlled state process in the unconstrained problem is the reflected Brownian motion on $\langle -\frac{3}{2}, \infty)$ with possible initial jump to $-\frac{3}{2}$ if the starting

position $x < -\frac{3}{2}$. Hence, the optimal controlled state process for the finite fuel problem can be summarized as follows:
• jump immediately to $-\frac{3}{2}$, if $x \in \langle -\frac{3}{2} - z, -\frac{3}{2})$,
• jump immediately to $x + z$, if $x < -\frac{3}{2} - z$,
• remain in the interval $\langle -\frac{3}{2}, \infty)$ by reflection until $z - \xi_t > 0$, if $x \geq -\frac{3}{2}$.

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