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Homogenization of random diffusions in non-stationary environments

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HOMOGENIZATION OF RANDOM DIFFUSIONS IN NON-STATIONARY ENVIRONMENTS

MARCIN BORYC AND TOMASZ KOMOROWSKI

ABSTRACT. In the present paper we prove homogenization for random diffusions with null drift and non-stationary diffusivity that satisfies the *local ergodicity* assumption. The results generalizes earlier work of Papanicolaou and Varadhan, see [11] and Yurinskii [15].

1. INTRODUCTION

Let us consider an $\mathbb{R}^d$-valued diffusion $X^x_{t,\varepsilon}$, defined over a probability space $(\Sigma, \mathcal{M}, P)$, that starts at x and whose generator is of the form

$$L^{(\varepsilon)}_{\tilde{\omega}} = \frac{1}{2}\sum_{i,j=1}^{d} a^{(\varepsilon)}_{ij}\left(\frac{x}{\varepsilon}, \tilde{\omega}\right)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Here, for a given $\varepsilon > 0$, the matrix of coefficients $a^{(\varepsilon)}(x;\tilde{\omega}) = [a^{(\varepsilon)}_{ij}(x;\tilde{\omega})]$ is a random field over another probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$ such that it is symmetric and uniformly positive definite (both in x and ω). We are interested in the limiting behavior of the diffusion over the product probability space $(\Sigma \times \tilde{\Omega}, \mathcal{M} \otimes \mathcal{F}, P \otimes \mathbb{P})$, as the parameter ε vanishes. Such a limit, if exists, is called the *homogenized diffusion* and the procedure itself is called the *homogenization*. When $a^{(\varepsilon)}(x;\tilde{\omega}) \equiv a(x;\tilde{\omega})$ for all $\varepsilon > 0$ and $a(x;\omega)$ is a stationary and ergodic random field, with sufficiently regular realizations then one can prove the convergence in law of $X^x_{t,\varepsilon}$ to a Brownian motion. This result has been obtained by Papanicolaou and Varadhan in [11] and independently by Yurinskii in [15]. A discrete version of the result has been shown in [9], see also [16] where the assumption of the uniform ellipticity has been relaxed and the quenched version of the result has been established in the discrete setting.

In our present paper we wish to depart from the assumption of stationarity of the coefficients, assuming though that the respective random fields are not that far from being stationary and ergodic. The prototype of this type of fields is given by $a^{(\varepsilon)}(x;\tilde{\omega}) = a(\varepsilon x, x;\tilde{\omega})$, where $a(y, x;\tilde{\omega})$ is continuous in the y variable and stationary and ergodic in the x variable, see Example 2.4 below. More generally, we shall work with the class of fields that are *locally*

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ergodic. This notion has been introduced in [10] in the context of random walks on a random lattice, see Definition 2.3 below for a precise formulation of the property. Intuitively, such fields can be described as follows: given x, the average of a local functional of the field $a^{(\varepsilon)}(y)$ over the ball $|y-x| \le R$, where $R \ll \varepsilon^{-1}$ can be approximated by a mean with respect to a certain stationary and homogeneous measure $\bar{\mu}_x$, sometimes also called a *local equilibrium.* In our main result, see Theorem 2.5, we establish the convergence in law of $\left(X^x_{t,\varepsilon}\right)$, as $\varepsilon \to 0+$, to a diffusion with the generator $\bar{L} = (1/2)\sum_{i,j=1}^d \bar{a}_{ij}\,(x)\,\partial^2_{ij}$ for some deterministic, symmetric, uniformly positive definite matrix valued function $\bar{a}(x) = [\bar{a}_{ij}\,(x)]$.

Locally ergodic environments in the sense considered in this paper have been used in statistical mechanics, see [6] and the relevant references therein. The homogenization of a symmetric random walk on a one dimensional integer lattice $\mathbb{Z}$, whose jump rates $\left(a^{(\varepsilon)}(x)\right)_{x\in\mathbb{R}}$ are given by a locally ergodic random field, has been shown in [10]. The dimensional restriction is essential, because in one dimension there is an explicit formula for the respective *corrector* that is used in the martingale argument for the identification of the diffusion that is the limit of the random walk.

In the continuum setting some results are available for diffusions, in an arbitrary dimension $d \ge 1$, whose generators are in the divergence form, i.e. can be written as

$$L^{(\varepsilon)}_{\tilde{\omega}} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a^{(\varepsilon)}_{ij}\left(\frac{x}{\varepsilon},\tilde{\omega}\right)\frac{\partial}{\partial x_j}\right).$$

When $a^{(\varepsilon)}(x) := a(\varepsilon x, x)$ and $a(y,x)$ is a non-random, matrix valued function that is uniformly positive definite in y, x (but not necessarily symmetric) and periodic in the x variable, the homogenization has been shown in [2], see Section 3.6. The random case has been investigated in [13], where somewhat different notion of the local ergodicity has been used. Namely, the processes $a^{(\varepsilon)}(x,\tilde{\omega})$ have been assumed to be of the form $a(\varepsilon x, x;\tilde{\omega})$, where the field $(a(y,x;\tilde{\omega}))_{(y,x)\in\mathbb{R}^{2d}}$ satisfies the following conditions: 1) its finite dimensional statistics are invariant under the shifts by vectors of the form $(0,h)$, for an arbitrary $h \in \mathbb{R}^d$, 2) the field is ergodic in the x variable for any fixed finite ensemble of y-s. In fact, as it is shown in Example 2.4, such fields are locally ergodic in our sense.

The method of the proof, we adopt in our present paper, makes use of the fact that the diffusions $\left(X^x_{t,\varepsilon}\right)$ are martingales with the quadratic covariation between i-th and j-th component given by $\int_0^t a^{(\varepsilon)}_{ij}(X^x_{s,\varepsilon}/\varepsilon)ds$. This immediately allows us to conclude tightness of their law in the space of continuous functions. To identify the limit we adapt the argument from [10] that relies on proving that an arbitrary limiting law has to satisfy the martingale problem corresponding to a certain second order differential operator whose coefficients can be determined by the local ergodic theorem, see Proposition 3.1 below.

As far as the structure of the paper is concerned it is as follows. In Section 2 we formulate basic notions used in the paper, see in particular Definition 2.3 for the notion of local ergodicity, give examples of random fields with such property, see Example 2.4 and state the main result of the article, see Theorem 2.5. In Section 3 we prove the result.

2. Preliminaries and the formulation of the main result

In this section we recall some fundamental facts needed in the paper and formulate our main result.

2.1. **Diffusions with null drift.** Suppose that $\lambda > 0$. Let $S_d^+(\lambda)$ be the space of symmetric positive semi-definite $d \times d$ matrices $a = [a_{ij}]$ satisfying

$$\lambda \sum_{j=1}^{d} \xi_j^2 \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le \frac{1}{\lambda}\sum_{j=1}^{d} \xi_j^2, \quad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \tag{1}$$

Assume that $a : \mathbb{R}^d \to S_d^+(\lambda)$ is of class C^2. In addition, we shall require that each entry of the matrix $a(x) = [a_{ij}(x)]$ is twice continuously differentiable and satisfies

$$\sum_{i,j,k=1}^{d} \left|\frac{\partial}{\partial x_k} a_{ij}(x)\right| \le \lambda, \quad \forall\, x \in \mathbb{R}^d. \tag{2}$$

Let (X_t^x) be a diffusion over a certain probability space $(\Sigma, \mathcal{M}, P)$ that starts at x and with the generator of the form

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2 f(x), \quad \forall\, f \in C_0^2(\mathbb{R}^d),$$

where $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$. It is well known that (X_t^x) is a square integrable, continuous trajectory martingale with respect to its natural filtration $(\mathcal{M}_t)$. Its mean equals x while the quadratic covariation equals

$$\langle X^{x,i}, X^{x,j}\rangle_t = \int_0^t a_{ij}(X_s^x)ds, \quad i, j = 1, \ldots, d, \tag{3}$$

see e.g. [14], Corollary 4.2.2, p. 90. The diffusion can be also represented as the solution of an Itô stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)dW_s\ , \tag{4}$$

where $\sigma(x) = a^{1/2}(x)$ and process (W_t) is a d-dimensional standard Brownian motion non-anticipative, with respect to the filtration $(\mathcal{M}_t)$. By the quadratic variation of (X_t) we shall understand

$$\langle X^x\rangle_t := \sum_{j=1}^{d} \langle X^{x,j}, X^{x,j}\rangle_t. \tag{5}$$

2.2. **Diffusions with fast oscillating, locally ergodic coefficients.**

2.2.1. *Stationary and ergodic, continuous random coefficients.* First we recall the notion of stationarity and ergodicity of a random field. Suppose that $(a(x;\tilde{\omega}))_{x\in\mathbb{R}^d}$ is a continuous trajectory, $S_d^+(\lambda)$-valued random field defined over a probability space $(\tilde{\Omega},\mathcal{F},\mathbb{P})$. The law of $(a(x;\tilde{\omega}))_{x\in\mathbb{R}^d}$ can be considered over some compact, Fréchet metric space Ω. Indeed, we can define Ω as the closure, in the topology of the uniform convergence on compacts, of the subset of $C(\mathbb{R}^d;S_d^+(\lambda))$ consisting of those ω that satisfy conditions (1) and (2). Let μ be the law of $(a(x;\tilde{\omega}))_{x\in\mathbb{R}^d}$ in this space. We say that the random field is *stationary*, when its law is *homogeneous*, i.e. $\mu(\tau_x(A))=\mu(A)$ for any $x\in\mathbb{R}^d$ and $A\in\mathcal{B}$. Here $\tau_x:\Omega\to\Omega$ is given by $\tau_x(\omega)(\cdot):=\omega(x+\cdot)$, for any $\omega\in\Omega$ and $\mathcal{B}$ denotes the σ-algebra of Borel sets in Ω. It is called *ergodic* if it is stationary and any set A that satisfies $\mu(\tau_x(A)\bigtriangleup A)=0$ for all $x\in\mathbb{R}^d$ is μ-trivial, i.e. $\mu(A)=0$, or 1. Here $\bigtriangleup$ denotes the symmetric difference of sets. In consequence of the ergodic theorem, see e.g. [7] Theorem 11.18, p. 370, we have then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} g(y)F(\tau_{y/\varepsilon}a(\cdot))dy=\int_{\mathbb{R}^d} g(y)dy\int_\Omega F d\mu$$

for any g belonging to the space $C_0(\mathbb{R}^d)$ (continuous and compactly supported functions) and $F:\Omega\to\mathbb{R}$ that is bounded and *local*, i.e. there exists $R>0$ such that $F(\omega)=F(\omega_{|B_R})$, where $\omega_{|B_R}$ denotes the restriction of ω to the ball B_R of radius R and centered at 0. The convergence holds both $\mathbb{P}$ a.s. and in any L^p sense for $p\in[1,+\infty)$.

2.2.2. *Locally ergodic coefficients.* We wish to generalize the notion of ergodicity to incorporate fields that are not stationary but only "locally stationary". For that purpose we recall the notion of local ergodicity defined in [10]. Before its formulation we introduce the following two definitions.

Definition 2.1. A function $F:\mathbb{R}^d\times\Omega\to\mathbb{R}$ is called *admissible* if it is bounded, uniformly local, i.e. there exists $R>0$ such that $F(z,\omega)=F(z,\omega_{|B_R})$ for all $(z,\omega)\in\mathbb{R}^d\times\Omega$ and continuous in both variables.

Definition 2.2. A family of Borel probability measures $(\nu_x)_{x\in\mathbb{R}^d}$ on Ω is called *regular* if

1) the function $x\mapsto\nu_x(A)$ is measurable for any $A\in\mathcal{B}$,
2) for any continuous (thus necessarily bounded) function $F:\Omega\to\mathbb{R}$ the function $x\mapsto\int F d\nu_x$ is continuous.

The notion of local ergodicity can be formulated as follows.

Definition 2.3. Suppose that $(a^{(\varepsilon)}(x;\tilde{\omega}))$, $\varepsilon\in(0,1]$, is a family of continuous trajectory, $S_d^+(\lambda)$-valued random fields defined over $(\tilde{\Omega},\mathcal{F},\mathbb{P})$. We say that it is *locally ergodic*, when there exists a regular family of homogeneous and ergodic Borel laws $(\bar{\mu}_z)_{z\in\mathbb{R}^d}$ on Ω such that for any admissible function $F:\mathbb{R}^d\times\Omega\to\mathbb{R}$ and $g\in C_0(\mathbb{R}^d)$ we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d} g(z)F(z,\tau_{z/\varepsilon}a^{(\varepsilon)}(\cdot))dz=\int_{\mathbb{R}^d} g(z)\bar{F}(z)dz, \tag{6}$$

where

$$\bar{F}(z) = \int_\Omega F(z,\omega)\bar{\mu}_z(d\omega) \tag{7}$$

and the convergence in (6) holds in probability.

Example 2.4. Suppose that $(a(y,x;\tilde{\omega}))_{(y,x)\in\mathbb{R}^{2d}}$ is an $S_d^+(\lambda)$ valued random field such that for any fixed y the field $(a(y,x;\tilde{\omega}))_{x\in\mathbb{R}^d}$ is stationary and ergodic in the x variable. In addition we assume that

$$\lim_{\delta\to 0}\sup_{|y-y'|<\delta}\sup_{x\in\mathbb{R}^d}|a(y',x)-a(y,x)| = 0 \tag{8}$$

in $\mathbb{P}$-probability. Denote by $\bar{\mu}_y$ the law of $a(y,\cdot)$ over the space of Ω. It is clear that the family $(\bar{\mu}_y)_{y\in\mathbb{R}^d}$ is regular in the sense of Definition 2.2. Define $a^{(\varepsilon)}(x) := a(\varepsilon x, x)$. We claim that the above family of fields is locally ergodic. Indeed, suppose that $F:\mathbb{R}^d\times\Omega\to\mathbb{R}$ is admissible and such that $\bar{F}(y)$, given by (7), vanishes and $g\in C_0(\mathbb{R}^d)$ with the support contained in a ball of radius R centered at the origin. Here $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. We need to show that

$$\lim_{\varepsilon\to 0+}\mathbb{E}\left|\int_{\mathbb{R}^d} g(z)F(z,\tau_{z/\varepsilon}a^{(\varepsilon)}(\cdot))dz\right| = 0. \tag{9}$$

Denote $\square := [0,1)^d$. Also, given $a>0$ and $x\in\mathbb{R}^d$, we let $\square_{x,a} := ax + a\square$. Assume that $\rho>0$ is arbitrary. Using (8) and continuity of the admissible function $F(z,\omega)$ we conclude that there exists $\delta>0$ and $\varepsilon_0\in(0,1]$ such that

$$\sup_{z\in\square_{k,\delta}}\mathbb{E}|F(z,\tau_{z/\varepsilon}a^{(\varepsilon)}(\cdot)) - F(k\delta,\tau_{z/\varepsilon}a(k\delta,\cdot))| < \rho,\ \forall\ \varepsilon\in(0,\varepsilon_0],\ k\in\mathbb{Z}^d,\ \text{ s.t. } |k|\delta<R.$$

There exists therefore a constant $C>0$ independent of δ and ε such that

$$\mathbb{E}\left|\int_{\mathbb{R}^d} g(z)F(z,\tau_{z/\varepsilon}a^{(\varepsilon)}(\cdot))dz - \sum_{k\in\mathbb{Z}^d}\int_{\square_{k,\delta}} g(z)F(k\delta,\tau_{z/\varepsilon}a(k\delta,\cdot))dz\right| < C\rho.$$

Passing to the limit in ε (with δ fixed), remembering that $a(k\delta,\cdot)$ is ergodic, we conclude from the individual ergodic theorem that

$$\lim_{\varepsilon\to 0+}\int_{\square_{k,\delta}} g(z)F(k\delta,\tau_{z/\varepsilon}a(k\delta,\cdot)))dz = 0, \quad \mathbb{P}\text{ a.s.}$$

Therefore,

$$\limsup_{\varepsilon\to 0+}\mathbb{E}\left|\int_{\mathbb{R}^d} g(z)F(z,\tau_{z/\varepsilon}a^{(\varepsilon)}(\cdot))dz\right| \le C\rho$$

and, since $\rho>0$ has been arbitrary, we get (9).

2.2.3. *Diffusions with locally ergodic coefficients and the statement of the main result.* Suppose that $\lambda > 0$ and $\left(a^{(\varepsilon)}(x)\right)$, $\varepsilon \in (0,1]$ is a locally ergodic family of $S_d^+(\lambda)$-valued, C^2-regular random fields that satisfy conditions (1) and (2) for all $\varepsilon \in (0,1]$, $\mathbb{P}$ a.s. Assume furthermore that $(X_{t,\varepsilon}^x)$, $\varepsilon > 0$ are diffusions starting at x whose generators equal

$$L_{\tilde{\omega}}^{(\varepsilon)} f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}^{(\varepsilon)}\left(\frac{x}{\varepsilon};\tilde{\omega}\right)\partial_{ij}^2 f(x), \quad \forall\, f \in C_0^2(\mathbb{R}^d).$$

Consider the process $X_{t,\varepsilon}^x$ over the product probability space $(\Sigma \times \tilde{\Omega}, \mathcal{M} \otimes \mathcal{F}, P \otimes \mathbb{P})$ and denote by $\gamma_\varepsilon^{(x)}$ its law over the Fréchet space $\mathcal{C} := C([0,+\infty);\mathbb{R}^d)$. Our main result can be now formulated as follows.

Theorem 2.5. *For any $x \in \mathbb{R}^d$ the laws $\gamma_\varepsilon^{(x)}$ converge, as $\varepsilon \to 0$, to the law of the diffusion that starts at x and corresponds to a well posed martingale problem for a second order differential operator of the form*

$$\bar{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} \bar{a}_{ij}\,(x)\,\partial_{ij}^2 f(x), \quad \forall\, f \in C_0^2(\mathbb{R}^d) \tag{10}$$

for some deterministic $S_d^+(\lambda)$-valued function $\bar{a}(x) = [\bar{a}_{ij}\,(x)]$.

3. Proof of Theorem 2.5

For convenience sake we reformulate the problem at hand in order to work in the space $\mathcal{C} \times \Omega$. By $\mathcal{B}_{\mathcal{C}}$, $\mathcal{B}_\Omega$ we denote the Borel σ-algebras of subsets of $\mathcal{C}$ and Ω, respectively.

Instead of random field $(a^{(\varepsilon)}(x))$ we consider the canonical field $\pi : \mathbb{R}^d \times \Omega \to S_d^+(\lambda)$, given by $\pi(x;\omega) = [\pi_{ij}(x;\omega)]$, where $\pi_{ij}(x) := \omega_{ij}(x)$ defined over the space $(\Omega, \mathcal{B}_\Omega, \mu_\varepsilon)$, where μ_ε is the law of $\left(a^{(\varepsilon)}(x)\right)_{x\in\mathbb{R}^d}$. Note that $\pi(x;\omega) = \hat{\pi}(\tau_x\omega)$, where $\hat{\pi} := \pi(0;\omega)$. The definition of the local ergodicity of the field in question can be now translated into the following form

$$\lim_{\epsilon\to 0} \int_\Omega \Big| \int_{\mathbb{R}^d} g(z) F(z, \tau_{z/\varepsilon}\omega) dz - \int_{\mathbb{R}^d} g(z)\bar{F}(z) dz \Big| \mu_\varepsilon(d\omega) = 0 \tag{11}$$

for any admissible $F(z,\omega)$ and $g \in C_0(\mathbb{R}^d)$. Here $\bar{F}(z)$ is given by (7).

Denote by $P_{x,\omega}$ the law on $\mathcal{C}$ that corresponds to the diffusion that starts at x and with the generator given by

$$L_\omega f(x) = \frac{1}{2}\sum_{i,j=1}^{d} \hat{\pi}_{ij}(\tau_x\omega)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d).$$

We shall omit writing subscript x in the notation of the path measure when the diffusion starts at 0.

The uniqueness of solutions of the corresponding martingale problem implies that the transition probability densities $p_t^\omega(x,y)$ associated with the diffusion satisfy

$$p_t^{\tau_z\omega}(x,y) = p_t^\omega(x+z, y+z), \quad \forall\, \omega \in \Omega,\ x,y,z \in \mathbb{R}^d. \tag{12}$$

In consequence

$$P_{x,\tau_z\omega}[A] = P_{x+z,\omega}[\theta_z(A)], \quad \forall\, A \in \mathcal{B}_{\mathcal{C}}. \tag{13}$$

Here the shift operator $\theta_z : \mathcal{C} \to \mathcal{C}$ is given by $\theta_z(\sigma)(t) := z + \sigma(t)$.

Let $(X(t))$ be the canonical process $X : [0, +\infty) \times \mathcal{C} \to \mathbb{R}^d$ given by $X(t;\sigma) := \sigma(t)$, $\sigma \in \mathcal{C}$. It is well known, see Corollary 8.1.8 of [14], that $P_{x,\omega}$ is the unique solution of the martingale problem corresponding to the functional

$$M_{t,\omega}(f) := f(X(t)) - \int_0^t L_\omega f(X(s))ds.$$

The above means that $P_{x,\omega}$ is the unique measure such that $(M_{t,\omega}(f))$ is a martingale with respect to the filtration $\mathcal{M}_t := \sigma(X(s),\, 0 \le s \le t)$ for an arbitrary $f \in C_0^2(\mathbb{R}^d)$. Then, $\gamma_\varepsilon^{(x)}$ coincides with the law of the process

$$X_\varepsilon(t) := \varepsilon X(t/\varepsilon^2) \tag{14}$$

considered over the product probability space $(\mathcal{C} \times \Omega, \mathcal{B}_{\mathcal{C}} \otimes \mathcal{B}_\Omega, P_{x/\varepsilon} \otimes \mu_\varepsilon)$. To simplify our considerations we assume in what follows that $x = 0$. Keeping up with our convention we shall write then γ_ε instead of $\gamma_\varepsilon^{(0)}$.

Let $\zeta_t(\omega) := \tau_{X(t)}\omega$. It is well known, see e.g. Section 9.4. of [7], that (ζ_t) is an Ω-valued, continuous trajectory, Markov process over $(\mathcal{C}, \mathcal{B}_{\mathcal{C}}, P_\omega)$. Its transition probabilities are given by

$$Q_t(\omega, A) = \int_{\mathbb{R}^d} p_t^\omega(0, y) 1_A(\tau_y\omega)dy, \quad \forall\, A \in \mathcal{B}_\Omega,\, \omega \in \Omega.$$

According to Theorem 2.1 of [11] there exists a unique $\Phi_{*,x}$ that is strictly positive $\bar\mu_x$ a.s. and such that measure $\lambda_x(d\omega) := \Phi_{*,x}(\omega)\bar\mu_x(d\omega)$ is ergodic and invariant under (ζ_t). In addition, $\Phi_{*,x} \in L^{d/(d-1)}(\bar\mu_x)$ and its $L^{d/(d-1)}(\bar\mu_x)$-norms satisfy

$$\Gamma_* := \sup_{x\in\mathbb{R}^d} \|\Phi_{*,x}\|_{L^{d/(d-1)}(\bar\mu_x)} < +\infty.$$

Define

$$\bar a_{ij}(x) := \int \hat\pi_{ij}(\omega)\Phi_{*,x}(\omega)\bar\mu_x(d\omega). \tag{15}$$

From the individual ergodic theorem we conclude also that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \hat\pi_{ij}(\zeta_t)dt = \bar a_{ij}(x), \quad \bar\mu_x \text{ a.s. and in } L^1(\bar\mu_x),\ i,j = 1,\ldots,d,\ x \in \mathbb{R}^d. \tag{16}$$

3.1. **Tightness.** Given $\varepsilon \in (0, 1]$ process $(X_\varepsilon(t))$ (see (14)) considered over $(\mathcal{C}, \mathcal{B}_{\mathcal{C}}, P_\omega)$, is a diffusion with the generator

$$L_\omega^{(\varepsilon)} f(x) = \frac{1}{2}\sum_{i,j=1}^d \hat\pi_{ij}(\tau_{x/\varepsilon}\omega)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d) \tag{17}$$

for μ_ε a.s. ω. It is therefore a square integrable martingale under P_ω whose quadratic variation satisfies

$$\langle X_\varepsilon\rangle_t - \langle X_\varepsilon\rangle_s = \int_s^t \operatorname{tr}\pi\left(\frac{X_\varepsilon(r)}{\varepsilon}\right)dr \le \frac{d}{\lambda}(t-s), \quad \forall\, t > s \ge 0.$$

Tightness of (γ_ε), $\varepsilon \in (0,1]$ is then a consequence of Theorem VI.5.17, p. 365 of [5].

3.2. **Limit identification.** Suppose that $\varepsilon_n \to 0$, as $n \to +\infty$, is a sequence such that γ_{ε_n} converge weakly over $\mathcal{C}$ to a certain limiting measure γ_*. We shall show that the limiting measure is a solution of a well posed martingale problem corresponding to an operator of the form given in (10). Assume that $f \in C_0^2(\mathbb{R}^d)$, $N \ge 1$, $0 \le s_1 \le \ldots \le s_N \le s < t$ and $\psi : \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function. Let

$$\Psi_\varepsilon := \psi(X_\varepsilon(s_1), \ldots, X_\varepsilon(s_N)).$$

We shall also write $\Psi := \Psi_1$. Using the fact that

$$M_{t,\omega}^{(\varepsilon)}(f) := f(X_\varepsilon(t)) - \int_0^t L_\omega^{(\varepsilon)} f(X_\varepsilon(r))dr$$

is a martingale under P_ω we can write

(18)

$$\int \left[f(X_{\varepsilon_n}(t)) - f(X_{\varepsilon_n}(s)) - \frac{1}{2}\sum_{i,j=1}^d \int_s^t \hat{\pi}_{ij}(\zeta_r^{(\varepsilon_n)})\partial_{ij}^2 f(X_{\varepsilon_n}(r))dr\right]\Psi_{\varepsilon_n} P_\omega(d\sigma)\mu_{\varepsilon_n}(d\omega) = 0$$

for each n. Here $\zeta_t^{(\varepsilon)} := \tau_{X_\varepsilon(t)/\varepsilon}\omega$. Thanks to the assumed weak convergence of (γ_{ε_n}) we have

$$(19)\qquad \lim_{n\to+\infty}\int \left[f(X_{\varepsilon_n}(t)) - f(X_{\varepsilon_n}(s))\right]\Psi_{\varepsilon_n} P_\omega(d\sigma)\mu_{\varepsilon_n}(d\omega)$$

$$= \lim_{n\to+\infty}\int \left[f(X(t)) - f(X(s))\right]\Psi\gamma_{\varepsilon_n}(d\sigma) = \int \left[f(X(t)) - f(X(s))\right]\Psi\gamma_*(d\sigma).$$

To deal with the time integral term appearing in (18) and thus with the problem of the limit identification we shall need the following result.

Proposition 3.1. 1) *Functions $x \mapsto \bar{a}_{ij}(x)$ are continuous for each $i,j = 1,\ldots,d$. In addition, each matrix $\bar{a}(x) = [\bar{a}_{ij}(x)]$ belongs to $S_d^+(\lambda)$.*

2) *Let $\Psi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be an admissible function such that*

$$(20)\qquad \int \Psi(x;\omega)\Phi_{*,x}(\omega)\bar{\mu}_x(d\omega) = 0, \quad \forall\, x \in \mathbb{R}^d.$$

Then,

$$(21)\qquad \lim_{\epsilon\to 0}\mathbb{E}_x^\epsilon\left[\sup_{0\le t\le T}\left|\int_0^t \Psi(X_\varepsilon(s), \tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|\right] = 0, \quad \forall\, T > 0.$$

Here $\mathbb{E}_x^\epsilon$ denotes the expectation with respect to the measure $\mathbb{P}_x^\varepsilon(d\sigma, d\omega) := P_{x/\varepsilon,\omega}(d\sigma)\mu_\varepsilon(d\omega)$ (we omit the subscript if $x = 0$).

We postpone for a moment the proof of the proposition and use it first to finish the demonstration of the theorem. For that purpose we note first that

$$\lim_{n\to+\infty} \mathbb{E}^{\varepsilon_n}\left|\int_0^t \hat{\pi}_{ij}(\zeta_r^{(\varepsilon_n)})\partial_{ij}^2 f(X_{\varepsilon_n}(r))dr - \int_0^t \bar{a}_{ij}(X_{\varepsilon_n}(r))\partial_{ij}^2 f(X_{\varepsilon_n}(r))dr\right| = 0. \tag{22}$$

The above equality follows from a direct application of part 2) of Proposition 3.1, where $\Psi(x,\omega) := [\hat{\pi}_{ij}(\omega) - \bar{a}_{ij}(x)]\partial_{ij}^2 f(x)$. From this and (18) together with (19) we conclude that γ_* satisfies the martingale problem corresponding to the operator

$$\bar{L}f(x) := \frac{1}{2}\sum_{i,j=1}^d \bar{a}_{ij}(x)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d). \tag{23}$$

To finish the proof of the theorem it suffices therefore to prove that the problem is well-posed. The latter is a consequence of the continuity of the entries of matrix $\bar{a}(x)$ and Theorem 7.2.1 of [14]. The only thing that remains yet to be shown is therefore Proposition 3.1.

Proof of Proposition 3.1. Part 1. Suppose that $x_n \to x_\infty$, as $n \to +\infty$. We prove that

$$\lim_{n\to+\infty} \bar{a}(x_n) = \bar{a}(x_\infty). \tag{24}$$

Since Ω is compact the sequence (λ_{x_n}) is tight. Assume that λ_∞ is the weak limit of a certain subsequence, which for convenience sake shall be denoted also by (λ_{x_n}). To show that (24) holds it suffices only to demonstrate that

$$\int_\Omega \hat{\pi}(\omega)\lambda_{x_\infty}(d\omega) = \int_\Omega \hat{\pi}(\omega)\lambda_\infty(d\omega). \tag{25}$$

Since the process (ζ_t) is Feller and each λ_{x_n} is its invariant probability measure, so is also the weak limit λ_∞. In fact, we prove the following result.

Lemma 3.2. *For any $\rho > 0$ there exists $\delta > 0$ such that for any function $\phi \in C(\Omega)$ satisfying*

$$0 \le \phi \le 1 \quad \textit{and} \quad \int_\Omega \phi(\omega)\bar{\mu}_{x_\infty}(d\omega) < \delta \tag{26}$$

we have

$$\int_\Omega \phi(\omega)\lambda_\infty(d\omega) < \rho. \tag{27}$$

Proof. Choose arbitrary $\rho > 0$ and ϕ as in (27). Parameter $\delta > 0$ is to be adjusted later on. Since $\bar{\mu}_{x_n}$ converge weakly to $\bar{\mu}_{x_\infty}$ we can find n_0 such that

$$\int_\Omega \phi(\omega)\bar{\mu}_{x_n}(d\omega) < 2\delta, \quad \forall\, n \ge n_0.$$

Note that, by Hölder inequality, (recall that $0 \le \phi \le 1$)

$$\int_\Omega \phi(\omega)\lambda_{x_n}(d\omega) = \int_\Omega \phi(\omega)\Phi_{*,x_n}(\omega)\bar{\mu}_{x_n}(d\omega) \le \|\Phi_{*,x}\|_{L^{d/(d-1)}(\bar{\mu}_x)}\|\phi\|_{L^d(\bar{\mu}_x)} \le \Gamma_*(2\delta)^{1/d}$$

and
$$\int_\Omega \phi(\omega)\lambda_\infty(d\omega) = \lim_{n\to+\infty}\int_\Omega \phi(\omega)\lambda_{x_n}(d\omega) \le \Gamma_*(2\delta)^{1/d}.$$
The conclusion of the proposition follows upon the choice $\delta := (\rho/\Gamma_*)^d/2$. □

Our next lemma is a slightly modification of the Radon-Nikodym theorem in case of Borel σ-algebra of a given metric space.

Lemma 3.3. *Suppose that $\mathcal{X}$ is a metric space and μ, ν are two Borel, probability measures satisfying: for any $\rho > 0$ there exists $\delta > 0$ such that for any function $\phi \in C(\Omega)$ satisfying*

$$0 \le \phi \le 1 \quad \textit{and} \quad \int_\Omega \phi(\omega)\mu(d\omega) < \delta \tag{28}$$

we have

$$\int_\Omega \phi(\omega)\nu(d\omega) < \rho. \tag{29}$$

Then, measure ν is absolutely continuous with respect to μ.

Proof. If ν had a non-trivial singular part with respect to μ we could find a Borel set A such that $\nu(A) > 0$ and $\mu(A) = 0$. Choose $\rho := \nu(A)/2$ and find $\delta > 0$ as in (28) and (29). Using regularity property of Borel measures on metric spaces, see e.g. Theorem 7.1.3 of [3], one can find an open set G and a closed one K such that $K \subset A \subset G$ and $\mu(G) < \delta/2$, and $\nu(A \setminus K) < \rho/4$. Suppose that $\phi : \mathcal{X} \to [0,1]$ is a continuous function such that $\phi \equiv 1$ on K and $\phi \equiv 0$ on G^c. We have
$$\int_\Omega \phi(\omega)\mu(d\omega) \le \mu(G) < \delta.$$
Therefore, using (29), we get
$$\nu(A) \le \int_\Omega \phi(\omega)\nu(d\omega) + \nu(A\setminus K) < \rho + \frac{\rho}{4} = \frac{5}{4}\rho = \frac{5}{8}\nu(A),$$
which is impossible. Thus, we have concluded that the assumption about the existence of the singular part of ν lead us to a contradiction. □

As a corollary of Lemmas 3.2 and 3.3 we conclude that λ_∞ is absolutely continuous with respect to $\bar{\mu}_{x_\infty}$. Since it is stationary under (ζ_t), the limit of $T^{-1}\int_0^T \hat{\pi}(\zeta_t)dt$ exists $P_\omega(d\sigma)\otimes\lambda_\infty(d\omega)$ a.s., thanks to the Birkhoff Individual Ergodic Theorem (Theorem 2.3. p. 9 of [8]). Thanks to measure theoretic equivalence of λ_{x_∞} and $\bar{\mu}_{x_\infty}$ and another application of the Birkhoff Individual Ergodic Theorem, this time with respect to the ergodic invariant measure λ_{x_∞}, we conclude that in fact

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \hat{\pi}(\zeta_t)dt = \int_\Omega \hat{\pi}\, d\lambda_{x_\infty} = \bar{a}(x_\infty), \quad P_\omega(d\sigma)\otimes\lambda_{x_\infty}(d\omega) \text{ a.s.} \tag{30}$$

Using stationarity of λ_∞, (30) and the Lebesgue dominated convergence theorem we conclude therefore that
$$\int_\Omega \hat{\pi}(\omega)\lambda_\infty(d\omega) = \lim_{T\to+\infty}\int_\Omega \lambda_\infty(d\omega)E_\omega\left[\frac{1}{T}\int_0^T \hat{\pi}(\zeta_t)dt\right] = \bar{a}(x_\infty)$$

and (25) follows.

Part 2. Fix any $\rho > 0$. Using tightness of $(X_\varepsilon(t))$ and continuity of the admissible function in the first variable we can find $R > 0$ sufficiently large and $r > 0$, both parameters are to be adjusted further later on, and ε_0 sufficiently small such that

$$\mathbb{E}_x^\epsilon\left[\sup_{0\le t\le T}\left|\int_0^t \Psi(X_\varepsilon(s),\tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|\right] \le \sum_{k=k_0}^{K}\mathbb{E}^\epsilon\left[\left|\int_{k\eta}^{(k+1)\eta}\Psi(X_\varepsilon(k\eta),\tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|,\ |X_\varepsilon(k\eta)|\le R\right]+\rho\|\Psi\|_\infty \tag{31}$$

for all $\varepsilon \in (0,\varepsilon_0]$. Here $\eta := r\varepsilon^2$, $K := [T/\eta]$ and $k_0 := [\rho/(2\eta)] + 1$. Invoking the Markov property of $X_\varepsilon(t)$ with respect to measure P_ω we can write that the expectation on the right hand side of (31) equals

$$\int_{|y|\le R}\int_\Omega p^\omega_{k\eta,\varepsilon}(0,y)E_{y/\varepsilon,\omega}\left|\int_0^\eta \Psi(y,\tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|dy\mu_\varepsilon(d\omega). \tag{32}$$

Here $p^\omega_{t,\varepsilon}(x,y)$ is the transition probability density of the diffusion corresponding to generator (17) and $E_{y,\omega}$ is the expectation with respect to $P_{y,\omega}$ (with the convention of suppressing y in case it equals 0). According to the results of Section 4 of [4] we can find constants $C > 0$, $\alpha_0 > d/(d-1)$ independent of ω and ε such that for each $\alpha \in [1,\alpha_0]$ we have

$$\sup_{x\in\mathbb{R}^d}\|p^\omega_{t,\varepsilon}(x,\cdot)\|_{L^\alpha(\mathbb{R}^d)} \le \frac{C}{t^{d/(2\alpha')}},\quad \forall\, t>0, \tag{33}$$

where $1/\alpha' = 1 - 1/\alpha$. From the above bound and remembering that $\eta = r\varepsilon^2$ we conclude that there exists a deterministic constant $C_1 > 0$ that is independent of ε, r, R and ρ such that the expression in (32) can be estimated by (remember that $k\eta \ge \rho/2$ for $k \ge k_0$)

$$\frac{C_1}{\rho^{d/(2\alpha')}}\int_\Omega\left\{\int_{|y|\le R}\left[E_{y/\varepsilon,\omega}\left|\int_0^\eta\Psi(y,\tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|\right]^{\alpha'}dy\right\}^{1/\alpha'}\mu_\varepsilon(d\omega) \le \frac{C_1\varepsilon^2}{\rho^{d/(2\alpha')}}\left\{\int_{|y|\le R}dy\left\{\int_\Omega\left[E_{y/\varepsilon,\omega}\left|\int_0^r\Psi(y,\tau_{X(s)}\omega)ds\right|\right]^{\alpha'}\mu_\varepsilon(d\omega)\right\}\right\}^{1/\alpha'}. \tag{34}$$

The last estimate following from the change of variables $s' := s/\varepsilon^2$ and Hölder inequality. Applying (13) we can rewrite the right hand side in the form

$$\frac{C_1\varepsilon^2}{\rho^{d/(2\alpha')}}\left\{\int_{|y|\le R}dy\left\{\int_\Omega\left[E_{\tau_{y/\varepsilon}\omega}\left|\int_0^r\Psi(y,\tau_{X(s)}\tau_{y/\varepsilon}\omega)ds\right|\right]^{\alpha'}\mu_\varepsilon(d\omega)\right\}\right\}^{1/\alpha'} \tag{35}$$

Now, recall the well known exponential martingale inequality, see e.g. p. 78 of [1], that states

$$P_{y,\omega}\left[\sup_{t\in[0,T]}|X(t)-y|\ge R\right]\le 2\exp\left\{-\frac{\lambda R^2}{2T}\right\},\quad \forall\, R,T>0,\ y\in\mathbb{R}^d,\ \omega\in\Omega.$$

Applying the above estimate we obtain

$$P_\omega\left[A(R)\right]\le 2\exp\left\{-\frac{\lambda R}{2}\right\},\quad \forall\, R>0,\ \omega\in\Omega, \tag{36}$$

where $A(R):=[\sigma:\ \sup_{t\in[0,R]}|X(t;\sigma)|\ge R]$. Dividing the domain of the path integration over $A(R)$ and its complement and using (36) to estimate the integral corresponding to $A(R)$ we conclude that there exists a constant $C_2>0$, independent of ε, r,R and ρ, such that expression (35) can be bounded by

$$\frac{C_2\eta}{\rho^{d/(2\alpha')}}\left[\int_{\mathbb{R}^d}\int_\Omega g_R(y)\Theta_{r,R}(y,\tau_{y/\varepsilon}\omega)dy\mu_\varepsilon(d\omega)\right]^{1/\alpha'}+\frac{C_2\eta}{\rho^{d/(2\alpha')}}R^{d/\alpha'}e^{-\lambda R/(2\alpha')}\|\Psi\|_\infty,$$

where $g_R:\mathbb{R}^d\to[0,1]$ is C^∞ smooth, equals 1 on the ball $[|z|\le R]$ and vanishes outside the complement of the concentric ball of radius $R+1$. Here also

$$\Theta_{r,R}(y,\omega):=\left[E_\omega\left|\frac{G_R}{r}\int_0^r\Psi(y,\tau_{X(s)}\omega)ds,\right|\right]^{\alpha'},$$

where $G_R(\sigma):=h_R(\sup_{t\in[0,R]}|\sigma(t)|)$ and $h_R:\mathbb{R}\to[0,1]$ is C^∞ smooth, equals 1 on $[-R,R]$ and vanishes outside the complement of $[-R-1,R+1]$. It is straightforward to note that each $g_R(y)\Theta_{r,R}(\omega)$ is admissible. Summarizing, we have shown so far that

$$\begin{aligned}\mathbb{E}^\epsilon\left[\sup_{0\le t\le T}\left|\int_0^t\Psi(X_\varepsilon(s),\tau_{X_\varepsilon(s)/\varepsilon}\omega)ds\right|\right]&\le\frac{C_2T}{\rho^{d/(2\alpha')}}\left[\int_{\mathbb{R}^d}\int_\Omega g_R(y)\Theta_{r,R}(y,\tau_{y/\varepsilon}\omega)dy\mu_\varepsilon(d\omega)\right]^{1/\alpha'}\\&+\frac{C_2T}{\rho^{d/(2\alpha')}}R^{d/\alpha'}e^{-\lambda R/(2\alpha')}\|\Psi\|_\infty+\rho\|\Psi\|_\infty.\end{aligned}\tag{37}$$

Taking the limits, as $\varepsilon\to0$ and using (11) we conclude that the first term on the right hand side tends to

$$\frac{C_2T}{\rho^{d/(2\alpha')}}\left[\int_{\mathbb{R}^d}g_R(y)\bar\Theta_{r,R}(y)dy\right]^{1/\alpha'},$$

with

$$\bar\Theta_{r,R}(y):=\int_\Omega\Theta_{r,R}(y,\omega)\bar\mu_y(d\omega).$$

Sending next $r \to +\infty$ we obtain, by virtue of the mean and individual ergodic theorems used relative to measure λ_y,

$$\lim_{r\to+\infty} \bar{\Theta}_{r,R}(y) = \int_\Omega \left[E_\omega \left| G_R \bar{\Psi}(y) \right| \right]^{\alpha'} \bar{\mu}_y(d\omega) = 0,$$

where

$$\bar{\Psi}(x) := \int \Psi(x;\omega)\lambda_x(d\omega) = 0.$$

We obtain therefore

$$\limsup_{\varepsilon\to 0} \mathbb{E}_x^\epsilon \left[\sup_{0\le t\le T} \left| \int_0^t \Psi(X_\varepsilon(s), \tau_{X_\varepsilon(s)/\varepsilon}\omega) ds \right| \right]$$

$$\le \frac{C_2 T}{\rho^{d/(2\alpha')}} R^{d/\alpha'} e^{-2R/(\lambda\alpha')} \|\Psi\|_\infty + \rho\|\Psi\|_\infty. \tag{38}$$

Sending first $R \to +\infty$ and then $\rho \to 0$ (in this order) we conclude that

$$\lim_{\varepsilon\to 0} \mathbb{E}_x^\epsilon \left[\sup_{0\le t\le T} \left| \int_0^t \Psi(X_\varepsilon(s), \tau_{X_\varepsilon(s)/\varepsilon}\omega) ds \right| \right] = 0$$

finishing in this way the proof of the theorem.

4. Appendix

In this Appendix we give a proof of the Lemma 3.3 from [11]. The autors explain that this is a well-known fact from the theory of stationaty process and reffer to [12] for the proof. In fact the proof given in [12] is not exactly the proof of the Lemma 3.3.

At first we cite notations and assumptions given by G. C. Papanicolau and S. R. S. Varadhan in [11]. Let $d \in \mathbb{N}$, S_d be the space of symmetric positive semi-definite matrices of size d and Ω denote the space of all continous maps $\omega : \mathbb{R}^d \to S_d$ satisfying

$$\exists_{C_1,C_2>0} \quad \forall_{x,\xi\in\mathbb{R}^d} \quad C_1 \sum_{j=1}^d \xi_j^2 \le \sum_{i,j=1}^d \omega_{ij}(x)\xi_i\xi_j \le C_2 \sum_{j=1}^d \xi_j^2 \ . \tag{39}$$

We will use also the customary notation $a_{ij}(x,\omega)$ to denote $\omega_{ij}(x)$. We also have the translations maps $\tau_x : \Omega \to \Omega$ defined by:

$$a(y, \tau_x\omega) = a(x+y, \omega), \quad \text{for all } x, y \in \mathbb{R}^d, \omega \in \Omega.$$

There is a canonical Markov process with Ω as state space. Indeed, let us fix $\omega \in \Omega$. Then we have the following diffusion operator on $\mathbb{R}^d$:

$$L_\omega = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x,\omega) \frac{\partial^2}{\partial x_i \partial x_j} \ .$$

This operator generates a diffusion process in $\mathbb{R}^d$

$$\forall_{t\geq 0} \quad X_t = \int_0^t \sigma(X_s,\omega)dW_s \ ,$$

where $\sigma : \mathbb{R}^d \times \Omega \to \mathbb{R}^{d\times N}$, $a(x,\omega) = \sigma(x,\omega)\sigma^T(x,\omega)$ and process W is N-dimensional standard Brownian motion, $N \in \mathbb{N}$. We can treat this process as a random variable with values in the space $C\left(\langle 0,\infty);\mathbb{R}^d\right)$. Hence, we have a diffusion probability measure P_ω on the space $C\left(\langle 0,\infty);\mathbb{R}^d\right)$ of continuous trajectories on $\langle 0,\infty)$ with values in $\mathbb{R}^d$ starting at $t = 0$ from $x = 0$.

Given a trajectory X_t with $X_0 = 0$ we can map it onto a trajectory in the Ω space by setting

$$\omega_t = \tau_{X_t}\omega \ .$$

This induces a probability measure Q_ω on the space of trajectories in Ω starting from ω at time 0. In this way we get in fact a Markov process $(\omega_t)_{t\geq 0}$ with Ω as a state space.

Lemma 3.3. Let μ be a stationary measure on Ω and $(l_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $l_n \to \infty$. Then we can find periodic functions $\omega_n \in \Omega$ of period l_n in each variable such that the measure μ_n on Ω, obtained as the distribution of $\tau_X\omega_n$ where X is random and distributed uniformly on the cube of size l_n, converges weakly to μ as $n \to \infty$.

We explain that μ is stationary measure on $(\Omega,\mathcal{F})$ if

$$\forall_{x\in\mathbb{R}^d}\ \forall_{A\in F} \quad \mu(\tau_x A) = \mu(A)$$

and the measure μ_n could be written in the following way

$$\forall_{A\in F} \quad \mu_n(A) = \mathbb{P}_n\left(x \in \langle -l_n, l_n\rangle^d : \tau_x\omega_n \in A\right),$$

where $\mathbb{P}_n$ is the uniform probability measure on the cube $\langle -l_n, l_n\rangle^d$.

Proof: For simplicity, we consider the case when $d = 1$ and $l_n > 2n$.

Let $f : \Omega \to \mathbb{R}$ be any continuous bounded function (we shall write $f \in CB(\Omega)$). Because μ is stationary measure and only periodic functions with period x are invariant with respect to τ_x, we conclude that the mapping τ_x is ergodic for every $x \in \mathbb{R}$. So from the ergodic theorem we have

$$\lim_{n\to\infty} \frac{1}{2n}\int_{-n}^{n} f(\tau_x\omega)dx = \int_\Omega f d\mu, \text{ for } \mu\text{-almost all } \omega \in \Omega. \tag{40}$$

We observe that the left-hand side of (40) could be written as some expected value, so

$$\forall_{f\in CB(\Omega)} \quad \lim_{n\to\infty} \mathbb{E}f(\tau_{nX}\omega) = \int_\Omega f d\mu, \text{ for } \mu\text{-almost all } \omega \in \Omega,$$

where X is the uniform distributed random variable on the interval $\langle -1, 1\rangle$.

Now we consider only $f : \Omega \to \mathbb{R}$ of the form

$$f(\omega) = \prod_{i=1}^{N} \phi_i \left(\omega(x_i)\right),$$

where $N \in \mathbb{N}$, $x_i \in \mathbb{Q}$ and ϕ_i are polynomials with rationals coefficients. We denote the set of function of this form by $\mathcal{G}$. Of course the set $\mathcal{G}$ is countable. Hence there exists $\omega_0 \in \Omega$ such that

$$\forall_{f\in\mathcal{G}} \quad \lim_{n\to\infty} \mathbb{E}f(\tau_{nX}\omega_0) = \int_\Omega f d\mu. \tag{41}$$

Next we can choose a sequence $(\omega_n)_{n\in\mathbb{N}}$ such that ω_n is l_n-periodic and

$$\sup_{|x|\leq n} |\omega_n(x) - \omega_0(x)| \leq \frac{1}{n} \ . \tag{42}$$

For example we can take

$$\omega_n(x) = \begin{cases} \omega_0(-n) & , \ x \in \langle -l_n, -n\rangle \\ \omega_0(x) & , \ x \in \langle -n, n\rangle \\ \frac{l_n - x}{l_n - n}\omega_0(n) + \frac{x-n}{l_n-n}\omega_0(-n) & , \ x \in \langle n, l_n\rangle \end{cases} .$$

Let fix $f \in G$. For quite big $n \in \mathbb{N}$ the points $x_1 + y, x_2 + y, ..., x_N + y \in (-l_n, l_n)$, where $|y| \leq n$. So from (42) and definition of τ_x we conclude that for $x \in \langle -1, 1\rangle$

$$\lim_{n\to\infty} |\tau_{nx}\omega_n(x_i) - \tau_{nx}\omega_0(x_i)| = \lim_{n\to\infty} |\omega_n(nx + x_i) - \omega_0(nx + x_i)| = 0.$$

Because f is continuous we have

$$\lim_{n\to\infty} \left(f(\tau_{nx}\omega_n) - f(\tau_{nx}\omega_0)\right) = 0.$$

Taking expected value of both sides we get

$$\lim_{n\to\infty} \mathbb{E}\left(f(\tau_{nX}\omega_n) - f(\tau_{nX}\omega_0)\right) = 0,$$

where X is the uniform distributed random variable on the interval $\langle -1, 1\rangle$. So using definition of μ_n and (41) we can state

$$\forall_{f\in\mathcal{G}} \quad \lim_{n\to\infty} \int_\Omega f d\mu_n = \int_\Omega f d\mu.$$

Ending we observe that ω_0 is bounded, because the condition (39) must be satisfied. We know from the Weierstrass theorem that any real function on compact set could be approximated by polynomials. So we conclude that if $f \in CB(\Omega)$ then we can approximate f by g, where $g \in \mathcal{G}$. Hence we have

$$\forall_{f\in CB(\Omega)} \quad \lim_{n\to\infty} \int_\Omega f d\mu_n = \int_\Omega f d\mu$$

and weak convergence μ_n to μ is proved. □

References

[1] Bass, R., *Probabilistic techniques in analysis. Probability and its Applications.* Springer-Verlag, New York, 1995.

[2] Bensoussan, A., Lions, J-L., Papanicolaou, G., *Asymptotic analysis for periodic structures. Studies in Mathematics and its Applications*, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[3] Dudley, R. M. *Real analysis and probability.* Cambridge Studies in Advanced Mathematics, **74**. Cambridge University Press, Cambridge, 2002.

[4] E. B. Fabes, D. W. Stroock, *The L^p integrability of Green's functions and fundamental solutions for elliptic and parabolic equations.* Duke Math. Journ. **51**, 997-1016.

[5] Jacod, J., Shiryaev, A., *Limit theorems for stochastic processes.* Second edition. Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag, Berlin, 2003.

[6] Kipnis, C., Landim, C., *Scaling limits of interacting particle systems.* Grundlehren der Mathematischen Wissenschaften, 320. Springer-Verlag, Berlin, 1999.

[7] Komorowski, T., Landim, C., Olla, S. *Fluctuations in Markov processes. Time symmetry and martingale approximation.* Grundlehren der Mathematischen Wissenschaften, 345. Springer, Heidelberg, 2012.

[8] Krengel U., *Ergodic Theorems*, New York, (1985).

[9] Lawler, G.F. *Weak convergence of a random walk in a random environment.* Comm. Math. Phys. **87**, 81-87 (1982)

[10] S. Olla, P. Siri, *Homogenization of a bond diffusion in a locally ergodic random environment*, Stochastic Processes and their Applications 109, pp. 317-326, 2004.

[11] G. C. Papanicolau and S. R. S. Varadhan, *Diffusions with random coefficients*, Statistics and Probability: Essays in Honor of C. R. Rao, pp. 547-552, North Holland, 1982.

[12] K. R. Parthasarathy, *On the category of ergodic measures*, Illinois Journal of Mathematics, Volume 5, pp. 648-656, 1961.

[13] Rhodes, R., *Diffusion in a locally stationary random environment.* Probab. Theory Related Fields **143** (2009), no. 3-4, 545568.

[14] Stroock, Daniel W.; Varadhan, S. R. Srinivasa Multidimensional diffusion processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 233. Springer-Verlag, Berlin-New York, 1979.

[15] V. V. Yurinskii, *Averaging of second-order non-divergent equations with random coefficients.* Sibirsk. Mat. Zh., **23** (2), pp. 176-188, 1982.

[16] Guo, X., Zeitouni, O., *Quenched invariance principle for random walks in balanced random environment.* Probab. Theory Relat. Fields (2012) **152**, 207-230

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