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# Some properties of the value function in a singular stochastic control problem with time-dependent coefficients

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# Some properties of the value function in a singular stochastic control problem with time-dependent coefficients

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**Abstract**

We prove some estimates and basic properties for the value function of a singular stochastic control problem in $n$-dimensions with time-dependent coefficients in a finite time horizon.

## Introduction.

We consider a $n$-dimensional singular stochastic control problem on a finite time horizon in which state is governed by a linear stochastic differential equation with time-dependent coefficients. We provide some estimates for the corresponding value function. These estimates imply that the value function has generalized derivatives of the second order with respect to the space variable and of the first order with respect to the time variable. These properties are needed to consider the value function as a solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation in some generalized sense.

Similar estimates for the value function have been shown in Theorem 2.1 from [1] but only in the one-dimensional case. Estimates for the value function in a multidimensional singular stochastic control problem have been considered in Theorem 1 from [7], however in that work the time horizon is infinite and coefficients of the drift, covariance, costs are time-independent.

The structure of this paper is as follows. In Section 1 we pose the singular stochastic control problem, give definitions and prove lemmas needed in further considerations. In Section 2 we prove estimates for the value function. In Section 3 we consider the Bellman's dynamic programming principle (DPP) and the HJB equation related to this problem.

## 1 Notation, assumptions and lemmas.

Let $(W_t, t \geq 0)$ be a standard $n$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t, t \geq 0)$ be the augmentation of the filtration generated by $W$ (see [4] p. 89).

Let $T > 0$ be a fixed number representing our time horizon. Denote by $\mathcal{V}$ the set of controls $v$ which are left-continuous, progressively measurable random processes acting from $\langle 0, T\rangle$ into $\mathbb{R}^n$, with $P$-a.s. bounded variation and s.t. $v(0) = 0$ $P$-a.s.. As it is customary in singular stochastic control theory (see e.g. [5]), we write

$$v(t) = \int_0^t \gamma(s) d\xi(s),$$

where $|\gamma(t)| = 1$ for every $t \in \langle 0, T\rangle$ and $\xi$ is nondecreasing and left-continuous. In other words, $\xi(t)$ is the total variation of $v$ on the time interval $\langle 0, t\rangle$ and $\gamma(t)$ is the Radon-Nikodym derivative of the vector-valued measure induced by $v$ on $\langle 0, T\rangle$ with respect to its total variation $\xi$.

Let $\mathbb{M}^{n\times n}$ denotes the set of matrices of dimension $n\times n$ with the maximum norm, i.e. if $A = [a_{ij}] \in \mathbb{M}^{n\times n}$, then $||A|| = \max_{1\le i,j\le n} |a_{ij}|$. Consider the state process described by the stochastic integral equation

$$y_{xt}(s) = x + \int_t^s \Big(a(r)y_{xt}(r) + b(r)\Big)dr + \int_t^s \sigma(r)dW_{r-t} + v(s-t), \quad s \in \langle t, T\rangle, \tag{1}$$

where $t \in \langle 0, T\rangle$ is an initial time, $x \in \mathbb{R}^n$ is an initial position, $a : \langle 0, T\rangle \to \mathbb{R}$, $b : \langle 0, T\rangle \to \mathbb{R}^n$ and $\sigma : \langle 0, T\rangle \to \mathbb{M}^{n\times n}$ stand for the drift and the covariance terms. Note that $(y_{xt}(s))_{s\in\langle t,T\rangle}$ is a random process adapted to $(\mathcal{F}_{s-t})_{s\in\langle t,T\rangle}$.

To each control $v \in \mathcal{V}$, we associate a cost given by the payoff functional

$$J_{xt}(v) = \mathbb{E}\left\{\int_t^T f(y_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds + \int_t^T c(s)e^{-\int_t^s \alpha(r)dr}d\xi(s-t)\right\}, \tag{2}$$

where $f$, $\alpha$ and $c$ are respectively the running cost, the discount factor and the instantaneous cost per unit of "fuel".

Our purpose is to characterize the optimal cost, the so called value function

$$u(x,t) = \inf\{J_{xt}(v) : v \in \mathcal{V}\}. \tag{3}$$

It is often convenient to consider the following penalized problem associated with (3):

$$u_\epsilon(x,t) = \inf\{J_{xt}(v) : v \in \mathcal{V}_\epsilon\}, \tag{4}$$

where $\epsilon > 0$ and $\mathcal{V}_\epsilon$ is the set of all controls $v \in \mathcal{V}$ which are Lipschitz continuous and $|\frac{dv}{dt}(t)| \le \frac{1}{\epsilon}$ for almost every $t \in \langle 0, T\rangle$ almost surely.

Definition 1.1. We say that the finite time horizon stochastic control problem has the *dynamic programming property in the weak sense* if for every $x \in \mathbb{R}^n$, $t, t' \in \langle 0, T\rangle$ s.t. $t < t'$ and $y^0_{xt}(s)$ given by (1) with $v \equiv 0$ we have

$$u(x,t) \le \mathbb{E}\left\{\int_t^{t'} f(y^0_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds + u(y^0_{xt}(t'), t')e^{-\int_t^{t'} \alpha(r)dr}\right\}. \tag{5}$$

Let us assume the following:

- $a,\ \alpha,\ c$ are Lipschitz continuous from $\langle 0, T\rangle$ into $\mathbb{R}$ with constant $L > 0$ ,
- $b$ is Lipschitz continuous from $\langle 0, T\rangle$ into $\mathbb{R}^n$ with the same constant $L > 0$ ,
- $\sigma$ is Lipschitz continuous from $\langle 0, T\rangle$ into $\mathbb{M}^{n\times n}$ with the same constant $L > 0$ ,
- there exists $c_0 > 0$ such that $c(t) \ge c_0$ for all $t \in \langle 0, T\rangle$,

- there exists $\alpha_0 > 0$ such that $\alpha(t) \geq \alpha_0$ for all $t \in \langle 0, T\rangle$ ,
- $f : \mathbb{R}^n \times \langle 0, T\rangle \to \langle 0, \infty)$ and there exist constants $p > 1$, $C_0$, $\tilde{C}_0 > 0$ such that for all $t, t' \in \langle 0, T\rangle$, $x, x' \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ we have

$$\tilde{C}_0|x|^p - C_0 \leq f(x,t) \leq C_0(1+|x|^p) \ , \tag{6}$$

$$|f(x,t) - f(x+x',t)| \leq C_0\big(1 + f(x,t) + f(x+x',t)\big)^{1-1/p}|x'| \ , \tag{7}$$

$$|f(x,t) - f(x,t')| \leq C_0(1+|x|^p)|t-t'| \ , \tag{8}$$

$$0 < f(x+\lambda x',t) - 2f(x,t) + f(x-\lambda x',t) \leq C_0\lambda^2(1+f(x,t))^q \ , \ q = \left(1 - \frac{2}{p}\right)^+ . \tag{9}$$

Note that the last assumption implies stricly convexity of the running cost function $f$ with respect to $x$.

Let denote by $c_{\max}$ and $\alpha_{\max}$ the maximum of the function $c$, $\alpha$ respectively.

Now we give lemmas needed for the proof of the Theorem 2.1.

LEMMA 1.1.

$$\forall_{x,y\geq 0}\ \forall_{p\geq 1} \quad x^p + y^p \leq (x+y)^p \leq 2^{p-1}(x^p+y^p), \tag{10}$$

$$\forall_{x,y\geq 0}\ \forall_{p\in(0,1)} \quad 2^{p-1}(x^p+y^p) \leq (x+y)^p \leq x^p + y^p. \tag{11}$$

*Proof:* First we prove $(x+y)^p \leq 2^{p-1}(x^p+y^p)$ for $p \geq 1$. Indeed, the function $(\cdot)^p$ is convex, so from the Jensen's inequality

$$\left(\frac{x+y}{2}\right)^p \leq \frac{1}{2}x^p + \frac{1}{2}y^p.$$

Multiplying both sides by $2^p$ we get the conlusion.

Similarly $2^{p-1}(x^p+y^p) \leq (x+y)^p$ for $p \in (0,1)$ because then $(\cdot)^p$ is concave.

To prove the remaining inequalities we consider the function

$$f(x,y) = (x+y)^p - x^p - y^p, \ x, y \geq 0.$$

We observe that $f(0,y) = f(x,0) = 0$. Moreover for $x, y > 0$ and $p \geq 1$ we have

$$f_x(x,y) = p\left((x+y)^{p-1} - x^{p-1}\right) > 0.$$

From symetry $f_y(x,y) > 0$. So we conclude that $f(x,y) \geq 0$ for $p \geq 1$.

Similarly, if $p \in (0,1)$, we have $f(x,y) \leq 0$ because then $f_x(x,y), f_y(x,y) < 0$. □

LEMMA 1.2. (See [6] Corollary 2.5.12 page 86). Consider an $n$-dimensional process described by the stochastic integral equation

$$x(t) = x_0 + \int_0^t g(x(s),s)ds + \int_0^t h(x(s),s)dW_s \ , t \geq 0 \ ,$$

where $x_0 \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \langle 0, \infty) \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \langle 0, \infty) \to \mathbb{M}^{n\times n}$. We assume that there exists a constant $C$ such that for all $x \in \mathbb{R}^n$ and $t \geq 0$

$$||h(x,t)|| + |g(x,t)| \leq C(1+|x|). \tag{12}$$

Then for every $q > 0$ there exists a constant $N$ depending only on $q, C$ such that for all $t \geq 0$

$$\mathbb{E} \sup_{0 \leq s \leq t} |x(s)|^q \leq N\, e^{Nt}(1 + |x_0|)^q. \tag{13}$$

REMARK 1.1. Observe that in our stochastic problem (1) with $v \equiv 0$ the assumption (12) holds. Indeed, $\sigma$ is Lipschitz continuous, independent on $x$ and defined on a finite time interval $\langle 0, T \rangle$, so it is bounded. We conclude the same about $a, b$, so $|g(x,t)| = |a(t) \cdot x + b(t)| \leq C(1 + |x|)$ where $C = \max\{|a(t)|, |b(t)| : t \in \langle 0, T \rangle\}$.

LEMMA 1.3. For all $x, x' \in \mathbb{R}^n$, $t \in \langle 0, T \rangle$ and $s \in \langle t, T \rangle$

$$y_{xt}(s) - y_{x't}(s) = e^{\int_t^s a(r)dr} \cdot (x - x') \ . \tag{14}$$

*Proof:* We denote $g(s) = y_{xt}(s) - y_{x't}(s)$. From (1) we have

$$g(s) = x - x' + \int_t^s a(r)(y_{xt}(r) - y_{x't}(r))dr = x - x' + \int_t^s a(r)g(r)dr.$$

Taking the derivative $d/ds$ of both sides, we get the differential equation $\frac{dg}{ds}(s) = a(s)g(s)$ with initial data $g(t) = x - x'$. The solution of this problem is $g(s) = e^{\int_t^s a(r)dr}(x - x')$. □

LEMMA 1.4. We suppose that for some $x \in \mathbb{R}^n,\ t \in \langle 0, T \rangle,\ v \in \mathcal{V}$ we have

$$\mathbb{E} \int_t^T f(y_{xt}(s), s) e^{-\int_t^s \alpha(r)dr} ds \leq C(1 + |x|^p)$$

for a suitable constant $C > 0$ independent on $x, t$. Then

$$\mathbb{E} \int_t^T f(y_{xt}(s), s)ds \leq C_4(1 + |x|^p), \text{ where } C_4 = C \cdot e^{\int_0^T \alpha(r)dr}.$$

*Proof:* Indeed, multiplying both sides of our assumption by $e^{\int_t^T \alpha(r)dr}$ we get

$$\mathbb{E} \int_t^T f(y_{xt}(s), s) e^{\int_s^T \alpha(r)dr} ds \leq C e^{\int_t^T \alpha(r)dr}(1 + |x|^p) \leq C_4(1 + |x|^p).$$

Of course, the left-hand side is greater than $\mathbb{E} \int_t^T f(y_{xt}(s), s)ds$. □

LEMMA 1.5. (compare a statement in [8] page 181). The function $J_{xt}(v)$ is convex with respect to $(x, v)$, more precisely, for all $x_1, x_2 \in \mathbb{R}^n$, $t \in \langle 0, T \rangle$, $v_1, v_2 \in \mathcal{V}$ and $\theta \in \langle 0, 1 \rangle$ the following holds

$$J_{\theta x_1 + (1-\theta)x_2, t}(\theta v_1 + (1 - \theta)v_2) \leq \theta J_{x_1,t}(v_1) + (1 - \theta)J_{x_2,t}(v_2) \ . \tag{15}$$

*Proof:* First, we note that the set $\mathcal{V}$ is obviously convex. Let $y^v_{xt}(s)$ be the solution of (1) corresponding to a control $v$. Denote $v_0 = \theta v_1 + (1 - \theta)v_2$ and $x_0 = \theta x_1 + (1 - \theta)x_2$. In view of the definition of $J_{xt}(v)$, it suffices to prove two following inequalities

$$f\big(y^{v_0}_{x_0,t}(s), s\big) \leq \theta f(y^{v_1}_{x_1,t}(s), s) + (1 - \theta) f(y^{v_2}_{x_2,t}(s), s),\ \ s \in \langle t, T \rangle \ , \tag{16}$$

$$\int_t^T d\xi_0(s - t) \leq \theta \int_t^T d\xi_1(s - t) + (1 - \theta) \int_t^T d\xi_2(s - t) \ , \tag{17}$$

where $\xi_0,\ \xi_1,\ \xi_2$ are the total variations of $v_0,\ v_1,\ v_2$ respectively.

The latter inequality is a consequence of the fact that the variation of the sum of functions is not greater than the sum of their variations. So $\xi_0 \le \theta\xi_1 + (1-\theta)\xi_2$. Because $\xi_0,\ \xi_1,\ \xi_2$ are nondecreasing and $\xi_0(0)=\xi_1(0)=\xi_2(0)=0$ $P$-a.s., we conclude that (17) is true.

To prove (16) we show first that

$$y^{v_0}_{x_0,t}(s) = \theta y^{v_1}_{x_1,t}(s) + (1-\theta) y^{v_2}_{x_2,t}(s)\ . \tag{18}$$

Indeed, using (1) we get

$$y^{v_i}_{x_i,t}(s) = x_i + \int_t^s \big(a(r) y^{v_i}_{x_i,t}(r) + b(r)\big)\,dr + \int_t^s \sigma(r) dW_{r-t} + v_i(s-t)\ ,\ i=0,1,2.$$

Let $g(s) = y^{v_0}_{x_0,t}(s) - \theta y^{v_1}_{x_1,t}(s) - (1-\theta) y^{v_2}_{x_2,t}(s)$. Then

$$g(s) = \int_t^s a(r)\Big(y^{v_0}_{x_0,t}(r) - \theta y^{v_1}_{x_1,t}(r) - (1-\theta) y^{v_2}_{x_2,t}(r)\Big) dr = \int_t^s a(r) g(r) dr\ .$$

Taking the derivative $d/ds$ of both sides, we get the differential equation $\frac{dg}{ds}(s) = a(s)g(s)$ with initial data $g(t) = x_0 - \theta x_1 - (1-\theta)x_2 = 0$. The solution of this problem is $g(s)\equiv 0$, so (18) holds.

Using (18) and the convexity of $f$ we have

$$f\big(y^{v_0}_{x_0,t}(s), s\big) = f\big(\theta y^{v_1}_{x_1,t}(s) + (1-\theta) y^{v_2}_{x_2,t}(s), s\big) \le \theta f(y^{v_1}_{x_1,t}(s), s) + (1-\theta) f(y^{v_2}_{x_2,t}(s), s)$$

and (16) is proved. □

LEMMA 1.6. We suppose that for some $t' \in \langle 0, T\rangle,\ x\in\mathbb{R}^n,\ v\in\mathcal{V}$ we have

$$\mathbb{E}\int_0^{T-t'} c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} d\xi(s) \le C(1+|x|^p)$$

for a suitable constant $C>0$ independent on $x, t'$. Then there exists a constant $C_6 \ge 0$ independent on $x,\ t'$ such that

$$\mathbb{E}\xi(T-t') \le C_6(1+|x|^p).$$

*Proof:* Indeed, multiplying both sides of our assumption by $e^{\int_0^{T-t'}\alpha(t'+r)dr}$ and using lower bound of $c$ we get

$$c_0 \mathbb{E}\xi(T-t') = c_0\mathbb{E}\int_0^{T-t'} d\xi(s) \le \mathbb{E}\int_0^{T-t'} c(t'+s) e^{\int_s^{T-t'}\alpha(t'+r)dr} d\xi(s) \le C e^{\int_0^T \alpha(r)dr}(1+|x|^p).\quad \square$$

LEMMA 1.7. We suppose that for some $x\in\mathbb{R}^n,\ t\in\langle 0,T\rangle,\ v\in\mathcal{V}$ we have

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t+r)dr} ds \le C(1+|x|^p)$$

for a suitable constant $C>0$ independent on $x, t$. Then there exists a constant $C_7 \ge 0$ independent on $x,\ t$ such that

$$\mathbb{E}\int_0^{T-t} (1+|y_{xt}(t+s)|^p) ds \le C_7(1+|x|^p).$$

*Proof:* From Lemma 1.4 we know that

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \le C_4(1+|x|^p).$$

Using (6) we get

$$\mathbb{E}\int_0^{T-t} \Big(\tilde{C}_0|y_{xt}(t+s)|^p - C_0\Big)ds \le C_4(1+|x|^p).$$

Hence

$$\tilde{C}_0\mathbb{E}\int_0^{T-t} |y_{xt}(t+s)|^p ds \le (C_4 + C_0T)(1+|x|^p)$$

and finally

$$\tilde{C}_0\mathbb{E}\int_0^{T-t}(1+|y_{xt}(t+s)|^p)ds \le (C_4 + C_0T + \tilde{C}_0T)(1+|x|^p)\ . \quad \square$$

LEMMA 1.8. Let $0 \le t' \le t \le T$ and suppose that for some $x \in \mathbb{R}^n,\ v \in \mathcal{V}$ we have

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \le C(1+|x|^p)$$

for suitable constant $C > 0$ independent on $x, t, t'$. Then there exists a constant $C_8 \ge 0$ independent on $x,\ t,\ t'$ such that

$$\mathbb{E}\int_0^{T-t} f(y_{xt'}(t'+s), t+s)ds \le C_8(1+|x|^p).$$

*Proof:* We observe that using (8) we have

$$f(y_{xt'}(t'+s), t+s) \le |f(y_{xt'}(t'+s), t+s) - f(y_{xt'}(t'+s), t'+s)| + f(y_{xt'}(t'+s), t'+s) \le$$
$$\le C_0|t-t'|(1+|y_{xt'}(t'+s)|^p) + f(y_{xt'}(t'+s), t'+s).$$

Hence, in view of Lemma 1.4 and Lemma 1.7, we get

$$\mathbb{E}\int_0^{T-t} f(y_{xt'}(t'+s), t+s)ds \le C_0|t-t'|C_7(1+|x|^p) + C_4(1+|x|^p) \le C_8(1+|x|^p),$$

where $C_8 = C_0TC_7 + C_4$. $\square$

LEMMA 1.9. (Compare [6] Corollary 2.5.5 page 80). Let $x, \tilde{x}$ be $n$-dimensional processes described by the stochastic integral equations

$$x(t) = z(t) + \int_0^t g(x(s), s)ds + \int_0^t h(x(s), s)dW_s\ , t \ge 0\ ,$$

$$\tilde{x}(t) = z(t) + \int_0^t \tilde{g}(\tilde{x}(s), s)ds + \int_0^t \tilde{h}(\tilde{x}(s), s)dW_s\ , t \ge 0\ ,$$

where $z(t), g(x,t), \tilde{g}(x,t)$ are random vectors in $\mathbb{R}^n$ and $h(x,t), \tilde{h}(x,t)$ are random matrices of dimension $n \times n$ for $x \in \mathbb{R}^n,\ t \ge 0$. Suppose that for each $t \in \langle 0, T\rangle$ the functions $g, \tilde{g}, h, \tilde{h}$ are Lipschitz continuous with respect to the variable $x$ with the same constant $K \ge 0$.

Then for each $q \geq 2$, $T > 0$ there exist constant $N$ depending only on $q, K, T$ such that for all $t \in \langle 0, T \rangle$

$$\mathbb{E}|x(t) - \tilde{x}(t)|^q \leq N\mathbb{E} \int_0^t \Big( |g(\tilde{x}(s), s) - \tilde{g}(\tilde{x}(s), s)|^q + ||h(\tilde{x}(s), s) - \tilde{h}(\tilde{x}(s), s)||^q \Big) ds \; .$$

LEMMA 1.10. Let $0 \leq t' \leq t \leq T$, $x \in \mathbb{R}^n$, $v \in \mathcal{V}$. Assume that

$$\mathbb{E} \int_0^{T-t'} f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} ds \leq C(1 + |x|^p)$$

for a suitable constant $C > 0$ independent on $x, t', t$. Then there exists constant $C_{10} > 0$ independent on $x, t', t$ such that for all $s \in \langle 0, T - t \rangle$ we have

$$\mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p \leq C_{10}|t - t'|^p(1 + |x|^p) \; . \tag{19}$$

*Proof:* First we assume that $p \geq 2$. We note that processes $y_{xt}(t+s), y_{xt'}(t'+s)$ for $s \in \langle 0, T-t \rangle$ satisfy assumptions of Lemma 1.9. Indeed, we can write them in the form

$$y_{xt}(t+s) = x + \int_0^s \big( a(t+r)y_{xt}(t+r) + b(t+r) \big) dr + \int_0^s \sigma(t+r)dW_r + v(s),$$

$$y_{xt'}(t'+s) = x + \int_0^s \big( a(t'+r)y_{xt'}(t'+r) + b(t'+r) \big) dr + \int_0^s \sigma(t'+r)dW_r + v(s).$$

So, in the notation of Lemma 1.9 we have

$$g(x,s) = a(t+s)x + b(t+s), \quad \tilde{g}(x,s) = a(t'+s)x + b(t'+s),$$

$$h(x,s) = \sigma(t+s), \quad \tilde{h}(x,s) = \sigma(t'+s).$$

Of course, the above functions are Lipschitz continuous with respect to variable $x$ with constant $K = \max_{s \in \langle 0,T \rangle} |a(s)|$. Hence we can conclude that

$$\mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p \leq N\mathbb{E} \int_0^s \Big( |a(t+r)y_{xt'}(t'+r) - a(t'+r)y_{xt'}(t'+r) + b(t+r) - b(t'+r)|^p +$$

$$+ ||\sigma(t+r) - \sigma(t'+r)||^p \Big) dr \; .$$

Lemma 1.1, Lemma 1.7 and Lipschitz continuity of $a, b, \sigma$ imply

$$\mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p \leq N\mathbb{E} \int_0^s \left\{ 2^{p-1}(L|t-t'||y_{xt'}(t'+r)|)^p + 2^{p-1}(L|t-t'|)^p + (L|t-t'|)^p \right\} dr$$

$$\leq (2^p + 1)NL^p|t-t'|^p \; \mathbb{E} \int_0^s (1 + |y_{xt'}(t'+r)|^p) dr \leq C_{10}|t-t'|^p(1 + |x|^p) \; ,$$

where $C_{10} = (2^p + 1)NL^pC_7$.

For $p \in (1, 2)$ we use the Hölder's inequality with exponent $2/p$ together with (19) for $p = 2$, and get

$$\mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p \leq \big( \mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^2 \big)^{p/2} \leq$$

$$\leq \Big\{C_{10}|t-t'|^2(1+|x|^2)\Big\}^{p/2} \leq C_{10}^{p/2}\ |t-t'|^p(1+|x|)^p \leq 2^{p-1}\ C_{10}^{p/2}\ |t-t'|^p(1+|x|^p).\ \Box$$

LEMMA 1.11. We suppose that for some $x \in \mathbb{R}^n,\ t \in \langle 0, T\rangle,\ v \in \mathcal{V}$ we have

$$\mathbb{E}\int_t^T f(y_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds \leq C(1+|x|^p)$$

for suitable constant $C > 0$ independent on $x, t$. Then there exists a constant $C_{11} \geq 0$ independent on $x,\ x',\ t$ such that for every $x' \in \mathbb{R}^n$

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq C_{11}(1+|x|^p+|x+x'|^p).$$

*Proof:* From (6) and Lemma 1.1 we have

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq \mathbb{E}\int_t^T C_0(1+|y_{x+x',t}(s)|^p)ds \leq$$

$$\leq TC_0 + C_0 2^{p-1}\mathbb{E}\int_t^T |y_{x+x',t}(s) - y_{x,t}(s)|^p ds + C_0 2^{p-1}\mathbb{E}\int_t^T |y_{x,t}(s)|^p ds\ .$$

Now using Lemma 1.3, Lemma 1.7 and Lemma 1.1 again, we get

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq TC_0 + C_0 2^{p-1} T e^{p\int_0^T a(r)dr}|x'|^p + C_0 2^{p-1} C_7(1+|x|^p) \leq$$

$$\leq TC_0 + C_0 2^{2p-2} T e^{p\int_0^T a(r)dr}\big(|x'+x|^p+|x|^p\big) + C_0 2^{p-1}C_7(1+|x|^p) \leq C_{11}(1+|x|^p+|x+x'|^p),$$

where $C_{11} = C_0\big(T + 2^{2p-2}Te^{p\int_0^T a(r)dr} + 2^{p-1}C_7\big)$. $\Box$

LEMMA 1.12. We suppose that for some $x \in \mathbb{R}^n,\ t' \in \langle 0, T\rangle,\ v \in \mathcal{V}$ we have

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \leq C(1+|x|^p)$$

for suitable constant $C > 0$ independent on $x, t'$. Then there exists a constant $C_{12} \geq 0$ independent on $x,\ t',\ t$ such that for every $t \in \langle t', T\rangle$

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \leq C_{12}(1+|x|^p).$$

*Proof:* Using (6) and Lemma 1.1 we have

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \leq \mathbb{E}\int_0^{T-t} C_0(1+|y_{xt}(t+s)|^p)ds \leq$$

$$\leq C_0 T + 2^{p-1}C_0\mathbb{E}\int_0^{T-t} |y_{xt'}(t'+s)|^p ds + 2^{p-1}C_0\mathbb{E}\int_0^{T-t} |y_{xt}(t+s) - y_{xt'}(t'+s)|^p ds\ .$$

In view of Lemma 1.7, the Fubini's theorem and Lemma 1.10 we get

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \leq C_0T + 2^{p-1}C_0C_7(1+|x|^p) + 2^{p-1}C_0TC_{10}|t-t'|^p(1+|x|^p) \leq$$

$$\le C_{12}(1+|x|^p), \quad \text{where } C_{12} = C_0\big(T + 2^{p-1}C_7 + 2^{p-1}T^{p+1}C_{10}\big). \ \square$$

The next two definitions and lemma refer to the mollification of a given function (see [2] pages 629-630).

DEFINITION 1.2. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} \hat{C}\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

where the constant $\hat{C} > 0$ is selected so that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

For each $m \in \mathbb{N}$ set

$$\eta_m(x) = m^n \cdot \eta(mx).$$

We call $\eta$ the standard mollifier. Moreover the functions $\eta_m$ are $C^\infty$ and satisfy $\int_{\mathbb{R}^n} \eta_m(x)dx = 1$.

DEFINITION 1.3. Fix $t' \in \langle 0,T\rangle$. For each $m \in \mathbb{N}$ we define mollification of the function $u(\cdot,t')$ by

$$u_m(x) = \int_{B(0,\frac{1}{m})} \eta_m(y)u(x-y,t')dy, \quad x \in \mathbb{R}^n,$$

where $B\big(0,\frac{1}{m}\big) = \big\{x \in \mathbb{R}^n : |x| < \frac{1}{m}\big\}$.

LEMMA 1.13. For each $m \in \mathbb{N}$ we have $u_m \in C^\infty(\mathbb{R}^n)$. Moreover if $u(\cdot,t')$ is continuous, then $u_m(x) \to u(x,t')$ uniformly on compact subsets of $\mathbb{R}^n$ as $m \to \infty$.

# 2 Estimates for the value function.

THEOREM 2.1. Let the assumptions from page 2 of Section 1 hold. Then the value function $u$ is a nonnegative function such that for some constants $C, C_1, C_2 > 0$, the same $p > 1$ as in the assumptions $(6)-(9)$ and every $t,t' \in \langle 0,T\rangle$, $x,x' \in \mathbb{R}^n$ and $\lambda \in (0,1)$ the following estimates hold:

$$0 \le u(x,t) \le C(1+|x|^p)\ , \tag{20}$$

$$|u(x,t)-u(x+x',t)| \le C_1(1+|x|^{p-1}+|x+x'|^{p-1})|x'|\ , \tag{21}$$

$$0 \le u(x+\lambda x',t) - 2u(x,t) + u(x-\lambda x',t) \le C_2\lambda^2(1+|x|)^{(p-2)^+}. \tag{22}$$

Moreover if the dynamic programming property in the weak sense is satisfied (Def. 1.1), then for some constant $C_3 > 0$ we have

$$|u(x,t)-u(x,t')| \le C_3(1+|x|^p)|t-t'|\ . \tag{23}$$

*Proof of* (20)*:* Nonnegativity of $u$ is consequence of nonnegativity of $f$ and $c$. Next, taking the control $v \equiv 0$ and using (6), the Fubini's theorem, Lemma 1.2 and Lemma 1.1, we get

$$u(x,t) \le J_{xt}(0) = \mathbb{E}\int_t^T f(y^0_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds \le \mathbb{E}\int_t^T C_0(1+|y^0_{xt}(s)|^p)e^{-\alpha_0(s-t)}ds \le$$

$$\leq C_0 \int_t^T \mathbb{E}(1+|y^0_{xt}(s)|^p)ds \leq C_0 \int_t^T \mathbb{E}\left(1+Ne^{N(s-t)}(1+|x|)^p\right)ds \leq$$

$$\leq C_0 \int_0^T (1+Ne^{NT}2^{p-1})(1+|x|^p)ds = C_0T(1+Ne^{NT}2^{p-1})(1+|x|^p) = C(1+|x|^p),$$

where $C$ depends only on $C_0, T, N, p$, so (20) is proved.

*Proof of* (21)*:* Now we note that

$$u(x+x',t)-u(x,t) = \inf_{v'\in\mathcal{V}} \sup_{v\in\mathcal{V}} \Big(J_{x+x',t}(v') - J_{x,t}(v)\Big) \leq \sup_{v\in\mathcal{V}} \Big(J_{x+x',t}(v) - J_{x,t}(v)\Big).$$

Hence

$$u(x+x',t)-u(x,t) \leq \sup_{v\in\mathcal{V}} |J_{xt}(v) - J_{x+x',t}(v)| \leq$$

$$\leq \sup_{v\in\mathcal{V}} \mathbb{E}\int_t^T |f(y_{xt}(s),s) - f(y_{x+x',t}(s),s)| e^{-\int_t^s \alpha(r)dr} ds.$$

Applying (7) we can estimate the last expression from above by

$$\sup_{v\in\mathcal{V}} \mathbb{E}\int_t^T C_0\big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)^{1-1/p} \cdot |y_{xt}(s) - y_{x+x',t}(s)|ds.$$

Using Lemma 1.3 we have

$$u(x+x',t)-u(x,t) \leq \sup_{v\in\mathcal{V}} C_0 \mathbb{E}\int_t^T \big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)^{1-1/p} \cdot |e^{\int_t^s a(r)dr} x'|ds \leq$$

$$\leq \sup_{v\in\mathcal{V}} C_0K|x'| \cdot \mathbb{E}\int_t^T \big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)^{\frac{p-1}{p}} ds,$$

where $K = e^{\int_0^T a(r)dr}$. Now we use the Hölder's inequality with exponent $\frac{p}{p-1}$ to get

$$u(x+x',t)-u(x,t) \leq \sup_{v\in\mathcal{V}} C_0K|x'| \cdot \left(\mathbb{E}\int_t^T \big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)ds\right)^{\frac{p-1}{p}} T^{1/p}. \tag{24}$$

By virtue of (20) we can consider only those controls $v$ for which

$$\mathbb{E}\int_t^T f(y_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds \leq (C+\epsilon)(1+|x|^p)$$

for an arbitrary $\epsilon > 0$. From (24), Lemma 1.4 and Lemma 1.11 we see that

$$u(x+x',t)-u(x,t) \leq C_0K|x'| \cdot \Big(T + C_4(1+|x|^p) + C_{11}(1+|x|^p+|x+x'|^p)\Big)^{\frac{p-1}{p}} T^{1/p} \leq$$

$$\leq C_1|x'|\big(1+|x|^p+|x+x'|^p\big)^{\frac{p-1}{p}},$$

where $C_1 = T^{1/p}C_0K(T+C_4+C_{11})^{1-1/p}$. Finally using Lemma 1.1 we get

$$u(x+x',t)-u(x,t) \leq C_1(1+|x|^{p-1}+|x+x'|^{p-1})|x'|.$$

In an analogous manner we can have

$$u(x,t) - u(x+x',t) \le C_1(1+|x|^{p-1}+|x+x'|^{p-1})|x'|.$$

The last two inequalities are equivalent to (21).

*Proof of* (22)*:* We observe that

$$u(x+\lambda x',t)+u(x-\lambda x',t)-2u(x,t) = \inf_{v_1\in\mathcal{V}}\inf_{v_2\in\mathcal{V}}\sup_{v\in\mathcal{V}} J_{x+\lambda x',t}(v_1)+J_{x-\lambda x',t}(v_2)-2J_{xt}(v) \le$$

$$\le \sup_{v\in\mathcal{V}} J_{x+\lambda x',t}(v)+J_{x-\lambda x',t}(v)-2J_{xt}(v) =$$

$$= \sup_{v\in\mathcal{V}} \mathbb{E}\int_t^T \Big(f(y_{x+\lambda x',t}(s),s)+f(y_{x-\lambda x',t}(s),s)-2f(y_{xt}(s),s)\Big)e^{-\int_t^s \alpha(r)dr}ds \ .$$

In view of (14) we can apply (9) to get

$$u(x+\lambda x',t)+u(x-\lambda x',t)-2u(x,t) \le \sup_{v\in\mathcal{V}}\mathbb{E}\int_t^T C_0\lambda^2\Big(1+f(y_{xt}(s),s)\Big)^{(1-2/p)^+}ds \ .$$

If $p\le 2$ we have $u(x+\lambda x',t)+u(x-\lambda x',t)-2u(x,t)\le C_0T\lambda^2$. If $p>2$ we use Hölder inequality with exponent $\frac{p}{p-2}$ to get

$$u(x+\lambda x',t)+u(x-\lambda x',t)-2u(x,t) \le \sup_{v\in\mathcal{V}} C_0\lambda^2\Big(\mathbb{E}\int_t^T (1+f(y_{xt}(s),s)ds\Big)^{1-2/p}T^{2/p} \ .$$

By virtue of (20) we can consider only those controls $v$ for which

$$\mathbb{E}\int_t^T f(y_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds \le (C+\epsilon)(1+|x|^p)$$

for an arbitrary $\epsilon>0$. From Lemma 1.4 and Lemma 1.1 we see that

$$u(x+\lambda x',t)+u(x-\lambda x',t)-2u(x,t) \le C_0\lambda^2\Big(T+C_4(1+|x|^p)\Big)^{1-2/p}T^{2/p} \le$$

$$\le C_2\lambda^2(1+|x|^p)^{1-2/p} \le C_2\lambda^2(1+|x|)^{p-2},$$

where $C_2 = T^{2/p}C_0(T+C_4)^{1-2/p}$. In this manner we have proved the upper bound of (22).

To prove the lower bound of (22), it clearly suffices to prove the convexity of $u(x,t)$ with respect to first variable. In view of definition of $u$ we know that for every $\epsilon>0$, $x_1,x_2\in\mathbb{R}^n$, $t\in\langle 0,T\rangle$, $\theta\in\langle 0,1\rangle$ there exist $v_1,v_2\in\mathcal{V}$ such that

$$J_{x_i,t}(v_i)\le u(x_i,t)+\epsilon, \quad i=1,2.$$

Using Lemma 1.5 we get

$$u(\theta x_1+(1-\theta)x_2,t) \le J_{\theta x_1+(1-\theta)x_2,t}(\theta v_1+(1-\theta)v_2) \le$$

$$\le \theta J_{x_1,t}(v_1)+(1-\theta)J_{x_2,t}(v_2) \le \theta u(x_1,t)+(1-\theta)u(x_2,t)+\epsilon \ .$$

Because $\epsilon>0$ is arbitrary, we get convexity of $u(x,t)$ with respect to first variable and (22) is proved.

*Proof of* (23)*:* We note that

$$u(x,t) - u(x,t') = \inf_{v\in\mathcal{V}} \sup_{v'\in\mathcal{V}} \Big(J_{xt}(v) - J_{xt'}(v')\Big) \le \sup_{v'\in\mathcal{V}} \Big(J_{xt}(v') - J_{xt'}(v')\Big).$$

For $t' \le t$ we have

$$J_{xt}(v) - J_{xt'}(v) = \mathbb{E}\Bigg\{\int_0^{T-t} f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t+r)dr} ds + \int_0^{T-t} c(t+s) e^{-\int_0^s \alpha(t+r)dr} d\xi(s)$$

$$- \int_0^{T-t'} f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} ds - \int_0^{T-t'} c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} d\xi(s)\Bigg\} =$$

$$= \mathbb{E}\Bigg\{\int_0^{T-t} \Big(f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t+r)dr} - f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr}\Big) ds +$$

$$+ \int_0^{T-t} \Big(c(t+s) e^{-\int_0^s \alpha(t+r)dr} - c(t'+s) e^{-\int_0^s \alpha(t'+r)dr}\Big) d\xi(s)$$

$$- \int_{T-t}^{T-t'} f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} ds - \int_{T-t}^{T-t'} c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} d\xi(s)\Bigg\} \ .$$

Let us denote the expection of the first integral of the last expression by $A$ and the second by $B$. Because the last two integrals are nonnegative we get

$$J_{xt}(v) - J_{xt'}(v) \le A + B \ . \tag{25}$$

We can estimate $B$ as follows:

$$B \le \mathbb{E} \int_0^{T-t} \Big| c(t+s) e^{-\int_0^s \alpha(t+r)dr} - c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} \Big| d\xi(s) \ .$$

Adding and subtracting $c(t+s)e^{-\int_0^s \alpha(t'+r)dr}$ under the absolute value sign and using the triangle inequality and positivity of $\alpha$ we get

$$B \le \mathbb{E} \int_0^{T-t} \Big( c_{\max} | e^{-\int_0^s \alpha(t+r)dr} - e^{-\int_0^s \alpha(t'+r)dr} | + |c(t+s) - c(t'+s)| \Big) d\xi(s).$$

Because $|e^x - e^y| \le |x-y|$ for $x, y \le 0$ and $c,\ \alpha$ are Lipschitz continuous, we have

$$B \le \mathbb{E} \int_0^{T-t} \Big( c_{\max} \int_0^s |\alpha(t+r) - \alpha(t'+r)| dr + |c(t+s) - c(t'+s)| \Big) d\xi(s) \le$$

$$\le (c_{\max}T + 1) L |t-t'| \mathbb{E} \int_0^{T-t} d\xi(s) = (c_{\max}T+1) L |t-t'| \mathbb{E}\xi(T-t).$$

By virtue of (20) we can consider only those controls $v$ for which

$$\mathbb{E} \int_0^{T-t'} c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} d\xi(s) \le (C+\epsilon)(1+|x|^p)$$

for an arbitrary $\epsilon > 0$. Using Lemma 1.6 we get $\mathbb{E}\xi(T-t) \le \mathbb{E}\xi(T-t') \le C_6(1+|x|^p)$ and

$$B \le \ C_{26}\ |t-t'|(1+|x|^p)\ ,\ \text{ where } C_{26} = (c_{\max}T+1)LC_6\ . \tag{26}$$

Now we estimate $A$:

$$A \le \mathbb{E}\int_0^{T-t} \left| f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t+r)dr} - f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t'+r)dr} \right| ds +$$

$$+ \mathbb{E}\int_0^{T-t} \left| f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t'+r)dr} - f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} \right| ds = A_1 + A_2 \ .$$

Using again the inequality $|e^x - e^y| \le |x-y|$ for $x, y \le 0$ we get

$$A_1 \le \mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s) \left( \int_0^s |\alpha(t+r) - \alpha(t'+r)| dr \right) ds \le \tag{27}$$

$$\le TL|t-t'| \ \mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s) ds \ .$$

By virtue of (20) we can consider only those controls $v$ for which

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} ds \le (C+\epsilon)(1+|x|^p) \tag{28}$$

for an arbitrary $\epsilon > 0$. Using (27) and Lemma 1.12 we get

$$A_1 \le \ C_{29} \ |t-t'|(1+|x|^p) \ , \ \text{where } C_{29} = TLC_{12} \ . \tag{29}$$

To estimate $A_2$ we use (7) – (8) and we have

$$A_2 \le \mathbb{E}\int_0^{T-t} \left| f(y_{xt}(t+s), t+s) - f(y_{xt'}(t'+s), t+s) + f(y_{xt'}(t'+s), t+s) - f(y_{xt'}(t'+s), t'+s) \right| ds \le$$

$$\le \mathbb{E}\int_0^{T-t} C_0 \Big( 1 + f(y_{xt}(t+s), t+s) + f(y_{xt'}(t'+s), t+s) \Big)^{1-1/p} |y_{xt'}(t'+s) - y_{xt}(t+s)| ds +$$

$$+ \mathbb{E}\int_0^{T-t} C_0 (1 + |y_{xt'}(t'+s)|^p) |t-t'| ds = A_3 + A_4 \ .$$

Using the Hölder's inequality and the Fubini's theorem we get

$$A_3 \le C_0 \left\{ \mathbb{E}\int_0^{T-t} \Big( 1 + f(y_{xt}(t+s), t+s) + f(y_{xt'}(t'+s), t+s) \Big) ds \right\}^{1-1/p} \cdot$$

$$\cdot \left\{ \int_0^{T-t} \mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p ds \right\}^{1/p} \ .$$

From this together with (28), Lemma 1.12, Lemma 1.8 and Lemma 1.10 we have

$$A_3 \le C_0 \left\{ (T + C_{12} + C_8)(1+|x|^p) \right\}^{1-1/p} \cdot \left\{ TC_{10}|t-t'|^p (1+|x|^p) \right\}^{1/p} .$$

Because $1 + |x|^p \le (1+|x|)^p$ we get

$$A_3 \le C_0 (T + C_{12} + C_8)^{1-1/p} (1+|x|)^{p-1} \cdot (TC_{10})^{1/p} |t-t'| (1+|x|).$$

Hence, from Lemma 1.1

$$A_3 \le C_{30}|t-t'|(1+|x|^p), \ \text{where } C_{30} = C_0 (T + C_{12} + C_8)^{1-1/p} (TC_{10})^{1/p} \ 2^{p-1}. \tag{30}$$

Furthermore, from Lemma 1.7 we get

$$A_4 \le C_{31}|t-t'|(1+|x|^p), \text{ where } C_{31}=C_0C_7. \tag{31}$$

In view of (25)-(26) and (29)-(31) we get for $t' \le t$

$$u(x,t)-u(x,t') \le C_{32}|t-t'|(1+|x|^p), \text{ where } C_{32}=C_{26}+C_{29}+C_{30}+C_{31}\ . \tag{32}$$

To obtain a similar inequality for $t < t'$ we do as follows. Let $\Big(y^0_{xt}(s)\Big)_{s\in\langle t,T\rangle}$ be a solution of (1) with $v\equiv 0$. We can write the $i$-th coordinate of $y^0_{xt}(s)$ as follows

$$y^0_{xt}(s)_i = x_i + \int_t^s \big(a(r)y^0_{xt}(r)_i + b_i(r)\big)\,dr + \sum_{j=1}^n \int_t^s \sigma_{ij}(r)dW^j_{r-t}, \quad i=1,...,n\ , \tag{33}$$

where $x=\big(x_i\big)_{i=1,\dots,n}$ , $b(r)=\big(b_i(r)\big)_{i=1,\dots,n}$ , $\sigma(r)=\big[\sigma_{ij}\big]_{i,j=1,\dots,n}$ and $W_r=\big(W^j_r\big)_{j=1,\dots,n}$ .

Let $\{u_m(\cdot)\}_{n\in\mathbb{N}}$ be a sequence of mollifications of the function $u(\cdot,t')$ (see Def. 1.3). Applying the Itô's formula ([4] Th. 3.3.6) we get

$$\begin{aligned}\mathbb{E}u_m(y^0_{xt}(t')) &= u_m(x) + \mathbb{E}\sum_{i=1}^n \int_t^{t'} \frac{\partial u_m(y^0_{xt}(s))}{\partial x_i}\Big(a(s)y^0_{xt}(s)_i + b_i(s)\Big)ds\ + \\ &+ \mathbb{E}\sum_{i=1}^n \int_t^{t'} \frac{\partial u_m(y^0_{xt}(s))}{\partial x_i}\sum_{j=1}^n \sigma_{ij}(s)dW^j_{s-t}\ + \\ &+ \frac{1}{2}\,\mathbb{E}\sum_{i,j=1}^n \int_t^{t'} \frac{\partial^2 u_m(y^0_{xt}(s))}{\partial x_i \partial x_j} d\langle y^0_{xt}(s)_i, y^0_{xt}(s)_j\rangle = u_m(x) + \mathbb{A} + \mathbb{B} + \mathbb{C}\ .\end{aligned} \tag{34}$$

We need here the following Lemma.

Lemma 2.2. We assume $(20)-(22)$. Let $t'\in\langle 0,T\rangle$ be fixed. Then there exist constants $\hat{C}_1,\hat{C}_2>0$ such that for all $x\in\mathbb{R}^n$, $m\in\mathbb{N}$ and $i,j\in\{1,...,n\}$

$$\lim_{m\to\infty} u_m(x) = u(x,t'), \tag{35}$$

$$\left|\frac{\partial u_m(x)}{\partial x_i}\right| \le \hat{C}_1(1+|x|)^{p-1}, \tag{36}$$

$$0 \le \frac{\partial^2 u_m(x)}{\partial x_i \partial x_j} \le \hat{C}_2(1+|x|^p). \tag{37}$$

We estimate $\mathbb{A}$ as follows

$$\begin{aligned}\mathbb{A} &\le \mathbb{E}\sum_{i=1}^n \int_t^{t'} \left|\frac{\partial u_m(y^0_{xt}(s))}{\partial x_i}\right|\Big|a(s)y^0_{xt}(s)_i + b_i(s)\Big|ds \le \\ &\le \mathbb{E}\sum_{i=1}^n \int_t^{t'} \left|\frac{\partial u_m(y^0_{xt}(s))}{\partial x_i}\right|\Big(|a(s)||y^0_{xt}(s)| + |b(s)|\Big)ds.\end{aligned}$$

Denoting $a_{\max} = \max_{s\in\langle 0,T\rangle} |a(s)|$, $b_{\max} = \max_{s\in\langle 0,T\rangle} |b(s)|$ and using Lemma 2.2 and Lemma 1.1 we have

$$\mathbb{A} \leq \sum_{i=1}^{n} \mathbb{E}\int_t^{t'} \hat{C}_1(1+|y_{xt}^0(s)|)^{p-1}(a_{\max}+b_{\max})(1+|y_{xt}^0(s)|)ds \leq C_{38}\mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|^p)ds \ , \tag{38}$$

where $C_{38} = n\hat{C}_1(a_{\max}+b_{\max})2^{p-1}$.

Now we show that $\mathbb{B} = 0$. Indeed

$$\mathbb{B} = \mathbb{E}\sum_{i,j=1}^{n} Z_{ij}(t'), \text{ where } Z_{ij}(s) = \int_t^s \frac{\partial u_m(y_{xt}^0(r))}{\partial x_i}\sigma_{ij}(r)dW_{r-t}^j \text{ for } s \in \langle t,t'\rangle.$$

From properties of the Itô's integrals (see [4] section 3.2) the process $\big(Z_{ij}(s)\big)_{s\in\langle t,t'\rangle}$ is a martingale provided that

$$\mathbb{E}\int_t^{t'}\left(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\right)^2 ds \leq \infty.$$

Denoting $\sigma_{\max} = \max_{s\in\langle 0,T\rangle} ||\sigma(s)||$ and using Lemma 2.2 and Lemma 1.1 we have

$$\mathbb{E}\int_t^{t'}\left(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\right)^2 ds \leq \mathbb{E}\int_t^{t'} \hat{C}_1^2(1+|y_{xt}^0(s)|)^{2p-2}\ \sigma_{\max}^2 ds \leq$$

$$\leq \hat{C}_1^2\ \sigma_{\max}^2\ \mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|)^{2p}ds \leq \hat{C}_1^2\ \sigma_{\max}^2 2^{2p-1}\ \mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|^{2p})ds$$

Using the Fubini's theorem and Lemma 1.2 we get

$$\mathbb{E}\int_t^{t'}\left(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\right)^2 ds \leq \hat{C}_1^2\ \sigma_{\max}^2 2^{2p-1}\int_t^{t'}\Big(1+Ne^{NT}(1+|x|)^{2p}\Big)ds < \infty.$$

Hence $Z_{ij}$ is a martingale and $\mathbb{E}Z_{ij}(t') = \mathbb{E}Z_{ij}(t) = 0$. So

$$\mathbb{B} = \mathbb{E}\sum_{i,j=1}^{n} Z_{ij}(t') = 0. \tag{39}$$

Now, using the conventional "multiplication rules" (see [4] page 154), we know that

$$dsds = 0, \qquad dsdW_s^i = 0, \qquad dW_s^i dW_s^i = ds, \qquad dW_s^i dW_s^j = 0 \quad \text{for } i \neq j.$$

So in view of (33) we can write

$$d\langle y_{xt}^0(s)_i, y_{xt}^0(s)_j\rangle = \sum_{k=1}^{n}\sigma_{ik}(s)dW_{s-t}^k \cdot \sum_{l=1}^{n}\sigma_{jl}(s)dW_{s-t}^l = \sum_{k=1}^{n}\sigma_{ik}(s)\sigma_{jk}(s)ds.$$

From Lemma 2.2 we have

$$\mathbb{C} = \frac{1}{2}\sum_{i,j=1}^{n}\mathbb{E}\int_t^{t'}\frac{\partial^2 u_m(y_{xt}^0(s))}{\partial x_i\partial x_j}\sum_{k=1}^{n}\sigma_{ik}(s)\sigma_{jk}(s)ds \leq \frac{1}{2}\sum_{i,j=1}^{n}\mathbb{E}\int_t^{t'}\hat{C}_2(1+|y_{xt}^0(s)|^p)n\sigma_{\max}^2 ds =$$

$$= C_{40}\ \mathbb{E}\int_t^{t'}(1+|y^0_{xt}(s)|^p)ds,\ \text{ where } C_{40}=\frac{1}{2}\hat{C}_2\sigma^2_{\max}n^3. \tag{40}$$

In summary, in view of (34) and (38) – (40)

$$\mathbb{E}u_m(y^0_{xt}(t'))\le u_m(x)+(C_{38}+C_{40})\mathbb{E}\int_t^{t'}(1+|y^0_{xt}(s)|^p)ds.$$

Taking the limit with $n\to\infty$ we get

$$\mathbb{E}u(y^0_{xt}(t'),t')\le u(x,t')+C_{41}\mathbb{E}\int_t^{t'}(1+|y^0_{xt}(s)|^p)ds,\quad C_{41}=C_{38}+C_{40}\ . \tag{41}$$

Furthermore, from Lemma 1.1 and Lemma 1.2 we have for each $s\in\langle t,T\rangle$

$$\mathbb{E}\big(1+|y^0_{xt}(s)|^p\big)\le C_{42}(1+|x|^p),\quad C_{42}=Ne^{NT}2^{p-1}+1\ . \tag{42}$$

Next, from (5), (6), (41), (42) and the Fubini's theorem we conclude

$$u(x,t)\le\mathbb{E}\left\{\int_t^{t'}f(y^0_{xt}(s),s)e^{-\int_t^s\alpha(r)dr}ds+u(y^0_{xt}(t'),t')e^{-\int_t^{t'}\alpha(r)dr}\right\}\le$$

$$\le\int_t^{t'}\mathbb{E}\big(1+|y^0_{xt}(s)|^p\big)ds+\mathbb{E}u(y^0_{xt}(t'),t')\le C_{42}|t-t'|(1+|x|^p)+u(x,t')+C_{41}C_{42}|t-t'|(1+|x|^p).$$

Hence, for $t<t'$

$$u(x,t)-u(x,t')\le C_{43}|t-t'|(1+|x|^p),\quad C_{43}=C_{42}(C_{41}+1)\ . \tag{43}$$

It is clear that (32) and (43) imply (23). □

Now we give the proof of Lemma 2.2.

*Proof of* (35)*:* The continuity of $u(\cdot,t')$ is a consequence of (21). So in view of Lemma 1.13 we conclude that $\lim_{m\to\infty}u_m(x)=u(x,t')$.

*Proof of* (36)*:* Let $x\in\mathbb{R}^n$, $0\le|x'|<1$. From Def. 1.2, Def. 1.3 and (21) we get

$$|u_m(x)-u_m(x+x')|=\left|\int_{B(0,\frac{1}{m})}\eta_m(y)\big(u(x-y,t')-u(x+x'-y,t')\big)dy\right|\le$$

$$\le\int_{B(0,\frac{1}{m})}m^n\cdot\eta(my)|u(x-y,t')-u(x+x'-y,t')|dy\le$$

$$\le\hat{C}C_1|x'|m^n\int_{B(0,\frac{1}{m})}\Big(1+|x-y|^{p-1}+|x+x'-y|^{p-1}\Big)dy.$$

Because $|x'|<1$ and $|y|\le\frac{1}{m}\le 1$, we have

$$1+|x-y|^{p-1}+|x+x'-y|^{p-1}\le 1+(1+|x|)^{p-1}+(2+|x|)^{p-1}\le(2+2^{p-1})(1+|x|)^{p-1}.$$

Furthermore (see [2] page 615)

$$\int_{B(0,\frac{1}{m})}dy=\frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\cdot\frac{1}{m^n},\ \text{ where }\Gamma(t)=\int_0^\infty s^{t-1}e^{-s}ds,\ \text{ for } t>0.$$

In summary

$$\frac{|u_m(x)-u_m(x+x')|}{|x'|}\leq \hat{C}C_1\frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}(2+2^{p-1})(1+|x|)^{p-1}.$$

Taking the limit by $|x'|\to 0$ on both sides, we can conclude (36).

*Proof of* (37)*:* Let $x'\in\mathbb{R}^n$ and $\lambda\in(0,1)$. We have

$$u_m(x+\lambda x')-2u_m(x)+u_m(x-\lambda x')=\int_{B(0,\frac{1}{m})}\eta_m(y)\big(u(x+\lambda x',t')-2u(x,t')+u(x-\lambda x',t')\big)dy.$$

From (22) and nonnegativity of $\eta_m$ we conclude that $\frac{\partial^2 u_m(x)}{\partial x_i \partial x_j}\geq 0$. On the other hand, using (22) and mimicking the proof of (36) we see that

$$u_m(x+\lambda x')-2u_m(x)+u_m(x-\lambda x')\leq \int_{B(0,\frac{1}{m})} m^n\cdot\eta(my)C_2\lambda^2(1+|x|)^{(p-2)^+}dy\leq$$

$$\leq \lambda^2\,\hat{C}C_2\,\frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,(1+|x|)^{(p-2)^+}.$$

For $p\in(1,2\rangle$

$$(1+|x|)^{(p-2)^+}=1\leq(1+|x|^p)\leq 2^{p-1}(1+|x|^p).$$

For $p>2$, in view of Lemma 1.1

$$(1+|x|)^{(p-2)^+}=(1+|x|)^{p-2}\leq(1+|x|)^p\leq 2^{p-1}(1+|x|^p).$$

In summary, we have for all $p>1$

$$\frac{u_m(x+\lambda x')-2u_m(x)+u_m(x-\lambda x')}{\lambda^2}\leq \hat{C}C_2\,\frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}2^{p-1}(1+|x|^p).$$

Taking the limit by $\lambda\to 0$ to both sides, we can conclude (37). □

REMARK 2.1. Theorem 2.1 is true for functions $u_\epsilon$ (see (4)) instead $u$. Indeed, in view of the proof we see, that the constants $C, C_1, C_2, C_3$ do not depend on $\epsilon$.

REMARK 2.2. Theorem 2.1 implies that the value function $u(x,t)$ has generalized derivatives of first order with respect to variable $t$ and second order with respect to variable $x$. Moreover, these generalized derivatives belongs to the space $L^\infty_{loc}(\mathbb{R}^n\times\langle 0,T\rangle)$ of all functions essentionally bounded on every open bounded subset of the domain.

PROPOSITION 2.3. There exists a constant $\tilde{C}>0$ such that for all $x\in\mathbb{R}^n$ and $t\in\langle 0,T\rangle$

$$u(x,t)\leq\tilde{C}(1+|x|).$$

*Proof:* Let $x'\in\mathbb{R}^n$ be arbitrary. Consider controls for which $\lim_{s\to 0+}v_s=x$. In view of (2) and (3) we have

$$u(x',t)=\inf\{J_{x't}(v):v\in\mathcal{V}\}\leq c(t)|x|+\inf\{J_{x+x',t}(v):v\in\mathcal{V}\}=c(t)|x|+u(x+x',t).$$

So $u(x', t) - u(x + x', t) \leq c(t)|x|$. Similarly $u(x + x', t) - u(x', t) \leq c(t)|x|$, so

$$|u(x + x', t) - u(x', t)| \leq c(t)|x|. \tag{44}$$

Taking $x' = 0$ we get

$$|u(x, t) - u(0, t)| \leq c(t)|x| \;\Leftrightarrow\; -c(t)|x| \leq u(x, t) - u(0, t) \leq c(t)|x|.$$

From (20) we know that $0 \leq u(0, t) \leq C$ so

$$u(x, t) \leq c(t)|x| + u(0, t) \leq c_{\max}|x| + C \leq \tilde{C}(1 + |x|), \tag{45}$$

where $\tilde{C} = C + c_{\max}$. □

Remark 2.3. The proof of Proposition 2.2 is not valid for $u_\epsilon$ instead $u$, because if a control $v \in \mathcal{V}_\epsilon$, then it is continuous, so the condition $\lim_{s \to 0+} v_s = x$ is invalid for $x \neq 0$.

Remark 2.4. We observe that the value function $u(x, t)$ satisfies

$$\forall_{x \in \mathbb{R}^n} \; \forall_{t \in \langle 0, T \rangle} \; |Du(x, t)| \leq c(t),$$

where $Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$. Indeed, the gradient exists in view of Remark 2.2. From (44) we see that the first derivative of $u(x, t)$ with respect to $x$ in any direction is bounded by $c(t)$. Hence, the norm of the gradient $Du(x, t)$ is bounded by $c(t)$, too.

# 3 Dynamic Programming Principle and HJB equation.

To consider DPP and HJB equation for our problem we must first prove the pointwise convergence of $u_\epsilon$ to $u$ if $\epsilon \to 0^+$. For this purpose we need the integral form of the Gronwall's inequality with locally finite measures.

Lemma 3.1. *The Gronwall's inequality with locally finite measures* (see [9]). Let $\mu$ be a locally finite measure on the Borel $\sigma$-algebra of $\langle t, T \rangle$, where $0 \leq t \leq T$. We consider a measurable function $\phi$ defined on $\langle t, T \rangle$ such that $\int_t^T |\phi(r)|\mu(dr) < \infty$. We assume that there exists a Borel function $\psi \geq 0$ on $\langle t, T \rangle$ such that

$$\forall_{s \in \langle t, T \rangle} \quad \phi(s) \leq \psi(s) + \int_{\langle t, s)} \phi(r)\mu(dr).$$

Then

$$\forall_{s \in \langle 0, T \rangle} \quad \phi(s) \leq \psi(s) + \int_{\langle t, s)} \psi(r) e^{\mu(\langle r, s))}\mu(dr).$$

Theorem 3.2.

$$\forall_{x \in \mathbb{R}^n} \; \forall_{t \in \langle 0, T \rangle} \quad \lim_{\epsilon \to 0^+} u_\epsilon(x, t) = u(x, t).$$

*Proof:* Let fix $x \in \mathbb{R}^n$ and $t \in \langle 0, T \rangle$. Consider arbitrary $v \in \mathcal{V}$ such that $J_{xt}(v) < \infty$.

*Step 1:* We show first that $v \in L^p(\Omega \times \langle 0, T-t\rangle, P \otimes \mu_{Leb})$, where $\mu_{Leb}$ denotes the Lebesgue measure. Because $J_{xt}(v) < \infty$ we have $\mathbb{E}\int_t^T f(y_{xt}(s), s)ds < \infty$ and from (6) we get

$$\mathbb{E}\int_t^T |y_{xt}(s)|^p ds < \infty. \tag{46}$$

From (1) we can write for $s \in \langle t, T\rangle$

$$v(s-t) = y_{xt}(s) - x - \int_t^s b(r)dr - \int_t^s \sigma(r)dW_{r-t} - \int_t^s a(r)y_{xt}(r)dr. \tag{47}$$

Using (46) and properties of the normal distribution we know that each term from the line above, maybe except for the last term, belongs to the space $L^p(\Omega \times \langle 0, T-t\rangle)$. But the last term belongs to this space, too. Indeed, denoting $a_{\max} = \max_{s\in\langle 0,T\rangle} |a(r)|$ we get

$$\mathbb{E}\int_t^T \left|\int_t^s a(r)y_{xt}(r)dr\right|^p ds \leq a_{\max}^p \ \mathbb{E}\int_t^T \left(\int_t^T |y_{xt}(r)|dr\right)^p ds.$$

Using the Hölder's inequality and (46) we can estimate the last expression above by

$$a_{\max}^p \ \mathbb{E}\int_t^T \left(\int_t^T |y_{xt}(r)|^p dr \cdot |T-t|^{p/q}\right) ds \leq a_{\max}^p \ T^{1+p/q} \ \mathbb{E}\int_t^T |y_{xt}(r)|^p dr < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, from (47) we see that $v \in L^p(\Omega \times \langle 0, T-t\rangle)$.

*Step 2:* Now we define a sequence of bounded controls $\{v_R, \ R > 0\}$ such that $v_R$ is convergent to $v$ in the space $L^p(\Omega \times \langle 0, T-t\rangle)$ and the variation of $v_R$ is pointwise convergent to the variation of $v$ from below. Let

$$v_R(s) = \begin{cases} v(s), & |v(s)| \leq R \\ \frac{v(s)}{|v(s)|} \cdot R, & |v(s)| > R \ . \end{cases}$$

We see that

$$\forall_{s\in\langle 0,T-t\rangle} \quad \lim_{R\to\infty} v_R(s) = v(s) \quad \text{and} \quad |v_R(s)| \leq |v(s)|.$$

Hence, from Lemma 1.1 and Step 1

$$\mathbb{E}\int_0^{T-t} |v(s) - v_R(s)|^p ds \leq 2^p \mathbb{E}\int_0^{T-t} |v(s)|^p ds < \infty$$

and using the Lebesgue's dominated convergence theorem we get

$$\lim_{R\to\infty} \mathbb{E}\int_0^{T-t} |v(s) - v_R(s)|^p ds = \mathbb{E}\int_0^{T-t} \lim_{R\to\infty} |v(s) - v_R(s)|^p ds = 0,$$

so the convergence in $L^p$ space is proved. Moreover, if $\xi(s), \xi_R(s)$ denote the variations on the interval $\langle 0, s\rangle$ of the functions $v, v_R$ respectively, then

$$\forall_{s\in\langle 0,T-t\rangle} \quad \xi_R(s) \leq \xi(s) \text{ and } \lim_{R\to\infty} \xi_R(s) = \xi(s). \tag{48}$$

*Step 3:* Let $y^v_{xt}$, $y^{v_R}_{xt}$ denote the state processes (see (1)) corresponding to the controls $v$, $v_R$ respectively. We want to show that $\{y^{v_R}_{xt}\}$ is convergent to $y^v_{xt}$ in the space $L^p(\Omega \times \langle t, T\rangle)$. First we observe that for $s \in \langle t, T\rangle$

$$y^v_{xt}(s) - y^{v_R}_{xt}(s) = \int_t^s a(r)\big(y^v_{xt}(r) - y^{v_R}_{xt}(r)\big)dr + v(s-t) - v_R(s-t).$$

Denoting $z_R(s) = y^v_{xt}(s) - y^{v_R}_{xt}(s)$ and $u_R(s) = v(s-t) - v_R(s-t)$ we can rewrite the last equality in the form

$$z_R(s) = \int_t^s a(r) z_R(r)dr + u_R(s).$$

Hence

$$|z_R(s)| \le \int_t^s |z_R(r)| a_{\max} dr + |u_R(s)|.$$

Using Lemma 3.1 with $\phi = |z_R|$, $\psi = |u_R|$ and $\mu = a_{\max} \cdot \mu_{Leb}$ we get

$$|z_R(s)| \le |u_R(s)| + \int_t^s |u_R(r)| e^{a_{\max}(s-r)} dr \le |u_R(s)| + C_{49} \int_t^s |u_R(r)| dr, \tag{49}$$

where $C_{49} = e^{a_{\max} T}$. So from Lemma 1.1 and the Hölder's inequality

$$|z_R(s)|^p \le 2^{p-1} \left\{ |u_R(s)|^p + C^p_{51} \left( \int_t^s |u_R(r)| dr \right)^p \right\} \le$$

$$\le 2^{p-1} \left\{ |u_R(s)|^p + C^p_{49} (s-t)^{p/q} \int_t^s |u_R(r)|^p dr \right\} \le C_{50} \left\{ |u_R(s)|^p + \int_t^s |u_R(r)|^p dr \right\}, \tag{50}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $C_{50} = 2^{p-1}(1 + C^p_{49} T^{p/q})$. Finally in view of Step 2 we have

$$\lim_{R\to\infty} \mathbb{E} \int_t^T |z_R(s)|^p ds \le \lim_{R\to\infty} C_{50} \mathbb{E} \int_t^T \left\{ |u_R(s)|^p + \int_t^T |u_R(r)|^p dr \right\} ds \le$$

$$\le \lim_{R\to\infty} \left\{ C_{50} \mathbb{E} \int_t^T |u_R(s)|^p ds + C_{50} T \, \mathbb{E} \int_t^T |u_R(r)|^p dr \right\} = 0.$$

*Step 4:* The next step is to show that $J_{xt}(v_R) \to J_{xt}(v)$ if $R \to \infty$. Indeed

$$|J_{xt}(v) - J_{xt}(v_R)| \le \left| \mathbb{E} \int_t^T \Big( f(y^v_{xt}(s), s) - f(y^{v_R}_{xt}(s), s) \Big) e^{-\int_t^s \alpha(r)dr} ds \right| +$$

$$+ \left| \mathbb{E} \int_t^T c(s) e^{-\int_t^s \alpha(r)dr} d(\xi - \xi_R)(s-t) \right| \le A_R + B_R.$$

We note that in view of (48)

$$B_R \le c_{\max} \left| \mathbb{E} \int_t^T d(\xi - \xi_R)(s-t) \right| = c_{\max} \mathbb{E}\big(\xi(T-t) - \xi_R(T-t)\big).$$

Using (48) again and the assumption that $J_{xt}(v) < \infty$ we see that $\mathbb{E}\big(\xi(T-t)-\xi_R(T-t)\big) \leq \mathbb{E}\xi(T-t) < \infty$. Hence, from the Lebesgue's dominated convergence theorem we get

$$\lim_{R\to\infty} B_R \leq \lim_{R\to\infty} c_{\max}\mathbb{E}\big(\xi(T-t)-\xi_R(T-t)\big) = c_{\max}\mathbb{E}\lim_{R\to\infty}\big(\xi(T-t)-\xi_R(T-t)\big) = 0. \quad (51)$$

Using (7) and the Hölder's inequality we have

$$A_R \leq \mathbb{E}\int_t^T \big|f(y_{xt}^v(s),s) - f(y_{xt}^{v_R}(s),s)\big|ds \leq$$

$$\leq \mathbb{E}\int_t^T \big(1 + f(y_{xt}^v(s),s) + f(y_{xt}^{v_R}(s),s)\big)^{1-1/p}|y_{xt}^v(s) - y_{xt}^{v_R}(s)|ds \leq$$

$$\leq \left\{\mathbb{E}\int_t^T \big(1 + f(y_{xt}^v(s),s) + f(y_{xt}^{v_R}(s),s)\big)ds\right\}^{1-1/p} \cdot \left\{\mathbb{E}\int_t^T |y_{xt}^v(s) - y_{xt}^{v_R}(s)|^p\right\}^{1/p}.$$

In view of Step 3 the second factor in the last expression goes to 0 if $R\to\infty$. We must show that the first factor is bounded. Indeed, from (6) and Lemma 1.1 we can write

$$\mathbb{E}\int_t^T \big(1 + f(y_{xt}^v(s),s) + f(y_{xt}^{v_R}(s),s)\big)ds \leq (1+C_0)\mathbb{E}\int_t^T \big(2 + |y_{xt}^v(s)|^p + |y_{xt}^{v_R}(s)|^p\big)ds \leq$$

$$\leq (1+C_0)\mathbb{E}\int_t^T \Big(2 + |y_{xt}^v(s)|^p + 2^{p-1}|y_{xt}^v(s)|^p + 2^{p-1}|y_{xt}^v(s) - y_{xt}^{v_R}(s)|^p\Big)ds.$$

Using (46) and Step 3 again we conclude that the last expression is bounded uniformly in $R$. Hence

$$\lim_{R\to\infty} A_R = 0. \quad (52)$$

Summarizing Steps 1-4, from (51) and (52) we know that $J_{xt}(v_R)$ goes to $J_{xt}(v)$ if $R\to\infty$, so we can consider only bounded controls.

*Step 5:* Let consider $v \in \mathcal{V}$ such that $||v||_\infty < R$ for some $R > 0$. Now we will construct a sequence of controls $\{v_n, n\in\mathbb{N}\}$ such that $v_n \in V_{1/(2nR)}$ and which is convergent to $v$ in the space $L^p(\Omega\times\langle 0, T-t\rangle)$. Besides we shall prove that the variation of $v_n$ is pointwise convergent to the variation of $v_R$ from below. Let

$$v_n(s) = n\int_{(s-1/n)\vee 0}^s v(r)dr, \quad s\in\langle 0, T-t\rangle.$$

We observe that $v_n$ is progressively measurable continuous random process such that $||v_n||_\infty \leq R$, so $v_n \in L^p(\Omega\times\langle 0,T-t\rangle)$. Besides, from left-continuity of $v$ we know that

$$\forall_{\omega\in\Omega}\ \forall_{s\in\langle 0,T-t\rangle} \quad \lim_{n\to\infty} v_n(s) = v(s). \quad (53)$$

Using the Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty}\mathbb{E}\int_o^{T-t} |v(s) - v_n(s)|^p ds = \mathbb{E}\int_o^{T-t}\lim_{n\to\infty}|v(s)-v_n(s)|^p ds = 0,$$

so $L^p$-convergence is proved.

Now we want to check that $v_n \in V_{1/(2nR)}$. Indeed,

$$\Big|\frac{d}{ds}v_n(s)\Big| = \left|\frac{d}{ds}\left(n\int_{(s-1/n)\vee 0}^{s} v(r)dr\right)\right| = n\Big|v(s) - v\big((s-1/n)\vee 0\big)\Big| \le 2nR.$$

Let $\xi_n(s), \xi(s)$ denote the variations on the interval $\langle 0,s\rangle$ of the functions $v_n, v$ respectively. For convenience, we define $v(r)\equiv 0$ for $r<0$. Then

$$v_n(s) = n\int_{s-1/n}^{s} v(r)dr, \quad s\in\langle 0, T-t\rangle.$$

Fix $\omega\in\Omega,\ s\in\langle 0,T-t\rangle$. Let $\Pi=\{s_0,s_1,...,s_k\}$ be a partition of the interval $\langle 0,s\rangle$, where $0=s_0<s_1<...<s_k=s$. Then

$$\sum_{i=1}^{k}|v_n(s_i)-v_n(s_{i-1})| = n\sum_{i=1}^{k}\left|\int_{s_i-1/n}^{s_i} v(r)dr - \int_{s_{i-1}-1/n}^{s_{i-1}} v(r)dr\right| =$$

$$= n\sum_{i=1}^{k}\left|\int_0^{1/n}\Big(v(s_i+r-1/n) - v(s_{i-1}+r-1/n)\Big)dr\right| \le$$

$$\le n\sum_{i=1}^{k}\int_0^{1/n}\Big|v(s_i+r-1/n) - v(s_{i-1}+r-1/n)\Big|dr =$$

$$= n\int_0^{1/n}\sum_{i=1}^{k}\Big|v(s_i+r-1/n) - v(s_{i-1}+r-1/n)\Big|dr \le n\int_0^{1/n}\xi(s)dr = \xi(s).$$

Taking $||\Pi||\to 0$ we get

$$\xi_n(s)\le\xi(s). \tag{54}$$

On the other hand, from (53) we see that

$$\sum_{i=1}^{k}|v(s_i)-v(s_{i-1})| = \sum_{i=1}^{k}\Big|\lim_{n\to\infty}v_n(s_i) - \lim_{n\to\infty}v_n(s_{i-1})\Big| =$$

$$= \lim_{n\to\infty}\sum_{i=1}^{k}|v_n(s_i)-v_n(s_{i-1})| \le \liminf_{n\to\infty}\xi_n(s).$$

Taking $||\Pi||\to 0$ and using (54) we have

$$\xi(s)\le\liminf_{n\to\infty}\xi_n(s)\le\limsup_{n\to\infty}\xi_n(s)\le\xi(s) \quad\Leftrightarrow\quad \lim_{n\to\infty}\xi_n(s)=\xi(s). \tag{55}$$

*Step 6:* In view of Step 5 we can mimick Steps 3 and 4 to conclude that $J_{xt}(v_n)\to J_{xt}(v)$ if $n\to\infty$, where $||v||_\infty<R$ for some $R>0$. From this and Step 4, remembering that $v_n\in\mathcal{V}_{1/(2nR)}$, we can write

$$\inf_{v\in\mathcal{V}} J_{xt}(v) = \inf_{v\in\bigcup_{\epsilon>0}\mathcal{V}_\epsilon} J_{xt}(v) \tag{56}$$

and $\lim_{\epsilon\to 0^+}u_\epsilon(x,t)=u(x,t)$. $\square$

THEOREM 3.3. *Bellman's dynamic programming principle.* Let $x \in \mathbb{R}^n$, $t \in \langle 0, T\rangle$ and $y_{xt}^v$ denotes the state process corresponding to a control $v \in \mathcal{V}$. Let $\tau \in \langle 0, T-t\rangle$ be a Markov time with respect to $\{\mathcal{F}_t\}$. Then

$$u(x,t) = \inf_{v \in \mathcal{V}} \mathbb{E}\Bigg\{ \int_t^{t+\tau} f(y_{xt}^v(s), s) e^{-\int_t^s \alpha(r)dr} ds + \int_t^{t+\tau} c(s) e^{-\int_t^s \alpha(r)dr} d\xi(s-t) + \\ + u(y_{xt}^v(t+\tau), t+\tau)\Bigg\}.$$

*Proof:* For convenience let us denote

$$J_{xt}(v,\tau) = \mathbb{E}\Bigg\{ \int_t^{t+\tau} f(y_{xt}^v(s), s) e^{-\int_t^s \alpha(r)dr} ds + \int_t^{t+\tau} c(s) e^{-\int_t^s \alpha(r)dr} d\xi(s-t)\Bigg\}.$$

It is known fact that DPP holds for regular stochastic control problems (see e.g. [6] Th. 3.1.6). Hence we have for each $\epsilon > 0$

$$u_\epsilon(x,t) = \inf_{v \in \mathcal{V}_\epsilon} \Big\{ J_{xt}(v,\tau) + \mathbb{E}u_\epsilon(y_{xt}^v(t+\tau), t+\tau)\Big\}. \tag{57}$$

Considering any $\tilde{v} \in \mathcal{V}_\epsilon$ we have

$$u_\epsilon(x,t) \le J_{xt}(\tilde{v},\tau) + \mathbb{E}u_\epsilon(y_{xt}^{\tilde{v}}(t+\tau), t+\tau).$$

If $\epsilon \longrightarrow 0^+$, in view of Theorem 3.2 and the Lebesgue's dominated convergence theorem we get

$$u(x,t) \le J_{xt}(\tilde{v},\tau) + \mathbb{E}u(y_{xt}^{\tilde{v}}(t+\tau), t+\tau).$$

Because $\epsilon > 0$ and $\tilde{v} \in \mathcal{V}_\epsilon$ are arbitrary we can conclude that

$$u(x,t) \le \inf_{v \in \bigcup_{\epsilon>0} \mathcal{V}_\epsilon} \Big\{ J_{xt}(v,\tau) + \mathbb{E}u(y_{xt}^v(t+\tau), t+\tau)\Big\}. \tag{58}$$

On the other hand, from (57)

$$u_\epsilon(x,t) \ge \inf_{v \in \bigcup_{\tilde{\epsilon}>0} \mathcal{V}_{\tilde{\epsilon}}} \Big\{ J_{xt}(v,\tau) + \mathbb{E}u(y_{xt}^v(t+\tau), t+\tau)\Big\}.$$

Letting $\epsilon \longrightarrow 0^+$ we get

$$u(x,t) \ge \inf_{v \in \bigcup_{\tilde{\epsilon}>0} \mathcal{V}_{\tilde{\epsilon}}} \Big\{ J_{xt}(v,\tau) + \mathbb{E}u(y_{xt}^v(t+\tau), t+\tau)\Big\}. \tag{59}$$

(58), (59) and an argument similar to the proof of Theorem 3.2 imply that

$$u(x,t) = \inf_{v \in \mathcal{V}} \Big\{ J_{xt}(v,\tau) + \mathbb{E}u(y_{xt}^v(t+\tau), t+\tau)\Big\}. \quad \square$$

COROLLARY 3.4. The dynamic programming property in the weak sense holds (see Def. 1.1) and hence tha value function satisfies (23).

THEOREM 3.5. *The HJB equation.* Denote $\beta(t)=\sigma(t)\sigma^T(t)$ and let

$$Au(x,t)=\frac{-\partial u(x,t)}{\partial t}-\frac{1}{2}\beta(t)\circ D^2u(x,t)-\Big(a(t)x+b(t)\Big)\circ Du(x,t)+\alpha(t)u(x,t),$$

where $Du=\left(\frac{\partial u}{\partial x_i}\right)_{i=1,...,n}$, $D^2u=\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)_{i,j=1,...,n}$ and $\circ$ denotes the scalar product of vectors and matrices respectively. The value function $u$ satisfies for almost all $(x,t)\in\mathbb{R}^n\times\langle 0,T\rangle$ the following second-order differential equation:

$$\max\Big\{Au(x,t)-f(x,t)\ ,\ |Du(x,t)|-c(t)\Big\}=0. \tag{60}$$

*Proof:* An application of the DPP for regular stochastic control problem yields for $\epsilon>0$ the following equation (see [3] Chapter IV.3):

$$Au_\epsilon(x,t)+\frac{1}{\epsilon}\Big(|Du_\epsilon(x,t)|-c(t)\Big)^+=f(x,t). \tag{61}$$

In view of Theorem 2.1, Theorem 3.2 and the Arzela-Ascoli's theorem we see that $u_\epsilon\to u$ uniformly on every compact set if $\epsilon\to 0^+$.

Fix $t\in\langle 0,T\rangle$. From (22) $D^2u_\epsilon(\cdot,t)$ are locally uniformly bounded for all $\epsilon>0$, so using the Arzela-Ascoli's theorem from every sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$ convergent to 0 above we can choose a subsequence $\{\tilde\epsilon_m\}_{m\in\mathbb{N}}$ such that

$$Du_{\tilde\epsilon_m}(\cdot,t)\to v=(v_1,...,v_n)\quad\text{almost uniformly if } m\to\infty.$$

But $v$ must be equal to $Du(\cdot,t)$ in the distribution sense. Indeed, for any function $\phi\in C_c^\infty(\mathbb{R}^n)$ and any $k=1,...,n$ we have

$$\int_{\mathbb{R}^n}\frac{\partial\phi(x)}{\partial x_k}u_{\tilde\epsilon_m}(x,t)dx=-\int_{\mathbb{R}^n}\phi(x)\frac{\partial u_{\tilde\epsilon_m}(x,t)}{\partial x_k}dx.$$

Letting $m\to\infty$ we get

$$\int_{\mathbb{R}^n}\frac{\partial\phi(x)}{\partial x_k}u(x,t)dx=-\int_{\mathbb{R}^n}\phi(x)v_k(x)dx,$$

so $v_k(\cdot)=\frac{\partial u(\cdot,t)}{\partial x_k}$ almost surely. Since $\frac{\partial u}{\partial x_k}$ and $v_k$ are Lipschitz continuous, the equality holds for all $x\in\mathbb{R}^n$. Thus, $v$ does not depend on choosing the subsequence $\{\tilde\epsilon_m\}_{m\in\mathbb{N}}$, so

$$\forall_{t\in\langle 0,T\rangle}\quad Du_\epsilon(\cdot,t)\to D_u(\cdot,t)\quad\text{almost uniformly if }\ \epsilon\to 0^+. \tag{62}$$

Let $\psi=(1+|x|)^{-2p-n-1}$. From (21)-(23) we get

$$|Au_\epsilon(x,t)|\le(C_3+C\alpha_{\max})(1+|x|^p)+\frac{1}{2}||\beta||_\infty n^2C_2(1+|x|)^{(p-2)^+}+\big(||a||_\infty|x|+||b||_\infty\big)nC_1(1+2|x|^{p-1})$$

for almost every $(x,t)\in\mathbb{R}^n\times\langle 0,T\rangle$. Using Lemma 1.1 we have the estimate

$$|Au_\epsilon(x,t)|\le C_{63}(1+|x|)^p\quad a.s. \tag{63}$$

for some constant $C_{63}>0$ depending only on parameters $C,C_1,C_2,C_3,n,p$ and the $L^\infty$-bounds for the function $a,b,\beta,\alpha$. Taking the last inequality to the power 2 we get

$$|Au_\epsilon(x,t)|^2\le C_{63}^2(1+|x|)^{2p}\quad a.s.$$

Hence

$$|Au_\epsilon(x,t)|^2\psi(x) \leq \frac{C_{63}^2(1+|x|)^{2p}}{(1+|x|)^{2p+n+1}} = \frac{C_{63}^2}{(1+|x|)^{n+1}} \quad a.s.$$

The same estimate holds for $u$ instead $u_\epsilon$. So $|Au_\epsilon|^2\psi,\ |Au|^2\psi \in L^1(\mathbb{R}^n\times\langle 0,T\rangle)$. Moreover, $Au_\epsilon,\ Au$ are uniformly bounded in the space $L^2_\psi$, where

$$L^2_\psi = \Big\{v : v^2\psi \in L^1(\mathbb{R}^n \times \langle 0,T\rangle)\Big\} = L^2_{\psi\cdot\mu_{Leb}}(\mathbb{R}^n \times \langle 0,T\rangle).$$

From the Banach-Alaoglu theorem we know that balls in the space $L^2$ are weakly compact. So for each sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$ convergent to 0 above, there exists a subsequence $\{\tilde{\epsilon}_m\}_{m\in\mathbb{N}}$ such that

$$Au_{\tilde{\epsilon}_m} \rightharpoonup v \quad \text{in } L^2_\psi \text{ if } m\to\infty.$$

We show that $v = Au$ in the distribution sense. Indeed, for any function $\phi \in C_c^\infty(\mathbb{R}^n \times \langle 0,T\rangle)$ we have

$$\int_0^T\int_{\mathbb{R}^n}(Au_{\tilde{\epsilon}_m})\phi dxdt = \int_0^T\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial t}u_{\tilde{\epsilon}_m}dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\Big(\beta(t)\circ D^2\phi\Big)u_{\tilde{\epsilon}_m}dxdt +$$

$$+\int_0^T\int_{\mathbb{R}^n}\Big(\Big(a(t)x+b(t)\Big)\circ D\phi\Big)u_{\tilde{\epsilon}_m}dxdt + n\int_0^T\int_{\mathbb{R}^n}\Big(a(t)u_{\tilde{\epsilon}_m}\Big)\phi dxdt + \int_0^T\int_{\mathbb{R}^n}\alpha(t)\phi u_{\tilde{\epsilon}_m}dxdt.$$

Letting $m\to\infty$, we get

$$\int_0^T\int_{\mathbb{R}^n}v\phi dxdt = \int_0^T\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial t}udxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\Big(\beta(t)\circ D^2\phi\Big)udxdt +$$

$$+\int_0^T\int_{\mathbb{R}^n}\Big(\Big(a(t)x+b(t)\Big)\circ D\phi\Big)udxdt + n\int_0^T\int_{\mathbb{R}^n}\Big(a(t)u\Big)\phi dxdt +$$

$$+\int_0^T\int_{\mathbb{R}^n}\alpha(t)\phi udxdt = \int_0^T\int_{\mathbb{R}^n}(Au)\phi dxdt.$$

Hence

$$Au_{\tilde{\epsilon}_m} \rightharpoonup Au \quad \text{in } L^2_\psi \text{ if } m\to\infty.$$

From uniquess of the limit we conclude

$$Au_\epsilon \rightharpoonup Au \quad \text{in } L^2_\psi \text{ if } \epsilon\to 0^+. \tag{64}$$

In view of (61) we have $Au_\epsilon \leq f$ a.s. From this and (64) we see that

$$Au(x,t) \leq f(x,t) \quad a.s.$$

This, together with Remark 2.4 ensure us that

$$\max\Big\{Au(x,t)-f(x,t)\ ,\ |Du(x,t)|-c(t)\Big\} \leq 0 \quad a.s. \tag{65}$$

We suppose that for some $(x_0,t_0)\in\mathbb{R}^n\times\langle 0,T\rangle$ such that $\frac{\partial u(x_0,t_0)}{\partial t}$ exists, we have $|Du(x_0,t_0)| < c(t_0)$. Then from (62) we have $|Du_\epsilon(x_0,t_0)| < c(t_0)$ for $\epsilon>0$ small enough. Because $Du_\epsilon$ are locally Lipschitz continuous (uniformly in $\epsilon$) so there exists $\delta>0$ independent on $\epsilon$ such that for any $x\in B(x_0,\delta)=\{x\in\mathbb{R}^n : |x-x_0|<\delta\}$ we have $|Du_\epsilon(x,t_0)| < c(t_0)$. Using (61) we get $Au_\epsilon(x,t_0)=f(x,t_0)$ for almost every $x\in B(x_0,\delta)$.

In other words

$$\mathbb{I}_{\{|Du(x,t)|<c(t)\}} A u_\epsilon(x,t) = \mathbb{I}_{\{|Du(x,t)|<c(t)\}} f(x,t) \quad a.s.$$

Letting $\epsilon \to 0^+$ from (64) we have

$$\mathbb{I}_{\{|Du(x,t)|<c(t)\}} A u(x,t) = \mathbb{I}_{\{|Du(x,t)|<c(t)\}} f(x,t) \quad a.s.$$

This allows us to write the equality in (65). □

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