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Multidimensional singular stochastic control problems on a finite time horizon

PhD thesis

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Contents

1 Preliminaries **2**

1.1 Introduction 2

1.2 Notation and assumptions 4

2 Existence and uniqueness of the optimal policy **7**

2.1 Preliminary lemmas 7

2.2 Estimates for the value function. 14

2.3 Dynamic Programming Principle and HJB equation. 24

2.4 Existence and uniqueness of the optimal control. 31

3 Characterization of the optimal policy **36**

3.1 The Skorokhod problem and the main theorem 36

3.2 Existence of ε-optimal solutions of the Skorokhod problems 38

3.3 The optimal policy keeps the process in $\overline{\mathcal{D}}$ 47

3.4 The optimal policy acts only on $\partial^*\mathcal{D}$ and its push direction is $-Du$ 47

3.5 The optimally controlled Brownian motion is a Markov process 50

3.6 Possible jumps of the optimal policy 53

Chapter 1

Preliminaries

1.1 Introduction

Singular stochastic control is a class of problems in which one is allowed to change the drift of a Markov process (usually a diffusion) at a price proportional to the variation of the control used. Admissible controls do not have to be absolutely continuous with respect to the Lebesgue measure and they may have jumps. This setup is natural for many problems of practical interest, including portfolio selection in finance, control of queueing networks and spacecraft control, to mention just a few examples. The reader is referred to Chapter VIII of [8] for more information and basic references.

One-dimensional singular stochastic control problems are well understood by now, see, e.g., [4] and the references given there. In this case, if the running cost is convex, the optimal control makes the underlying process a reflected diffusion at the boundary of the so-called *nonaction region* $\mathcal{D}$. In the case of a diffusion with time-independent coefficients and discounted cost on the infinite time horizon, $\mathcal{D}$ is just an interval and the value function enjoys C^2 regularity (*smooth fit*). Both C^2-regularity of the value function and the characterization of the optimally controled process have been extended to the case of singular control for the two-dimensional Brownian motion [24]. In $n \geq 3$ dimensions, except for "close to one-dimensional" cases of a single push direction [25, 26] and the radially symmetric running cost [13], only partial results are known. For example, for optimal control of the Brownian motion on the infinite time horizon, regularity of the boundary of $\mathcal{D}$ away from some "corner points" was shown in [27] and a characterization of the optimal control as a solution of the corresponding modified Skorokhod problem was given in [12].

In this thesis we consider a n-dimensional singular stochastic control problem on a finite time horizon in which state is governed by a linear stochastic differential equation with time-dependent coefficients, the running cost is convex and controls may act in any direction. We provide estimates for the corresponding value function. These estimates imply that the value function has locally bounded generalized derivatives of the second order with respect to the space variable and of the first order with respect to the time variable. These properties are needed to consider the value function as a solution of the corresponding parabolic Hamilton-Jacobi-Bellman (HJB) equation in some generalized sense and to show existence and uniqueness of an optimal control.

Similar results have been shown in Theorem 2.2.1 and Theorem 3.4 from [4] in the one-dimensional case with a single push direction. The corresponding results for a multidimensional singular stochastic control problem on the infinite time horizon with time-independent drift, covariance, cost (i.e., for the elliptic case) can be found in [18]. The first part of this thesis contains a generalization (or adjustment) of the approach of [4, 18] to a n-dimensional parabolic problem. It turns out that while the main ideas from that papers may be applied in our case, a mathematically rigorous analysis of our problem is somewhat delicate and needs rather careful arguments.

Existence results for multidimensional singular control problems closely related to our work may be found in [1, 6, 10]. Apparently, in spite of their considerable generality, none of them contains our existence result as a special case. Indeed, in these papers optimal *weak solutions* to the corresponding SDEs are constructed, while we are concerned about finding an optimal *strong solution*, i.e., for the given (as opposed to some) flitration and underlying Brownian motion. Moreover, the problem considered in [1] is elliptic and the allowable control directions lie in a cone, the opening of which cannot be too wide. In [6, 10] the time horizon is finite, but the problem considered in [6] has the final cost instead of the running cost, while in [10] the drift of the controlled diffusion is bounded, which excludes its linear dependence on the state.

The second part of the thesis is devoted to a characterization of the optimal policy for an n-dimensional Brownian motion in the parabolic case as a unique solution to the corresponding Skorokhod problem for the nonaction region $\mathcal{D}$, having time dependent (moving) boundary. This result is an analog of the main theorem from [12]. It resolves a long-standing open problem on the structure of the optimal control in the case under consideration.

The Skorokhod problem for domain with time-dependent boundaries have been considered, inter alia, in [2, 3, 19, 23]. In these papers, restrictive assumptions on regularity of the boundary of the domain and the directions of reflection are necessary for existence of a solution to the problem. In our case, explicit regularity results for $\partial\mathcal{D}$ are not available and the assumptions on $\mathcal{D}$ are hidden in the very nature of our stochastic control problem. Therefore, we use a direct probabilistic control theoretic approach, similar to the one applied in [12]. Our problem, however, is notably more difficult than the elliptic one, considered in [12], because time-dependence in the value function and the nonaction region creates serious technical problems, which have to be overcome in our analysis.

The structure of this thesis is as follows. In Section 1.2 we give the notation, assumptions and pose the singular stochastic control problem. In Chapter 2 we first prove estimates for the value function. Next we consider the Bellman's dynamic programming principle (DPP) and the HJB equation related to this problem. The final section of this chapter contains proofs of existence and uniqueness of an optimal control. In Chapter 3 we define the notion of a solution of the Skorokhod problem in a time-dependent domain. Next we prove assumptions needed to define a sequence of suitable ε-optimal policies as solutions to the corresponding Skorokhod problems. The final result is a characterization of the optimal policy as a unique solution of the modified Skorokhod problem.

1.2 Notation and assumptions

Let $\mathbb{M}^{n\times n}$ denote the set of matrices of dimension $n\times n$ with the operator norm, i.e. $||A|| = \sup\{|Ax| : x\in\mathbb{R}^n,\ |x|=1\}$.

Let $m\geq 1$. For vectors $v, w\in\mathbb{R}^m$, let $v\circ w$ denote the inner product of the vectors v, w and let $|v|$ denote the Euclidean norm of v. For $x\in\mathbb{R}^m$ and $r>0$, let $B(x,r)=\{y\in\mathbb{R}^m : |x-y|<r\}$.

Let $T>0$ be a fixed number representing our time horizon and let $n\in\mathbb{N}$. For a real-valued function $u=u(x,t)$, $(x,t)\in\mathbb{R}^n\times[0,T]$, we denote the gradient and the Hessian of u with respect to the space variables (i.e., x_i) by Du and D^2u, respectively. More generally, for $m\geq 1$, $D^m u$ denotes the collection of all partial derivatives of u of the m-th order with respect to the space variables. By u_t (or $\frac{\partial u}{\partial t}$), u_{x_i}, $i=1,2,...,n$, we denote the partial derivative of u with respect to the time variable and the i-th space variable respectively. Let G be a relatively open subset of $\mathbb{R}^n\times[0,T]$. We write $u\in C^{2,1}(G)$ if D^2u, u_t exist and are continuous in G. For $\alpha\in(0,1)$, we write $u\in C_\alpha(G)$ if u is Hölder continuous in G with exponent α, with respect to the parabolic distance

$$d(P,Q)=\sqrt{|x-\bar{x}|^2+|t-\bar{t}|}, \qquad P=(x,t), Q=(\bar{x},\bar{t}),$$

(see, e.g., [9] for more details). By ∂G we denote the boundary of G. Moreover if

$$G=\bigcup_{t\in[a,b]}\left(G_t\times\{t\}\right),$$

where $G_t=\{x\in\mathbb{R}^n : (x,t)\in G\}$ are nonempty, open, connected subset of $\mathbb{R}^n$, we define "the lateral boundary" of G by

$$\partial^* G=\partial G\setminus\Big\{\big(G_a\times\{a\}\big)\cup\big(G_b\times\{b\}\big)\Big\}.$$

We introduce the abbrevations: almost surely (a.s.), almost everywhere (a.e.) and almost all (a.a.).

Let $W=(W_t, t\geq 0)$ be a standard n-dimensional Brownian motion relative to a filtration $(\mathcal{F}_t, t\geq 0)$ satisfying the usual conditions, defined on a complete probability space $(\Omega,\mathcal{F},P)$. In particular, W is adapted to $(\mathcal{F}_t, t\geq 0)$ and for every $0\leq s\leq t$ the increment W_t-W_s is independent on $\mathcal{F}_s$. By $\mathbb{I}[A]$ we denote the indicator of the event $A\in\mathcal{F}$. Denote by $\mathcal{V}$ the set of controls v which are left-continuous, adapted to the filtration $(\mathcal{F}_t, t\in[0,T])$ random processes acting from $[0,T]$ into $\mathbb{R}^n$, with P-a.s. bounded variation and such that $v_0=0$ P-a.s.. We note that these processes are also progressively measurable (see [11], Th. 1.1.13). As it is customary in singular stochastic control theory (see, e.g., [12]), for $v\in\mathcal{V}$ we write

$$v_t=\int_0^t \gamma_s d\xi_s, \tag{1.1}$$

where $|\gamma_t| = 1$ for every $t \in [0,T]$ and ξ is nondecreasing and left-continuous. In other words, ξ_t is the total variation of v on the time interval $[0,t]$ and γ_t is the Radon-Nikodym derivative of the vector-valued measure induced by v on $[0,T]$ with respect to its total variation ξ. We identify a control process v with the corresponding pair of processes (N,ξ).

Consider the state process described by the stochastic integral equation

$$y_{xt}(s) = x + \int_t^s \Big(a(r)y_{xt}(r) + b(r)\Big)dr + \int_t^s \sigma(r)dW_{r-t} + v(s-t), \quad s \in [t,T], \tag{1.2}$$

where $t \in [0,T]$ is an initial time, $x \in \mathbb{R}^n$ is an initial position, $b : [0,T] \to \mathbb{R}^n$ and $a, \sigma : [0,T] \to \mathbb{M}^{n\times n}$ stand for the drift and the covariance terms. Note that $(y_{xt}(s))_{s\in[t,T]}$ is a random process adapted to $(\mathcal{F}_{s-t})_{s\in[t,T]}$.

With each control $v \in \mathcal{V}$, we associate a cost given by the payoff functional

$$J_{xt}(v) = \mathbb{E}\left\{\int_t^T f(y_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds + \int_t^T c(s)e^{-\int_t^s \alpha(r)dr}d\xi(s-t)\right\}, \tag{1.3}$$

where f, α and c are respectively the running cost, the discount factor and the instantaneous cost per unit of "fuel".

Our purpose is to characterize the optimal cost, the so called value function

$$u(x,t) = \inf\{J_{xt}(v) : v \in \mathcal{V}\}. \tag{1.4}$$

If this infimum is attained for some $v^* \in \mathcal{V}$, we say that v^* is an optimal policy (for given t and x).

It is often convenient to consider the following penalized problem associated with (1.4):

$$u_\epsilon(x,t) = \inf\{J_{xt}(v) : v \in \mathcal{V}_\epsilon\}, \tag{1.5}$$

where $\epsilon > 0$ and $\mathcal{V}_\epsilon$ is the set of all controls $v \in \mathcal{V}$ which are Lipschitz continuous and $|\frac{dv}{dt}(t)| \le \frac{1}{\epsilon}$ for almost every $t \in [0,T]$ almost surely.

Definition 1.2.1. We say that the finite time horizon stochastic control problem has the dynamic programming property in the weak sense if for every $x \in \mathbb{R}^n$, $t,t' \in [0,T]$ s.t. $t < t'$ and $y^0_{xt}(s)$ given by (1.2) with $v \equiv 0$ we have

$$u(x,t) \le \mathbb{E}\left\{\int_t^{t'} f(y^0_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds + u(y^0_{xt}(t'),t')e^{-\int_t^{t'} \alpha(r)dr}\right\}. \tag{1.6}$$

Let us assume the following:

- α, c are Lipschitz continuous from $[0,T]$ into $[0,\infty)$ with constant $L > 0$,
- b is Lipschitz continuous from $[0,T]$ into $\mathbb{R}^n$ with the same constant $L > 0$,

- a, σ are Lipschitz continuous from $[0,T]$ into $\mathbb{M}^{n\times n}$ with the same constant $L > 0$,
- there exists $c_0 > 0$ such that $c(t) \geq c_0$ for all $t \in [0,T]$,
- $f : \mathbb{R}^n \times [0,T] \to [0,\infty)$ and there exist constants $p > 1$, C_0, $\tilde{C}_0 > 0$ such that for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^n$ and $\lambda \in (0,1)$ we have

$$\tilde{C}_0|x|^p - C_0 \leq f(x,t) \leq C_0(1+|x|^p) \ , \tag{1.7}$$

$$|f(x,t) - f(x+x',t)| \leq C_0\big(1 + f(x,t) + f(x+x',t)\big)^{1-1/p}|x'| \ , \tag{1.8}$$

$$|f(x,t) - f(x,t')| \leq C_0(1+|x|^p)|t-t'| \ , \tag{1.9}$$

$$0 < f(x+\lambda x',t) - 2f(x,t) + f(x-\lambda x',t) \leq C_0\lambda^2(1+f(x,t))^q \ , \ q = (1-2/p)^+. \tag{1.10}$$

The last assumption implies strict convexity of the function f with respect to x.

Let us denote by $c_{\max}$ and $\alpha_{\max}$ the maximum of the function c, α, respectively. Moreover, by $a_{\max}$, $\sigma_{\max}$, $\beta_{\max}$ and $b_{\max}$ we denote the maximum over $t \in [0,T]$ of the norms of the matrices $a(t), \sigma(t), \beta(t)$ and the vector $b(t)$ respectively, where $\beta(t) = \sigma(t)\sigma^T(t)$.

Chapter 2

Existence and uniqueness of the optimal policy

This chapter is devoted to derivation of some estimates for the value function u and the corresponding HJB equation. Moreover we prove that the optimal policy in our singular stochastic control problem (1.2), (1.3), (1.4) exists and is unique.

2.1 Preliminary lemmas

In this section we give lemmas needed for the proofs of the Theorems 2.2.1 and 2.2.2, to follow.

Lemma 2.1.1. *For all $x, y \geq 0$ we have*

$$x^p + y^p \leq (x+y)^p \leq 2^{p-1}(x^p + y^p), \;\; \textit{if} \;\; p \geq 1,$$

$$2^{p-1}(x^p + y^p) \leq (x+y)^p \leq x^p + y^p, \;\; \textit{if} \;\; p \in (0,1).$$

Proof. First we prove $(x+y)^p \leq 2^{p-1}(x^p+y^p)$ for $p \geq 1$. Indeed, the function $(\cdot)^p$ is convex, so from the Jensen's inequality

$$\left(\frac{x+y}{2}\right)^p \leq \frac{1}{2}x^p + \frac{1}{2}y^p.$$

Multiplying both sides by 2^p we get the conlusion.

Similarly $2^{p-1}(x^p + y^p) \leq (x+y)^p$ for $p \in (0,1)$ because then $(\cdot)^p$ is concave.

To prove the remaining inequalities we consider the function

$$f(x,y) = (x+y)^p - x^p - y^p, \;\; x, y \geq 0.$$

We observe that $f(0,y) = f(x,0) = 0$. Moreover for $x, y > 0$ and $p \geq 1$ we have

$$f_x(x,y) = p\left((x+y)^{p-1} - x^{p-1}\right) > 0.$$

From symetry $f_y(x,y) > 0$. So we conclude that $f(x,y) \geq 0$ for $p \geq 1$.

Similarly, if $p \in (0,1)$, we have $f(x,y) \leq 0$ because then $f_x(x,y), f_y(x,y) < 0$. □

Lemma 2.1.2. *(See [14], Corollary 2.5.12). Consider an n-dimensional process described by a stochastic integral equation*

$$x(t) = x_0 + \int_0^t g(x(s), s)ds + \int_0^t h(x(s), s)dW_s, \quad t \geq 0,$$

where $x_0 \in \mathbb{R}^n$, $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times [0, \infty) \to \mathbb{M}^{n\times n}$. We assume that there exists a constant C such that for all $x \in \mathbb{R}^n$ and $t \geq 0$

$$||h(x, t)|| + |g(x, t)| \leq C(1 + |x|). \tag{2.1}$$

Then for every $q > 0$ there exists a constant $C_{2.2} > 0$ depending only on q, C such that for all $t \geq 0$

$$\mathbb{E} \sup_{0\leq s\leq t} |x(s)|^q \leq C_{2.2}\, e^{C_{2.2}t}(1 + |x_0|)^q. \tag{2.2}$$

Remark 2.1.3. For the process y_{xt} defined by (1.2) with $v \equiv 0$ the assumption (2.1) holds. Indeed, σ is Lipschitz continuous, independent on x and defined on a finite time interval $[0, T]$, so it is bounded. We conclude the same about a, b, so $|g(x, t)| = |a(t) \circ x + b(t)| \leq C(1 + |x|)$, where $C = \max\{||a(t)||, |b(t)| : t \in [0, T]\}$.

Lemma 2.1.4. *Let $x, x' \in \mathbb{R}^n$, $t \in [0, T]$ and $g(s) = y_{xt}(s) - y_{x't}(s)$ for $s \in [t, T]$. Then*

$$\frac{dg}{ds}(s) = a(s)g(s), \qquad |g(s)| \leq C_{2.3}|x - x'|, \qquad s \in [t, T], \tag{2.3}$$

where $C_{2.3} = \big(1 + a_{\max}Te^{a_{\max}T}\big)$.

Proof. In view of (1.2) we have

$$g(s) = x - x' + \int_t^s a(r)(y_{xt}(r) - y_{x't}(r))dr = x - x' + \int_t^s a(r)g(r)dr.$$

Taking the derivative d/ds of both sides, we get the differential equation $\frac{dg}{ds}(s) = a(s)g(s)$ with initial data $g(t) = x - x'$. The solution of this problem satisfies

$$|g(s)| \leq |x - x'| + \int_t^s |a(r)g(r)|dr \leq |x - x'| + a_{\max} \int_t^s |g(r)|dr.$$

Using the Gronwall's inequality (see [7], p. 625) we get the second part of (2.3). □

Lemma 2.1.5. *Suppose that for some $x \in \mathbb{R}^n$, $t \in [0, T]$, $v \in \mathcal{V}$ we have*

$$\mathbb{E} \int_t^T f(y_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds \leq C(1 + |x|^p)$$

for a suitable constant $C > 0$ independent on x, t. Then

$$\mathbb{E} \int_t^T f(y_{xt}(s), s)ds \leq C_{2.4}(1 + |x|^p), \text{ where } C_{2.4} = C \cdot e^{\int_0^T \alpha(r)dr}. \tag{2.4}$$

Proof. Indeed, multiplying both sides of our assumption by $e^{\int_t^T \alpha(r)dr}$ we get

$$\mathbb{E}\int_t^T f(y_{xt}(s),s)e^{\int_s^T \alpha(r)dr}ds \le Ce^{\int_t^T \alpha(r)dr}(1+|x|^p) \le C_{2.4}(1+|x|^p).$$

Of course, the left-hand side is greater than $\mathbb{E}\int_t^T f(y_{xt}(s),s)ds$. □

Lemma 2.1.6. *(Compare a statement in [27], p. 181). The function $J_{xt}(v)$ is convex with respect to (x,v), more precisely, for all $x_1,x_2\in\mathbb{R}^n$, $t\in[0,T]$, $v_1,v_2\in\mathcal{V}$ and $\theta\in[0,1]$,*

$$J_{\theta x_1+(1-\theta)x_2,t}(\theta v_1+(1-\theta)v_2) \le \theta J_{x_1,t}(v_1)+(1-\theta)J_{x_2,t}(v_2)\ .$$

Proof. First, we note that the set $\mathcal{V}$ is obviously convex. Let $y^v_{xt}(s)$ be the solution of (1.2) corresponding to a control v. Denote $v_0=\theta v_1+(1-\theta)v_2$ and $x_0=\theta x_1+(1-\theta)x_2$. In view of the definition of $J_{xt}(v)$, it suffices to prove two following inequalities

$$f\big(y^{v_0}_{x_0,t}(s),s\big) \le \theta f(y^{v_1}_{x_1,t}(s),s)+(1-\theta)f(y^{v_2}_{x_2,t}(s),s),\ \ s\in[t,T]\ , \tag{2.5}$$

$$\int_t^T d\xi_0(s-t) \le \theta\int_t^T d\xi_1(s-t)+(1-\theta)\int_t^T d\xi_2(s-t)\ , \tag{2.6}$$

where $\xi_0,\ \xi_1,\ \xi_2$ are the total variations of $v_0,\ v_1,\ v_2$ respectively.

The latter inequality is a consequence of the fact that the variation of the sum of functions is not greater than the sum of their variations. So $\xi_0\le\theta\xi_1+(1-\theta)\xi_2$. Because $\xi_0,\ \xi_1,\ \xi_2$ are nondecreasing and $\xi_0(0)=\xi_1(0)=\xi_2(0)=0$ P-a.s., we conclude that (2.6) is true.

To prove (2.5) we show first that

$$y^{v_0}_{x_0,t}(s)=\theta y^{v_1}_{x_1,t}(s)+(1-\theta)y^{v_2}_{x_2,t}(s)\ . \tag{2.7}$$

Indeed, using (1.2) we get

$$y^{v_i}_{x_i,t}(s)=x_i+\int_t^s\big(a(r)y^{v_i}_{x_i,t}(r)+b(r)\big)\,dr+\int_t^s\sigma(r)dW_{r-t}+v_i(s-t)\ ,\ \ i=0,1,2.$$

Let $g(s)=y^{v_0}_{x_0,t}(s)-\theta y^{v_1}_{x_1,t}(s)-(1-\theta)y^{v_2}_{x_2,t}(s)$. Then

$$g(s)=\int_t^s a(r)\Big(y^{v_0}_{x_0,t}(r)-\theta y^{v_1}_{x_1,t}(r)-(1-\theta)y^{v_2}_{x_2,t}(r)\Big)dr=\int_t^s a(r)g(r)dr\ .$$

Taking the derivative d/ds of both sides, we get the differential equation $\frac{dg}{ds}(s)=a(s)g(s)$ with initial data $g(t)=x_0-\theta x_1-(1-\theta)x_2=0$. The solution of this problem is $g(s)\equiv 0$, so (2.7) holds. Using (2.7) and convexity of f we have (2.5) . □

Lemma 2.1.7. *Suppose that for some $t' \in [0,T],\ x \in \mathbb{R}^n,\ v \in \mathcal{V}$ we have*

$$\mathbb{E}\int_0^{T-t'} c(t'+s)e^{-\int_0^s \alpha(t'+r)dr} d\xi(s) \le C(1+|x|^p)$$

for a suitable constant $C > 0$ independent on x, t'. Then there exists a constant $C_{2.8} > 0$ independent on $x,\ t'$ such that

$$\mathbb{E}\xi(T-t') \le C_{2.8}(1+|x|^p). \tag{2.8}$$

Proof. Indeed, multiplying both sides of our assumption by $e^{\int_0^{T-t'} \alpha(t'+r)dr}$ and using the lower bound of c we get

$$c_0\mathbb{E}\xi(T-t') = c_0\mathbb{E}\int_0^{T-t'} d\xi(s) \le \mathbb{E}\int_0^{T-t'} c(t'+s)e^{\int_s^{T-t'} \alpha(t'+r)dr} d\xi(s) \le Ce^{\int_0^T \alpha(r)dr}(1+|x|^p).$$

□

Lemma 2.1.8. *Suppose that for some $x \in \mathbb{R}^n,\ t \in [0,T],\ v \in \mathcal{V}$ we have*

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)e^{-\int_0^s \alpha(t+r)dr} ds \le C(1+|x|^p)$$

for a suitable constant $C > 0$ independent on x, t. Then there exists a constant $C_{2.9} > 0$ independent on $x,\ t$ such that

$$\mathbb{E}\int_0^{T-t} (1+|y_{xt}(t+s)|^p)ds \le C_{2.9}(1+|x|^p). \tag{2.9}$$

Proof. From Lemma 2.1.5 we know that

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \le C_{2.4}(1+|x|^p).$$

Using (1.7) we get

$$\mathbb{E}\int_0^{T-t} \Big(\tilde{C}_0|y_{xt}(t+s)|^p - C_0\Big)ds \le C_{2.4}(1+|x|^p).$$

Hence

$$\tilde{C}_0\mathbb{E}\int_0^{T-t} |y_{xt}(t+s)|^p ds \le (C_{2.4}+C_0T)(1+|x|^p)$$

and finally

$$\tilde{C}_0\mathbb{E}\int_0^{T-t}(1+|y_{xt}(t+s)|^p)ds \le (C_{2.4}+C_0T+\tilde{C}_0T)(1+|x|^p)\ .$$

□

Lemma 2.1.9. *Let $0 \leq t' \leq t \leq T$ and suppose that for some $x \in \mathbb{R}^n$, $v \in \mathcal{V}$ we have*

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \leq C(1+|x|^p)$$

for a suitable constant $C > 0$ independent on x, t, t'. Then there exists a constant $C_{2.10} > 0$ independent on x, t, t' such that

$$\mathbb{E}\int_0^{T-t} f(y_{xt'}(t'+s), t+s)ds \leq C_{2.10}(1+|x|^p). \tag{2.10}$$

Proof. We observe that using (1.9) we have

$$f(y_{xt'}(t'+s), t+s) \leq |f(y_{xt'}(t'+s), t+s) - f(y_{xt'}(t'+s), t'+s)| + f(y_{xt'}(t'+s), t'+s) \leq$$

$$\leq C_0|t-t'|(1+|y_{xt'}(t'+s)|^p) + f(y_{xt'}(t'+s), t'+s).$$

Hence, in view of Lemma 2.1.5 and Lemma 2.1.8, we get

$$\mathbb{E}\int_0^{T-t} f(y_{xt'}(t'+s), t+s)ds \leq C_0|t-t'|C_{2.9}(1+|x|^p) + C_{2.4}(1+|x|^p) \leq C_{2.10}(1+|x|^p),$$

where $C_{2.10} = C_0 T C_{2.9} + C_{2.4}$. □

Lemma 2.1.10. *Let $0 \leq t' \leq t \leq T$, $x \in \mathbb{R}^n$, $v \in \mathcal{V}$. Assume that*

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \leq C(1+|x|^p)$$

for a suitable constant $C > 0$ independent of x, t', t. Then there exists a constant $C_{2.11} > 0$ independent on x, t', t such that for all $s \in [0, T-t]$ we have

$$\mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p \leq C_{2.11}|t-t'|^p(1+|x|^p)\ . \tag{2.11}$$

Proof. For $s \in [0, T-t]$, we have

$$y_{xt}(t+s) = x + \int_0^s \big(a(t+r)y_{xt}(t+r) + b(t+r)\big)dr + \int_0^s \sigma(t+r)dW_r + v(s),$$

$$y_{xt'}(t'+s) = x + \int_0^s \big(a(t'+r)y_{xt'}(t'+r) + b(t'+r)\big)dr + \int_0^s \sigma(t'+r)dW_r + v(s),$$

so

$$y_{xt'}(t'+s) - y_{xt}(t+s) = A_s + B_s + M_s, \tag{2.12}$$

where

$$
\begin{aligned}
A_s &= \int_0^s \big(a(t'+r)y_{xt'}(t'+r) - a(t+r)y_{xt}(t+r)\big)dr = A_s^1 + A_s^2,\\
A_s^1 &= \int_0^s a(t+r)\big(y_{xt'}(t'+r) - y_{xt}(t+r)\big)dr,\\
A_s^2 &= \int_0^s \big(a(t'+r) - a(t+r)\big)y_{xt'}(t'+r)dr,\\
B_s &= \int_0^s \big(b(t'+r) - b(t+r)\big)dr,\\
M_s &= \int_0^s \big(\sigma(t'+r) - \sigma(t+r)\big)dW_r.
\end{aligned}
$$

Recall that a, b, σ are Lipschitz continuous with the constant L. The process M_s is a martingale with quadratic variation

$$
[M]_s = \int_0^s \big(\sigma(t'+r) - \sigma(t+r)\big)^2 dr \leq L^2|t-t'|^2 s.
$$

This, together with the Burkholder-Davis-Gundy inequalities (see [11], Theorem 3.3.28), implies the existence of a constant C_p, depending only on p, such that

$$
\mathbb{E} \sup_{0\leq s\leq T-t} |M_s|^p \leq C_p L^p T^{\frac{p}{2}} |t-t'|^p. \tag{2.13}
$$

Clearly,

$$
\sup_{0\leq s\leq T-t} |B_s| \leq LT|t-t'|. \tag{2.14}
$$

By the Hölder's inequality, for $q = p/(p-1)$ we have

$$
\begin{aligned}
|A_s^1|^p &\leq (a_{max}^q s)^{\frac{p}{q}} \int_0^s |y_{xt'}(t'+r) - y_{xt}(t+r)|^p dr, &(2.15)\\
|A_s^2|^p &\leq \big((L|t-t'|)^q s\big)^{\frac{p}{q}} \int_0^s |y_{xt'}(t'+r)|^p dr. &(2.16)
\end{aligned}
$$

By Lemma 2.1.8, the inequality (2.9) holds for every $t \in [0,T]$. Lemma 2.1.1 and the relations (2.9), (2.12)-(2.16) imply that the random variable

$$
\sup_{0\leq s\leq T-t} |y_{xt'}(t'+s) - y_{xt}(t+s)|^p
$$

is integrable and hence, by the Lebesgue dominated convergence theorem, the function $F(s) = \mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p$ is continuous on $[0, T-t]$. From Lemma 2.1.1 and (2.9), (2.12)-(2.16) we also have, for each $s \in [0, T-t]$,

$$
F(s) \leq c_1|t-t'|^p(1+|x|^p) + c_2\int_0^s F(r)dr,
$$

where $c_1 = 2^{2p-2}(L^pT^p + C_pL^pT^{\frac{p}{2}} + (LT)^{\frac{p}{q}}C_{2.9})$, $c_2 = 2^{2p-2}a^p_{max}T^{\frac{p}{q}}$. This, together with the Gronwall's inequality (see, e.g., [11], Problem 5.2.7), implies that for all $s \in [0, T-t]$,

$$F(s) \leq c_1|t-t'|^p(1+|x|^p)\left(1 + c_2\int_0^s e^{c_2(s-r)}dr)\right).$$

We have obtained (2.11) with $C_{2.11} = c_1(1 + c_2\int_0^T e^{c_2(T-r)}dr)$. □

Lemma 2.1.11. *Suppose that for some $x \in \mathbb{R}^n$, $t \in [0,T]$, $v \in \mathcal{V}$ we have*

$$\mathbb{E}\int_t^T f(y_{xt}(s), s)e^{-\int_t^s \alpha(r)dr}ds \leq C(1+|x|^p)$$

for a suitable constant $C > 0$ independent on x, t. Then there exists a constant $C_{2.17} > 0$ independent on x, x', t such that for every $x' \in \mathbb{R}^n$

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq C_{2.17}(1 + |x|^p + |x+x'|^p). \tag{2.17}$$

Proof. From (1.7) and Lemma 2.1.1 we have

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq \mathbb{E}\int_t^T C_0(1 + |y_{x+x',t}(s)|^p)ds \leq$$

$$\leq TC_0 + C_0 2^{p-1}\mathbb{E}\int_t^T |y_{x+x',t}(s) - y_{x,t}(s)|^p ds + C_0 2^{p-1}\mathbb{E}\int_t^T |y_{x,t}(s)|^p ds\ .$$

Now using Lemma 2.1.4, Lemma 2.1.8 and Lemma 2.1.1 again, we get

$$\mathbb{E}\int_t^T f(y_{x+x',t}(s), s)ds \leq TC_0 + C_0 2^{p-1}T \cdot C^p_{2.3}|x'|^p + C_0 2^{p-1}C_{2.9}(1+|x|^p) \leq$$

$$\leq TC_0 + C_0 2^{2p-2}T \cdot C^p_{2.3}\big(|x'+x|^p + |x|^p\big) + C_0 2^{p-1}C_{2.9}(1+|x|^p) \leq C_{2.17}(1+|x|^p+|x+x'|^p),$$

where $C_{2.17} = C_0\big(T + 2^{2p-2}T \cdot C^p_{2.3} + 2^{p-1}C_{2.9}\big)$. □

Lemma 2.1.12. *Suppose that for some $x \in \mathbb{R}^n$, $t' \in [0,T]$, $v \in \mathcal{V}$ we have*

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s), t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \leq C(1+|x|^p)$$

for a suitable constant $C > 0$ independent on x, t'. Then there exists a constant $C_{2.18} > 0$ independent on x, t', t such that for every $t \in [t', T]$

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s), t+s)ds \leq C_{2.18}(1+|x|^p). \tag{2.18}$$

Proof. Using (1.7) and Lemma 2.1.1 we have

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s),t+s)ds \le \mathbb{E}\int_0^{T-t} C_0(1+|y_{xt}(t+s)|^p)ds \le$$

$$\le C_0T + 2^{p-1}C_0\mathbb{E}\int_0^{T-t}|y_{xt'}(t'+s)|^p ds + 2^{p-1}C_0\mathbb{E}\int_0^{T-t}|y_{xt}(t+s)-y_{xt'}(t'+s)|^p ds\ .$$

In view of Lemma 2.1.8, the Fubini's theorem and Lemma 2.1.10 we get

$$\mathbb{E}\int_0^{T-t} f(y_{xt}(t+s),t+s)ds \le C_0T + 2^{p-1}C_0C_{2.9}(1+|x|^p) + 2^{p-1}C_0TC_{2.11}|t-t'|^p(1+|x|^p) \le$$

$$\le C_{2.18}(1+|x|^p), \quad \text{where } C_{2.18} = C_0\big(T + 2^{p-1}C_{2.9} + 2^{p-1}T^{p+1}C_{2.11}\big).$$

□

The next two definitions and lemma refer to mollification of a given function (see [7], p. 629-630).

Definition 2.1.13. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C_{2.19}\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x|\ge 1, \end{cases} \tag{2.19}$$

where the constant $C_{2.19}$ is selected so that $\int_{\mathbb{R}^n}\eta(x)dx = 1$. For each $m \in \mathbb{N}$ set $\eta_m(x) = m^n \cdot \eta(mx)$. We call η the standard mollifier. The functions η_m belong to the class $C^\infty(\mathbb{R}^n)$ and satisfy $\int_{\mathbb{R}^n}\eta_m(x)dx = 1$.

Definition 2.1.14. Fix $t' \in [0,T]$. For each $m \in \mathbb{N}$ we define mollification of the function $u(\cdot,t')$ by

$$u_m(x) = \int_{B(0,\frac{1}{m})}\eta_m(y)u(x-y,t')dy, \quad x\in\mathbb{R}^n, \text{ where } B(0,r) = \{x\in\mathbb{R}^n : |x|<r\}.$$

Lemma 2.1.15. *For each $m \in \mathbb{N}$ we have $u_m \in C^\infty(\mathbb{R}^n)$. Moreover if $u(\cdot,t')$ is continuous, then $u_m(x)\to u(x,t')$ uniformly on compact subsets of $\mathbb{R}^n$ as $m\to\infty$.*

2.2 Estimates for the value function.

Let the assumptions from Section 1.2 appearing immediately after Definition 1.2.1 hold.

Theorem 2.2.1. *Let u be the value function defined by* (1.4). *Then for some positive constants $C_{2.20}, C_{2.21}, C_{2.22}$, the same $p>1$ as in the assumptions* (1.7) - (1.10) *and every $t\in[0,T]$, $x,x'\in\mathbb{R}^n$ and $\lambda\in(0,1)$, the following estimates hold:*

$$0 \le u(x,t) \le C_{2.20}(1+|x|^p)\ , \tag{2.20}$$

$$|u(x,t)-u(x+x',t)| \le C_{2.21}(1+|x|^{p-1}+|x+x'|^{p-1})|x'|\ , \tag{2.21}$$

$$0 \le u(x+\lambda x',t) - 2u(x,t) + u(x-\lambda x',t) \le C_{2.22}\lambda^2(1+|x|)^{(p-2)^+}. \tag{2.22}$$

Proof of (2.20). Nonnegativity of u is consequence of nonnegativity of f and c. Next, taking the control $v \equiv 0$ and using (1.7), nonnegativity of α, the Fubini's theorem, Lemma 2.1.2 and Lemma 2.1.1, we get

$$u(x,t) \le J_{xt}(0) = \mathbb{E}\int_t^T f(y^0_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds \le \mathbb{E}\int_t^T C_0(1+|y^0_{xt}(s)|^p)ds \le$$

$$\le C_0\int_t^T \mathbb{E}(1+|y^0_{xt}(s)|^p)ds \le C_0\int_t^T \mathbb{E}\left(1+C_{2.2}e^{C_{2.2}(s-t)}(1+|x|)^p\right)ds \le$$

$$\le C_0\int_0^T (1+C_{2.2}e^{C_{2.2}T}2^{p-1})(1+|x|^p)ds = C_0T(1+C_{2.2}e^{C_{2.2}T}2^{p-1})(1+|x|^p) = C_{2.20}(1+|x|^p),$$

where $C_{2.20}$ depends only on $C_0, T, C_{2.2}, p$, so (2.20) is proved. □

Proof of (2.21). Now we note that

$$u(x+x',t)-u(x,t) = \inf_{v'\in\mathcal{V}}\sup_{v\in\mathcal{V}}\Big(J_{x+x',t}(v')-J_{x,t}(v)\Big) \le \sup_{v\in\mathcal{V}}\Big(J_{x+x',t}(v)-J_{x,t}(v)\Big).$$

Hence

$$u(x+x',t)-u(x,t) \le \sup_{v\in\mathcal{V}}\ |J_{xt}(v)-J_{x+x',t}(v)| \le$$

$$\le \sup_{v\in\mathcal{V}}\mathbb{E}\int_t^T |f(y_{xt}(s),s)-f(y_{x+x',t}(s),s)|e^{-\int_t^s \alpha(r)dr}ds.$$

Applying (1.8) we can estimate the last expression from above by

$$\sup_{v\in\mathcal{V}}\ \mathbb{E}\int_t^T C_0\big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)^{1-1/p}\cdot|y_{xt}(s)-y_{x+x',t}(s)|ds.$$

Using Lemma 2.1.4 we have

$$u(x+x',t)-u(x,t) \le \sup_{v\in\mathcal{V}} C_0C_{2.3}|x'|\cdot\mathbb{E}\int_t^T \big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)^{\frac{p-1}{p}}ds.$$

We use the Hölder's inequality with exponent $\frac{p}{p-1}$ to estimate the last expression above by

$$\sup_{v\in\mathcal{V}} C_0C_{2.3}|x'|\cdot\left(\mathbb{E}\int_t^T \big(1+f(y_{xt}(s),s)+f(y_{x+x',t}(s),s)\big)ds\right)^{\frac{p-1}{p}}T^{\frac{1}{p}}. \tag{2.23}$$

By virtue of (2.20) we can consider only those controls v for which

$$\mathbb{E}\int_t^T f(y_{xt}(s),s)e^{-\int_t^s \alpha(r)dr}ds \le (C_{2.20}+\epsilon)(1+|x|^p)$$

for some arbitrary $\epsilon > 0$. From (2.23), Lemma 2.1.5 and Lemma 2.1.11 we see that

$$u(x+x',t) - u(x,t) \le C_0 C_{2.3}|x'|\Big(T + C_{2.4}(1+|x|^p) + C_{2.17}(1+|x|^p+|x+x'|^p)\Big)^{\frac{p-1}{p}} T^{\frac{1}{p}} \le$$

$$\le C_{2.21}|x'|\big(1+|x|^p+|x+x'|^p\big)^{\frac{p-1}{p}} \ ,$$

where $C_{2.21} = T^{1/p} \cdot C_0 C_{2.3}(T + C_{2.4} + C_{2.17})^{1-1/p}$. Finally using Lemma 2.1.1 we get

$$u(x+x',t) - u(x,t) \le C_{2.21}(1+|x|^{p-1}+|x+x'|^{p-1})|x'|.$$

In an analogous manner we get the same estimate for $u(x,t) - u(x+x',t)$. □

Proof of (2.22). We observe that

$$u(x+\lambda x',t) + u(x-\lambda x',t) - 2u(x,t) \le \sup_{v\in\mathcal{V}} \left(J_{x+\lambda x',t}(v) + J_{x-\lambda x',t}(v) - 2J_{xt}(v)\right) =$$

$$= \sup_{v\in\mathcal{V}} \mathbb{E}\int_t^T \Big(f(y_{x+\lambda x',t}(s),s) + f(y_{x-\lambda x',t}(s),s) - 2f(y_{xt}(s),s)\Big)e^{-\int_t^s \alpha(r)dr} ds \ .$$

In view of (2.3) we can apply (1.10) to get

$$u(x+\lambda x',t) + u(x-\lambda x',t) - 2u(x,t) \le \sup_{v\in\mathcal{V}} \mathbb{E}\int_t^T C_0\lambda^2\Big(1+f(y_{xt}(s),s)\Big)^{(1-2/p)^+} ds \ .$$

If $p \le 2$ we have $u(x+\lambda x',t) + u(x-\lambda x',t) - 2u(x,t) \le C_0 T\lambda^2$. If $p > 2$ we use Hölder inequality with exponent $\frac{p}{p-2}$ to get

$$u(x+\lambda x',t) + u(x-\lambda x',t) - 2u(x,t) \le \sup_{v\in\mathcal{V}} C_0\lambda^2\Big(\mathbb{E}\int_t^T (1+f(y_{xt}(s),s)ds\Big)^{1-2/p} T^{2/p} \ .$$

By virtue of (2.20) we can consider only those controls v for which

$$\mathbb{E}\int_t^T f(y_{xt}(s),s)e^{-\int_t^s \alpha(r)dr} ds \le (C_{2.20}+\epsilon)(1+|x|^p)$$

for some arbitrary $\epsilon > 0$. From Lemma 2.1.5 and Lemma 2.1.1 we see that

$$u(x+\lambda x',t) + u(x-\lambda x',t) - 2u(x,t) \le C_0\lambda^2\Big(T + C_{2.4}(1+|x|^p)\Big)^{1-2/p} T^{2/p} \le$$

$$\le C_{2.22}\lambda^2(1+|x|^p)^{1-2/p} \le C_{2.22}\lambda^2(1+|x|)^{p-2},$$

where $C_{2.22} = T^{2/p}C_0(T + C_{2.4})^{1-2/p}$. We have proved the upper bound of (2.22).

To prove the lower bound of (2.22), it clearly suffices to prove convexity of $u(x,t)$ with respect to the first variable. In view of the definition of u we know that for every $\epsilon > 0$, $x_1, x_2 \in \mathbb{R}^n$, $t \in [0,T]$, $\theta \in [0,1]$ there exist $v_1, v_2 \in \mathcal{V}$ such that $J_{x_i,t}(v_i) \le u(x_i,t) + \epsilon$, $i = 1,2$. Using Lemma 2.1.6 we get

$$u(\theta x_1 + (1-\theta)x_2, t) \le J_{\theta x_1 + (1-\theta)x_2, t}(\theta v_1 + (1-\theta)v_2) \le$$

$$\le \theta J_{x_1,t}(v_1) + (1-\theta)J_{x_2,t}(v_2) \le \theta u(x_1,t) + (1-\theta)u(x_2,t) + \epsilon \ .$$

Because $\epsilon > 0$ is arbitrary, we get convexity of $u(x,t)$ with respect to the first variable. □

Theorem 2.2.2. *Let the assumptions of Theorem 2.2.1 be satisfied and* $t' \in [0,T]$. *Assume that the dynamic programming property in the weak sense holds (Def. 1.2.1). Then for some constant* $C_{2.24} > 0$ *we have*

$$|u(x,t) - u(x,t')| \le C_{2.24}(1 + |x|^p)|t - t'| \ . \tag{2.24}$$

Proof. We note that

$$u(x,t) - u(x,t') = \inf_{v\in\mathcal{V}} \sup_{v'\in\mathcal{V}} \Big(J_{xt}(v) - J_{xt'}(v') \Big) \le \sup_{v'\in\mathcal{V}} \Big(J_{xt}(v') - J_{xt'}(v') \Big).$$

For $t' \le t$ the difference $J_{xt}(v) - J_{xt'}(v)$ is equal to

$$\mathbb{E}\Bigg\{ \int_0^{T-t} \Big(f(y_{xt}(t+s), t+s) e^{-\int_0^s \alpha(t+r)dr} - f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} \Big) ds \ +$$

$$+ \int_0^{T-t} \Big(c(t+s) e^{-\int_0^s \alpha(t+r)dr} - c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} \Big) d\xi(s)$$

$$- \int_{T-t}^{T-t'} f(y_{xt'}(t'+s), t'+s) e^{-\int_0^s \alpha(t'+r)dr} ds - \int_{T-t}^{T-t'} c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} d\xi(s) \Bigg\} \ .$$

Let us denote the expectations of the first two integrals in the last expression by A and B, respectively. Because the last two integrals are nonnegative we get

$$J_{xt}(v) - J_{xt'}(v) \le A + B \ . \tag{2.25}$$

We can estimate B as follows:

$$B \le \mathbb{E} \int_0^{T-t} \Big| c(t+s) e^{-\int_0^s \alpha(t+r)dr} - c(t'+s) e^{-\int_0^s \alpha(t'+r)dr} \Big| d\xi(s) \ .$$

Adding and subtracting $c(t+s)e^{-\int_0^s \alpha(t'+r)dr}$ under the absolute value sign and using the triangle inequality and positivity of α we get

$$B \le \mathbb{E} \int_0^{T-t} \Big(c_{\max} |e^{-\int_0^s \alpha(t+r)dr} - e^{-\int_0^s \alpha(t'+r)dr}| + |c(t+s) - c(t'+s)| \Big) d\xi(s).$$

Because $|e^x - e^y| \le |x - y|$ for $x, y \le 0$ and $c, \ \alpha$ are Lipschitz continuous, we have

$$B \le \mathbb{E} \int_0^{T-t} \Big(c_{\max} \int_0^s |\alpha(t+r) - \alpha(t'+r)| dr + |c(t+s) - c(t'+s)| \Big) d\xi(s) \le$$

$$\le (c_{\max} T + 1) L |t - t'| \mathbb{E} \int_0^{T-t} d\xi(s) = (c_{\max} T + 1) L |t - t'| \mathbb{E}\xi(T - t).$$

By virtue of (2.20) we can consider only those controls v for which

$$\mathbb{E}\int_0^{T-t'} c(t'+s)e^{-\int_0^s \alpha(t'+r)dr}d\xi(s) \le (C_{2.20}+\epsilon)(1+|x|^p)$$

for some arbitrary $\epsilon > 0$. Using Lemma 2.1.7 we get $\mathbb{E}\xi(T-t) \le \mathbb{E}\xi(T-t') \le C_{2.8}(1+|x|^p)$ and

$$B \le \ C_{2.26}\ |t-t'|(1+|x|^p)\ ,\ \text{ where } C_{2.26} = (c_{\max}T+1)LC_{2.8}\ . \tag{2.26}$$

Now we estimate A:

$$A \le \mathbb{E}\int_0^{T-t} \left| f(y_{xt}(t+s),t+s)e^{-\int_0^s \alpha(t+r)dr} - f(y_{xt}(t+s),t+s)e^{-\int_0^s \alpha(t'+r)dr}\right| ds\ +$$

$$+\ \mathbb{E}\int_0^{T-t} \left| f(y_{xt}(t+s),t+s)e^{-\int_0^s \alpha(t'+r)dr} - f(y_{xt'}(t'+s),t'+s)e^{-\int_0^s \alpha(t'+r)dr}\right| ds = A_1 + A_2\ .$$

Using the inequality $|e^x - e^y| \le |x-y|$ for $x, y \le 0$ again, we get

$$A_1 \le \mathbb{E}\int_0^{T-t} f(y_{xt}(t+s),t+s)\left(\int_0^s |\alpha(t+r)-\alpha(t'+r)|dr\right) ds \le \tag{2.27}$$

$$\le TL|t-t'|\ \mathbb{E}\int_0^{T-t} f(y_{xt}(t+s),t+s)ds\ .$$

By virtue of (2.20) we can consider only those controls v for which

$$\mathbb{E}\int_0^{T-t'} f(y_{xt'}(t'+s),t'+s)e^{-\int_0^s \alpha(t'+r)dr}ds \le (C_{2.20}+\epsilon)(1+|x|^p) \tag{2.28}$$

for some arbitrary $\epsilon > 0$. Using (2.27) and Lemma 2.1.12 we get

$$A_1 \le \ C_{2.29}\ |t-t'|(1+|x|^p)\ ,\ \text{ where } C_{2.29} = TLC_{2.18}\ . \tag{2.29}$$

To estimate A_2 we use (1.8)-(1.9) and we have that A_2 is less than or equal to

$$\mathbb{E}\int_0^{T-t}\left| f(y_{xt}(t+s),t+s) - f(y_{xt'}(t'+s),t+s) + f(y_{xt'}(t'+s),t+s) - f(y_{xt'}(t'+s),t'+s)\right| ds \le$$

$$\le \mathbb{E}\int_0^{T-t} C_0\Big(1+f(y_{xt}(t+s),t+s)+f(y_{xt'}(t'+s),t+s)\Big)^{1-1/p}|y_{xt'}(t'+s)-y_{xt}(t+s)|ds\ +$$

$$+\ \mathbb{E}\int_0^{T-t} C_0(1+|y_{xt'}(t'+s)|^p)|t-t'|ds = A_3 + A_4\ .$$

Using the Hölder's inequality and the Fubini's theorem we get

$$A_3 \le C_0\left\{\mathbb{E}\int_0^{T-t}\Big(1+f(y_{xt}(t+s),t+s)+f(y_{xt'}(t'+s),t+s)\Big)ds\right\}^{1-1/p}\cdot$$

$$\cdot \left\{ \int_0^{T-t} \mathbb{E}|y_{xt'}(t'+s) - y_{xt}(t+s)|^p ds \right\}^{1/p} .$$

From this together with (2.28), Lemma 2.1.12, Lemma 2.1.9 and Lemma 2.1.10 we have

$$A_3 \le C_0 \left\{ (T + C_{2.18} + C_{2.10})(1 + |x|^p) \right\}^{1-1/p} \cdot \left\{ T C_{2.11} |t - t'|^p (1 + |x|^p) \right\}^{1/p} .$$

Because $1 + |x|^p \le (1 + |x|)^p$, we get

$$A_3 \le C_0 (T + C_{2.18} + C_{2.10})^{1-1/p} (1 + |x|)^{p-1} \cdot (T C_{2.11})^{1/p} |t - t'| (1 + |x|).$$

Hence, from Lemma 2.1.1,

$$A_3 \le C_{2.30} |t - t'| (1 + |x|^p), \text{ where } C_{2.30} = C_0 (T + C_{2.18} + C_{2.10})^{1-1/p} (T C_{2.11})^{1/p} \, 2^{p-1}. \quad (2.30)$$

Furthermore, from Lemma 2.1.8 we get

$$A_4 \le C_{2.31} |t - t'| (1 + |x|^p), \text{ where } C_{2.31} = C_0 C_{2.9}. \quad (2.31)$$

In view of (2.25)-(2.26) and (2.29)-(2.31) we get for $t' \le t$,

$$u(x,t) - u(x,t') \le C_{2.32} |t - t'| (1 + |x|^p), \text{ where } C_{2.32} = C_{2.26} + C_{2.29} + C_{2.30} + C_{2.31} \ . \quad (2.32)$$

To obtain a similar inequality for $t < t'$ we proceed as follows. Let $\left(y^0_{xt}(s) \right)_{s \in [t,T]}$ be a solution of (1.2) with $v \equiv 0$. We can write the i-th coordinate of $y^0_{xt}(s)$ as follows

$$y^0_{xt}(s)_i = x_i + \int_t^s \left(\sum_{j=1}^n a_{ij}(r) y^0_{xt}(r)_j + b_i(r) \right) dr + \sum_{j=1}^n \int_t^s \sigma_{ij}(r) dW^j_{r-t}, \quad i = 1, ..., n \ , \quad (2.33)$$

where subscripts denote the corresponding coordinates. Let $\{u_m(\cdot)\}_{n \in \mathbb{N}}$ be a sequence of mollifications of the function $u(\cdot, t')$ (see Def. 2.1.14). Applying the Itô's formula ([11], Th. 3.3.6) we get

$$\begin{aligned} \mathbb{E} u_m(y^0_{xt}(t')) &= u_m(x) + \mathbb{E} \sum_{i=1}^n \int_t^{t'} \frac{\partial u_m(y^0_{xt}(s))}{\partial x_i} \left(\sum_{j=1}^n a_{ij}(s) y^0_{xt}(s)_j + b_i(s) \right) ds \ + \\ &+ \mathbb{E} \sum_{i=1}^n \int_t^{t'} \frac{\partial u_m(y^0_{xt}(s))}{\partial x_i} \sum_{j=1}^n \sigma_{ij}(s) dW^j_{s-t} \ + \\ &+ \frac{1}{2} \, \mathbb{E} \sum_{i,j=1}^n \int_t^{t'} \frac{\partial^2 u_m(y^0_{xt}(s))}{\partial x_i \partial x_j} d[y^0_{xt}(s)_i, y^0_{xt}(s)_j] = u_m(x) + \mathbb{A} + \mathbb{B} + \mathbb{C} \ . \end{aligned} \quad (2.34)$$

We need the following lemma.

Lemma 2.2.3. *We assume* (2.20)-(2.22). *Let* $t' \in [0,T]$ *be fixed. Then there exist constants* $C_{2.36}, C_{2.37} > 0$ *such that for all* $x \in \mathbb{R}^n$, $m \in \mathbb{N}$ *and* $i, j \in \{1, ..., n\}$,

$$\lim_{m\to\infty} u_m(x) = u(x,t'), \tag{2.35}$$

$$\left|\frac{\partial u_m(x)}{\partial x_i}\right| \le C_{2.36}(1+|x|)^{p-1}, \tag{2.36}$$

$$0 \le \frac{\partial^2 u_m(x)}{\partial x_i \partial x_j} \le C_{2.37}(1+|x|^p). \tag{2.37}$$

We estimate $\mathbb{A}$ as follows

$$\mathbb{A} \le \mathbb{E}\sum_{i=1}^{n}\int_t^{t'} \left|\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\right| \cdot \left|\sum_{j=1}^{n} a_{ij}(s) y_{xt}^0(s)_j + b_i(s)\right| ds \le$$

$$\le \mathbb{E}\sum_{i=1}^{n}\int_t^{t'} \left|\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\right|\Big(n||a(s)|| \cdot |y_{xt}^0(s)| + |b(s)|\Big) ds.$$

Using Lemma 2.2.3 and Lemma 2.1.1 we see that $\mathbb{A}$ is not greater than

$$\sum_{i=1}^{n}\mathbb{E}\int_t^{t'} C_{2.36}(1+|y_{xt}^0(s)|)^{p-1} n(a_{\max}+b_{\max})(1+|y_{xt}^0(s)|)ds \le C_{2.38}\mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|^p)ds, \tag{2.38}$$

where $C_{2.38} = n^2 C_{2.36}(a_{\max}+b_{\max})2^{p-1}$.

Now we show that $\mathbb{B} = 0$. Indeed,

$$\mathbb{B} = \mathbb{E}\sum_{i,j=1}^{n} Z_{ij}(t'), \text{ where } Z_{ij}(s) = \int_t^s \frac{\partial u_m(y_{xt}^0(r))}{\partial x_i}\sigma_{ij}(r) dW_{r-t}^j \text{ for } s \in [t,t'].$$

From properties of the Itô's integrals (see [11], section 3.2) the process $\big(Z_{ij}(s)\big)_{s\in[t,t']}$ is a martingale provided that $\mathbb{E}\int_t^{t'}\Big(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\Big)^2 ds \le \infty$. Using Lemma 2.2.3 and Lemma 2.1.1, we have

$$\mathbb{E}\int_t^{t'}\left(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\right)^2 ds \le \mathbb{E}\int_t^{t'} C_{2.36}^2(1+|y_{xt}^0(s)|)^{2p-2}\ \sigma_{\max}^2 ds \le$$

$$\le C_{2.36}^2\ \sigma_{\max}^2\ \mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|)^{2p} ds \le C_{2.36}^2\ \sigma_{\max}^2 2^{2p-1}\ \mathbb{E}\int_t^{t'}(1+|y_{xt}^0(s)|^{2p})ds.$$

Using the Fubini's theorem and Lemma 2.1.2, we get

$$\mathbb{E}\int_t^{t'}\left(\frac{\partial u_m(y_{xt}^0(s))}{\partial x_i}\sigma_{ij}(s)\right)^2 ds \le C_{2.36}^2\ \sigma_{\max}^2 2^{2p-1}\int_t^{t'}\Big(1+C_{2.2}e^{C_{2.2}T}(1+|x|)^{2p}\Big)ds < \infty.$$

Hence Z_{ij} is a martingale and $\mathbb{E}Z_{ij}(t') = \mathbb{E}Z_{ij}(t) = 0$. So

$$\mathbb{B} = \mathbb{E}\sum_{i,j=1}^{n} Z_{ij}(t') = 0. \tag{2.39}$$

Now, using the conventional "multiplication rules" (see [11], p. 154), we know that

$$dsds = 0, \qquad dsdW_s^i = 0, \qquad dW_s^i dW_s^i = ds, \qquad dW_s^i dW_s^j = 0 \quad \text{for } i \neq j.$$

So in view of (2.33) we can write

$$d[y_{xt}^0(s)_i, y_{xt}^0(s)_j] = \sum_{k=1}^{n} \sigma_{ik}(s) dW_{s-t}^k \cdot \sum_{l=1}^{n} \sigma_{jl}(s) dW_{s-t}^l = \sum_{k=1}^{n} \sigma_{ik}(s)\sigma_{jk}(s) ds.$$

From Lemma 2.2.3 we have

$$\mathbb{C} = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{E}\int_t^{t'} \frac{\partial^2 u_m(y_{xt}^0(s))}{\partial x_i \partial x_j} \sum_{k=1}^{n} \sigma_{ik}(s)\sigma_{jk}(s) ds \leq \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{E}\int_t^{t'} C_{2.37}(1 + |y_{xt}^0(s)|^p) n\sigma_{\max}^2 ds =$$

$$= C_{2.40}\ \mathbb{E}\int_t^{t'} (1 + |y_{xt}^0(s)|^p) ds, \text{ where } C_{2.40} = \frac{1}{2} C_{2.37}\sigma_{\max}^2 n^3. \tag{2.40}$$

In summary, in view of (2.34) and (2.38)-(2.40)

$$\mathbb{E}u_m(y_{xt}^0(t')) \leq u_m(x) + (C_{2.38} + C_{2.40})\mathbb{E}\int_t^{t'} (1 + |y_{xt}^0(s)|^p) ds.$$

Taking the limit as $n \to \infty$ and using the Fatou's lemma we get

$$\mathbb{E}u(y_{xt}^0(t'), t') \leq u(x, t') + C_{2.41}\mathbb{E}\int_t^{t'} (1 + |y_{xt}^0(s)|^p) ds, \quad C_{2.41} = C_{2.38} + C_{2.40}\ . \tag{2.41}$$

Furthermore, from Lemma 2.1.1 and Lemma 2.1.2 we have for each $s \in [t, T]$

$$\mathbb{E}\big(1 + |y_{xt}^0(s)|^p\big) \leq C_{2.42}(1 + |x|^p), \quad C_{2.42} = C_{2.2} e^{C_{2.2}T} 2^{p-1} + 1\ . \tag{2.42}$$

Next, from (1.6), (1.7), (2.41), (2.42) and the Fubini's theorem we conclude

$$u(x, t) \leq \mathbb{E}\left\{ \int_t^{t'} f(y_{xt}^0(s), s) e^{-\int_t^s \alpha(r)dr} ds + u(y_{xt}^0(t'), t') e^{-\int_t^{t'} \alpha(r)dr} \right\} \leq$$

$$\leq C_0 \int_t^{t'} \mathbb{E}\big(1 + |y_{xt}^0(s)|^p\big) ds + \mathbb{E}u(y_{xt}^0(t'), t') \leq \big(C_0 C_{2.42} + C_{2.41} C_{2.42}\big)\big(|t - t'|(1 + |x|^p)\big) + u(x, t').$$

Hence, for $t < t'$

$$u(x, t) - u(x, t') \leq C_{2.43}|t - t'|(1 + |x|^p), \quad C_{2.43} = C_{2.42}(C_0 + C_{2.41})\ . \tag{2.43}$$

It is clear that (2.32) and (2.43) imply (2.24). □

Now we give the proof of Lemma 2.2.3.

Proof of (2.35). The continuity of $u(\cdot, t')$ is a consequence of (2.21). So in view of Lemma 2.1.15 we conclude that $\lim_{m\to\infty} u_m(x) = u(x,t')$. $\square$

Proof of (2.36). Let $x \in \mathbb{R}^n$, $0 \le |x'| < 1$. From Definitions 2.1.13, 2.1.14 and (2.21) we get

$$|u_m(x) - u_m(x+x')| = \left| \int_{B(0,\frac{1}{m})} \eta_m(y)\big(u(x-y,t') - u(x+x'-y,t')\big)dy \right| \le$$

$$\le \int_{B(0,\frac{1}{m})} m^n \cdot \eta(my)|u(x-y,t') - u(x+x'-y,t')|dy \le$$

$$\le C_{2.19}C_{2.21}|x'|m^n \int_{B(0,\frac{1}{m})} \Big(1 + |x-y|^{p-1} + |x+x'-y|^{p-1}\Big)dy.$$

Because $|x'| < 1$ and $|y| \le \frac{1}{m} \le 1$, we have

$$1 + |x-y|^{p-1} + |x+x'-y|^{p-1} \le 1 + (1+|x|)^{p-1} + (2+|x|)^{p-1} \le (2+2^{p-1})(1+|x|)^{p-1}.$$

Furthermore (see [7], p. 615),

$$\int_{B(0,\frac{1}{m})} dy = \frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot \frac{1}{m^n}, \text{ where } \Gamma(t) = \int_0^\infty s^{t-1}e^{-s}ds, \text{ for } t > 0.$$

In summary,

$$\frac{|u_m(x) - u_m(x+x')|}{|x'|} \le C_{2.19}C_{2.21}\frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}(2+2^{p-1})(1+|x|)^{p-1}.$$

Taking the limit as $|x'| \to 0$ on both sides, we conclude (2.36). $\square$

Proof of (2.37). Let $x' \in \mathbb{R}^n$ and $\lambda \in (0,1)$. We have

$$u_m(x+\lambda x') - 2u_m(x) + u_m(x-\lambda x') = \int_{B(0,\frac{1}{m})} \eta_m(y)\big(u(x+\lambda x',t') - 2u(x,t') + u(x-\lambda x',t')\big)dy.$$

From (2.22) and nonnegativity of η_m we conclude that $\frac{\partial^2 u_m(x)}{\partial x_i \partial x_j} \ge 0$. On the other hand, using (2.22) and mimicking the proof of (2.36) we see that

$$u_m(x+\lambda x') - 2u_m(x) + u_m(x-\lambda x') \le \int_{B(0,\frac{1}{m})} m^n \cdot \eta(my)C_{2.22}\lambda^2(1+|x|)^{(p-2)^+}dy \le$$

$$\le \lambda^2\, C_{2.19}C_{2.22}\, \frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\, (1+|x|)^{(p-2)^+}.$$

For $p \in (1,2]$, $(1+|x|)^{(p-2)^+} = 1 \le (1+|x|^p) \le 2^{p-1}(1+|x|^p)$. For $p > 2$, in view of Lemma 2.1.1, $(1+|x|)^{(p-2)^+} = (1+|x|)^{p-2} \le (1+|x|)^p \le 2^{p-1}(1+|x|^p)$. Thus, for all $p > 1$ we have

$$\frac{u_m(x+\lambda x') - 2u_m(x) + u_m(x-\lambda x')}{\lambda^2} \le C_{2.19}C_{2.22}\ \frac{\Pi^{n/2}}{\Gamma(\frac{n}{2}+1)}2^{p-1}(1+|x|^p).$$

Taking the limit as $\lambda \to 0$, we can conclude (2.37). $\square$

Remark 2.2.4. Theorems 2.2.1 and 2.2.2 are true for functions u_ϵ (see (1.5)) instead of u. Indeed, in view of the proof we see that the constants $C_{2.20}, C_{2.21}, C_{2.22}, C_{2.24}$ do not depend on ϵ.

Remark 2.2.5. It follows from (2.35)-(2.37) that for every $t' \in [0,T]$ $Du_m(\cdot\,; t')$ converges to $Du(\cdot, t')$ the distributional gradient of u with respect to x almost uniformly as $m \to \infty$ (see the proof of Theorem 2.3.5, to follow, for a similiar argument, with u_m replaced by u_{ϵ_m}). This implies differentiability of u with respect to x in the classical sense (see, e.g., Theorem 7.17 in [21]), so Du is the classical gradient of u with respect to x at any point $(x,t') \in \mathbb{R}^n \times [0,T]$. Moreover by (2.37) Du_m are locally Lipschitz in x uniformly in m, so Du is also locally Lipschitz in x. Thus Theorems 2.2.1, 2.2.2 and their proofs imply that the value function $u(x,t)$ has generalized derivatives of the first order with respect to t and of the second order with respect to x. These generalized derivatives belongs to the space $L^\infty_{loc}(\mathbb{R}^n \times [0,T])$ of all functions essentially bounded on every open bounded subset of the domain.

Proposition 2.2.6. *For all $x \in \mathbb{R}^n$ and $t \in [0,T]$ we have $u(x,t) \le (c_{\max} + C_{2.20})(1+|x|)$.*

Proof. Let $x' \in \mathbb{R}^n$ be arbitrary. Consider controls for which $\lim_{s\to 0+} v_s = x$. In view of (1.3) and (1.4) we have

$$u(x',t) = \inf\{J_{x't}(v) : v \in \mathcal{V}\} \le c(t)|x| + \inf\{J_{x+x',t}(v) : v \in \mathcal{V}\} = c(t)|x| + u(x+x',t).$$

So $u(x',t) - u(x+x',t) \le c(t)|x|$. Similarly $u(x+x',t) - u(x',t) \le c(t)|x|$, so

$$|u(x+x',t) - u(x',t)| \le c(t)|x|. \tag{2.44}$$

Taking $x' = 0$ we get $|u(x,t) - u(0,t)| \le c(t)|x|$. From (2.20) we see that $u(0,t) \le C_{2.20}$ so $u(x,t) \le c(t)|x| + u(0,t) \le c_{\max}|x| + C_{2.20} \le (c_{\max} + C_{2.20})(1+|x|)$. $\square$

Remark 2.2.7. The proof of Proposition 2.2.6 is not valid for u_ϵ instead u, because if a control $v \in \mathcal{V}_\epsilon$, then it is continuous, so the condition $\lim_{s\to 0+} v_s = x$ is invalid for $x \neq 0$.

Remark 2.2.8. The value function $u(x,t)$ satisfies $|Du(x,t)| \le c(t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Indeed, the gradient exists for all $(x,t) \in \mathbb{R}^n \times [0,T]$ in view of Remark 2.2.5. From (2.44) we see that the first derivative of $u(x,t)$ with respect to x in any direction is bounded by $c(t)$. Hence, the norm of the gradient $Du(x,t)$ is bounded by $c(t)$, too.

2.3 Dynamic Programming Principle and HJB equation.

To consider the DPP and the HJB equation for our problem we will first prove the pointwise convergence of u_ϵ to u if $\epsilon \to 0^+$. For this purpose we need an integral form of the Gronwall's inequality with locally finite measures.

Lemma 2.3.1. *(see [28]). Let μ be a locally finite measure on the Borel σ-algebra of $[t,T]$, where $0 \le t \le T$. We consider a measurable function ϕ defined on $[t,T]$ such that $\int_t^T |\phi(r)|\mu(dr) < \infty$. We assume that there exists a Borel function $\psi \ge 0$ on $[t,T]$ such that for all $s \in [t,T]$,*

$$\phi(s) \le \psi(s) + \int_{[t,s)} \phi(r)\mu(dr).$$

Then for all $s \in [t,T]$,

$$\phi(s) \le \psi(s) + \int_{[t,s)} \psi(r)e^{\mu([r,s))}\mu(dr).$$

Theorem 2.3.2. *For all $(x,t) \in \mathbb{R}^n \times [0,T]$ we have $\lim_{\epsilon\to 0^+} u_\epsilon(x,t) = u(x,t)$.*

Proof. Fix $x \in \mathbb{R}^n$ and $t \in [0,T]$. Consider an arbitrary $v \in \mathcal{V}$ such that $J_{xt}(v) < \infty$.

Step 1. We show first that $v \in L^p(\Omega \times [0,T-t], P \otimes \mu_{Leb})$, where μ_{Leb} denotes the Lebesgue's measure. Since $J_{xt}(v) < \infty$, we have $\mathbb{E}\int_t^T f(y_{xt}(s),s)ds < \infty$ and from (1.7) we get

$$\mathbb{E}\int_t^T |y_{xt}(s)|^p ds < \infty. \tag{2.45}$$

From (1.2) we can write for $s \in [t,T]$

$$v(s-t) = y_{xt}(s) - x - \int_t^s b(r)dr - \int_t^s \sigma(r)dW_{r-t} - \int_t^s a(r)y_{xt}(r)dr. \tag{2.46}$$

Using (2.45) and properties of the normal distribution we know that each term from the line above, maybe except for the last one, belongs to the space $L^p(\Omega \times [0,T-t])$. But the last term belongs to this space, too. Indeed,

$$\mathbb{E}\int_t^T \left|\int_t^s a(r)y_{xt}(r)dr\right|^p ds \le a_{\max}^p \,\mathbb{E}\int_t^T \left(\int_t^T |y_{xt}(r)|dr\right)^p ds.$$

Using the Hölder's inequality and (2.45) we can estimate the last expression above by

$$a_{\max}^p \,\mathbb{E}\int_t^T \left(\int_t^T |y_{xt}(r)|^p dr \cdot |T-t|^{p/q}\right) ds \le a_{\max}^p \, T^{1+p/q}\,\mathbb{E}\int_t^T |y_{xt}(r)|^p dr < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, from (2.46) we see that $v \in L^p(\Omega \times [0,T-t])$.

Step 2. Now we define a sequence of bounded controls $\{v_R,\ R>0\}$ such that v_R is convergent to v in the space $L^p(\Omega\times[0,T-t])$ and the total variation of v_R is pointwise convergent to the total variation of v from below. Let

$$v_R(s)=\begin{cases} v(s), & |v(s)|\le R\\ \frac{v(s)}{|v(s)|}\cdot R, & |v(s)|>R\ . \end{cases}$$

We see that for all $s\in[0,T-t]$ $\lim_{R\to\infty}v_R(s)=v(s)$ and $|v_R(s)|\le|v(s)|$. Hence, from Lemma 2.1.1 and Step 1,

$$\mathbb{E}\int_0^{T-t}|v(s)-v_R(s)|^p ds\le 2^p\mathbb{E}\int_0^{T-t}|v(s)|^p ds<\infty$$

and using the Lebesgue's dominated convergence theorem, we get

$$\lim_{R\to\infty}\ \mathbb{E}\int_0^{T-t}|v(s)-v_R(s)|^p ds=\mathbb{E}\int_0^{T-t}\lim_{R\to\infty}|v(s)-v_R(s)|^p ds=0.$$

The convergence in L^p is proved. Moreover, if $\xi(s),\xi_R(s)$ denote the total variations on the interval $[0,s]$ of the functions v,v_R respectively, then for all $s\in[0,T-t]$,

$$\xi_R(s)\le\xi(s)\ \text{and}\ \lim_{R\to\infty}\ \xi_R(s)=\xi(s). \tag{2.47}$$

Step 3. Let y^v_{xt}, $y^{v_R}_{xt}$ denote the state processes (see (1.2)) corresponding to the controls v, v_R respectively. We want to show that $\{y^{v_R}_{xt}\}$ is convergent to y^v_{xt} in the space $L^p(\Omega\times[t,T])$. First we observe that for $s\in[t,T]$,

$$y^v_{xt}(s)-y^{v_R}_{xt}(s)=\int_t^s a(r)\big(y^v_{xt}(r)-y^{v_R}_{xt}(r)\big)dr+v(s-t)-v_R(s-t).$$

Denoting $z_R(s)=y^v_{xt}(s)-y^{v_R}_{xt}(s)$ and $u_R(s)=v(s-t)-v_R(s-t)$ we can rewrite the last equality in the form $z_R(s)=\int_t^s a(r)z_R(r)dr+u_R(s)$. Hence $|z_R(s)|\le\int_t^s|z_R(r)|a_{\max}dr+|u_R(s)|$. Using Lemma 2.3.1 with $\phi=|z_R|$, $\psi=|u_R|$ and $\mu=a_{\max}\cdot\mu_{Leb}$, we get

$$|z_R(s)|\le|u_R(s)|+\int_t^s|u_R(r)|e^{a_{\max}(s-r)}dr\le|u_R(s)|+C_{2.48}\int_t^s|u_R(r)|dr, \tag{2.48}$$

where $C_{2.48}=e^{a_{\max}T}$. So from Lemma 2.1.1 and the Hölder's inequality

$$|z_R(s)|^p\le 2^{p-1}\left\{|u_R(s)|^p+C^p_{2.48}\left(\int_t^s|u_R(r)|dr\right)^p\right\}\le \tag{2.49}$$

$$\le 2^{p-1}\left\{|u_R(s)|^p+C^p_{2.48}(s-t)^{p/q}\int_t^s|u_R(r)|^p dr\right\}\le C_{2.49}\left\{|u_R(s)|^p+\int_t^T|u_R(r)|^p dr\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $C_{2.49}=2^{p-1}(1+C_{2.48}^p T^{p/q})$. Finally, in view of Step 2 we have

$$\lim_{R\to\infty}\ \mathbb{E}\int_t^T |z_R(s)|^p ds \le \lim_{R\to\infty}\ C_{2.49}\mathbb{E}\int_t^T \left\{|u_R(s)|^p+\int_t^T |u_R(r)|^p dr\right\}ds \le$$

$$\le \lim_{R\to\infty}\ \left\{C_{2.49}\mathbb{E}\int_t^T |u_R(s)|^p ds + C_{2.49}T\ \mathbb{E}\int_t^T |u_R(r)|^p dr\right\}=0.$$

Step 4. The next step is to show that $J_{xt}(v_R)\to J_{xt}(v)$ if $R\to\infty$. Indeed,

$$|J_{xt}(v)-J_{xt}(v_R)| \le \left|\mathbb{E}\int_t^T \Big(f(y_{xt}^v(s),s)-f(y_{xt}^{v_R}(s),s)\Big)e^{-\int_t^s \alpha(r)dr}ds\right| +$$

$$+\left|\mathbb{E}\int_t^T c(s)e^{-\int_t^s \alpha(r)dr}d(\xi-\xi_R)(s-t)\right| = A_R+B_R.$$

In view of (2.47),

$$B_R \le c_{\max}\left|\mathbb{E}\int_t^T d(\xi-\xi_R)(s-t)\right| = c_{\max}\mathbb{E}\big(\xi(T-t)-\xi_R(T-t)\big).$$

Using (2.47) again and the assumption that $J_{xt}(v)<\infty$ we see that $\mathbb{E}\big(\xi(T-t)-\xi_R(T-t)\big)\le \mathbb{E}\xi(T-t)<\infty$. Hence, from the Lebesgue's dominated convergence theorem we get

$$\lim_{R\to\infty} B_R \le \lim_{R\to\infty} c_{\max}\mathbb{E}\big(\xi(T-t)-\xi_R(T-t)\big) = c_{\max}\mathbb{E}\lim_{R\to\infty}\big(\xi(T-t)-\xi_R(T-t)\big)=0.$$

Using (1.8) and the Hölder's inequality we have

$$A_R \le \mathbb{E}\int_t^T \big|f(y_{xt}^v(s),s)-f(y_{xt}^{v_R}(s),s)\big|ds \le$$

$$\le \mathbb{E}\int_t^T \big(1+f(y_{xt}^v(s),s)+f(y_{xt}^{v_R}(s),s)\big)^{1-1/p}|y_{xt}^v(s)-y_{xt}^{v_R}(s)|ds \le$$

$$\le \left\{\mathbb{E}\int_t^T \big(1+f(y_{xt}^v(s),s)+f(y_{xt}^{v_R}(s),s)\big)ds\right\}^{1-1/p}\cdot\left\{\mathbb{E}\int_t^T |y_{xt}^v(s)-y_{xt}^{v_R}(s)|^p\right\}^{1/p}.$$

In view of Step 3, the second factor in the last expression goes to 0 if $R\to\infty$. We must show that the first factor is bounded. Indeed, from (1.7) and Lemma 2.1.1 we can write

$$\mathbb{E}\int_t^T \big(1+f(y_{xt}^v(s),s)+f(y_{xt}^{v_R}(s),s)\big)ds \le (1+C_0)\mathbb{E}\int_t^T \big(2+|y_{xt}^v(s)|^p+|y_{xt}^{v_R}(s)|^p\big)ds \le$$

$$\le (1+C_0)\mathbb{E}\int_t^T \Big(2+|y_{xt}^v(s)|^p+2^{p-1}|y_{xt}^v(s)|^p+2^{p-1}|y_{xt}^v(s)-y_{xt}^{v_R}(s)|^p\Big)ds.$$

Using (2.45) and Step 3 again we conclude that the last expression is bounded uniformly in R. Hence $\lim_{R\to\infty} A_R = 0$.

Summarizing Steps 1-4, we know that $J_{xt}(v_R)$ goes to $J_{xt}(v)$ if $R\to\infty$, so we can consider only bounded controls.

Step 5. Consider $v \in \mathcal{V}$ such that $||v||_\infty < R$ for some $R > 0$. We will construct a sequence of controls $\{v_n, n\in\mathbb{N}\}$ convergent to v in $L^p(\Omega\times[0,T-t])$ and such that $v_n \in V_{1/(2nR)}$ for all n. Besides we shall prove that the variation of v_n is pointwise convergent to the variation of v from below. Let $v_n(s) = n\int_{(s-1/n)\vee 0}^{s} v(r)dr$, $s\in[0,T-t]$. We observe that v_n is a progressively measurable continuous random process such that $||v_n||_\infty \le R$, so $v_n \in L^p(\Omega\times[0,T-t])$. From left-continuity of v we know that

$$\forall_{\omega\in\Omega}\ \forall_{s\in[0,T-t]}\quad \lim_{n\to\infty} v_n(s) = v(s). \tag{2.50}$$

Using the Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty}\ \mathbb{E}\int_0^{T-t} |v(s)-v_n(s)|^p ds = \mathbb{E}\int_0^{T-t} \lim_{n\to\infty} |v(s)-v_n(s)|^p ds = 0,$$

so L^p-convergence is proved.

Now we want to check that $v_n \in V_{1/(2nR)}$. Indeed,

$$\Big|\frac{d}{ds}v_n(s)\Big| = \left|\frac{d}{ds}\left(n\int_{(s-1/n)\vee 0}^{s} v(r)dr\right)\right| = n\Big|v(s) - v\big((s-1/n)\vee 0\big)\Big| \le 2nR.$$

Let $\xi_n(s), \xi(s)$ denote the variations on the interval $[0,s]$ of the functions v_n, v respectively. For convenience, we define $v(r)\equiv 0$ for $r<0$. Then $v_n(s) = n\int_{s-1/n}^{s} v(r)dr$, $s\in[0,T-t]$. Fix $\omega\in\Omega,\ s\in(0,T-t]$. Let $\Pi = \{s_0, s_1, ..., s_k\}$ be a partition of the interval $[0,s]$, where $0 = s_0 < s_1 < ... < s_k = s$. Then

$$\sum_{i=1}^{k} |v_n(s_i) - v_n(s_{i-1})| = n\sum_{i=1}^{k}\left|\int_{s_i-1/n}^{s_i} v(r)dr - \int_{s_{i-1}-1/n}^{s_{i-1}} v(r)dr\right| =$$

$$= n\sum_{i=1}^{k}\left|\int_0^{1/n}\Big(v(s_i+r-1/n) - v(s_{i-1}+r-1/n)\Big)dr\right| \le$$

$$\le n\int_0^{1/n}\sum_{i=1}^{k}\Big|v(s_i+r-1/n) - v(s_{i-1}+r-1/n)\Big|dr \le n\int_0^{1/n}\xi(s)dr = \xi(s).$$

Letting $||\Pi||\to 0$, we get

$$\xi_n(s) \le \xi(s). \tag{2.51}$$

On the other hand, from (2.50) we see that

$$\sum_{i=1}^{k} |v(s_i) - v(s_{i-1})| = \sum_{i=1}^{k}\Big|\lim_{n\to\infty} v_n(s_i) - \lim_{n\to\infty} v_n(s_{i-1})\Big| =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{k} |v_n(s_i) - v_n(s_{i-1})| \le \liminf_{n\to\infty} \xi_n(s).$$

Letting $||\Pi|| \to 0$ and using (2.51), we have

$$\xi(s) \le \liminf_{n\to\infty} \xi_n(s) \le \limsup_{n\to\infty} \xi_n(s) \le \xi(s) \quad \Leftrightarrow \quad \lim_{n\to\infty} \xi_n(s) = \xi(s).$$

Step 6. In view of Step 5 we can mimic Steps 3 and 4 to conclude that $J_{xt}(v_n) \to J_{xt}(v)$ if $n \to \infty$, where $||v||_\infty < R$ for some $R > 0$. From this and Step 4, remembering that $v_n \in \mathcal{V}_{1/(2nR)}$, we can write

$$\inf_{v\in\mathcal{V}} J_{xt}(v) = \inf_{v\in\bigcup_{\epsilon>0}\mathcal{V}_\epsilon} J_{xt}(v) \tag{2.52}$$

and $\lim_{\epsilon\to 0^+} u_\epsilon(x,t) = u(x,t)$. □

Theorem 2.3.3 (Bellman's dynamic programming principle). *Let $x \in \mathbb{R}^n$, $t \in [0,T]$ and let y^v_{xt} denote the state process corresponding to a control $v \in \mathcal{V}$. Let $\tau \in [0, T-t]$ be a Markov time with respect to $\{\mathcal{F}_t\}$. Then*

$$u(x,t) = \inf_{v\in\mathcal{V}} \mathbb{E}\Bigg\{ \int_t^{t+\tau} f(y^v_{xt}(s), s) e^{-\int_t^s \alpha(r)dr} ds \ + \int_t^{t+\tau} c(s) e^{-\int_t^s \alpha(r)dr} d\xi(s-t) \ + \\ + \ e^{-\int_t^{t+\tau} \alpha(r)dr} u(y^v_{xt}(t+\tau), t+\tau) \Bigg\}.$$

Proof. For convenience let us denote

$$J_{xt}(v,\tau) = \mathbb{E}\Bigg\{ \int_t^{t+\tau} f(y^v_{xt}(s), s) e^{-\int_t^s \alpha(r)dr} ds + \int_t^{t+\tau} c(s) e^{-\int_t^s \alpha(r)dr} d\xi(s-t) \Bigg\}.$$

It is known that DPP holds for regular stochastic control problems (see, e.g., [14], Th. 3.1.6). Hence we have for each $\epsilon > 0$,

$$u_\epsilon(x,t) = \inf_{v\in\mathcal{V}_\epsilon} \Big\{ J_{xt}(v,\tau) + \ \mathbb{E}\ e^{-\int_t^{t+\tau} \alpha(r)dr} u_\epsilon(y^v_{xt}(t+\tau), t+\tau) \Big\}. \tag{2.53}$$

Considering any $\tilde{v} \in \mathcal{V}_\epsilon$ we have

$$u_\epsilon(x,t) \le J_{xt}(\tilde{v},\tau) + \ \mathbb{E}\ e^{-\int_t^{t+\tau} \alpha(r)dr} u_\epsilon(y^{\tilde{v}}_{xt}(t+\tau), t+\tau).$$

If $\epsilon \to 0^+$, from Theorem 2.3.2 and the Lebesgue's dominated convergence theorem we get

$$u(x,t) \le J_{xt}(\tilde{v},\tau) + \ \mathbb{E}\ e^{-\int_t^{t+\tau} \alpha(r)dr} u(y^{\tilde{v}}_{xt}(t+\tau), t+\tau).$$

Because $\epsilon > 0$ and $\tilde{v} \in \mathcal{V}_\epsilon$ are arbitrary we can conclude that

$$u(x,t) \le \inf_{v\in\bigcup_{\epsilon>0}\mathcal{V}_\epsilon} \Big\{ J_{xt}(v,\tau) \ + \ \mathbb{E}\ e^{-\int_t^{t+\tau} \alpha(r)dr} u(y^v_{xt}(t+\tau), t+\tau) \Big\}. \tag{2.54}$$

On the other hand, from (2.53)

$$u_\epsilon(x,t) \geq \inf_{v\in\bigcup_{\epsilon>0}\mathcal{V}_\epsilon} \Big\{J_{xt}(v,\tau) \ + \ \mathbb{E}\, e^{-\int_t^{t+\tau}\alpha(r)dr} u(y_{xt}^v(t+\tau),t+\tau)\Big\}.$$

Letting $\epsilon \to 0^+$ we get

$$u(x,t) \geq \inf_{v\in\bigcup_{\epsilon>0}\mathcal{V}_\epsilon} \Big\{J_{xt}(v,\tau) \ + \ \mathbb{E}\, e^{-\int_t^{t+\tau}\alpha(r)dr} u(y_{xt}^v(t+\tau),t+\tau)\Big\}. \tag{2.55}$$

The inequalities (2.54), (2.55) and an argument similar to the proof of Theorem 2.3.2 (see (2.52)) imply that

$$u(x,t) = \inf_{v\in\mathcal{V}} \Big\{J_{xt}(v,\tau) \ + \ \mathbb{E}\, e^{-\int_t^{t+\tau}\alpha(r)dr} u(y_{xt}^v(t+\tau),t+\tau)\Big\}.$$

□

Corollary 2.3.4. *The dynamic programming property in the weak sense holds (see Def. 1.2.1) and hence the value function satisfies* (2.24).

Denote

$$Au(x,t) = \frac{-\partial u(x,t)}{\partial t} - \frac{1}{2}\beta(t)\circ D^2u(x,t) - \Big(a(t)x+b(t)\Big)\circ Du(x,t) + \alpha(t)u(x,t),$$

where $\circ$ denotes the scalar product of vectors and matrices respectively.

Theorem 2.3.5 (The HJB equation)**.** *The value function u satisfies almost everywhere (a.e.) the following second-order differential equation:*

$$\max\Big\{Au(x,t) - f(x,t)\ ,\ |Du(x,t)| - c(t)\Big\} = 0. \tag{2.56}$$

Proof. An application of the DPP for regular stochastic control problems yields for $\epsilon > 0$ the following equation (see [8], Chapter IV.3):

$$Au_\epsilon(x,t) + \frac{1}{\epsilon}\Big(|Du_\epsilon(x,t)| - c(t)\Big)^+ = f(x,t) \quad a.e.. \tag{2.57}$$

In view of Theorems 2.2.1, 2.2.2, Remark 2.2.4, Theorem 2.3.2, Corollary 2.3.4 and the Arzela-Ascoli's theorem we see that $u_\epsilon \to u$ uniformly on every compact set if $\epsilon \to 0^+$.

Fix $t \in [0,T]$. From (2.22) and Remark 2.2.4 we see that $D^2u_\epsilon(\cdot,t)$ are locally uniformly bounded for all $\epsilon > 0$ in their domains, so using the Arzela-Ascoli's theorem from every sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$ convergent to 0 we can choose a subsequence $\{\tilde{\epsilon}_m\}_{m\in\mathbb{N}}$ such that

$$Du_{\tilde{\epsilon}_m}(\cdot,t) \to v = (v_1,...,v_n) \quad \text{almost uniformly if } m \to \infty.$$

But v must be equal to $Du(\cdot,t)$ in the distribution sense. Indeed, for any function $\phi \in C_c^\infty(\mathbb{R}^n)$ and any $k = 1,...,n$ we have

$$\int_{\mathbb{R}^n} \frac{\partial\phi(x)}{\partial x_k} u_{\tilde{\epsilon}_m}(x,t)dx = -\int_{\mathbb{R}^n} \phi(x)\frac{\partial u_{\tilde{\epsilon}_m}(x,t)}{\partial x_k}dx.$$

Letting $m \to \infty$ we get

$$\int_{\mathbb{R}^n} \frac{\partial \phi(x)}{\partial x_k} u(x,t)dx = -\int_{\mathbb{R}^n} \phi(x) v_k(x) dx,$$

so $v_k(\cdot) = \frac{\partial u(\cdot,t)}{\partial x_k}$ almost everywhere. Since $\frac{\partial u}{\partial x_k}$ and v_k are Lipschitz continuous, the equality holds for all $x \in \mathbb{R}^n$. Thus, v does not depend on the choice of the subsequence $\{\tilde{\epsilon}_m\}$, so

$$\forall_{t\in[0,T]} \quad Du_\epsilon(\cdot,t) \to D_u(\cdot,t) \quad \text{almost uniformly if } \ \epsilon \to 0^+. \tag{2.58}$$

Let $\psi = (1+|x|)^{-2p-n-1}$. From (2.20)-(2.24) we conclude that $|Au_\epsilon(x,t)|$ is not greater than

$$(C_{2.24}+C_{2.20}\alpha_{\max})(1+|x|^p)+\frac{1}{2}\beta_{\max}n^2C_{2.22}(1+|x|)^{(p-2)^+}+\big(a_{\max}|x|+b_{\max}\big)nC_{2.21}(1+2|x|^{p-1})$$

for almost every $(x,t) \in \mathbb{R}^n \times [0,T]$. Using Lemma 2.1.1, we have the estimate

$$|Au_\epsilon(x,t)| \le C_{2.59}(1+|x|)^p \quad a.e. \tag{2.59}$$

for some constant $C_{2.59} > 0$ depending only on $C_{2.20}, C_{2.21}, C_{2.22}, C_{2.24}, n, p, a_{\max}, b_{\max}, \alpha_{\max}, \beta_{\max}$. Hence

$$|Au_\epsilon(x,t)|^2\psi(x) \le \frac{C_{2.59}^2(1+|x|)^{2p}}{(1+|x|)^{2p+n+1}} = \frac{C_{2.59}^2}{(1+|x|)^{n+1}} \quad a.e..$$

The same estimate holds for u instead u_ϵ. So $|Au_\epsilon|^2\psi,\ |Au|^2\psi \in L^1(\mathbb{R}^n \times [0,T])$. Moreover, $Au_\epsilon,\ Au$ are uniformly bounded in the space L^2_ψ, where

$$L^2_\psi = \Big\{ v : v^2\psi \in L^1(\mathbb{R}^n \times [0,T]) \Big\} = L^2_{\psi\cdot\mu_{Leb}}(\mathbb{R}^n \times [0,T]).$$

From the Banach-Alaoglu theorem we know that balls in the space L^2 are weakly compact. So for each sequence $\{\epsilon_m\}_{m\in\mathbb{N}}$ convergent to 0, there exists a subsequence $\{\tilde{\epsilon}_m\}_{m\in\mathbb{N}}$ such that $Au_{\tilde{\epsilon}_m} \rightharpoonup v$ in L^2_ψ if $m \to \infty$. We will show that $v = Au$ in the distribution sense. Indeed, for any function ϕ belonging to the class $C_c^\infty(\mathbb{R}^n \times [0,T])$, we have

$$\int_0^T \int_{\mathbb{R}^n} (Au_{\tilde{\epsilon}_m})\phi dxdt = \int_0^T \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t} u_{\tilde{\epsilon}_m} dxdt - \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \Big(\beta(t)\circ D^2\phi\Big)u_{\tilde{\epsilon}_m}dxdt+$$

$$+\int_0^T \int_{\mathbb{R}^n} \bigg(\Big(a(t)x+b(t)\Big)\circ D\phi\bigg)u_{\tilde{\epsilon}_m}dxdt+\int_0^T \int_{\mathbb{R}^n} \operatorname{tr}\big(a(t)\big)u_{\tilde{\epsilon}_m}\phi dxdt+\int_0^T \int_{\mathbb{R}^n} \alpha(t)\phi u_{\tilde{\epsilon}_m}dxdt.$$

Letting $m \to \infty$, we get

$$\int_0^T \int_{\mathbb{R}^n} v\phi dxdt = \int_0^T \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t} udxdt - \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \Big(\beta(t)\circ D^2\phi\Big)udxdt+$$

$$+\int_0^T\int_{\mathbb{R}^n}\Big(\Big(a(t)x+b(t)\Big)\circ D\phi\Big)udxdt+\int_0^T\int_{\mathbb{R}^n}\mathrm{tr}\big(a(t)\big)u\phi dxdt+$$

$$+\int_0^T\int_{\mathbb{R}^n}\alpha(t)\phi udxdt=\int_0^T\int_{\mathbb{R}^n}(Au)\phi dxdt.$$

Hence $Au_{\tilde{\epsilon}_m}\rightharpoonup Au\quad$ in L^2_ψ if $m\to\infty$. From uniqess of the limit we conclude

$$Au_\epsilon\rightharpoonup Au\quad \text{in } L^2_\psi \text{ if } \epsilon\to 0^+. \tag{2.60}$$

In view of (2.57) we have $Au_\epsilon\le f$ a.e.. From this and (2.60) we see that $Au(x,t)\le f(x,t)$ *a.e.*. This, together with Remark 2.2.8 ensure us that

$$\max\Big\{Au(x,t)-f(x,t)\ ,\ |Du(x,t)|-c(t)\Big\}\le 0\quad a.e.. \tag{2.61}$$

Take a sequence $\epsilon_n\to 0$ and let $\mathcal{D}$ be the set of $(x,t)\in\mathbb{R}^n\times[0,T]$ such that (2.57) holds at (x,t) for all ϵ_n. Then $\mu_{Leb}\big((\mathbb{R}^n\times[0,T])\setminus\mathcal{D}\big)=0$. Choose $t\in[0,T]$ such that for almost every $x\in\mathbb{R}^n$ we have $(x,t)\in\mathcal{D}$. Since $|Du_{\varepsilon_n}(x,t)|\to|Du(x,t)|$ as $n\to\infty$ (see (2.58)), $\mathbb{I}_{\{|Du_{\varepsilon_n}(x,t)|<c(t)\}}=\mathbb{I}_{\{|Du(x,t)|<c(t)\}}$ for n large enough (depending on (x,t)). We have from this and (2.57) that

$$\mathbb{I}_{\{|Du(x,t)|<c(t)\}}Au_{\varepsilon_n}(x,t)\to\mathbb{I}_{\{|Du(x,t)|<c(t)\}}f(x,t)\quad a.e.. \tag{2.62}$$

On the other hand, (2.60) yields

$$\mathbb{I}_{\{|Du(x,t)|<c(t)\}}Au_{\varepsilon_n}(x,t)\rightharpoonup\mathbb{I}_{\{|Du(x,t)|<c(t)\}}Au(x,t)\quad\text{in } L^2_\psi, \tag{2.63}$$

so the sequence $\big\{\mathbb{I}_{\{|Du(x,t)|<c(t)\}}Au_{\varepsilon_n}(x,t)\big\}$ is bounded in L^2_ψ and thus it is uniformly integrable in L^1_ψ. This, together with (2.62), implies that for every $\phi\in C_c^\infty(\mathbb{R}^n\times[0,T])\subset L^2_\psi$

$$\int_0^T\int_{\mathbb{R}^n}\mathbb{I}_{\{|Du(x,t)|<c(t)\}}(Au_{\varepsilon_n}\phi\psi)(x,t)dxdt\to\int_0^T\int_{\mathbb{R}^n}\mathbb{I}_{\{|Du(x,t)|<c(t)\}}(f\phi\psi)(x,t)dxdt, \tag{2.64}$$

which, together with (2.63) implies that

$$\mathbb{I}_{\{|Du(x,t)|<c(t)\}}Au(x,t)=\mathbb{I}_{\{|Du(x,t)|<c(t)\}}f(x,t)\quad a.e..$$

□

2.4 Existence and uniqueness of the optimal control.

The results of this section are analogous to Theorems 7 and 8 from [18].

Fix $(t,x)\in[0,T)\times\mathbb{R}^n$ (for $t=T$ the only admissible control is $v(0)=0$ a.s.). Let m_t be the measure on $\big([t,T]\times\Omega,\mathcal{B}([t,T])\otimes\mathcal{F}\big)$ equal to the product of the Lebesgue's measure and P.

Remark 2.4.1. If a process X is a modification of a process Y and both processes have left-continuous sample paths a.s., then the processes X, Y are indistinguishable (compare Problem 1.1.5, [11]).

Theorem 2.4.2. *The optimal control $v^* \in \mathcal{V}$, if it exists, is unique up to the indistinguishability.*

Proof. Suppose there are $v_1, v_2 \in \mathcal{V}$ for which $u(x,t) = J_{xt}(v_1) = J_{xt}(v_2)$. Put $v_0 = (v_1 + v_2)/2$. Of course $v_0 \in \mathcal{V}$. From Lemma 2.1.6 we have

$$u(x,t) - J_{xt}(v_0) = \frac{1}{2}\big(J_{xt}(v_1) + J_{xt}(v_2)\big) - J_{xt}(v) \geq 0. \tag{2.65}$$

Let $y_{xt}^0, y_{xt}^1, y_{xt}^2$ be the solutions of (1.2) corresponding to $v = v_0, v_1, v_2$ respectively. In view of the proof of Lemma 2.1.6 and strict convexity of the runnig cost function f, we have

$$f\big(y_{xt}^0(s), s\big) < \frac{1}{2} f\big(y_{xt}^1(s), s\big) + \frac{1}{2} f\big(y_{xt}^2(s), s\big) \tag{2.66}$$

provided that $y_{xt}^1(s) \neq y_{xt}^2(s)$.

Assume that v_1, v_2 are not indistinguishable. Then there exists $s' \in (t, T]$ such that $P(A) > 0$, where $A = \{v_1(s') \neq v_2(s')\}$ (see Remark 2.4.1). Because v_1, v_2 have left-continuous sample paths a.s., there exists $s''(\omega) \in (t, s')$ such that $v_1(s) \neq v_2(s)$ for all $s \in [s''(\omega), s']$, $\omega \in A$. Thus, $y_{xt}^1(s) \neq y_{xt}^2(s)$ on some m_t-nonzero set. This fact together with (2.66) and the definition of J_{xt} imply that the inequality (2.65) is strict, so we get a contradiction. We conclude that v_1, v_2 must be indistinguishable. □

Lemma 2.4.3. *Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $L^p(m_t)$. If $z_n \to 0$ in $L^p(m_t)$, then $\mathcal{T} z_n \to 0$ in $L^p(m_t)$, where*

$$\mathcal{T} z_n(s,\omega) = z_n(s,\omega) - \int_t^s a(r) z_n(r,\omega) dr.$$

Proof. By the Hölder's inequality, the function $g(s,\omega) = \int_t^s a(r) z_n(r,\omega) dr$ satisfies

$$||g||_{L^p}^p \leq a_{\max}^p \mathbb{E} \int_t^T \int_t^T |z_n(r,\omega)|^p \cdot T^{p/q} dr ds \leq a_{\max}^p \cdot T^{1+p/q} \cdot ||z_n||_{L^p}^p,$$

so $\mathcal{T}$ is a bounded operator from $L^p(m_t)$ into $L^p(m_t)$. □

Theorem 2.4.4. *There exists an optimal control $v^* \in \mathcal{V}$.*

Proof. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of admissible controls such that $J_{xt}(v_k) \to u(x,t)$ as $k \to \infty$ and let y_{xt}^k be the solution of (1.2) corresponding to $v = v_k$. Then $J_{xt}(v_k)$ are uniformly bounded in k. By Lemma 2.1.8 the sequence $\{y_{xt}^k\}_{k\in\mathbb{N}}$ is bounded in $L^p(m_t)$ and hence, by the Banach-Alaoglu theorem, there exists a subsequence (still denoted by $\{y_{xt}^k\}$) and a process y_{xt} such that $y_{xt}^k \rightharpoonup y_{xt}$ in $L^p(m_t)$.

Fix $k \in \mathbb{N}$. Since the sequence $\{y^i_{xt}\}_{i \geq k}$ is also convergent to y_{xt}, by the Mazur theorem there exists

$$z^k_{xt} = \sum_{i=k}^{n(k)} \alpha_{k,i} \cdot y^i_{xt}, \quad \alpha_{k,i} \geq 0, \ \sum_{i=k}^{n(k)} \alpha_{k,i} = 1, \ k \leq n(k) < \infty$$

such that $||z^k_{xt} - y_{xt}||_{L^p} \leq 1/k$. In particular $z^k_{xt} \to y_{xt}$ in $L^p(m_t)$. Let $\eta_k = \sum_{i=k}^{n(k)} \alpha_{k,i} \cdot v_i$ be the control corresponding to z^k_{xt} in (1.2). Then $\eta_k \in \mathcal{V}$. Moreover by Lemma 2.1.6,

$$u(x,t) \leq J_{xt}(\eta_k) \leq \sum_{i=k}^{n(k)} \alpha_{k,i} \cdot J_{xt}(v_i) \leq \max_{i=k,\dots,n(k)} J_{xt}(v_i) \overset{k\to\infty}{\longrightarrow} u(x,t).$$

For $s \in [t,T]$ we have

$$z^k_{xt}(s) - z^m_{xt}(s) - \int_t^s a(r)\big(z^k_{xt}(r) - z^m_{xt}(r)\big)dr = \eta_k(s-t) - \eta_m(s-t).$$

Because $\{z^k_{xt}\}$ is convergent in $L^p(m_t)$, $z^k_{xt} - z^m_{xt}$ goes to 0 in $L^p(m_t)$ as $k, m \to \infty$. Using Lemma 2.4.3 we conclude that $\{\eta_k(\cdot - t, \cdot)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^p(m_t)$ so it is convergent to a process $v \in L^p(m_t)$. Without loss of generality we may assume that $v(0) \equiv 0$.

Now we choose a subsequence (still denoted by k) such that $\eta_k(s,\omega) \to v(s,\omega)$ as $k \to \infty$ for $(s,\omega) \in \mathcal{A}$, where $(\mu_{Leb} \times P)(\mathcal{A}) = T - t$. For $\omega \in \Omega$ and $s \in [0, T-t]$, we define

$$\mathcal{A}_\omega = \big\{s \in [0,T-t] : (s,\omega) \in \mathcal{A}\big\}, \quad \mathcal{A}_s = \big\{\omega \in \Omega : (s,\omega) \in \mathcal{A}\big\}.$$

Note that $P(\mathcal{A}_0) = 1$ because $\eta_k(0) = v(0) = 0$ P-a.s.. Furthermore, let

$$\tilde{\Omega} = \big\{\omega \in \Omega : \mu_{Leb}(\mathcal{A}_\omega) = T - t\big\}, \quad \mathcal{S} = \big\{s \in [0,T-t] : P(\mathcal{A}_s) = 1\big\}.$$

Then $P(\tilde{\Omega}) = 1$ and $\mu_{Leb}(\mathcal{S}) = T - t$. Let $\mathcal{N}$ be a countable subset of $\mathcal{S}$, dense in $[0, T-t]$, including 0 and let $\mathcal{A}_\mathcal{N} = \bigcap_{s\in\mathcal{N}} \mathcal{A}_s$. We have $P(\mathcal{A}_\mathcal{N}) = 1$.

Let $\xi_k(s)$ denote the total variation of η_k on the interval $[0,s]$. Because $J_{xt}(\eta_k)$ are uniformly bounded in k, there exists a constant $C > 0$ such that $\mathbb{E}\xi_k(T-t) \leq C$ for all $k \in \mathbb{N}$. In view of the Fatou's lemma, $\mathbb{E} \liminf_{k\to\infty} \xi_k(T-t) \leq \liminf_{k\to\infty} \mathbb{E}\xi_k(T-t) \leq C$, so $\liminf_{k\to\infty} \xi_k(T-t)$ is finite a.s..

Fix $\omega \in \Omega$ and let $\Pi \subset \mathcal{A}_\omega$, $\Pi = \{t_0, t_1, ..., t_m\}$, $0 = t_0 < t_1 < ... < t_m \leq T - t$. Let $k_n = k_n(\omega) \to \infty$ be a sequence of natural numbers such that $\lim_{k_n\to\infty} \xi_{k_n}(T-t) = \liminf_{k\to\infty} \xi_k(T-t)$. Then

$$\sum_{i=0}^{m-1} \big|v(t_{i+1}) - v(t_i)\big| = \lim_{k_n\to\infty} \sum_{i=0}^{m-1} \big|\eta_{k_n}(t_{i+1}) - \eta_{k_n}(t_i)\big| \leq \lim_{k_n\to\infty} \xi_{k_n}(T-t) = \liminf_{k\to\infty} \xi_k(T-t).$$

Thus, $v_{|\mathcal{A}_\omega}$ has bounded variation and hence it has left-hand and right-hand limits at each point. Let $v^* = 0$ on the P-zero set $\Omega \setminus (\mathcal{A}_\mathcal{N} \cap \tilde{\Omega})$. On $\mathcal{A}_\mathcal{N} \cap \tilde{\Omega}$ let

$$v^*(s) = \begin{cases} 0 = v(0), & s = 0 \\ \lim\limits_{\mathcal{A}_\omega \ni u \uparrow s} v(u) = \lim\limits_{\mathcal{N} \ni u \uparrow s} v(u), & s \in (0, T-t] \end{cases}.$$

Then v^* is progressively measurable, left-continous and $v^*(0) = 0$. Moreover, for $\omega \in \mathcal{A}_{\mathcal{N}} \cap \tilde{\Omega}$ and for each partition $\Pi = \{t_0, t_1, ..., t_m\}$, $0 = t_0 < t_1 < ... < t_m \leq T - t$, we can choose $\{t_i^k\}_{k\in\mathbb{N}} \subset \mathcal{A}_\omega$ such that $t_i^k \uparrow t_i$ as $k \to \infty$, $i = 1, 2, ..., m$. Therefore

$$\sum_{i=0}^{m-1} \left|v^*(t_{i+1}) - v^*(t_i)\right| = \lim_{k\to\infty} \sum_{i=0}^{m-1} \left|v(t_{i+1}^k) - v(t_i^k)\right| \leq Var(v, [0, T-t] \cap \mathcal{A}_\omega) \leq \liminf_{k\to\infty} \xi_k(T-t),$$

so v^* has bounded variation. This ensures us that $v^* \in \mathcal{V}$.

For $\omega \in \mathcal{A}_{\mathcal{N}} \cap \tilde{\Omega}$, the set $A_\omega \cap \{s \in [0, T-t] : v(s,\omega) \neq v^*(s,\omega)\}$ is countable, so its Lebesgue's measure is equal to 0. Therefore $v = v^*$ m_t-a.e.. In particular, $\eta_k \to v^*$ in $L^p(m_t)$. Proceeding as in Steps 3-4 in the proof of Theorem 2.3.2 we can show that $y_{xt}^{\eta_k} \to y_{xt}^{v^*}$ in $L^p(m_t)$ and hence

$$\lim_{k\to\infty} \mathbb{E} \int_t^T f(y_{xt}^{\eta_k}(s), s) e^{-\int_t^s \alpha(r)dr} ds = \mathbb{E} \int_t^T f(y_{xt}^{v^*}(s), s) e^{-\int_t^s \alpha(r)dr} ds. \tag{2.67}$$

To finish the proof we need to check that

$$\mathbb{E} \int_t^T c(s) e^{-\int_t^s \alpha(r)dr} d\xi^*(s-t) \leq \liminf_{k\to\infty} \mathbb{E} \int_t^T c(s) e^{-\int_t^s \alpha(r)dr} d\xi_k(s-t), \tag{2.68}$$

where ξ^* is the total variation of v^*. Fix $\omega \in \mathcal{A}_{\mathcal{N}} \cap \tilde{\Omega}$ and let $0 \leq s_1 \leq s_2 \leq T - t$. Let $\Pi = \{t_0, t_1, ..., t_m\}$, $s_1 = t_0 < t_1 < ... < t_m = s_2$ and let $\{t_i^k\}_{k\in\mathbb{N}} \subset \mathcal{A}_\omega$ be such that $t_i^k \uparrow t_i$ as $k \to \infty$, $i = 0, 1, ..., m$. Then for every $k_0 \in \mathbb{N}$

$$\sum_{i=0}^{m-1} \left|v^*(t_{i+1}) - v^*(t_i)\right| = \lim_{k\to\infty} \sum_{i=0}^{m-1} \left|v(t_{i+1}^k) - v(t_i^k)\right| \leq Var(v, [t_0^{k_0}, s_2] \cap \mathcal{A}_\omega).$$

Letting $k_0 \to \infty$ we get $\sum_{i=0}^{m-1} \left|v^*(t_{i+1}) - v^*(t_i)\right| \leq Var(v, [s_1, s_2] \cap \mathcal{A}_\omega)$ and hence

$$Var(v^*, [s_1, s_2]) \leq Var(v, [s_1, s_2] \cap \mathcal{A}_\omega). \tag{2.69}$$

Let $\xi(s) = Var(v, [0, s] \cap \mathcal{A}_\omega)$, $s \in [0, T-t]$. Restricting $\Pi = \{t_0, t_1, ..., t_m\}$, $s_1 = t_0 < t_1 < ... < t_m = s_2$ so that $\Pi \subset \mathcal{A}_\omega$ (in particular assuming $s_1, s_2 \in \mathcal{A}_\omega$) we get

$$\sum_{i=0}^{m-1} \left|v(t_{i+1}) - v(t_i)\right| = \lim_{k\to\infty} \sum_{i=0}^{m-1} \left|\eta_k(t_{i+1}) - \eta_k(t_i)\right| \leq \liminf_{k\to\infty} \big(\xi_k(s_2) - \xi_k(s_1)\big).$$

As $||\Pi|| \to 0$ we get

$$\xi(s_2) - \xi(s_1) \leq \liminf_{k\to\infty} \big(\xi_k(s_2) - \xi_k(s_1)\big). \tag{2.70}$$

Now take $\Pi = \{t_0, t_1, ..., t_m\}$, $0 = t_0 < t_1 < ... < t_m \leq T - t$, $\Pi \subset \mathcal{N}$. In particular, $\Pi \subset \mathcal{A}_\omega$ for all $\omega \in \mathcal{A}_{\mathcal{N}}$. For every interval $[t_i, t_{i+1}]$, $i = 0, 1, ..., m-1$, let $l_i = \min\left\{c(s)e^{-\int_t^s \alpha(r)dr} :\ s \in [t_i + t,\ t_{i+1} + t]\right\}$. For $\omega \in \mathcal{A}_{\mathcal{N}} \cap \tilde{\Omega}$, by (2.69)-(2.70), we have

$$\sum_{i=0}^{m-1} l_i \cdot \big(\xi^*(t_{i+1}) - \xi^*(t_i)\big) \leq \sum_{i=0}^{m-1} l_i \cdot \big(\xi(t_{i+1}) - \xi(t_i)\big) \leq \liminf_{k\to\infty} \sum_{i=0}^{m-1} l_i \cdot \big(\xi_k(t_{i+1}) - \xi_k(t_i)\big).$$

This together with the Fatou's lemma and the fact that $P(\mathcal{A}_{\mathcal{N}} \cap \tilde{\Omega}) = 1$ yields

$$\mathbb{E}\sum_{i=0}^{m-1} l_i \cdot \big(\xi^*(t_{i+1}) - \xi^*(t_i)\big) \leq \mathbb{E}\liminf_{k\to\infty}\sum_{i=0}^{m-1} l_i \cdot \big(\xi_k(t_{i+1}) - \xi_k(t_i)\big) \leq$$

$$\leq \liminf_{k\to\infty}\mathbb{E}\sum_{i=0}^{m-1} l_i \cdot \big(\xi_k(t_{i+1}) - \xi_k(t_i)\big) \leq \liminf_{k\to\infty}\mathbb{E}\int_t^T c(s)e^{-\int_t^s \alpha(r)dr} d\xi_k(s-t).$$

Letting $||\Pi|| \to 0,\ t_m \uparrow T-t$ so that each partition in the sequence is contained in the next one, by the monotone convergence theorem we get (2.68).

From (2.67) and (2.68) $J_{xt}(v^*) \leq \liminf_{k\to\infty} J_{xt}(\eta_k) = u(x,t)$. On the other hand, $J_{xt}(v^*) \geq u(x,t)$ because $v^* \in \mathcal{V}$ and hence $J_{xt}(v^*) = u(x,t)$ so v^* is an optimal control.

□

Chapter 3

Characterization of the optimal policy

In this chapter we assume that $a, b \equiv 0$, σ is constant and equal to $\sqrt{2}$ times the identity matrix with dimension n and $c, \alpha \equiv 1$. Then (1.2), (1.3) take forms

$$y_{xt}(s) = x + \sqrt{2}W_{s-t} + v_{s-t}, \quad s \in [t, T], \tag{3.1}$$

$$J_{xt}(v) = \mathbb{E}\left\{\int_t^T f(y_{xt}(s), s)e^{-(s-t)}ds + \int_t^T e^{-(s-t)}d\xi(s-t)\right\} \tag{3.2}$$

and the corresponding HJB equation is

$$\max\left\{u(x,t) - u_t(x,t) - \Delta u(x,t) - f(x,t)\ ,\ |Du(x,t)| - 1\right\} = 0, \tag{3.3}$$

where $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

We will characterize the optimal policy in the above singular stochastic control problem.

3.1 The Skorokhod problem and the main theorem

Let $\mathcal{D}$ be a relatively open, connected subset of $\mathbb{R}^n \times [0, T]$ such that

$$\mathcal{D} = \bigcup_{t\in[0,T]} (\mathcal{D}_t \times \{t\}),$$

where $\mathcal{D}_t = \{x \in \mathbb{R}^n : (x,t) \in \mathcal{D}\}$ are open, connected, nonempty subsets of $\mathbb{R}^n$. We will often indentify the domain $\mathcal{D}$ with the family $\{\mathcal{D}_t\}_{t\in[0,T]}$.

Definition 3.1.1. (compare [12] def. 2.4 and [19] def. 1.1). Let $x_0 \in \overline{\mathcal{D}_0}$. Let $\Gamma = \{\Gamma_t\}_{t\in[0,T]}$ where Γ_t is a continuous unit vector field defined on $\partial\mathcal{D}_t$ pointing inside $\mathcal{D}_t$ (in particular, nontangential to $\partial\mathcal{D}_t$). We say that a process $v \in \mathcal{V}$, $v_t = \int_0^t \gamma_s d\xi_s$, is a solution to the Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting at x_0 with reflection direction Γ if

(a) v is continuous,

(b) the process $X_t = x_0 + \sqrt{2}W_t + v_t$ satisfies $X_t \in \overline{\mathcal{D}_t}$ for $t \in [0,T]$ a.s.,

(c) for every $t \in [0,T]$

$$\xi_t = \int_0^t \mathbb{I}[X_s \in \partial\mathcal{D}_s,\ \gamma_s = \Gamma_s(X_s)]d\xi_s.$$

Now we shall give a slightly modified definition of a solution to the Skorokhod problem.

Definition 3.1.2. (compare [12] def. 2.5). Let $(x_0, 0) \in \overline{\mathcal{D}}$. Let $\Gamma = \{\Gamma_t(x),\ (x,t) \in \partial^*\mathcal{D}\}$ be a continuous unit vector field. We say that a process $v \in \mathcal{V}$, $v_t = \int_0^t \gamma_s d\xi_s$, is a solution to the modified Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting at x_0 with reflection direction Γ if

(a) the process $X_t = x_0 + \sqrt{2}W_t + v_t$ satisfies $(X_t, t) \in \overline{\mathcal{D}}$ for $t \in [0,T]$ a.s.,

(b) for every $t \in [0,T]$

$$\xi_t = \int_0^t \mathbb{I}[(X_s, s) \in \partial\mathcal{D},\ \gamma_s = \Gamma_s(X_s)]d\xi_s,$$

(c) with probability 1, for each $t \in [0,T]$, a possible jump of the process X at time t occurs on some interval $I \subset \mathbb{R}^n$ parallel to the vector field Γ_t on I (i.e. for all $x \in I$ $\Gamma_t(x)$ is parallel to I) and such that $I \times \{t\} \subseteq \partial^*\mathcal{D} \cap (\mathbb{R}^n \times \{t\})$. If X_t encounters such an interval I, it instantaneously jumps to its endpoint in the direction Γ_t on I.

The Main Theorem

Let

$$\mathcal{D} = \{(x,t) \in \mathbb{R}^n \times [0,T] : |Du(x,t)| < 1\}, \tag{3.4}$$

$$\mathcal{D}_t = \{x \in \mathbb{R}^n : |Du(x,t)| < 1\}, \tag{3.5}$$

$$\Gamma_t(\cdot) = \frac{-Du(\cdot,t)}{|Du(\cdot,t)|}. \tag{3.6}$$

Hölder continuity of $Du(x,t)$ implies that $\mathcal{D}$ is relatively open in $\mathbb{R}^n \times [0,T]$ and the vector field $\Gamma_t(x)$ is Hölder continuous jointly in (x,t). Note, by (3.3), that $\Gamma_t(x) = -Du(x,t)$ for $x \notin \mathcal{D}_t$.

The goal of this chapter is to prove the following theorem.

Theorem 3.1.3. *For every initial position $x_0 \in \mathbb{R}^n$ such that $(x_0, 0) \in \overline{\mathcal{D}}$, the optimal policy v^* for our singular stochastic control problem is a solution to the modified Skorokhod problem for the Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting from x_0 at time 0 with reflection direction Γ.*

Remark 3.1.4. An analogous statement is true for every initial time $t \in [0,T)$ and every initial position x_t instead of x_0, such that $(x_t, t) \in \overline{\mathcal{D}}$.

3.2 Existence of ε-optimal solutions of the Skorokhod problems

In this section we prove that there exist ε-optimal solutions of the Skorohod problems for domains approximating $\mathcal{D}$ in a suitable sense. We prove also that the optimal policy is the limit of these solutions.

Lemma 3.2.1. *For every m, k such that $0 \le m + 2k \le 7$, $0 \le k \le 3$, the function $D^m \frac{\partial^k}{\partial t^k} u$ is locally Hölder continuous with exponent α in $\mathcal{D} \cap (\mathbb{R}^n \times [0,T))$. Moreover, for each $t \in [0,T)$, $u(\cdot, t)$ is strictly convex on each convex subset of $\mathcal{D}_t$.*

Proof. In view of (3.3) the value function satisfies the equation

$$u(x,t) - u_t(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \mathcal{D}. \tag{3.7}$$

After substitutions $s = T - t$ and $w(x,s) = e^s \cdot u(x,t)$, the equation (3.7) takes the form

$$w_s(x,s) - \Delta w(x,s) = e^s \cdot f(x, T-s), \tag{3.8}$$

where $(x,s) \in \overleftarrow{\mathcal{D}} = \{(x,s) : (x, T-s) \in \mathcal{D}\}$. Let $X = (x,s) \in \overleftarrow{\mathcal{D}}$ be such that $s > 0$ and let $r > 0$, $\epsilon > 0$ be such that $Q = B(x,r) \times (s - \epsilon, s) \subseteq \overleftarrow{\mathcal{D}}$. Let $\mathcal{P}Q = \partial Q \setminus (B(x,r) \times \{s\})$ denote the parabolic boundary of Q. By Theorems 5.9, 5.10 and the proof of Lemma 3.25 in [17], there exists a function $\tilde{w} \in C^{2,1}(Q \setminus \mathcal{P}Q)$ continuous in $\overline{Q}$ and satisfying (3.8) in $Q \setminus \mathcal{P}Q$, such that $\tilde{w} = w$ on $\mathcal{P}Q$. Then Lemma 3.25 in [17] implies that $\tilde{w} \equiv w$ in $\overline{Q}$, so $w \in C^{2,1}(Q \setminus \mathcal{P}Q)$. Substituting back for w and s we obtain that $u \in C^{2,1}(\mathcal{D} \cap (\mathbb{R}^n \times [0,T))$, and hence it is a classical solution of (3.7). Thus, Theorem 11 and Corollary 1 on p. 74 of [9], together with the assumed regularity of f, imply the first assertion of the lemma.

Now we show that the second derivative $u_{\theta\theta} > 0$ in $\mathcal{D} \cap (\mathbb{R}^n \times [0,T))$ for each direction $\theta \in \mathbb{R}^n$. Indeed, differentiating (3.7) twice in the direction θ and taking $v = u_{\theta\theta}$, we get

$$v(x,t) - v_t(x,t) - \Delta v(x,t) = f_{\theta\theta}(x,t), \quad (x,t) \in \mathcal{D} \cap (\mathbb{R}^n \times [0,T)).$$

Substituting $s = T - t$ and $w(x,s) = v(x, T-s)$, we get

$$w(x,s) + w_s(x,s) - \Delta w(x,s) = f_{\theta\theta}(x, T-s), \quad (x,s) \in \overleftarrow{\mathcal{D}} \cap (\mathbb{R}^n \times (0,T]).$$

From (weak) convexity of u we know that $w(x,s) \ge 0$. If $w(\tilde{x}, s) = 0$ for some $(\tilde{x}, s) \in \overleftarrow{\mathcal{D}} \cap (\mathbb{R}^n \times (0,T])$, then w attains a nonpositive minimum in $\overleftarrow{\mathcal{D}} \cap (\mathbb{R}^n \times (0,T])$. Therefore, by the strong maximum principle (see, e.g., Theorem 2.7 in [17]), w vanishes in some closed cylinder $Q = \overline{B}(\tilde{x}, r) \times [s - \epsilon, s] \subseteq \overleftarrow{\mathcal{D}} \cap (\mathbb{R}^n \times (0,T])$, $r > 0$, $\epsilon > 0$. But then from strict convexity of f

$$0 = w_s(\tilde{x}, s) - \Delta w(\tilde{x}, s) = f_{\theta\theta}(\tilde{x}, T-s) > 0,$$

and we have a contradiction. Thus, $w(x,s) > 0$ for each $(x,s) \in \overleftarrow{\mathcal{D}} \cap (\mathbb{R}^n \times (0,T])$ and hence $u_{\theta\theta}(x,t) > 0$ for all $(x,t) \in \mathcal{D} \cap (\mathbb{R}^n \times [0,T))$ and $\theta \in \mathbb{R}^n$. □

Lemma 3.2.2. *Let $t \in [0, T)$. Then the set $\mathcal{D}_t$ is*

(a) nonempty,

(b) open,

(c) connected.

Moreover, there exists a unique $\tilde{x} = \tilde{x}(t) \in \mathcal{D}_t$ such that $|Du(\tilde{x}, t)| = 0$.

Proof. Fix $t \in [0, T)$.

Proof of (a). In view of Theorem 2.2.1 we know that u is a convex function with respect to the space variable. Suppose that

$$\lim_{|x|\to\infty} u(x, t) = \infty. \tag{3.9}$$

Then the function $u(\cdot, t)$ attains the global minimum in some point $x_0 \in \mathbb{R}^n$, so $Du(x_0, t) = 0$ and $x_0 \in \mathcal{D}_t$.

We must prove (3.9). Let $M > 1$ be such that $\frac{1}{2}\tilde{C}_0 M^p \le c_0 M^p - C_0$. Let $|x| = 3M$ and let τ be the time of the first exit of the Brownian motion $\sqrt{2}W_t$ starting at x from the ball $B(x, M)$. Without loss of generality we can take M so big that $P(\mathcal{A}) > \frac{3}{4}$, where $\mathcal{A} = \{\tau > T - t\}$. For a given $v = (N, \xi) \in \mathcal{V}$ we define an event $\mathcal{A}_v = \{\xi(T - t) \le M\}$.

First we consider $v \in \mathcal{V}$ such that $P(\mathcal{A}_v) \le \frac{1}{2}$, so $P(\mathcal{A}_v^C) \ge \frac{1}{2}$. Then from (3.2) we get

$$J_{xt}(v) \ge \mathbb{E}\int_t^T e^{-(s-t)} d\xi(s-t) \ge e^{-T}\mathbb{E}\xi(T-t) \ge e^{-T} \cdot P(\mathcal{A}_v^C) \cdot M \ge \frac{1}{2}e^{-T} \cdot M. \tag{3.10}$$

Now we consider $v \in \mathcal{V}$ such that $P(\mathcal{A}_v) > \frac{1}{2}$. Then $P(\mathcal{A}_v \cap \mathcal{A}) > \frac{1}{4}$ and we have

$$J_{xt}(v) \ge \mathbb{E}\int_t^{(t+\tau)\wedge T} f(y_{xt}(s), s) e^{-(s-t)} \cdot \mathbb{I}_{\mathcal{A}_v} \cdot \mathbb{I}_{\mathcal{A}} ds.$$

In view of the definitions of τ and $\mathcal{A}_v$ we know that $y_{xt}(s) \in B(x, 2M)$ on $\mathcal{A}_v$ for $s \in [t, (t+\tau) \wedge T]$, so $|y_{xt}(s)| \ge M$. From (1.7) we get

$$\begin{aligned} J_{xt}(v) &\ge \mathbb{E}\int_t^{(t+\tau)\wedge T} \frac{1}{2}\tilde{C}_0 M^p e^{-T} \mathbb{I}_{\mathcal{A}_v}\mathbb{I}_{\mathcal{A}} ds = \frac{1}{2}\tilde{C}_0 e^{-T} M^p \cdot \mathbb{E}\Big(\big(\tau \wedge (T-t)\big)\mathbb{I}_{\mathcal{A}_v}\mathbb{I}_{\mathcal{A}}\Big) = \\ &= \frac{1}{2}\tilde{C}_0 e^{-T} M^p (T-t)\mathbb{E}\mathbb{I}_{\mathcal{A}_v}\mathbb{I}_{\mathcal{A}} = \frac{1}{2}\tilde{C}_0 e^{-T} M^p (T-t) P(\mathcal{A}_v \cap \mathcal{A}) > \frac{1}{8}\tilde{C}_0 e^{-T} M^p (T-t). \end{aligned} \tag{3.11}$$

In view of (3.11), (3.10) and (1.4) taking $M \to \infty$ we get (3.9).

Proof of (b). The set $\mathcal{D}_t$ is open because of continuity of the function $|Du(\cdot, t)|$ and the fact that for a continuous function the inverse image of an open set is open.

Proof of (c). We show that $\mathcal{D}_t$ is a pathwise-connected set using a gradient flow argument (see [24], section 7).

In view of convexity of u (Lemma 3.2.1) and the proof of (a) we conclude that there exists $\tilde{x} \in \mathbb{R}^n$ which is the unique minimizer of $u(\cdot, t)$ and, in particular, $|Du(\tilde{x}, t)| = 0$. From this and strict convexity of u in each convex subset of $\mathcal{D}_t$ we can choose $\delta, \mu > 0$ such that

$$B(\tilde{x}, 2\delta) \subset \mathcal{D}_t,$$
$$D^2u(x,t)y \cdot y \geq \mu|y|^2 \quad \text{for all } x \in B(\tilde{x}, 2\delta),\ y \in \mathbb{R}^n,$$
$$\mu \leq |Du(x,t)|^2 \leq \frac{1}{2} \quad \text{for all } x \in \partial B(\tilde{x}, \delta),$$
$$Du(\tilde{x} + \delta\theta, t) \cdot \theta \geq \mu \quad \text{for all } \theta \in \partial B(0,1).$$

For $\theta \in \partial B(0,1)$ we define *the gradient flow* $\psi(s,\theta)$ to be unique solution to the differential equation

$$\frac{d}{ds}\psi(s,\theta) = Du\big(\psi(s,\theta), t\big),\ s \geq 0, \tag{3.12}$$

with the initial condition $\psi(0,\theta) = \tilde{x} + \delta\theta$. As in [24], Theorem 7.1, we can prove that the map ψ is a homeomorphism from $[0,\infty) \times \partial B(0,1)$ onto $\mathbb{R}^n \setminus B(\tilde{x}, \delta)$.

Now we take two points $x, y \in \mathcal{D}_t$. We will show that there exists a continuous path, contained in $\mathcal{D}_t$, connecting these points.

If $x, y \in \overline{B}(\tilde{x}, \delta)$ then we can take as this path a line segment from x to y.

If $x \in \overline{B}(\tilde{x}, \delta), y \notin \overline{B}(\tilde{x}, \delta)$ then there exist $s_y > 0,\ \theta_y \in \partial B(0,1)$ such that $\psi(s_y, \theta_y) = y$ and

$$\phi(s) = \begin{cases} (1-s)x + s(\tilde{x} + \delta\theta_y), & s \in [0,1] \\ \psi(s-1, \theta_y), & s \in [1, s_y + 1] \end{cases}$$

is a path connecting x to y.

If $x, y \notin \overline{B}(\tilde{x}, \delta)$ then there exist $s_x, s_y > 0, \theta_x,\ \theta_y \in \partial B(0,1)$ such that $\psi(s_x, \theta_x) = x$, $\psi(s_y, \theta_y) = y$ and

$$\phi(s) = \begin{cases} \psi(s_x - s, \theta_x), & s \in [0, s_x] \\ (1 + s_x - s)(\tilde{x} + \delta\theta_x) + (s - s_x)(\tilde{x} + \delta\theta_y), & s \in [s_x, s_x + 1] \\ \psi(s - s_x - 1, \theta_y), & s \in [s_x + 1, s_x + 1 + s_y] \end{cases}$$

is a path connecting x to y.

To complete the proof of (c) we must prove that the above paths are contained in $\mathcal{D}_t$. It is sufficient to show that $\big|Du\big(\psi(s,\theta),t\big)\big|^2$ is a nondecreasing function of s. Indeed, because of convexity of $u(\cdot, t)$, as long as $\psi(s,\theta) \in \mathcal{D}_t$, we have

$$\begin{aligned} \frac{d}{ds}\big|Du\big(\psi(s,\theta),t\big)\big|^2 &= 2D^2u\big(\psi(s,\theta),t\big)\ Du\big(\psi(s,\theta),t\big) \cdot \frac{d}{ds}\psi(s,\theta) = \\ &= 2Du\big(\psi(s,\theta),t\big)^T\ D^2u\big(\psi(s,\theta),t\big)\ Du\big(\psi(s,\theta),t\big) > 0. \end{aligned} \tag{3.13}$$

If the path, say $\psi(s,\theta_y)$, $0 \leq s \leq s_y$, is not contained in $\mathcal{D}_t$, then on this path there must exist a point which is the last one outside $\mathcal{D}_t$. More precisely, there exist $s_0 > 0$, $z_0 \in \partial\mathcal{D}_t$ such that $\psi(s_0,\theta) = z_0$ and $\psi(s,\theta) \in \mathcal{D}_t$ for $s > s_0$. But $\big|Du\big(\psi(s_0,\theta),t\big)\big|^2 = 1$ and $\big|Du\big(\psi(s,\theta),t\big)\big|^2 < 1$ for $s \in (s_0, s_y]$, which contradicts (3.13). $\square$

By definition, $u(x,T) = 0$ for each $x \in \mathbb{R}^n$, so $\mathcal{D}_T = \mathbb{R}^n$. Hence, $\mathcal{D}_T$ satisfies all the assertions of Lemma 3.2.2, except for the last one.

Corollary 3.2.3. *The set $\mathcal{D}$ is connected.*

Proof. For every $t \in [0,T]$, Hölder continuity of Du and the equality $|Du(\tilde{x}(t),t)| = 0$ imply the existence of $\epsilon = \epsilon(t)$ such that $|Du(\tilde{x}(t),\tilde{t})| \leq 1/2$ for every $\tilde{t} \in (t-\epsilon, t+\epsilon) \cap [0,T]$. This, together with the proof of Lemma 3.2.2 (c), shows that the set $A(t) = \mathcal{D} \cap (\mathbb{R}^n \times (t-\epsilon, t+\epsilon))$ is arcwise connected. The interval $[0,T]$ is compact, so there exists a finite subcovering $(t_i - \epsilon(t_i), t_i + \epsilon(t_i))$, $i = 1, ..., m$, of $[0,T]$. Without loss of generality we may assume that $t_1 < ... < t_m$ and $t_i + \epsilon(t_i) > t_{i+1} - \epsilon(t_{i+1})$, $i = 1, ..., m-1$. Consequently, $\mathcal{D} = \bigcup_{i=1}^m A(t_i)$, where $A(t_i) \cap A(t_{i+1}) \neq \emptyset$, $i = 1, ..., m-1$, and each $A(t_i)$ is arcwise connected. This clearly implies that $\mathcal{D}$ itself is arcwise connected, and hence it is connected. □

Define for $\varepsilon \in (0,1)$

$$\mathcal{D}^\varepsilon = \Big\{(x,t) \in \mathbb{R}^n \times [0,T] : \ |Du(x,t)| < 1 - \varepsilon\Big\}, \tag{3.14}$$

$$\mathcal{D}^\varepsilon_t = \Big\{x \in \mathbb{R}^n : \ |Du(x,t)| < 1 - \varepsilon\Big\}, \quad t \in [0,T]. \tag{3.15}$$

Remark 3.2.4. In view of the proof of Lemma 3.2.2, the sets $\mathcal{D}^\varepsilon_t$ are nonempty, open and connected. Hölder continuity of Du and the proof of Corollary 3.2.3 imply that the sets $\mathcal{D}^\varepsilon$ are relatively open and connected.

We see that

$$\mathcal{D}^\varepsilon = \bigcup_{t \in [0,T]} \Big(\mathcal{D}^\varepsilon_t \times \{t\}\Big)$$

and, by regularity of Du on $\mathcal{D} \supset \overline{\mathcal{D}^\varepsilon}$,

$$\partial^* \mathcal{D}^\varepsilon = \bigcup_{t \in [0,T]} \Big(\partial \mathcal{D}^\varepsilon_t \times \{t\}\Big).$$

We want to make sure that there exists the solution of the Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}^\varepsilon}$ starting at $x_0 \in \mathcal{D}^\varepsilon_0$ with reflection direction $\Gamma_t(\cdot) = \frac{-Du(\cdot,t)}{|Du(\cdot,t)|}$.

To this aim we will apply the Theorem 4.3 from [16]. The idea (see the proof of Theorem 2.1 from [2]) is to redefine our n-dimensional problem as an $(n+1)$-dimensional Skorokhod problem. More precisely, we want to use Theorem 4.3 from [16] for the space-time Brownian motion $(\sqrt{2}W_t, t)$. To do this, we must be sure that our domain and vector field are suitably smooth and bounded.

For $\eta \in (0, \frac{T}{2})$ and $M > 0$ we define

$$A^{\eta,M} = \inf_{\substack{M \leq |x| \leq M+1 \\ 0 \leq t \leq T-\eta}} |Du(x,t)|^2. \tag{3.16}$$

Lemma 3.2.5. *For given $\eta \in (0, \frac{T}{2})$ and for $M > 0$ large enough, we have*

$$A^{\eta,M} > 0. \tag{3.17}$$

Proof. In view of the proof of Lemma 3.2.2 (a), the convergence (3.9) is uniform with respect to $t \in [0, T-\eta]$ (see (3.10)-(3.11)). Thus, for any $A > 0$ there exists $B > 0$ such that for each $t \in [0, T-\eta]$ if $u(x,t) \leq A$ then $|x| \leq B$. Let $\tilde{x}(t)$ denote the unique minimizer of $u(\cdot, t)$. From estimates of the value function (see Theorem 2.2.1) $u(x,t) \leq A(1+|x|^p)$ for a suitable constant $A > 0$. So $u(\tilde{x}(t), t) \leq u(0,t) \leq A$. Hence there exists $B > 0$ such that $|\tilde{x}(t)| \leq B$ for each $t \in [0, T-\eta]$. Taking M greater than B, we get positivity of the infimum. □

Let $\varepsilon \in (0,1)$, $\eta \in (0, \frac{T}{2})$ be fixed. Next we choose $M > 0$ so big that the condition (3.17) holds. From the HJB equation (3.3) the value function u satisfies the parabolic PDE $u - u_t - \Delta u = f$ in $\overline{\mathcal{D}^\varepsilon}$. We note that $D(|Du(x,t)|^2) = 2D^2u(x,t)Du(x,t) \neq 0$ on $\partial\mathcal{D}^\varepsilon$. So from Lemma 3.2.1 we see that $\partial^*\mathcal{D}^\varepsilon$ is C^3-smooth.

We define

$$\mathcal{D}^{\varepsilon,\eta} = \bigcup_{t \in [0, T-\eta]} \Big(\mathcal{D}^\varepsilon_t \times \{t\}\Big) = \mathcal{D}^\varepsilon \cap \Big(\mathbb{R}^n \times [0, T-\eta]\Big) \tag{3.18}$$

and bounded sets

$$\mathcal{D}^{\varepsilon,\eta,M} = \mathcal{D}^{\varepsilon,\eta} \cap \Big(\overline{B(0,M)} \times [0,T]\Big). \tag{3.19}$$

Unfortunately, $\partial^*\mathcal{D}^{\varepsilon,\eta,M}$ is not necessarily smooth. To get a smooth bounded set, let $\mathcal{R}^{\varepsilon,\eta,M}$ be the region bounded by the hypersurface

$$\mathcal{S}^{\varepsilon,\eta,M} = \big\{(x,t) \in \mathbb{R}^n \times [0, T-\eta] : |x| \leq M+1,\ g^{\varepsilon,\eta,M}(x,t) = 1\big\}, \tag{3.20}$$

where

$$g^{\varepsilon,\eta,M}(x,t) = \frac{|Du(x,t)|^2}{(1-\varepsilon)^2}\phi^{\varepsilon,\eta,M}\big(|x|^2\big). \tag{3.21}$$

Here, $\phi^{\varepsilon,\eta,M} : [0, (M+1)^2] \to \mathbb{R}$ is a smooth function satisfying $\phi^{\varepsilon,\eta,M}(a) = 1$ for $a \in [0, M^2]$ and strictly increasing on $\big(M^2, (M+1)^2\big)$ with

$$\phi^{\varepsilon,\eta,M}\big((M+1)^2\big) = 2\frac{(1-\varepsilon)^2}{A^{\eta,M}}.$$

We note that for $|x| \leq M$, $t \in [0, T-\eta]$ we have

$$g^{\varepsilon,\eta,M}(x,t) = 1 \ \Leftrightarrow\ |Du(x,t)|^2 = (1-\varepsilon)^2 \ \Leftrightarrow\ (x,t) \in \partial^*\mathcal{D}^\varepsilon. \tag{3.22}$$

Hence the lateral boundary of $\mathcal{R}^{\varepsilon,\eta,M}$ coincides with the lateral boundary of $\mathcal{D}^\varepsilon$ for $|x| \leq M$. The set $\mathcal{R}^{\varepsilon,\eta,M}$ satisfies

$$\mathcal{D}^{\varepsilon,\eta,M} \subset \mathcal{R}^{\varepsilon,\eta,M} \subset \mathcal{D}^{\varepsilon,\eta,M+1} \text{ and } \partial^*\mathcal{R}^{\varepsilon,\eta,M} \text{ is } C^3\text{-smooth} \tag{3.23}$$

(the latter assumption appears in Remark 3.1 in [16]).

Let $\Gamma^{\varepsilon,\eta,M} : \partial^*\mathcal{R}^{\varepsilon,\eta,M} \to \mathbb{R}^{n+1}$ be given by

$$\Gamma^{\varepsilon,\eta,M}(x,t) = \Big(\Gamma_t(x), 0\Big). \tag{3.24}$$

Then $\Gamma^{\varepsilon,\eta,M}$ is C_b^2.

Lemma 3.2.6. *There exists $c > 0$ such that for every $(x,t) \in \partial^*\mathcal{R}^{\varepsilon,\eta,M}$*

$$\Gamma^{\varepsilon,\eta,M}(x,t) \circ n(x,t) \geq c, \tag{3.25}$$

where $n(x,t)$ is the unit inward normal vector to $\partial^*\mathcal{R}^{\varepsilon,\eta,M}$ at (x,t).

Proof. Because of smoothness of $\Gamma^{\varepsilon,\eta,M}$ on a smooth set $\partial^*\mathcal{R}^{\varepsilon,\eta,M}$ and boundedness of $\mathcal{R}^{\varepsilon,\eta,M}$ it is sufficient to prove that $\Gamma^{\varepsilon,\eta,M}(x,t)$ is not perpendicular to $n(x,t)$ at any $(x,t) \in \partial\mathcal{R}^{\varepsilon,\eta,M}$. Moreover, the last coordinate of $\Gamma^{\varepsilon,\eta,M}$ is equal to 0, so it is sufficient to show that $\Gamma_t(x)$ is not tangential to $\partial\mathcal{R}_t^{\varepsilon,\eta,M}$ at any $(x,t) \in \partial^*\mathcal{R}^{\varepsilon,\eta,M}$, where

$$\mathcal{R}_t^{\varepsilon,\eta,M} = \{x \in \mathbb{R}^n : (x,t) \in \mathcal{R}^{\varepsilon,\eta,M}\}. \tag{3.26}$$

Fix $t \in (0, T-\eta)$. The value function $u(\cdot,t)$ is stricly convex in every convex subset of $\overline{\mathcal{R}_t^{\varepsilon,\eta,M}}$ (see Lemma 3.2.1). Thus, by compactness, there exists $c > 0$ such that

$$\inf_{x \in \overline{\mathcal{R}_t^{\varepsilon,\eta,M}}} \quad \inf_{\theta \in \mathbb{R}^n, |\theta|=1} \quad D^2u(x,t)\,\theta \circ \theta \ \geq c. \tag{3.27}$$

For $x \in \partial\mathcal{R}_t^{\varepsilon,\eta,M}$, $|x| > M$, we note that

$$Dg^{\varepsilon,\eta,M}(x,t) = 2\frac{\phi^{\varepsilon,\eta,M}\big(|x|^2\big)}{(1-\varepsilon)^2} D^2u(x,t)Du(x,t) + 2\frac{|Du(x,t)|^2}{(1-\varepsilon)^2}\left(\phi^{\varepsilon,\eta,M}\right)'\left(|x|^2\right) x. \tag{3.28}$$

Consider the inner product $Dg^{\varepsilon,\eta,M}(x,t) \circ Du(x,t)$. We will show that it is positive on $\partial^*\mathcal{R}^{\varepsilon,\eta,M}$. For $|x| > M$ we know that $\phi^{\varepsilon,\eta,M}$ and $|Du(x,t)|$ are positive and $\left(\phi^{\varepsilon,\eta,M}\right)'$ is nonnegative. In view of (3.27)

$$D^2u(x,t)Du(x,t) \circ Du(x,t) > 0.$$

Finally, we observe that for $|x| > M$ we have $Du(x,t) \circ x > 0$. Indeed, in view of the proof of Lemma 3.2.5, there exist $A, B > 0$ such that $u(0,t) \leq A$ and if $u(x,t) \leq A$ then $|x| \leq B < M$. Taking $|x| > M$ we have $u(x,t) > A$. Suppose that $Du(x,t) \circ x \leq 0$. Then, by convexity of u, for $\nu = \frac{x}{|x|}$ we get

$$A \geq u(0,t) \geq -\frac{\partial u}{\partial \nu}(x,t)||x|| + u(x,t) \geq u(x,t) > A$$

and this is a contradiction. Hence, by (3.28)

$$\begin{aligned} Dg^{\varepsilon,\eta,M}(x,t) \circ Du(x,t) = 2\frac{\phi^{\varepsilon,\eta,M}\big(|x|^2\big)}{(1-\varepsilon)^2} D^2u(x,t)Du(x,t) \circ Du(x,t) + \\ + 2\frac{|Du(x,t)|^2}{(1-\varepsilon)^2} x \circ Du(x,t) > 0. \end{aligned}$$

In the case of $x \in \partial\mathcal{R}_t^{\varepsilon,\eta,M}$, $|x| \le M$ we can mimic the proof of Lemma 2.7 from [12]. Let $w(y) = |Du(y,t)|^2$. In view of (3.22) and the strict convexity of u in $\mathcal{D}^\varepsilon$ we have $Dw(y) = 2D^2u(y,t)\,Du(y,t) \neq 0$ for $y \in \partial\mathcal{R}_t^{\varepsilon,\eta,M}$. Set $\theta = \theta(x) = \Gamma_t(x)$. Then $w(x) = \left|\frac{\partial u}{\partial\theta}(x,t)\right|^2$ and $\frac{\partial^2 u}{\partial\theta^2}(y,t) \ge \frac{1}{2}c$ for some constant $c > 0$ and y near x. So for $\lambda > 0$ small enough we have

$$\sqrt{w(x+\lambda\theta)} = |Du(x+\lambda\theta,t)| \ge \frac{\partial u}{\partial\theta}(x+\lambda\theta,t) \ge \frac{\partial u}{\partial\theta}(x,t) + \lambda\frac{c}{2} = \sqrt{w(x)} + \lambda\frac{c}{2}. \quad (3.29)$$

Suppose that $\theta(x)$ at $x \in \partial\mathcal{R}_t^{\varepsilon,\eta,M}$ is tangential to $\partial\mathcal{R}_t^{\varepsilon,\eta,M}$ and $|x| \le M$. Then $x \in \partial\mathcal{D}_t^\varepsilon$, $|Du(x,t)| = 1-\varepsilon$ and we can write $w(x+\lambda\theta) = w(x) + O(\lambda^2)$ (see (3.22)) which contradicts (3.29). □

Let $x_0 \in \mathcal{D}_0$. We assume that $\varepsilon > 0$ is so small that $x_0 \in \mathcal{D}_0^\varepsilon$. We want to use Theorem 4.3 from [16] to conclude that there exists the unique solution of the Skorokhod problem for the space-time Brownian motion $\{(x_0+\sqrt{2}W_t, t)\}_{t\in[0,T-\eta]}$ in $\overline{\mathcal{R}^{\varepsilon,\eta,M}}$ with reflection direction $\Gamma^{\varepsilon,\eta,M}$. We need the smoothness of $\overline{\mathcal{R}^{\varepsilon,\eta,M}}$, the boundedness and smoothness of $\Gamma^{\varepsilon,\eta,M}$ and the condition (3.25). Because the last coordinate of our space-time Brownian motion is the time coordinate, the smoothness of $\partial^*\mathcal{R}^{\varepsilon,\eta,M}$ is sufficient. From the definition of $\mathcal{R}^{\varepsilon,\eta,M}$ and Lemma 3.2.6 we see that all needed assumptions are satisfied to conclude that the unique solution of the above Skorokhod problem exists. We denote this solution by $(v^{\varepsilon,\eta,M}, 0)$ and corresponding state process by $X^{\varepsilon,\eta,M}$, i.e.

$$X_t^{\varepsilon,\eta,M} = x_0 + \sqrt{2}W_t + v_t^{\varepsilon,\eta,M},\ t \in [0, T-\eta] \quad (3.30)$$

where the decomposition (1.1) of $v^{\varepsilon,\eta,M}$ has the form

$$v_t^{\varepsilon,\eta,M} = \int_0^t \Gamma_s(X_s^{\varepsilon,\eta,M})d\xi_s^{\varepsilon,\eta,M}. \quad (3.31)$$

Now we want to get a controlled process X^ε on the entire time interval $[0,T]$. To this end we take a decreasing sequence $\{\eta_k\}$ convergent to 0 and the corresponding increasing sequence of natural numbers $\{M_k\}$ convergent to $+\infty$ and such that pairs (η_k, M_k) satisfy the condition (3.17).

Denote

$$X^{\varepsilon,k} = X^{\varepsilon,\eta_k,M_k}, \quad v^{\varepsilon,k} = v^{\varepsilon,\eta_k,M_k}, \quad \xi^{\varepsilon,k} = \xi^{\varepsilon,\eta_k,M_k}.$$

Set the random time

$$\tau^{\varepsilon,k} = \inf_{t\in[0,T-\eta_k]} \left\{ \left|X_t^{\varepsilon,k}\right| = M_k \right\} \ \wedge\ (T-\eta_k).$$

The sequence $\{\tau^{\varepsilon,k}\}$, $k \ge 1$ is increasing and $X_t^{\varepsilon,k} = X_t^{\varepsilon,k+1}$ for $t \in [0,\tau^{\varepsilon,k}]$ a.s., because of uniqueness of the solution of the Skorokhod problem (Theorem 4.3 from [16]). We denote

$$X_t^\varepsilon = X_t^{\varepsilon,k}, \quad v_t^\varepsilon = v_t^{\varepsilon,k}, \quad \xi_t^\varepsilon = \xi_t^{\varepsilon,k} \text{ on } [0,\tau^{\varepsilon,k}]. \quad (3.32)$$

So X^ε, v^ε, ξ^ε are defined on $[0, \lim_{k\to\infty}\tau^{\varepsilon,k})$.

Lemma 3.2.7. $\lim_{k\to\infty} \tau^{\varepsilon,k} = T$ *a.s.*.

Proof. We use the Itô's formula for the process $X^{\varepsilon,k}$ on the time interval $[0,\tau^{\varepsilon,k}]$ and the function $e^{-t}u(x,t)$. Then

$$e^{-\tau^{\varepsilon,k}}\, u(X^{\varepsilon,k}_{\tau^{\varepsilon,k}},\, \tau^{\varepsilon,k}) =$$

$$= u(x_0,0) + \int_0^{\tau^{\varepsilon,k}} e^{-t}\left(u_t(X^{\varepsilon,k}_t,t) - u(X^{\varepsilon,k}_t,t) + \Delta u(X^{\varepsilon,k}_t,t)\right)dt +$$

$$+\sqrt{2}\sum_{i=1}^{n}\int_0^{\tau^{\varepsilon,k}} e^{-t}u_{x_i}(X^{\varepsilon,k}_t,t)dW^{(i)}_t + \int_0^{\tau^{\varepsilon,k}} e^{-t}Du(X^{\varepsilon,k}_t,t)\circ\Gamma_t(X^{\varepsilon,k}_t)d\xi^{\varepsilon,k}_t.$$

In view of (3.3) and (3.6) taking the expectation on both sides, we get

$$u(x_0,0) = \mathbb{E}\left[e^{-\tau^{\varepsilon,k}}\, u(X^{\varepsilon,k}_{\tau^{\varepsilon,k}},\, \tau^{\varepsilon,k})\right] +$$

$$+\mathbb{E}\int_0^{\tau^{\varepsilon,k}} e^{-t}f(X^{\varepsilon,k}_t,t)dt + \mathbb{E}\int_0^{\tau^{\varepsilon,k}} e^{-t}|Du(X^{\varepsilon,k}_t,t)|d\xi^{\varepsilon,k}_t.$$

From nonnegativity of f and u we have

$$u(x_0,0) \geq \mathbb{E}\int_0^{\tau^{\varepsilon,k}} e^{-t}|Du(X^{\varepsilon,k}_t,t)|d\xi^{\varepsilon,k}_t \geq e^{-T}(1-\varepsilon)\mathbb{E}\xi^{\varepsilon,k}_{\tau^{\varepsilon,k}}.$$

The last equality is implied by the definitions $X^{\varepsilon,k}$, $\xi^{\varepsilon,k}$ and $\tau^{\varepsilon,k}$.

Hence

$$\mathbb{E}\xi^{\varepsilon,k}_{\tau^{\varepsilon,k}} \leq \frac{e^T}{1-\varepsilon}u(x_0,0).$$

So the process ξ^ε has a.s. bounded variation on $[0,\tau^{\varepsilon,k}]$ uniformly for each $k>0$. Hence X^ε has no explosion on $[0,T-\eta_k]$ for each $k>0$ and hence $\lim_{k\to\infty}\tau^{\varepsilon,k} = T$ a.s.. □

Setting $X^\varepsilon_T = X^\varepsilon_{T-}$ and $v^\varepsilon_T = v^\varepsilon_{T-}$, we get the process $X^\varepsilon_t = x_0 + \sqrt{2}W_t + v^\varepsilon_t$ well-defined for all $t\in[0,T]$. Moreover, the process X^ε is the Brownian motion $(\sqrt{2}W_t)$ reflected at the boundary $\partial\mathcal{D}^\varepsilon$ in the direction Γ_t.

Lemma 3.2.8.

$$\lim_{\varepsilon\to 0^+} J_{x_0 0}(v^\varepsilon) = u(x_0,0). \tag{3.33}$$

Proof. We use the Itô's formula for the semimartingale X^ε on the time interval $[0,T]$ and the function $e^{-t}u(x,t)$. Because X^ε is the Brownian motion reflected at the boundary of $\partial\mathcal{D}^\varepsilon$ in the direction $\Gamma_t(\cdot)$ and the HJB equation (3.3) holds, we get (see the proof of Lemma 3.2.7)

$$u(x_0,0) = \mathbb{E}\left\{e^{-T}u(X^\varepsilon_T,T) + \int_0^T e^{-t}f(X^\varepsilon_t,t)dt + \int_0^T e^{-t}|Du(X^\varepsilon_t,t)|d\xi^\varepsilon_t\right\} =$$

$$= 0 + \mathbb{E}\left\{\int_0^T e^{-t} f(X_t^\varepsilon, t)dt + \int_0^T e^{-t}(1-\varepsilon)d\xi_t^\varepsilon\right\} \geq$$

$$\geq (1-\varepsilon)\mathbb{E}\left\{\int_0^T e^{-t} f(X_t^\varepsilon, t)dt + \int_0^T e^{-t}d\xi_t^\varepsilon\right\} = (1-\varepsilon)J_{x_0 0}(v^\varepsilon).$$

Taking $\varepsilon \to 0^+$ we get (3.33). □

Let m_0 be the measure on $\big([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}\big)$ equal to the product of the Lebesgue's measure and P.

Lemma 3.2.9. *There exists a sequence $\{\varepsilon_k\}$ decreasing to 0 such that $\{v^{\varepsilon_k}(t,\omega)\}$ is convergent to the optimal policy $v^*(t,\omega)$ for m_0-almost all $(t,\omega) \in [0,T] \times \Omega$.*

Proof. Using the arguments from the proof of Theorem 8 from [18] we can show that for any $\mu > 0$

$$\mathbb{E}\int_0^T \mathbb{I}\left[\left|X_t^{\varepsilon_k} - X_t^{\varepsilon_m}\right| > \mu\right] \to 0, \text{ as } k, m \to \infty.$$

Hence $\{X^{\varepsilon_k}\}$ and $\{v^{\varepsilon_k}\}$ are Cauchy sequences with respect to convergence in the measure m_0. So there exists a subsequence, still denoted by $\{v^{\varepsilon_k}\}$, which is convergent m_0-a.e. to some process v. Following the proof of Theorem 2.4.4, we construct the left-continuous, adapted process v^* with $v_0^* = 0$ a.s. and such that $v = v^*$ m_0-a.e. and, moreover, the total variation ξ^* of v^* satisfies for every $0 \leq s_1 \leq s_2 \leq T$

$$\xi_{s_2}^* - \xi_{s_1}^* \leq \liminf_{k\to\infty}\left(\xi_{s_2}^{\varepsilon_k} - \xi_{s_1}^{\varepsilon_k}\right) \quad a.s.. \tag{3.34}$$

In particular,

$$\mathbb{E}\int_0^T e^{-s} d\xi_s^* \leq \liminf_{k\to\infty} \mathbb{E}\int_0^T e^{-s} d\xi_s^{\varepsilon_k},$$

so $v^* \in \mathcal{V}$. Let $X_t^* = x_0 + \sqrt{2}W_t + v_t^*$, $t \in [0,T]$. Then $X^{\varepsilon_k} \to X^*$ m_0-a.e.. Thus, using the Fatou's lemma

$$\mathbb{E}\int_0^T e^{-s} f(X^*(s), s)ds \leq \liminf_{k\to\infty} \mathbb{E}\int_0^T e^{-s} f(X^{\varepsilon_k}(s), s)ds.$$

Hence $J_{x_0 0}(v^*) \leq u(x_0, 0)$. The opposite inequality holds in view of the definition of the value function (1.4). □

Remark 3.2.10. In Theorem 2.4.4 the state process has, in general, a nonzero drift. Therefore the Mazur theorem has been invoked to create a sequence $\{v_k^\varepsilon\}$ convergent to v^* m_0-a.e.. In our problem the state process (3.1) has no drift which allows for a simplification of the argument.

3.3 The optimal policy keeps the process in $\overline{\mathcal{D}}$

Theorem 3.3.1. *Let $x_0 \in \mathcal{D}_0$ and let X^* denote the optimally controlled state process with $X_0^* = x_0$, i.e. $X_t^* = x_0 + \sqrt{2}W_t + v_t^*$. For all $t \in [0,T]$ we have $(X_t^*, t) \in \overline{\mathcal{D}}$ a.s..*

Proof. We mimic the proof of Lemma 2.9 from [12]. Let $\{\varepsilon_k\}$ be the sequence satisfying Lemma 3.2.9. Define the events

$$A = \left\{\omega \in \Omega : \left(X_t^{\varepsilon_k}(\omega), t\right) \in \overline{\mathcal{D}}^{\varepsilon_k} \text{ for all } k > 0,\ t \in [0,T]\right\},$$

$$B = \left\{\omega \in \Omega : X_t^{\varepsilon_k} \to X_t^* \text{ for a.e. } t \in [0,T]\right\}.$$

From the definition of v^ε we know that $P(A) = 1$. For $\omega \in A$ and every $k > 0$, $t \in [0,T]$ we have $\left(X_t^{\varepsilon_k}(\omega), t\right) \in \overline{\mathcal{D}}^{\varepsilon_k} \subset \overline{\mathcal{D}}$. From Lemma 3.2.9 we get $P(B) = 1$. Hence for $\omega \in A \cap B$ the state process $\left(X_t^*(\omega), t\right) \in \overline{\mathcal{D}}$ for a.e. $t \in [0,T]$. But X^* is left continuous, so $\left(X_t^*(\omega), t\right) \in \overline{\mathcal{D}}$ for all $t \in [0,T]$. □

Remark 3.3.2. The statement of Theorem 3.3.1 is also true for $x_0 \in \partial\mathcal{D}_0$ (compare the remark in section 2.3 from [12]).

Proof. Let $\{x_k\}$ be a sequence in $\mathcal{D}_0$ convergent to x_0. Denote by v^k the optimal policy for our problem starting from x_k at time zero. We want to control the process starting at x_0. Policies

$$\bar{v}^k = v^k + x_k - x_0$$

jump at time 0 from the starting point x_0 to x_k and then follow v^k. Because $x_k \to x_0$ and u is continuous,

$$J_{x_0 0}(\bar{v}^k) = |x^k - x^0| + u(x_k, 0) \to u(x_0, 0).$$

Using the proof of Lemma 3.2.9 we see that there exists a subsequence of policies (still denoted by $\{\bar{v}^k\}$) convergent m_0-a.e. to v^0 - the optimal policy for our problem starting at x_0. In view of Theorem 3.3.1 the process $(X_t^k, t) \in \overline{\mathcal{D}}$ for all $t \in [0,T]$ a.s., where

$$X_t^k = x_0 + \sqrt{2}W_t + \bar{v}_t^k = x_k + \sqrt{2}W_t + v_t^k.$$

Repeating the proof of the last theorem we see that also the optimally control process for starting point x_0

$$X_t^* = x_0 + \sqrt{2}W_t + v_t^0$$

satisfies $(X_t^*, t) \in \overline{\mathcal{D}}$ for all $t \in [0,T]$ a.s.. □

3.4 The optimal policy acts only on $\partial^*\mathcal{D}$ and its push direction is $-Du$

Theorem 3.4.1. *The optimal policy $v_t^* = \int_0^t \gamma_s^* d\xi_s^*$ acts only on $\partial^*\mathcal{D}$, and its push direction is $-Du$, for every $t \geq 0$ a.s..*

Proof. We know that $(X_t^*, t) \in \overline{\mathcal{D}}$. We would like to use the Itô's formula for a semimartingale $e^{-t}u(X_t^*, t)$. The problem is that we cannot assume $u \in C^2$ on $\partial\mathcal{D}$. To overcome this difficulty we use a regularization of u by convolutions.

Let $\phi \in C^\infty(\mathbb{R}^{n+1})$, $\phi \geq 0$, $supp\,\phi \subset B(0,1)$, $\int_{\mathbb{R}^{n+1}} \phi = 1$. For $\delta > 0$ let

$$\phi^\delta(x,t) = \frac{1}{\delta^{n+1}}\phi\left(\frac{x}{\delta}, \frac{t}{\delta}\right)$$

and

$$u^\delta(x,t) = (u*\phi^\delta)(x,t) = \int_{\mathbb{R}^{n+1}} u(y,s)\phi^\delta(x-y,t-s)dyds = -\int_{\mathbb{R}^{n+1}} u(x-y,t-s)\phi^\delta(y,s)dyds,$$

where $u(x,t) = u(x,0)$ for $t < 0$ and $u(x,t) = 0$ for $t > T$. Then $u^\delta \in C^\infty$. Denote $f^\delta = f * \phi^\delta$, where $f(x,t) = f(x,0)$ for $t < 0$ and $f(x,t) = f(x,T)$ for $t > T$. In view of (3.3) we get

$$u - u_t - \Delta u \leq f, \quad |Du| \leq 1$$

so

$$u^\delta - (u^\delta)_t - \Delta u^\delta \leq f^\delta, \quad |Du^\delta| \leq 1. \tag{3.35}$$

For $M > 0$ we introduce the Markov time

$$\tau^M = \inf_{t\in[0,T]} \{|X_t^*| \geq M\}. \tag{3.36}$$

From the Itô's formula we get

$$\mathbb{E}\left[e^{-\tau^M}u^\delta(X_{\tau^M}^*, \tau^M)\right] = u^\delta(x,0) + \mathbb{E}\int_0^{\tau^M} e^{-t}(\Delta u^\delta + (u^\delta)_t - u^\delta)(X_t^*,t)dt +$$

$$+\mathbb{E}\int_0^{\tau^M} e^{-t}Du^\delta(X_t^*,t)\circ\gamma_t^*(X_t^*)d\xi_t^* +$$

$$+\mathbb{E}\sum_{0\leq t<\tau^M} e^{-t}\left\{u^\delta(X_{t+}^*, t+) - u^\delta(X_t^*,t) - Du^\delta(X_t^*,t)\circ\gamma_t^*(X_t^*)(\xi_{t+}^* - \xi_t^*)\right\}.$$

The last term keeps account of jumps of X_t^*.

By (3.35) we have

$$\mathbb{E}e^{-\tau^M}u^\delta(X_{\tau^M}^*,\tau^M) + \mathbb{E}\int_0^{\tau^M} e^{-t}f^\delta(X_t^*,t)dt - \mathbb{E}\int_0^{\tau^M} e^{-t}Du^\delta(X_t^*,t)\circ\gamma_t^*(X_t^*)d\xi_t^* + \tag{3.37}$$

$$-\mathbb{E}\sum_{0\leq t<\tau^M} e^{-t}\left\{u^\delta(X_{t+}^*,t) - u^\delta(X_t^*,t) - Du^\delta(X_t^*,t)\circ\gamma_t^*(X_t^*)(\xi_{t+}^* - \xi_t^*)\right\} \geq u^\delta(x,0).$$

From (3.36) and the continuity of u^δ, u, f^δ, f we get

$$u^\delta(X_t^*,t) \to u(X_t^*,t),\ Du^\delta(X_t^*,t) \to Du(X_t^*,t),\ f^\delta(X_t^*,t) \to f(X_t^*,t)$$

uniformly for $t \in [0, \tau^M]$ if $\delta \to 0$. Moreover v^* is the optimal control so in view of (3.2) and (1.4) we have

$$\mathbb{E} \int_0^T e^{-t} d\xi_t^* < \infty. \tag{3.38}$$

Taking in (3.37) $\delta \to 0$ and using the bounded convergence theorem we get

$$\mathbb{E}e^{-\tau^M} u(X^*_{\tau^M}, \tau^M) + \mathbb{E} \int_0^{\tau^M} e^{-t} f(X_t^*, t) dt - \mathbb{E} \int_0^{\tau^M} e^{-t} Du(X_t^*, t) \circ \gamma_t^*(X_t^*) d\xi_t^* + \tag{3.39}$$
$$-\mathbb{E} \sum_{0 \le t < \tau^M} e^{-t} \left\{ u(X^*_{t+}, t) - u(X_t^*, t) - Du(X_t^*, t) \circ \gamma_t^*(X_t^*)(\xi^*_{t+} - \xi_t^*) \right\} \ge u(x, 0).$$

We note that $\mathbb{E}\xi_T^* < \infty$ by (3.38). The process X^* is left-continuous with right-limits, so its trajectories are bounded on compact sets. Hence, $\tau^M \to T$ a.s.. So, taking $M \to \infty$ in (3.39), noting that $|Du| \le 1$ and using the bounded convergence theorem again, we have

$$\mathbb{E}e^{-T} u(X_T^*, T) + \mathbb{E} \int_0^T e^{-t} f(X_t^*, t) dt - \mathbb{E} \int_0^T e^{-t} Du(X_t^*, t) \circ \gamma_t^*(X_t^*) d\xi_t^* + \tag{3.40}$$
$$-\mathbb{E} \sum_{0 \le t < T} e^{-t} \left\{ u(X^*_{t+}, t+) - u(X_t^*, t) - Du(X_t^*, t) \circ \gamma_t^*(X_t^*)(\xi^*_{t+} - \xi_t^*) \right\} \ge u(x, 0).$$

The first term on left-hand side is equal to 0. Because of the strict convexity of u and the equation $\gamma_t(X_t^*)(\xi^*_{t+} - \xi_t^*) = X^*_{t+} - X_t^*$, the last term on left-hand side is nonpositive. Hence

$$\mathbb{E} \int_0^T e^{-t} f(X_t^*, t) dt - \mathbb{E} \int_0^T e^{-t} Du(X_t^*, t) \circ \gamma_t^*(X_t^*) d\xi_t^* \ge u(x, 0). \tag{3.41}$$

Because

$$u(x, 0) = \mathbb{E} \int_0^T e^{-t} f(X_t^*, t) dt + \mathbb{E} \int_0^T e^{-t} d\xi_t^*,$$

we get

$$-\mathbb{E} \int_0^T e^{-t} Du(X_t^*, t) \circ \gamma_t^*(X_t^*) d\xi_t^* \ge \mathbb{E} \int_0^T e^{-t} d\xi_t^*. \tag{3.42}$$

Hence

$$\mathbb{E} \int_0^T e^{-t} \left(1 + Du(X_t^*, t) \circ \gamma_t^*(X_t^*) d\xi_t^*\right) \le 0.$$

From (3.3) we have $|Du| \le 1$ and $|\gamma^*| = 1$, so $|Du \circ \gamma^*| \le 1$. Hence

$$-Du(X_t^*, t) \circ \gamma_t^*(X_t^*) = 1 \ d\xi \ a.e. \text{ on } [0, T] \ P \, a.s.. \tag{3.43}$$

Because $|Du| < 1$ in $\mathcal{D}$ and $(X_t^*, t) \in \overline{\mathcal{D}}$, so

$$(X_t^*, t) \in \partial^* \mathcal{D} \ d\xi \ a.e. \text{ on } [0, T] \ P \, a.s.. \tag{3.44}$$

But for $(X_t^*, t) \in \partial^* \mathcal{D}$ we have $|Du(X_t^*, t)| = 1$, so (3.43) holds only if

$$\gamma_t^*(X_t^*) = -Du(X_t^*, t) = \Gamma_t(X_t^*) \ d\xi \ a.e. \text{ on } [0, T] \ P \, a.s.. \tag{3.45}$$

□

3.5 The optimally controlled Brownian motion is a Markov process

Now we will show that the optimaly controlled process X^* is a Markov process. From Theorem 2.3.3 we know that for our singular stochastic control problem the Bellman principle holds, i.e. for every $x_0 \in \mathbb{R}^n$ and for every stopping time $\mathcal{T}$ of the filtration $\{\mathcal{F}_t\}$ such that $0 \leq \mathcal{T} \leq T$ we have

$$u(x_0, 0) = \inf_{v \in \mathcal{V}} \mathbb{E} \left\{ \int_0^{\mathcal{T}} e^{-t} f(X_t, t) dt + \int_0^{\mathcal{T}} e^{-t} d\xi_t + e^{-\mathcal{T}} u(X_{\mathcal{T}}, \mathcal{T}) \right\}. \tag{3.46}$$

We want to prove that this infimum is actually attained for the optimal policy.

Lemma 3.5.1. *Let $\mathcal{T}$ be a stopping time of the filtration $\{\mathcal{F}_t\}$ such that $0 \leq \mathcal{T} \leq T$ and let $v_t^* = \int_0^t \gamma_s^* d\xi_s^*$ be the optimal control for given $x_0 \in \mathbb{R}^n$. Then*

$$u(x_0, 0) = \mathbb{E} \left\{ \int_0^{\mathcal{T}} e^{-t} f(X_t^*, t) dt + \int_0^{\mathcal{T}} e^{-t} d\xi_t^* + e^{-\mathcal{T}} u(X_{\mathcal{T}}^*, \mathcal{T}) \right\}. \tag{3.47}$$

Proof. Because of (3.46) it is sufficient to show that inequality $\geq$ holds in (3.47) instead of equality. Let $\epsilon > 0$. Recall that $\mathcal{V}_\epsilon$ is the set of all controls $v \in \mathcal{V}$ which are Lipschitz continuous and $|\frac{dv}{dt}(t)| \leq \frac{1}{\epsilon}$ for almost every $t \in [0, T]$ almost surely. The value function u_ε for an approximating regular control problem have been defined by (1.5). By the classical control theory for diffusion processes, there exists a unique optimal policy $\tilde{v}^\varepsilon \in \mathcal{V}^\varepsilon$ for which the infimum in right-hand size in (1.5) is attained and

$$u_\varepsilon(x_0, 0) = \mathbb{E} \left\{ \int_0^{\mathcal{T}} e^{-t} f(\tilde{X}_t^\varepsilon, t) dt + \int_0^{\mathcal{T}} e^{-t} d\tilde{\xi}_t^\varepsilon + e^{-\mathcal{T}} u_\varepsilon(\tilde{X}_{\mathcal{T}}^\varepsilon, \mathcal{T}) \right\}, \tag{3.48}$$

where $\tilde{v}_t^\varepsilon = \int_0^t \tilde{\gamma}_s^\varepsilon d\tilde{\xi}_s^\varepsilon$ and $\tilde{X}_t^\varepsilon = x_0 + \sqrt{2} W_t + \tilde{v}_t^\varepsilon$. The proof of (3.48) can be obtained by suitable modification of the argument proving Lemma IV.3.1 in [8].

The value function $u_\varepsilon(x_0, 0)$ converges to $u(x_0, 0)$ as $\varepsilon \to 0^+$ (see Theorem 2.3.2). In view of the proof of Lemma 3.2.9 there exists a sequence $\{\varepsilon_k\}$ decreasing to 0 such that $\tilde{v}^{\varepsilon_k} \to v^*$ for m_0−almost all $(t, \omega) \in [0, T] \times \Omega$ and (3.34) holds.

Assume that $\tilde{v}_{\mathcal{T}}^{\varepsilon_k} \to v_{\mathcal{T}}^*$ a.s.. Then, letting $k \to \infty$ in (3.48) for $\varepsilon = \varepsilon_k$, we get, by (3.34) and Fatou's lemma,

$$\begin{aligned} u(x_0, 0) &= \liminf_{k \to \infty} u_{\varepsilon_k}(x_0, 0) = \\ &= \liminf_{k \to \infty} \mathbb{E} \left\{ \int_0^{\mathcal{T}} e^{-t} f(\tilde{X}_t^{\varepsilon_k}, t) dt + \int_0^{\mathcal{T}} e^{-t} d\tilde{\xi}_t^{\varepsilon_k} + e^{-\mathcal{T}} u_{\varepsilon_k}(\tilde{X}_{\mathcal{T}}^{\varepsilon_k}, \mathcal{T}) \right\} \geq \\ &\geq \mathbb{E} \liminf_{k \to \infty} \left\{ \int_0^{\mathcal{T}} e^{-t} f(\tilde{X}_t^{\varepsilon_k}, t) dt + \int_0^{\mathcal{T}} e^{-t} d\tilde{\xi}_t^{\varepsilon_k} + e^{-\mathcal{T}} u(\tilde{X}_{\mathcal{T}}^{\varepsilon_k}, \mathcal{T}) \right\} \geq \\ &\geq \mathbb{E} \left\{ \int_0^{\mathcal{T}} e^{-t} f(X_t^*, t) dt + \int_0^{\mathcal{T}} e^{-t} d\xi_t^* + e^{-\mathcal{T}} u(X_{\mathcal{T}}^*, \mathcal{T}) \right\}. \end{aligned} \tag{3.49}$$

Now consider the case when $\tilde{v}^{\varepsilon_k}_{\mathcal{T}}$ does not converge to $v^*_{\mathcal{T}}$ a.s.. We can take a sequence δ_i decreasing to 0 such that for each $i \in \mathbb{N}$ $\tilde{v}^{\varepsilon_k}_{\tau_i} \to v^*_{\tau_i}$ *a.s.*, where τ_i denotes the Markov time $\tau_i = (\tau + \delta_i) \wedge (T - \delta_i)$. Using (3.49) for τ_i instead of τ and letting $i \to \infty$, we get

$$u(x_0, 0) \geq \mathbb{E}\Bigg\{ \int_0^{\mathcal{T}} e^{-t} f(X_t^*, t)dt + \int_0^{\mathcal{T}+} e^{-t} d\xi_t^* \cdot \mathbb{I}[\tau < T] + \int_0^{\mathcal{T}} e^{-t} d\xi_t^* \cdot \mathbb{I}[\tau = T] +$$

$$+ e^{-\mathcal{T}} u(X^*_{\mathcal{T}+}, \mathcal{T}) \cdot \mathbb{I}[\tau < T] + e^{-\mathcal{T}} u(X^*_{\mathcal{T}}, \mathcal{T}) \cdot \mathbb{I}[\tau = T] \Bigg\} =$$

$$= \mathbb{E}\Bigg\{ \int_0^{\mathcal{T}} e^{-t} f(X_t^*, t)dt + \int_0^{\mathcal{T}} e^{-t} d\xi_t^* + e^{-\mathcal{T}} u(X^*_{\mathcal{T}}, \mathcal{T}) \Bigg\} +$$

$$+ \mathbb{E}\Big\{ e^{-\mathcal{T}} \mathbb{I}[\tau < T] \left(|X^*_{\mathcal{T}+} - X^*_{\mathcal{T}}| + u(X^*_{\mathcal{T}+}, \mathcal{T}) - u(X^*_{\mathcal{T}}, \mathcal{T}) \right) \Big\} \geq$$

$$\geq \mathbb{E}\left\{ \int_0^{\mathcal{T}} e^{-t} f(X_t^*, t)dt + \int_0^{\mathcal{T}} e^{-t} d\xi_t^* + e^{-\mathcal{T}} u(X^*_{\mathcal{T}}, \mathcal{T}) \right\}.$$

The last inequality holds because $|Du| \leq 1$. □

Lemma 3.5.2. *Let $\mathcal{T} \in (0, T)$ be constant. For $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$ the following is true:*

$$v_t^{\bar{x}, \mathcal{T}} = v^*_{t+\mathcal{T}} - v^*_{\mathcal{T}}, \quad t \in [0, T - \mathcal{T}] \tag{3.50}$$

is the optimal policy controlling $\bar{x} + \sqrt{2}\bar{W}_t$ on the time interval $[0, T - \mathcal{T}]$, where $\bar{W}_t = W_{t+\mathcal{T}} - W_{\mathcal{T}}$ is a Brownian motion starting from $\bar{x}$ under the measure $\bar{P}^{\bar{x}} = P(\cdot | X^*_{\mathcal{T}} = \bar{x})$. By this we mean the value of the regular conditional probability distribution of $(W_t, v_t^*, t \in [0, T])$ given the σ-field $\sigma(X^*_{\mathcal{T}})$ on the event $X^*_{\mathcal{T}} = \bar{x}$.

Proof. In view of definition the value function we have

$$u(x_0, 0) = \mathbb{E} \int_0^{\mathcal{T}} e^{-t} \big[f(X_t^*, t)dt + d\xi_t^* \big] + \mathbb{E}\left(e^{-\mathcal{T}} \mathbb{E}^{X^*_{\mathcal{T}}} \int_{\mathcal{T}}^{T} e^{-(t-\mathcal{T})} \big[f(X_t^*, t)dt + d\xi_t^* \big] \right), \tag{3.51}$$

where $\mathbb{E}^{X^*_{\mathcal{T}}} = \mathbb{E}(\cdot | X^*_{\mathcal{T}})$ is the conditional expectation operator. For simplicity, assume that v^* is left-continuous for all $\omega \in \Omega$. This can be achieved by modifying v^* on a set of measure zero, which does not lead to any difficulty. Let

$$Y = \int_0^{T-\mathcal{T}} e^{-t} \big[f(X^*_{t+\mathcal{T}}, t + \mathcal{T})dt + d\xi^*_{t+\mathcal{T}} \big]. \tag{3.52}$$

Y is a $\sigma(W_t, v_t, t \in [0, T])$-measurable random variable. It is well known, that there exists a Borel function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbb{E}^{X^*_{\mathcal{T}}} Y = g(X^*_{\mathcal{T}}) \tag{3.53}$$

for $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$. The proof of this fact for one-dimensional case is included in [22] (see Theorem 3, p.174), but this proof goes through also in n dimensions. In other words

$$g(\bar{x}) = \bar{\mathbb{E}}^{\bar{x}} \int_0^{T-\mathcal{T}} e^{-t}\big[f(X^*_{t+\mathcal{T}}, t+\mathcal{T})dt + d\xi^*_{t+\mathcal{T}}\big], \tag{3.54}$$

where the last expectation is the value of $\mathbb{E}^{X^*_{\mathcal{T}}}Y$ on $[X^*_{\mathcal{T}} = \bar{x}]$. Equation (3.51) combined with (3.52) and (3.53) yields

$$u(x_0, 0) = \mathbb{E}\int_0^{\mathcal{T}} e^{-t}\big[f(X^*_t, t)dt + d\xi^*_t\big] + \mathbb{E}\left(e^{-\mathcal{T}}g(X^*_{\mathcal{T}})\right). \tag{3.55}$$

We want to prove that $g(\bar{x}) = u(\bar{x}, \mathcal{T})$, for $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$. $\bar{W}_t$ is a Brownian motion independent on $\mathcal{F}_{\mathcal{T}}$, in particular on $X^*_{\mathcal{T}}$. Hence it may be seen that $g(\bar{x})$ is a payoff of the form (3.2) $J_{\bar{x},\mathcal{T}}(v^{\bar{x},\mathcal{T}})$ for the Brownian motion $\bar{W}_t$ on $(\Omega, \mathcal{F}_t, \bar{P}^{\bar{x}})$ starting at $\bar{x}$. Indeed, $\bar{\mathbb{E}}^{\bar{x}} \int_0^{T-\mathcal{T}} e^{-t}\big[f(X^*_{t+\mathcal{T}}, t+\mathcal{T})dt + d\xi^*_{t+\mathcal{T}}\big]$ defined as above is the expectation of Y under $\bar{P}^{\bar{x}}$ for $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$. This can be seen by approximating Y by suitable finite sums, as in the definition of an integral, evaluating their expectations under $\bar{P}^{\bar{x}}$, and then going to the limit, using the bounded and monotone convergence theorems. Also, $\xi^*_\cdot(\omega)$ is left continous for all $\omega \in \Omega$) by assumption. Moreover, the value function for the latter control problem is $u(\bar{x}, \mathcal{T})$

The above considerations lead us to

$$u(\bar{x}, \mathcal{T}) \le g(\bar{x}) \quad \text{for } P(X^*_{\mathcal{T}})^{-1} \text{ a.a. } \bar{x} \text{ such that } (\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}. \tag{3.56}$$

We want to show that, in fact, equality holds in (3.56). Suppose it is not true, i.e.,

$$P(X^*_{\mathcal{T}})^{-1}(A) > 0, \tag{3.57}$$

where

$$A = \{\bar{x} : (\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}},\ u(\bar{x}, \mathcal{T}) < g(\bar{x})\}. \tag{3.58}$$

Then (3.56), (3.57), (3.58) yield

$$\mathbb{E}e^{-\mathcal{T}}u(X^*_\tau, \mathcal{T}) < \mathbb{E}e^{-\mathcal{T}}g(X^*_\tau). \tag{3.59}$$

By Lemma 3.5.1, (3.59) and (3.55) we get

$$u(x_0, 0) < \mathbb{E}\int_0^{\mathcal{T}} e^{-t}\big[f(X^*_t, t)dt + d\xi^*_t\big] + \mathbb{E}\left(e^{-\mathcal{T}}g(X^*_{\mathcal{T}})\right) = u(x_0, 0), \tag{3.60}$$

a contradiction.

Thus, $u(\bar{x}, \mathcal{T}) = g(\bar{x})$ for $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$. But, by definitions of u and g, this means exactly that, for $P(X^*_{\mathcal{T}})^{-1}$ a.e. $\bar{x}$ such that $(\bar{x}, \mathcal{T}) \in \overline{\mathcal{D}}$, $v_t^{\bar{x},\mathcal{T}}$ defined by (3.50) is indeed the optimal policy for controlling $\bar{W}_t$ on the time horizon $[0, T - \mathcal{T}]$ under $\bar{P}^{\bar{x}}$. □

Remark 3.5.3. We note that the conclusion of Lemma 3.5.2 holds true if $\mathcal{T}$ is any stopping time of the filtration $\{\mathcal{F}_t\}$ bounded by T. Indeed, we can modify the above proof suitably, replacing $\sigma(X^*_{\mathcal{T}})$ by $\sigma(X^*_{\mathcal{T}}, \mathcal{T})$, $P(X^*_{\mathcal{T}})^{-1}$ by $P(X^*_{\mathcal{T}}, \mathcal{T})^{-1}$ and $\bar{P}^{\bar{x}}$ by $\bar{P}^{\bar{x},t}$. By this we mean the value of the regular conditional probability distribution of $(W_t, v^*_t, t \in [0,T])$ given $\sigma(X^*_{\mathcal{T}}, \mathcal{T})$ on $[X^*_{\mathcal{T}} = \bar{x}, \mathcal{T} = t]$. In this case, we must use the strong Markov property of the Brownian motion in the proof.

Corollary 3.5.4. *Set $X^*_t = W_t - W_T + X^*_T$ and $v^*_t = v^*_T$ for $t > T$. Then the process (X^*_t, t) is a strong Markov process with respect to the filtration $\{\mathcal{F}_t\}$.*

Indeed, for any $t \in [0,T)$ and the stopping time $\mathcal{T}$ bounded by $T - t$ we have

$$X^*_{t+\mathcal{T}} - X^*_{\mathcal{T}} = (W_{t+\mathcal{T}} - W_{\mathcal{T}}) + (v^*_{t+\mathcal{T}} - v^*_{\mathcal{T}}).$$

The Brownian increment is independent on $\mathcal{F}_{\mathcal{T}}$ and all relevant information about the increment of v^* that can be found in $\mathcal{F}_{\mathcal{T}}$ is, in view of Remark 3.5.3, the value of $(X^*_{\mathcal{T}}, \mathcal{T})$. Thus, all the information about

$$X^*_{t+\mathcal{T}} = X^*_{t+\mathcal{T}} - X^*_{\mathcal{T}} + X^*_{\mathcal{T}}$$

that can be found in $\mathcal{F}_{\mathcal{T}}$ is actually the value of $(X^*_{\mathcal{T}}, \mathcal{T})$.

3.6 Possible jumps of the optimal policy

Theorem 3.6.1. *Let $x_0 \in \mathcal{D}_0$. With probability 1, for each $t \in [0,T]$, a possible jump of the optimally controlled process X^* at time t occurs on some interval $I \subset \mathbb{R}^n$ parallel to the vector field Γ_t on I (i.e. for all $x \in I$ $\Gamma_t(x)$ is parallel to I) and such that $I \times \{t\} \subseteq \partial^*\mathcal{D} \cap (\mathbb{R}^n \times \{t\})$. If X^*_t encounters such an interval I, it instantaneously jumps to its endpoint in the direction Γ_t on I.*

Proof. Step 1. We show that if X^*_t starts from such an interval I, it instantaneously jumps to its endpoint in the direction Γ_t on I.

Fix $t \in (0,T)$. Suppose that $x \in \mathbb{R}^n$ satisfies $(x,t) \in \partial^*\mathcal{D}$ and has the following property: there exists an interval $I \subset \mathbb{R}^n$ such that $I \times \{t\} \subseteq \partial^*\mathcal{D} \cap (\mathbb{R}^n \times \{t\})$ and

$$I = \{a + r\eta : r \in [0,c]\} \tag{3.61}$$

for some $\eta, a \in \mathbb{R}^n$, $|\eta| = 1$, $c > 0$, $Du(\cdot,t) = \eta$ on I and $x \in I - \{a\}$.

We denote the set of all such x by $\mathcal{H}_t$ and let $\mathcal{H} = \bigcup_{t\in[0,T]} \big(\mathcal{H}_t \times \{t\}\big)$. We assume that I in the above definition is maximal, i.e. is the sum of all the intervals with such property.

Because

$$\frac{\partial u(\cdot,t)}{\partial \eta} = Du(\cdot,t) \circ \eta = |Du(\cdot,t)|^2 = 1 \text{ on } I$$

we get

$$u(a + r\eta, t) = u(a,t) + r, \quad r \in [0,c]. \tag{3.62}$$

Analytically, the set

$$\mathcal{H} = \bigcup_{i=1}^{\infty} \left\{ (x,t) \in \partial^* \mathcal{D} : \; u(x,t) - u\Big(x - \frac{1}{i} Du(x,t), t\Big) = \frac{1}{i} \right\}, \tag{3.63}$$

is a countable sum of closed sets. Then $\mathcal{H}$ and $\mathcal{H}_t$ are Borel measurable.

Let $v^{x,t}$ be the optimal policy for the process starting from $x \in \mathcal{H}_t$ at time t (see Lemma 3.5.2) and let $x = a + r\eta$ where a, r, η are as above. Then

$$v_s^{x,t} = \begin{cases} 0 & , \; s = 0 \\ a - x + v_s^{a,t} & , \; s \in (0, T-t] \end{cases} \tag{3.64}$$

i.e. the optimal policy first jumps from x to a and proceeds optimally thereafter.

Indeed, if we define $v^{x,t}$ by (3.64), then, by (3.62),

$$J_{xt}(v^{x,t}) = |a - x| + u(a,t) = r + u(a,t) = u(x,t). \tag{3.65}$$

Step 2. Now we want to prove that X_t^* jumps only when $X_t^* \in \mathcal{H}_t$, a.s.. Suppose it is not true. Because v^* is left-continuous, the only possible discontinuities of v^* are jumps. Suppose that v^* does have jumps and let

$$T^\varepsilon(\omega) = \inf\{t \in [0,T] : X_t^*(\omega) \notin \mathcal{H}_t, \; |X_t^*(\omega) - X_{t+}^*(\omega)| \geq \varepsilon\}; \tag{3.66}$$

i.e., T^ε is the first time when the process X^* undergoes a jump of magnitude at least ε starting out of $\mathcal{H}_{T^\varepsilon}$. Suppose ε is so small that $P[T^\varepsilon < T] > 0$. A slight modification of theorem in [5] page 84 guarantees that there is a sequence $T_1, T_2, ...$ of stopping times exhausting the jumps of X^*, i.e.,

$$\{(t,\omega) \in [0,T] \times \Omega : X_t^*(\omega) \neq X_{t+}^*(\omega)\} \subset \bigcup_{i=1}^{\infty} \{(t,\omega) \in [0,T] \times \Omega : T_i(\omega) = t\}. \tag{3.67}$$

Thus

$$T^\varepsilon(\omega) = \inf\{T_i(\omega) : X_{T_i(\omega)}^* \notin \mathcal{H}_{T_i(\omega)}, \; |X_{T_i(\omega)}^*(\omega) - X_{T_i(\omega)+}^*(\omega)| \geq \varepsilon\}. \tag{3.68}$$

Because the optimal control is a process of bounded variation a.s., there can be only finitely many jumps of magnitude at least ε a.s.. Hence in (3.68) we can use min instead of inf. Thus, T^ε is a stoppping time.

On $[T^\varepsilon < T]$ we have $\big(X_{T^\varepsilon}^*, T^\varepsilon\big) \in \overline{\mathcal{D}} \setminus \mathcal{H}$, $\big(X_{T^\varepsilon}^*, T^\varepsilon\big) \in \overline{\mathcal{D}}$, so, by $|Du| \leq 1$,

$$u(X_{T^\varepsilon+}^*(\omega), T^\varepsilon) + |X_{T^\varepsilon(\omega)}^*(\omega) - X_{T^\varepsilon(\omega)+}^*(\omega)| > u(X_{T^\varepsilon}^*(\omega), T^\varepsilon) \tag{3.69}$$

because $Du(\cdot, T^\varepsilon)$ is not identically equal to -1 on the interval joining $X_{T^\varepsilon}^*(\omega)$ to $X_{T^\varepsilon+}^*(\omega)$.

Using (3.47) for $T_n = \left(T^\varepsilon + \frac{1}{n}\right) \wedge T$, then letting $n \to \infty$ and using the bounded convergence theorem, we get

$$
\begin{aligned}
u(x_0,0) = \ & \mathbb{E}\left\{\int_0^{T^\varepsilon(\omega)} e^{-t} f(X_t^*,t)dt + \int_0^{T^\varepsilon(\omega)} e^{-t} d\xi_t^*\right\} + \\
& + \mathbb{E}\Big\{ \ \mathbb{I}[T^\varepsilon < T]\cdot e^{-T^\varepsilon(\omega)} |X^*_{T^\varepsilon(\omega)}(\omega) - X^*_{T^\varepsilon(\omega)+}(\omega)|\Big\} + \\
& + \mathbb{E}\Big\{ \ \mathbb{I}[T^\varepsilon < T]\cdot e^{-T^\varepsilon(\omega)} u(X^*_{T^\varepsilon(\omega)+}, T^\varepsilon(\omega))\Big\} > \\
& > \ \mathbb{E}\left\{\int_0^{T^\varepsilon(\omega)} e^{-t} f(X_t^*,t)dt + \int_0^{T^\varepsilon(\omega)} e^{-t} d\xi_t^*\right\} + \\
& + \mathbb{E}\Big\{ \ \mathbb{I}[T^\varepsilon < T]\cdot e^{-T^\varepsilon(\omega)} u(X^*_{T^\varepsilon(\omega)}, T^\varepsilon(\omega))\Big\} = u(x_0,0).
\end{aligned}
\tag{3.70}
$$

The above inequality follows from (3.69) and the last equality follows from (3.47). But (3.70) is a clear contradiction. Thus X_t^* jumps only when $X_t^* \in \mathcal{H}_t$.

Step 3. The fact that for $X_t^* \in \mathcal{H}_t$ the optimally controlled process X_t^* a.s. jumps to the endpoint of the interval I described above, where $X_t^* \in I$, follows from (3.64) and Lemma 3.5.2 combined with the fact that there are at most countably many jumps of X^* (see (3.67)).

□

Corollary 3.6.2. *If the set $\mathcal{H}$ is empty, then X^*, ξ^*, v^* are continuous.*

Remark 3.6.3. In view of Theorems 3.3.1, 3.4.1, 3.6.1 and Remark 3.3.2, the proof of the main theorem 3.1.3 in the case of $x_0 \in \overline{\mathcal{D}_0}$ is completed.

Remark 3.6.4. Let the starting point $x_0 \notin \overline{\mathcal{D}_0}$. In this case the optimal policy jumps immediately to some point $\hat{x} \in \partial\mathcal{D}_0$ and then follows the optimal policy starting at $\hat{x}$.

Proof. We again use the gradient flow argument (see the proof of c) in Lemma 3.2.2). Let ψ be the solution of (3.12) with with the initial condition $\psi(0,\theta) = \tilde{x} + \delta\theta$. Recall that the map ψ is a homeomorphism from $[0,\infty) \times \partial B(0,1)$ onto $\mathbb{R}^n \setminus B(\tilde{x},\delta)$.

Let $x_0 = (s_0, \theta_0)$ for some $s_0 > 0, \theta_0 \in \partial B(0,1)$ and let $s_1 = \inf\{s \leq s_0 : \psi(s,\theta_0) \in \partial\mathcal{D}_0\}$. Let $\hat{x} = \psi(s_1,\theta_0)$. Then $\hat{x} \in \partial\mathcal{D}_0$ so the vector $v = Du(\hat{x},0)$ has the norm 1. Let $L = \{\hat{x} + sv : s \geq 0\}$. Then, by (3.3) and the convexity of u, we have $|Du(\cdot,0)| = 1$ on L. Thus,

$$u(x_0,0) = u(\hat{x},0) + s_0 - s_1$$

and $|x_0 - \hat{x}| = s_0 - s_1$. Let v^x denote the optimal policy for the problem with initial position x at time 0. By an argument analogous to the one given in the proof of Theorem 3.6.1, we have

$$v_t^{x_0} = \hat{x} - x_0 + v_t^{\hat{x}} \quad \text{for a.e. } t \in [0,T] \ \ P - a.s..$$

□

It is clear that any process that solves the Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ with reflection direction $-Du$ in the sense of Definition 3.1.2 is an optimal policy for our problem (compare, e.g., the verification Theorem VIII.4.1 from [8]). Thus, uniqueness of the optimal policy implies uniquess of a solution to the modified Skorokhod problem for $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ with reflection direction $-Du$.

Summarizing, in view of the main theorem 3.1.3, if $x_0 \in \overline{\mathcal{D}_0}$, then the optimal policy in our singular stochastic control problem (3.1)-(3.2) is the solution of the modified Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ starting at x_0 with reflection direction $-Du$. For $x_0 \notin \overline{\mathcal{D}_0}$ the optimal policy jumps immediately in the direction $-Du$ to the nearest point $\hat{x} \in \partial\mathcal{D}_0$ and then follows the optimal policy starting at $\hat{x}$ (Remark 3.6.4).

Our conjecture is that the set $\mathcal{H}$ (see (3.63)) is empty and the optimal policy is the solution of the regular Skorokhod problem for a Brownian motion $\sqrt{2}W_t$ in $\overline{\mathcal{D}}$ with reflection direction $-Du$ (see Definition 3.1.1), with immediate jump in the direction $-Du$ to $\partial\mathcal{D}_0$ if the initial position $x_0 \notin \overline{\mathcal{D}_0}$.

Bibliography

[1] A. Budhiraja and K. Ross, *Existence of optimal controls for singular control problems with state constraints*, Ann. Appl. Probab., 16, No. 4, pp. 2235-2255, 2006.

[2] K. Burdzy, Z.-Q. Chen, J. Sylvester, *The heat equation and reflected Brownian motion in time-dependent domains*, The Annals of Probability, Vol. 32, No. 1B, pp. 775-804, 2004.

[3] K. Burdzy, W. Kang, K. Ramanan, *The Skorokhod problem in a time-dependent interval*, Stochastic Processes and their Applications, Vol. 119, Issue 2, pp. 428-452, 2009.

[4] P. L. Chow, J. L. Menaldi and M. Robin, *Additive control of stochastic linear systems with finite horizon*, SIAM J. Control Optim., 23, No.6, pp. 858-899, 1985.

[5] C. Dellacherie, *Capacités et Processus Stochastiques*, Springer-Verlag, Berlin 1972.

[6] F. Dufour and B. Miller, *Singular stichastic control problems*, SIAM J. Control Optim., 43, No. 2, pp. 708-730, 2004.

[7] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[8] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscocity Solutions*, Springer, 2006.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. J. 1964. Reprinted by Krieger, Malabar, Fl, 1989.

[10] U. G. Haussman and W. Suo, *Singular optimal stochastic controls. I. Existence*, SIAM J. Control Optim., 33, No. 3, pp. 916-936, 1995.

[11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York 1988.

[12] Ł. Kruk, *Optimal policies for n-dimensional singular stochastic control problems, Part I: The Skorokhod problem*, SIAM J. Control Optim., 38, No. 5, pp. 1603-1622, 2000.

[13] Ł. Kruk, *Optimal policies for n-dimensional singular stochastic control problems, Part II: The radially symmetric case. Ergodic control*, SIAM J. Control Optim., 39, No. 2, pp. 635-659, 2000.

[14] N. V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, New York 1980.

[15] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Izdat. "Nauka", Moscow, 1967 (Russian). English transl.: American Mathematical Society, Providence RI 1968.

[16] P. L. Lions and A. S. Sznitman, *Stochastic Differential Equations with Reflecting Boundary Conditions*, Comm. on Pure and Applied Mathematics, 37, pp. 511-537, 1984.

[17] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, Singapore 2005.

[18] J. L. Menaldi and M. I. Taksar, *Optimal correction problem of a multidimensional stochastic system*, Automatica, Vol.25, No.2, pp. 223-232, 1989.

[19] K. Nyström, T. Önskog, *The Skorokhod oblique reflection problem in time-dependent domains*, The Annals of Probability, Vol. 38, No. 6, pp. 2170-2223, 2010.

[20] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York 1991.

[21] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Company, New York 1976.

[22] A. N. Shiriaev, *Probability*, 2nd ed., Springer-Verlag, New York, 1996.

[23] M. Słaby, *Explicit solutions of the extended Skorokhod problems in time-dependent bounded regions with orthogonal reflection fields*, Probability and Mathematical Statistics, Vol. 33, Fasc. 1, pp. 121-148, 2013.

[24] H. M. Soner and S.E. Shreve, *Regularity of the value function for a two-dimensional singular stochastic control problem*, SIAM J. Control Optim., 27, pp. 876-907, 1989.

[25] H. M. Soner and S.E. Shreve, *A free boundary problem related to singular stochastic control*, Proceedings of Imperial College Workshop on Applied Stochastic Analysis, H. M. A. Davis, ed., pp.265-301, CRC Press, 1991.

[26] H. M. Soner and S.E. Shreve, *A free boundary problem related to singular stochastic control; the parabolic case*, Comm. PDE 16, pp. 373-424, 1991.

[27] S. A. Williams, P. L. Chow and J. L. Menaldi, *Regularity of the free boundary in singular stochastic control*, Journal of Differential Equations 111, pp. 175-201, 1994.

[28] *http://en.wikipedia.org/wiki/Gronwall's_inequality*, 24.09.2013.