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Hamilton cycles in random lifts of complete graphs

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HAMILTON CYCLES IN RANDOM LIFTS OF COMPLETE GRAPHS

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Abstract. We study asymptotic properties of random lifts – a model of random graph introduced by Amit and Linial. We show that if $h \geq 30$, then asymptotically almost surely the random lift of the complete graph K_h contains a Hamilton cycle.

1. Introduction

The notion of a covering map is a restriction of the general topological notion to homomorphisms of graphs. For graphs G and H, a map $\pi : V(H) \to V(G)$ is a *homomorphism* of H to G if it preserves the adjacency of vertices, i.e., we have $\{\pi(x), \pi(y)\} \in E(G)$ whenever $\{x, y\} \in E(H)$. We say that π is a *covering* if it is 'locally bijective', which means that for each vertex $v \in H$ there exists a bijection between $\widetilde{v}$ and $\pi(\widetilde{v})$, where for a vertex v by $\widetilde{v}$ we denote the star which consists of a vertex v and all edge incident to it. Hence, in particular, the degree of v must be the same as the degree of $\pi(v)$. For a covering $\pi : H \to G$ and a vertex $v \in G$ by the *fiber of* v we mean the set $\pi^{-1}(v)$.

A simple model for a random finite coverings $R(n, G)$ of G was proposed by Amit and Linial [1]. For a given graph G and a natural number n, the random graph $L(n, G)$ is defined in the following way. First we associate with each vertex v of G a set V_v on n vertices. The set $\bigcup_{v \in V(G)} V_v$ is the set of vertices of $L(n, G)$. Let us denote the family of all graphs which cover G and have fibers $\{V_v : v \in V(G)\}$ by $\mathcal{L}(n, G)$. Then, $L(n, G)$ is a graph chosen uniformly at random from the family $\mathcal{L}(n, G)$. In order to distinguish this from other notions of coverings in graphs, such as edge or cycle coverings, Amit and Linial called $L(n, G)$ the *random lift* of G. Let us observe that one can also construct the random lift $L(n, G)$ in the following way. For each edge $\{v, w\}$ of G we join sets V_v and V_w by a random matching, i.e., a matching choosing uniformly at random from all possible matchings between V_v and V_w. Equivalently, if $V_v = \{v_1, ..., v_n\}$, and $V_w = \{w_1, ..., w_n\}$ we should choose uniformly at random one of $n!$ permutations $\sigma_{vw} : [n] \to [n]$ and connect v_i with $w_{\sigma_{vw}(i)}$. Note that such permutations are chosen independently for each pair of edges of G.

As typical in random graph theory we are interested mainly in asymptotic properties of lifts of graphs when n is large. In particular, we say that a property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity.

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Random lifts are rather hard to study and there are only a handful of papers concerning this model of random graphs. Amit and Linial [1] proved that if G is a simple, connected graph with minimum degree $\delta \geq 3$, then its random lift $L(n, G)$ is aas δ-connected. They continued the study of random lifts in [3] where they proved expansion properties of lifts. The third in this series of papers on random lifts, written jointly with Matousek [2], deals with the independence and chromatic numbers of random lifts. In Linial and Rozenman [7] studied the properties of G which ensure that $L(n, G)$ has aas a perfect matching.

The problem of hamiltonicity of random lifts has been addressed by Linial in [6] where he asked the following two questions.

Problem 1. *Is there a zero-one law for Hamiltonicity? Namely, is it true that for every G almost every or almost none of the graphs in $\mathcal{L}_n(G)$ have a Hamilton cycle?*

Problem 2. *Let G be a d-regular graph with $d \geq 3$. Is it true that $L(n, G)$ is aas hamiltonian?*

Recently Burgin, Chebolu, Cooper and Frieze [4] show that there is a constant h_0 such that if $h \geq h_0$, then both $L(n, K_h)$ and $L(n, K_{h,h})$ are aas hamiltonian. Later on Chebolu and Frieze [5] proved the same property for an appropriately defined random lifts of complete directed graphs. Here we give alternative proof that random lift of K_h, where $h \geq 30$ aas contains a Hamilton cycle.

Theorem 1. *Let $h \geq 30$. Then $L(n, K_h)$ is aas hamiltonian.*

2. Asymptotic properties of lifts

Let us start with a few facts on the asymptotic properties of random lifts. It is easy to see that a lift of any path is a union of vertex disjoint paths while the lift of a cycle C_h is a sum of disjoint cycles. Our first result states that the number of cycles in $L(n, C_h)$ is not too large.

Lemma 2. *Let $h \geq 3$. Then $L(n, C_h)$ consists of at most $2 \log n$ cycles.*
Moreover, aas fewer than $5 \log \log n$ such cycles are shorter than $\log^4 n$.

Proof. Let us choose any edge $e = \{u, v\}$ of C_h and consider a path $P = C_h \backslash e$. Then, as we have already observed, the path P is lifted to n disjoint paths in $L(n, C_h)$. Lifting the edge e is equivalent to adding to it a random matching between the ends of the paths. The number of cycles in the resulting graph does not depend on the length of paths, so we can assume that all of them trivial, i.e., each of them consists of a single vertex. Hence, the number of cycles in $L(n, C_h)$ is the same as in the random permutation on set $[n] = \{1, 2, \ldots, n\}$ and the number of cycle of length at most $K = \log^4 n$ in $L(n, C_h)$ is bounded from above by the number of cycles of length at most K in a random permutation. The precise distribution of the number of cycles in random permutation is well known [8] but we estimate it here for the completeness of the argument.

Let $X_k = X_k(n)$ denote the number of cycles in the random permutation on $[n]$. Then, for the expectation of X_k we get

$$\mathbb{E}X_k = \binom{n}{k}(k-1)!\frac{(n-k)!}{n!} = \frac{1}{k} \tag{1}$$

Thus, for the total number cycles $X = X(n) = \sum_{k=1}^{n} X_k$ we have

$$\mathbb{E}X = \sum_{k=1}^{n} \mathbb{E}X_k = \sum_{k=1}^{n} \frac{1}{k} = \log n + O(1). \tag{2}$$

In order to compute the variance note that if we fix a cycle in a random permutation, each permutation on the remaining vertices is equally likely. Hence

$$\mathbb{E}X(n)[(X(n)-1)] = \sum_{k=1}^{n}\sum_{\ell=1}^{n-k} \mathbb{E}X_k(n)\mathbb{E}X_\ell(n-k) = \sum_{k=1}^{n}\sum_{\ell=1}^{n-k} \frac{1}{k\ell}\,. \tag{3}$$

Let $s = n\exp(-\sqrt{\log n})$. Then,

$$\begin{aligned}\mathbb{E}X[(X-1)] &= \sum_{k=1}^{s}\sum_{\ell=1}^{n-k} \frac{1}{k\ell} + \sum_{k=s+1}^{n}\sum_{\ell=1}^{n-k} \frac{1}{k\ell} \\ &= (\log s + O(1))(\log n + O(1)) + (\log(n/s) + O(1))O(\log n) \\ &= (\log n)^2 + O\big(\log^{3/2} n\big)\,.\end{aligned} \tag{4}$$

and

$$\mathbb{V}\mathrm{ar}X = \mathbb{E}X(X-1) + \mathbb{E}X - (\mathbb{E}X)^2 = O\big(\log^{3/2} n\big)\,.$$

Hence, from Chebyshev's inequality we get

$$\mathbb{P}(X \ge 2\log n) \le \mathbb{P}(|X - \mathbb{E}X| \ge 0.5\log n) \le \frac{4\mathbb{V}\mathrm{ar}X}{\log^2 n} = o(1)\,.$$

Now let $K = \log^4 n$ and $Y = \sum_{i=1}^{K} X_k$. Then, from (1),

$$\mathbb{E}Y = \sum_{k=1}^{K} \mathbb{E}X_k = \sum_{k=1}^{K} \frac{1}{k} \le 4\log\log n + O(1)\,.$$

Arguing as in (4) one can prove that $\mathbb{V}\mathrm{ar}Y = O\big((\log\log n)^{3/2}\big)$, so the second part of assertion follows from Chebyshev's inequality. □

Now let us choose a Hamilton cycle C_h in K_h and consider $L(n, C_h)$ as a subgraph of $L(n, K_h)$. Our next result states that short cycles in $L(n, C_h)$ are sparsely distributed around $L(n, K_h)$.

Lemma 3. *Let $h \ge 3$ and $C_h \subseteq K_h$. Then aas no pair of cycles contained in $L(n, C_h)$ which are shorter than $\log^4 n$ lie within distance 15 in $L(n, K_h)$.*

Proof. Let $L = \log^4 n$, and let $Z_m = Z_m(n, L)$, $m = 1, \ldots, 12$, counts the number of pairs of cycle of $L(n, C_h)$ which has at most L vertices and are connected by a path of m edges. Finally, let $Z = \sum_{m=1}^{15} Z_m$. Then, using an argument which led us to (1) we get

$$\begin{aligned}\mathbb{E}Z &\le \sum_{m=1}^{15}\sum_{k=3}^{L}\sum_{\ell=3}^{L}\frac{1}{k\ell}\binom{nh}{m-1}(m-1)!k\ell\Big(\frac{1}{n-2L-m}\Big)^m\\ &\le 16h^{15}L^2\log^2 L/n \le \log^3/n \to 0\,.\end{aligned}$$

Thus, from Markov's inequality,

$$\mathbb{P}(Z>0) \le \mathbb{E}Z = o(1)\,,$$

and the assertion follows. □

We shall also use the fact that small subsets of vertices of the random lift of a graph with large minimal degree has good expanding properties.

Lemma 4. *For every $h > \delta \ge 12$ and there exists a constant $\alpha > 0$ such that for each simple graph G with h vertices and minimum degree δ aas for every subset S of vertices of $L(n,G)$ with $|S| \le \alpha n$, we have*

$$|S \cup N(S)| > \frac{\delta}{3}|S|.$$

Proof. We show that there exists a constant α such that the probability that the size of the neighborhood of S is smaller than $\frac{\delta}{3}|S|$ tends to 0 as $n \to \infty$.

Let S be any subset of vertices from $L(n,G)$, and let $s = |S|$. For a given set of vertices T of $L(n,G)$, with $|T| = (\delta/3-1)s$, the probability that $N(S) \subseteq S \cup T$ is bounded from above by

$$\left(\frac{\delta s/3}{n-s}\right)^{\delta s/2},$$

since for each vertex $v \in S$ we have to choose all its neighbors in $S \cup T$, where each neighbor can be chosen from all vertices of appropriate fibers, except the ones which have been already selected in the previous steps. There are $\binom{n}{s}$ possible sets S with $|S| = s$ and $\binom{n}{(\delta/3-1)s}$ choices for T, so we need to show that

$$\sum_{s=1}^{\alpha n} B(s) = o(1), \tag{5}$$

where

$$B(s) = \binom{nh}{s}\binom{nh}{(\frac{\delta}{3}-1)s}\left(\frac{\delta s/3}{n}\right)^{\delta s/2}.$$

Using well known estimate

$$\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$$

we get

$$B(s) \le \Big(\frac{ehn}{s}\Big)^s \Big(\frac{ehn}{(\delta/3-1)s}\Big)^{(\delta/3-1)s} \Big(\frac{e\delta s/3}{n}\Big)^{\delta s/2}$$
$$\le \Big(\frac{\delta-3}{3}\Big(\frac{3eh}{\delta-3}\Big)^{\delta/3}\Big(\frac{s}{n}\Big)^{\delta/6}\Big)^s.$$

Hence, if $s \le \alpha n$, where $\alpha = \alpha(h,\delta)$ is chosen small enough so that

$$\Big(\frac{3eh}{\delta-3}\Big)^{\delta/3}\alpha^{\delta/6} < 0.99,$$

we have $B(s) = o(1/n)$. Hence, (5) holds and so the assertion follows. □

3. Pósa's Lemma

In our proof we shall use a path reversal technique of Pósa [9]. Let G be any connected graph and $P = (v_0v_1,...v_t)$ be a path of maximum length in G. If $1 \le i \le l-2$ and $\{v_l, v_i\}$ is an edge of G, then $P' = (v_0v_1...v_iv_lv_{l-1}...v_{i+1})$ is a path with the same vertex set as P, and thus also of maximum length. We call P' a rotation of P with preserves *starting point* v_0 and a *pivot* v_i. By $\mathcal{P}(v_0)$ we denote the set of all paths of G which can be obtained from P by rotations preserving the starting point of P. Most of results using rotation techniques are based on the following result of Pósa [9].

Theorem 5. *Let P denote a path of maximum length in G which starts at v_0. Moreover, let S denote the set of ends of paths in $\mathcal{P}(v_0)$ and by S^- and S^+ the sets of vertices immediately preceding and following the vertices of S on P, respectively. Then*

$$N(S) \subseteq S^- \cup S \cup S^+.$$

In our argument we shall need the following consequence of the above result.

Lemma 6. *Let G be a graph such that for every subset S of vertices of G with $|S| \le m$ we have*

$$|S \cup N(S)| > 4|S| + 1, \tag{6}$$

Furthermore, let P be a path of length ℓ in G, which starts at v_0 and $\mathcal{E} = e_1,...,e_t$ be a set of a 'special' edges of P such that each vertex $v \in G$ is adjacent to at most one of the ends of edges from $\mathcal{E}$. Then, either G contains a path P' longer than P which starts at v_0 and contains all edges $e_1,...,e_t$, or G contains a family $\mathcal{P}$ of at least $m = \lceil(\ell-t-1)/3\rceil$ paths of length ℓ such that all of them start at v_0, contains $e_1,...,e_t$, and have different ends.

Proof. Let $\overline{\mathcal{P}}(v_0)$ be the set of all paths from $\mathcal{P}(v_0)$ which contain all the edges $e_1,...,e_t$. If one of the paths from $\overline{\mathcal{P}}(v_0)$ can be extended to a path longer than P then we are done. Thus, let us assume that such a path does not exist. Denote by S the set of all ends of paths from $\overline{\mathcal{P}}(v_0)$ and let S^+ and S^- be respectively left and rights neighbors of vertices from S on the path P. Let W be a set of the left ends of the edges $e_1,...,e_t$ on the path P and the vertex v_0. By Theorem 5, for the path P we have

$S \cup N(S) \subseteq S \cup S^- \cup S^+ \cup K$. Hence $|S \cup N(S)| \leq 3|S| + |S| + 1$ (the last vertex of P, which belongs to S, has no following vertex so $|S^+| \leq |S| - 1$). Consequently, $|S| > m$.

Now choose a set $S' \subset S$ such that $|S'| = m$. Then $\ell = |P| \geq |S' \cup N(S')| \geq 3|S'| + |S| + 1$. Thus,

$$|S'| \leq \frac{\ell - |S| - 1}{3},$$

which implies $m \leq \lceil (\ell - t - 1)/3 \rceil$. □

Observe that by Lemma 4 aas a random n-lift of a graph with h vertices and minimum degree $\delta \geq 15$ satisfies condition (6) of Lemma 6 with $m = \alpha n$. Thus, aas such graphs contain long paths (at least as long as αn) starting at v_0 and having m possible endpoints.

4. Proof of Main Result

Proof of Theorem 1. Let us first outline the main idea of our argument. We first divide edges of our graph K_h into three parts: two complete graphs of roughly $h/2$ each and a complete bipartite graph which contains all the edges connecting these two subgraphs. We denote this parts as F_1, F_2 and D. Clearly, this partition induces a partition of edges of $L(n, K_h)$ into three parts $L(n, F_1)$, $L(n, F_2)$, and $L(n, D)$.

We choose Hamilton cycles in F_1 and F_2 and look at its lifts. Let H be a subgraph of $L(n, K_h)$ which consists of lifts of these Hamilton cycles. By Lemma 2 we know that H is aas a sum of at most $4 \log n$ disjoint cycles. We are going to merge them by edges of $L(n, D)$, and use Pósa rotation technique to transform H into another spanning 2-regular graph of $L(n, K_h)$ which has fewer cycles. Thus, after at most $4 \log n$ such procedures we will get a Hamilton cycle in $L(n, K_h)$.

We refer to edges in D we include to H as *bridge* edges. We create a Hamilton cycle in stages, so that after each of them we get a spanning subgraph of $L(n, K_h)$. Each stage consists of the five following steps.

(1) Choose a shortest cycle in H. Denote it by C.
(2) Choose a bridge edge e_b from $L(n, D)$ between a vertex v from C and a vertex u which is at distance at least 5 from every other bridge in H. Add e_b to H and delete from it one edge adjacent to v and one edge adjacent to u. Denote vertices of degree one in H which arise after this step as q and r.
(3) Let P be a path which starts at $q \in V(L(n, F_1))$ and ends at $r \in V(L(n, F_2))$. Using Pósa transformation expand path P such that it preserves bridges i.e., once we add a bridge edge into our path we cannot choose it ends as a pivot. Moreover, each time we reach some cycle from H, we add this whole cycle (minus one edge) to the path.
(4) For each possible end $x \in V(L(n, F_2))$ of a path P, reverse P and use Pósa transformations to generate a set of possible ends of maximal paths which start at x.
(5) Find a bridge edge between set of possible endpoints of P from $L(n, F_1)$ and possible ends of P from $L(n, F_2)$ which is at the distance

at least 12 from all short cycles and at distance at least 5 from all other bridges in H. Add this edge to H.

Since at the beginning there are at most $4\log n$ cycles in H, after $4\log n$ steps we should obtain a Hamilton cycle in $L(n, K_h)$. Now we have to show that the probability that our algorithm fails in one of the above steps is $o(1/\log n)$. The crucial fact for our analysis is that the above procedure is using only tiny fraction of edges of $L(n, D)$. More precisely, we say that at some stage of the algorithm a vertex of $L(n, K_h)$ is *explored* if we have generated at least one edge of $L(n, D)$ incident to it. We shall show that in every stage of the process we explore not more than $O(\log n)$.

The first step (1) of our procedure is, clearly, always possible, so we can choose a cycle C from 2-regular subgraph H of $L(n, K_h)$ we have constructed so far. Now we consider the two cases.

First we study the early stages of the algorithm, when a cycle C chosen in the first step is short, i.e., his length is smaller than $\log^4 n$. Since, as we will see shortly, each cycle which contains bridges must be of order n, C must be contained in one of the graphs $L(n, F_1)$ and $L(n, F_2)$, say in the former one. Choose any vertex v of C and generate an edge incident to it in $L(n, D)$. We denote this new bridge by $e = \{v, w\}$, and mark its ends, v and w, as explored. Note that each bridge generated so far either is far away from all short cycles (as the ones chosen in step (4)), or has one end in a short cycle. But due to Lemma 3, aas the distance between any two short cycles is at least 15. Hence, $\{v, w\}$ is far away from all other bridges and all short cycles (except of C itself).

Let us assume that P is the path constructed in the previous step which starts at $q \in V(L(n, F_1))$ and ends in $r \in V(L(n, F_2))$. For (3) we have to show that we can either enlarge P, or rotate it in a way which preserves bridges and get a lot of paths with the same starting vertex as P but different ends. By Lemma 4, for $h \geq 30$, the graph $L(n, F_2)$ has good expanding properties and fulfill conditions needed to apply Pósa transformations. However in our procedure we must preserve all bridges, i.e., no end of a bridge contained P can be used as a pivot. Note that this fact prevents us from getting a path P with both ends in the same F_1. Because we chose bridges that are far away from each other, by Lemma 6 we obtain a family of at least αn paths starting at vertex v which have the same set of vertices as P, contains all bridges, but have different ends in $L(n, F_2)$. One can argue in the same way to assure that in step (4), when we 'reverse' each such path, we generate a family of paths with at least αn different ends in $L(n, F_1)$.

In the last step we are going to connect the ends of some long path. In order to do that we choose randomly $2\log n$ possible ends of rotated P in $V(L(n, F_2))$. Note that the number of different ends of rotated P is at least αn while the number of bridges is smaller than $4\log n$, the number of explored vertices is smaller than $O(\log^2 n)$, and the number of vertices in small cycles is smaller than $\log^5 n$. Thus, the probability that more than $\log n$ selected vertices lie within distance 12 from one of the above is bounded from above by

$$2^{2\log n}\Big(\frac{h^{15}\log^5 n}{\alpha n}\Big)^{\log n} = o(1/\log n)\,.$$

For each of selected vertices which are unexplored and are far away from small cycles and bridges we generate all edges incident to it and mark as active their ends. The probability that in this way we do not close a cycle by a bridge which is far away from small cycles and the other bridges is smaller than

$$\Big(1 - \frac{\alpha n - h^{12}\log^5 n}{hn}\Big)^{\log n} = o(1/\log n)\,.$$

Now we study the late stages of the algorithm, when the cycle C is longer than $\log^4 n$. Let us assume that more than half of vertices of C belongs to $L(n, F_1)$. Observe that since $|V(C)| \le hn/2$ we must then have

$$|V(L(n, F_2)| - |V(C)| \ge n/5\,. \tag{7}$$

Now we have to show that at stage (2) we may find a suitable bridge which connects $V(C) \cap V(L_1(n, F_2))$ to the other part of the graph. Note that the number of vertices which have been explored so far is smaller than $O(\log^2 n)$, the number of ends of the bridges is $O(\log n)$ and so we can always find $\log n$ vertices of C which are at distance at least 12 to each of them. Find all edges incident to them in $L(n, D)$ and mark the vertices as well as its neighbors as explored. The probability that each of the neighbors found in this way is either inside C or is far away from every bridge is, by (7), bounded from above by

$$\left(\frac{O(\log^2 n) + 4n/5}{n - \log^3 n}\right)^{\log n} = o(1/\log n)\,.$$

Hence, with probability at least $1 - o(1/\log n)$ we can select one bridge and add it to H.

The rest of the argument which deal with steps (3), (4), and (5) is precisely the same as in the case of small cycles.

Thus, we prove that the probability that in one stage of the algorithm it will fail is smaller than $o(1/\log n)$. Since at each stage the total number of cycles decrease by at least 1, due to Lemma 2, the algorithm ends after at most $4\log n$ stages. Hence the probability that at some stage it fails is $o(1)$. □

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