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Marcin Witkowski

Uniwersytet A. Mickiewicza w Poznaniu

Topological cliques in random lifts of graphs

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Opiekun pracy: Tomasz Łuczak

KAPITAŁ LUDZKI
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TOPOLOGICAL CLIQUES IN RANDOM LIFTS OF GRAPHS

MARCIN WITKOWSKI

Abstract. In this note we study asymptotic properties of random lifts of graphs introduced by Amit and Linial as a new model of random graphs. Given a base graph G and an integer n, a random lift of G is generated by replacing each vertex of G by a set of n vertices, and joining these sets by random matchings whenever the corresponding vertices of G are adjacent. In this paper we study the size of the largest topological clique in typical random lifts, with G fixed and $n \to \infty$. We show that almost all lifts of G contains a topological clique of size equal to the maximum degree in the 2-core of G plus one.

1. Introduction

The concept of covering maps of graphs is essentially a restriction to the case of graphs of the general topological notion of covering maps. Let G, $\tilde{G}$ be finite graphs. A map f from $V(\tilde{G})$ to $V(G)$ is called a *covering map* if for every vertex $x \in \tilde{G}$, the mapping f maps the neighbors of x one-to-one onto the neighbors of $f(x)$. Note that every covering map is also a homomorphism of graphs, but the converse is not true.

Whenever there is a covering map f from $\tilde{G}$ to G, we say that $\tilde{G}$ is a *lift* of G; then G is called the *base graph*. The set of vertices which are mapped in f to a given vertex $v \in G$ is called a *fiber* related to v and denoted by $\tilde{G}_v$. One can visualize $\tilde{G}_v$ as a vertical stack of vertices above v. In particular, fibers of all vertices from G have the same cardinality (when G is connected). This common value is denoted by n and called the *degree of covering*. The edge set of an n-lift $\tilde{G}$ consists of perfect matchings between fibers $\tilde{G}_u$ and $\tilde{G}_w$ for each edge $(u, w) \in E(G)$. The set of all n-lifts of a fixed graph G is denoted $L_n(G)$.

A simple model for a random lift $R(n, G)$ of G was proposed by Amit and Linial [1]. The idea is to choose uniformly at random some graph from the set $L_n(K)$. It can be obtained in following way: for each edge $\{v, w\}$ of G we join sets $\tilde{G}_v$ and $\tilde{G}_w$ by a random matching, i.e., a matching choosing uniformly at random from all possible matchings between V_v and V_w. Equivalently, if $\tilde{G}_v = \{v_1, ..., v_n\}$, and $\tilde{G}_w = \{w_1, ..., w_n\}$ we may choose uniformly at random one of $n!$ permutations $\sigma_{vw} : [n] \to [n]$ and connect v_i with $w_{\sigma_{vw}(i)}$.

As typical in random graph theory we are interested mainly in asymptotic properties of lifts of graphs when n is large. In particular, we say that some property of a random lift of graph G holds for *almost every* random lift of G if a graph H drawn at random from $L_n(G)$ has this property with probability

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$1 - \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$. We say that a sequence of random events ε_n occurs with high probability (w.h.p.) if $\lim_{n\to\infty} \Pr[\varepsilon_n] = 1$.

There are only a handful of papers concerning this model of random graphs. Amit and Linial [1] proved that if G is a simple connected graph with minimum degree $\delta \geq 3$, then almost every random lift of G is δ-connected. They continued the study of random lifts in [3] proving that random lifts have good expanding properties. The third article in this series of papers on random lifts, written jointly with Matousek [2], deals with the independence and chromatic numbers of random lifts.

We say that some theorem is a *zero-one law* if it specifies that a certain type of event either w.h.p. happen or w.h.p. does not happen. That is, the probability of such an event occurs tends either to zero, or to one. Some part of research in the area of random lifts is connected with such theorems. Linial and Rozenman [9] showed that for any graph G its random lifts either w.h.p. has a perfect matching or w.h.p. does not have such a matching. Similar question has been raised for hamiltonicity, but there are only some partial results obtained for this property (see [4] and [5]).

Subdividing an edge uv in a graph means replacing the edge uv by a path uwv containing a new vertex w. Graph H is a *topological clique* if it can be obtained from a complete graph by series of edge subdivisions. The vertices of the original complete graph are called *branch vertices*. In this paper we focus on the typical size of the maximum topological clique in random lifts of a given graph.

Drier and Linial [6] studied existence of minors in random lifts of complete graphs. Among others they proved following theorems concerning topological cliques.

Theorem 1. *For almost every $H \in L_n(K_\ell)$, a maximum topological clique in H is smaller than $O(\sqrt{n\ell})$.*

Theorem 2. *If $\ell \geq \Omega(n)$ then for almost every $H \in L_n(K_\ell)$, the size of a maximum topological clique in H is greater than $\Omega(n)$.*

In those cases the degree of covering is a function of the size of complete graph. Our main result, Theorem 3, deals with the case when the size of the base graph is fixed and does not depend on n. In order to state it we need to introduce one more notion. For a graph $G = (V, E)$, the *core* of G, denoted as $core(G)$, is the maximal subgraph of G with minimum degree at least two. It is easy to see that the core of the lift of a graph G is the same as the lift of the core of G. Thus, the maximum size of the topological clique contained in the lift of the graph is bounded from above by $\Delta(core(G)) + 1$, where $\Delta(core(G))$ stands for the maximum degree in the core of G. We show that in a random lift of a given graph G with high probability we can find a topological clique of a maximum possible size.

Theorem 3. *For a given graph G almost every $H \in L_n(G)$ contains a topological clique of size $\Delta(core(G)) + 1$.*

In the following section we present a proof of this theorem. We conclude the note with a simple application of this result.

2. Proof of the Main Result

Proof of Theorem 3. Let G be a base graph and let H denote the core of G. By Δ_H we denote the maximum degree of vertices in H and let $v \in V(H)$ be a vertex with degree equal to Δ_H. We show that with probability tending to 1 there exists $\Delta_H + 1$ vertices from the fiber $\tilde{H}_v$ related to vertex v that form a topological clique.

If $\Delta_H = 2$ then the base graph G contains a cycle. The lift of this cycle is also a cycle, so the lift of G contains a topological clique of size 3. Therefore we may assume $\Delta_H > 2$ and, since we are considering the core of G, $\delta_H \geq 2$.

In our argument we use a family $\mathcal{W} = \{C_1, C_2, ..., C_{\Delta_H}\}$ of closed walks which starts and ends in v and are such that each edge incident in v is traversed from v to its neighbour by precisely one walk from $\mathcal{W}$ (for which it is, of course, the starting edge), and each walk from $\mathcal{W}$ form either a cycle, or a cycle connected by a path to v. Moreover, we require the family $\mathcal{W}$ to be *edge-distinct* meaning that each walk is either a reverse of some other walk from the family or have at least one edge which is not contained in any other walk from $\mathcal{W}$. It is easy to see that, because the minimum degree of H is at least two, such a family exists.

Note that for every walk from $C_i \in \mathcal{W}$ its lift $\tilde{C}_i$ is a set of disjoint walks which start and end at vertices of the fiber $\tilde{H}_v$. Major part of the proof requires, for a given $u \in \tilde{H}_v$, to recursively build sets of vertices of $\tilde{H}$ which can be reached from u; we do it in the following way. Let $T_0(u) = R_0(u) = u$. By $T_1(u)$ denote the set of vertices of the closed walks in $\tilde{H}$ which starts at u and are lifts of C_i, for $i \in \{1, 2, \ldots, \Delta_H\}$. Let $R_1 = T_1(u) \cap \tilde{H}_v$ denote the set of all vertices of a fiber above v in which those walks ends. Then, $T_2(u)$ be the set which consists of $T_1(u)$ and $T_1(u')$, for all $u' \in R_1(u) \backslash u$. Thus, let us recall, we start with a vertex u, use the lifts of each C_i, for $i \in \{1, 2, \ldots, \Delta_H\}$, to travel from u to the fiber $\tilde{H}$ and then use all the walks from $\mathcal{W}$ again to reach next vertices. Notice that this time there is no point in using edges by which we arrived to the points from $R_1(u)$, because they take us back to u using vertices which were already visited. In general we set $R_\ell(u) = T_\ell(u) \cap \tilde{H}_v$ and call it *ℓ-vicinity* of u. The set of vertices $T_\ell(u)$ is defined recursively, we take all vertices of $T_{\ell-1}(u)$ and add to them vertices of closed walks which covers walks from $\mathcal{W}$ and starts at vertices from $R_{\ell-1}(u) \backslash R_{\ell-2}(u)$. Again, whenever we branch from any one of them one direction point us to previously used path, so it does not add any new vertex to $T_\ell(u)$.

Note that the set $R_\ell(u)$ has a natural structure of a tree rooted at u, which has all leaves placed on the fiber $\tilde{H}_v$; we can think of the ordering vertices of this tree from the root to the leaves and for any $w \in R_\ell(u)$ we call the vertices w arrows the *successors* of w, i.e. each successors of w follows w in the lift of one of the walks from $\mathcal{W}$. Note that because $\delta(H) \geq 3$ the sizes of $R_\ell(u)$ are expected to grow exponentially with ℓ, at least for small ℓ.

Let us consider vertices from sets $R_\ell(u')$ and their successors on the tree rooted at u'. Notice that for a given closed walk C starting at v in a base graph G a mapping assigning in $\tilde{G}$ to each vertex of $\tilde{G}_v$ its closest successor on $\tilde{G}_v$ is a random permutation. Whenever we have in $\mathcal{W}$ two closed walks which are reverse of each other the corresponding permutations are inverse.

In the other case the assumption about edge-distinctness of closed walks let us treat any two permutations related to adequate walks as independent. We will not draw whole permutations at once but generate them in steps by choosing successively successors for selected vertex.

The idea of the proof is following. We choose $\Delta_H + 1$ vertices $U = \{u_1, u_2, \ldots, u_{\Delta_h+1}\}$ from the fiber $\tilde{H}_v$. Our aim is to connect each pair of the vertices from U one by one by paths. We do it in two phases; firstly for each $u \in U$ we generate $R_1(u)$ and then in the second phase we match in pairs vertices from 1-vicinity of each vertex with the particular vertices from 1-vicinities of other vertices. Then we try to connect each of such a pair by a path. In order to connect some u_i with u_j, $i < j$, we generate $R_\ell(u_i)$ and $R_\ell(u_j)$, where ℓ are chosen in such a way that $|R_\ell(u_i)|, |R_\ell(u_j)| = \Theta(\sqrt{n}\log n)$. Then, w.h.p. the sets $R_\ell(u_i)$ and $R_\ell(u_j)$ have a non-empty intersection and so there is a path connecting u_i and u_j in $\tilde{H}$. We repeat above argument for each pair of vertices matched at the beginning of second phase.

Note, however, that in the topological clique the paths which connect the vertices should be vertex disjoint. This is the main technical obstacle we should deal with in our argument. Thus, roughly speaking, in the process of generating sets $T_\ell(u')$, we omit the vertices which has been used in the sets $T_\ell(u'')$ we generate in the earlier stages of the algorithm. Then, clearly, the modified set $\hat{T}_\ell(u')$ generated in this way will be slightly smaller than $T_\ell(u')$, but we argue that w.h.p. the difference is not substantial and do not affect much the probability that two vertices are connected.

When we connect vertices of a clique by paths it is important that this paths do not cross at any place. In particular we want to avoid intersections on $R_\ell(u')$ as well as on the vertices from $T_\ell(u')$. Let us first observe that each time there is an intersection between some closed walks C_i and C_j in a base graph G, using a walk from the set of all walks that cover C_i as a part of the clique forbid us to use exactly one closed walk from the set of all walks that cover the walk C_j. Thus, in this case, we add the ends of the second walk to the set of already visited vertices in order to prevent them from being a part of vicinity for any prospective vertex. Let c denote the total number of intersections between walks $C_1, C_2, ..., C_{\Delta_H}$ apart from vertex v. Note that since G is a given graph, c is bounded from above by a constant which depends only on G but not on n.

After the above sketch of the idea of the proof let us be more specific. We assume that we start at some vertex u_1 from the fiber $\tilde{G}_v$. We are going follow the walks $C_1, C_2, ..., C_{\Delta_H}$ in $\tilde{G}$ and return at some vertices $u_1^1, u_1^2, ..., u_1^{\Delta_H}$ from $\tilde{G}_v$. As stated earlier this is the same as selecting adjacent elements in random permutations corresponding to those walks. The probability of the event $\mathcal{A}$ that any of the path is getting back to u_1 (i.e. that u_1 is a fixed point in any of the permutations) or that two paths ends at the same vertex (which means that adjacent elements for two different permutations are the same) is smaller than

$$\Pr(\mathcal{A}) \leq \Delta_H \frac{\Delta_H}{n} \xrightarrow{n\to\infty} 0.$$

When we also avoid possible intersections on cycles, the probability of the corresponding event $\hat{\mathcal{A}}$ is bounded by

$$\Pr(\hat{\mathcal{A}}) \leq \Delta_H \frac{2c\Delta_H}{n} \overset{n\to\infty}{\longrightarrow} 0,$$

since each intersection between the closed walks add two vertices to the set of vertices that we want to avoid in random choice. From this point we do not distinguished this two cases and always multiply the probability of bad events by c. Since we only assume that c can be bounded from above by a constant independent on n, in the following estimates we skip the 2 and any other possible constants and write just c.

The vertices of $R_1(u_1) = u_1^1, u_1^2, ..., u_1^{\Delta_H}$ are the 1-vicinity of u_1. Let S denote the set vertices which have been "already visited", meaning all vertices u' for which we have already generate $T_1(u')$. At this moment $S = \{u_1, u_1^1, u_1^2, ..., u_1^{\Delta_H}\}$ plus possible ends $\{u_p, .., u_q\}$ of other walks which cross with vertices from $T_1(u)$. In the next stage we look at the set $\tilde{G}_v - S$ and choose at random Δ_H vertices $u_2, u_3, ..., u_{\Delta_H+1}$ from it, then we generate sets $R_1(u_2), R_1(u_3), ..., R_1(u_{\Delta_H+1})$ in the same manner as for u_1. Again the probability of the event $\mathcal{B}$ that any of those vicinities is smaller than Δ_H, or contains already visited vertices, or that two of them intersect, is bounded from above by

$$\Pr(\mathcal{B}) \leq \Delta_H^2 \frac{c(\Delta_H + 1)^2}{n} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, with high probability we can choose a set of vertices $u_1, ..., u_{\Delta_H+1}$ together with disjoint sets $R_1(u_1), ..., R_1(u_{\Delta_H+1})$ of sizes Δ_H for each one of them. Connecting vertices from those 1-vicinities by a disjoint paths would give us a topological clique. Therefore we will take pairs of vertices $\{u_i^j, u_j^i\}$ for $i \neq j$ and $\{u_i^j, u_{\Delta_H+1}^i\}$ for $i = j$ and connect them with a set of disjoint paths.

Let us recall that in each of the forthcoming steps we avoid vertices visited earlier. One of the reason is that we do not want to cross with some path chosen earlier and the second is that we want vicinities to be generated uniformly at random independently of previously generated ones.

Consider the first pair of vertices $\{u_1^2, u_2^1\}$. At first we branch from vertex u_1^2 expanding set $T(u_1^2)$ in stages in a following manner. Let S_{C_i} denote the vertices already visited using the closed walk C_i or, if exist, a closed walk which is a reverse of C_i (i.e. vertices with generated in random permutation derived from C_i). In the j-th stage we generate $T_j(u_1^2)$; this is equivalent with choosing, for all of the vertices $u' \in R_{j-1}(u_1^2) \backslash R_{j-2}(u_1^2)$, and for all walks $C_k, k \in \{1, 2, ..., \Delta_H\}$, an element from respective set $\tilde{G}_v - S_{C_k}$ (set of unvisited vertices for the permutation corresponding to C_k) at random with uniform distribution. The selected vertex is added to the vicinity and to the sets S and S_{C_k}. Furthermore if a walk C_k crosses any other walk C_q we add adequate vertices from crossed walk $\tilde{C}_q$ to the sets S and S_{C_q}. We continue the expanding until $|R_\ell(u_1^2)| = \Theta(\sqrt{n}/\log n)$. The expected value of a number of times we choose some vertex which is already in S is bounded by

$$\frac{\sqrt{n}}{\log n} \cdot \frac{c\sqrt{n}}{n \log n} = \frac{c}{\log^2 n} \overset{n\to\infty}{\longrightarrow} 0.$$

It means that if we expand to fewer than $\sqrt{n}/\log n$ vertices, then the probability of choosing some of the already visited vertices tends to zero. Let Y be a random variable counting the number of times we select a vertex from S while expanding the vicinity of u_1^2 to size $|R_{\ell'}(u_1^2)| = \Theta(\sqrt{n}\log n)$.

Since

$$\mathrm{E}Y = \sqrt{n}\log n \cdot \frac{c\sqrt{n}\log n}{n} = c\log^2 n,$$

we can bound Y using Markov's inequality and get

$$\Pr(Y > \log^3 n) < \frac{c\log^2 n}{\log^3 n} \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, w.h.p. there would be no more than $\log^3 n$ times we have chosen already visited vertex. As we have already mentioned we do not add those vertices to $T_{\ell'}(u_1^2)$. Considering worst case scenario we may assume that we start from the set of size $\sqrt{n}/\log n - \log^3 n$ and in each stage we increase it $\Delta_H - 1$ times. Which implies that in $O(\log n)$ stages we expand the $|R_{\ell'}(u_1^2)|$ to at least $\sqrt{n}\log n$ vertices.

The same analysis can be done in respect to the vertex u_2^1. Again, during the expansion of the vicinity, we have to exclude all vertices from S (which contains now also the vertices from $R_{\ell'}(u_1^2)$, where $\ell' \approx \log n$). Thus, the expected number of times we choose some vertex from S, for the first $\sqrt{n}/\log n$ vertices equals

$$\begin{aligned} \frac{\sqrt{n}}{\log n} \cdot \frac{c\sqrt{n}}{(n-\sqrt{n}\log n)\log n} &= \frac{cn}{(n-\sqrt{n}\log n)\log^2 n} \\ &= \frac{c}{(1-\frac{\log n}{\sqrt{n}})\log^2 n} \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

Morover, as before, the number of bad choices which can occur during further expansion (from this point we want to avoid only vertices from the $R_\ell(u_2^1)$, since our aim is to connect with some vertex from $R'_\ell(u_1^2)$) can be bounded by

$$\Pr(Y > \log^3 n) < \frac{c\log^2 n}{\log^3 n} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally we expand both ℓ'-vicinities to the size of $\sqrt{n}\log n$. In order to connect vertices u_1^2 and u_2^1 by a path we need to find some vertex $w \in R_{\ell'}(u_2^1) \cap R_{\ell'}(u_1^2)$. Then the path $u_1^2 - w - u_2^1$ would connect u_1 with u_2. The probability that such a vertex does not exist can be bounded by

$$\begin{aligned}\frac{\binom{n-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{n}{\sqrt{n}\log n}} &= \frac{(n-\sqrt{n}\log n)!}{(\sqrt{n}\log n)!(n-2\sqrt{n}\log n)!}\frac{(\sqrt{n}\log n)!(n-\sqrt{n}\log n)!}{n!}\\ &= \frac{(n-2\sqrt{n}\log n)(n-2\sqrt{n}\log n+1)\cdot\ldots\cdot(n-\sqrt{n}\log n)}{(n-\sqrt{n}\log n)(n-\sqrt{n}\log n+1)\cdot\ldots\cdot n}\\ &\leq \left(1-\frac{\log n}{\sqrt{n}-\log n}\right)^{\sqrt{n}\log n}\\ &\leq \left(1-\frac{\log n}{\sqrt{n}\log n}\right)^{\sqrt{n}\log n} = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

The argument for other pairs of vertices is similar to the one above but after each phase we must take into account the fact that the size of the set S increases. Therefore the probability of success in connecting the last pair is smaller than the probability of success in connecting previous pairs. Since there are only $(\Delta_H+1)^2$ pairs, then if we show that probability of failure in connecting the last pair tends to zero as n goes to infinity, then we prove that w.h.p. we get success for all the events.

When we are connecting the last pair size of S is less than $S=(\Delta_H+1)^2\sqrt{n}\log n+\Delta_H+1$. Then the probability that we would be able to expand the vicinity of both vertices to the size of $\sqrt{n}/\log n$, without selecting some previously visited vertices, is bounded by

$$\begin{aligned}\frac{\sqrt{n}}{\log n}\cdot\frac{2c\sqrt{n}}{(n-(\Delta_H+1)^2\sqrt{n}\log n+\Delta_H+1)\log n}\\ \leq \frac{2cn}{(n-(\Delta_H+2)^2\sqrt{n}\log n)\log^2 n}\\ = \frac{2c}{(1-\frac{(\Delta_H+2)^2\log n}{\sqrt{n}})\log^2 n} \leq \frac{2c}{\log n} \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Now we estimate the number of bad choices during further expansion. Let Y be a random variable counting the number of times we select a vertex from S while expanding vicinities to the size of $\sqrt{n}\log n$. Then

$$\begin{aligned}\mathrm{E}Y &= \sqrt{n}\log n\cdot\frac{c\sqrt{n}\log n}{(n-(\Delta_H+1)^2\sqrt{n}\log n+\Delta_H+1)}\\ &\leq \frac{cn\log^2 n}{n-(\Delta_H+2)^2\sqrt{n}\log n} = \frac{c\log^2 n}{\frac{\sqrt{n}-(\Delta_H+2)^2\log n}{\sqrt{n}}}\\ &\leq cn\log^2 n\frac{\sqrt{n}}{\sqrt{n}-(\Delta_H+2)^2\log n} \leq cn\log^3 n.\end{aligned}$$

so

$$\Pr(Y>\log^4 n) < \frac{c\log^3 n}{\log^4 n} \leq \frac{c}{\log n} \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, w.h.p. there will be no more than $\log^4 n$ times we choose already visited vertex, i.e. w.h.p. in $O(\log n)$ stages we expand the vicinities of vertices from our pair to the size of $\sqrt{n}\log n$.

Finally the probability that we do not find a vertex which connect this two vicinities equals

$$\frac{\binom{m-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{m}{\sqrt{n}\log n}} = \frac{(m-\sqrt{n}\log n)!}{(\sqrt{n}\log n)!(m-2\sqrt{n}\log n)!}\frac{(\sqrt{n}\log n)!(m-\sqrt{n}\log n)!}{m!}$$
$$\leq \frac{1}{n-\sqrt{n}\log n} \overset{n\to\infty}{\longrightarrow} 0\,,$$

where $m = n - (\Delta_H + 1)^2\sqrt{n}\log n + \Delta_H + 1$.

Thus, we show that the probability of failure in connecting any pair is smaller than $o(1)$. Since there are finite number of pairs, the probability that we do not find a topological clique of size $\Delta_H + 1$ can be also bounded by $o(1)$. □

3. Applications

We can apply Theorem 3 to describe the connectivity properties of a random lift of a fixed graph G. The first researchers who studied the connectivity properties of random lifts were Amit and Linial [1]. They proved following theorem.

Theorem 4 ([1])**.** *Let G be a connected simple graph with minimum degree $\delta \geq 3$. Then almost every $H \in L_n(G)$ is δ-connected.*

Let us recall that graph with at least $2k$ vertices is said to be *k-linked* if for every $2k$ distinct vertices $s_1, s_2, ..., s_k, t_1, t_2, ..., t_k$ it contains k vertex-disjoint paths $P_1, P_2, ..., P_k$ such that P_i connects s_i to t_i, $1 \leq i \leq k$. Obviously, from Menger's theorem it follows that each k-linked graph is k-connected. The converse is far from being true. Jung [7] and, independently, Larman and Mani [8] proved that every $2k$-connected graph that contains a K_{3k} as a topological minor is k-linked. Combining their result with Theorem 1 and Theorem 2 we get the following theorem.

Theorem 5. *Let G be a connected graph with minimum degree δ. Then almost every $H \in L_n(G)$ is $\min\{\Delta(core(G))/3, \delta/2\}$-linked.*

Since the result of Larman and Mani is not sharp above theorem gives us only the lower bound on the maximum k for which almost every random lift $L_n(G)$ is k-linked. The question about the exact value of this parameter (or better upper and lower bounds for it) remains open.

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Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland