Properties of random coverings of graphs

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“ *When you are a Bear of Very Little Brain, and you Think of Things, you find sometimes that a Thing which seemed very Thingish inside you is quite different when it gets out into the open and has other people looking at it.*

Kiedy się jest Misiem o Bardzo Małym Rozumku i myśli się o Rozmaitych Rzeczach, to okazuje się czasami, że Rzecz, która zdawała się bardzo Prosta, gdy miało się ją w głowie, staje się całkiem inna, gdy wychodzi z głowy na świat i inni na nią patrzą. ”

A.A. Milne, *The House at Pooh Corner (Chatka Puchatka)*, 1928

Dla wszystkich, którzy przyczynili się do jej powstania
i tych, zainteresowanych na tyle, by ją przeczytać.
For those who helped make it possible
and those who are reading it.

> “ *I can no other answer make but thanks, and thanks.* ”
>
> William Shakespeare, *Twelfth Night*, 1602

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Abstract

In the thesis we study selected properties of random coverings of graphs introduced by Amit and Linial in 2002. A random n-covering of a graph G, denoted by $\tilde{G}$, is obtained by replacing each vertex v of G by an n-element set $\tilde{G}_v$ and then choosing, independently for every edge $e = \{x, y\} \in E(G)$, uniformly at random a perfect matching between $\tilde{G}_x$ and $\tilde{G}_y$.

The first problem we consider is the typical size of the largest topological clique in random covering of given graph G. We show that asymptotically almost surely a random n-covering of a graph G contains the largest possible topological clique.

The second property we examine is the existence of a Hamilton cycle in $\tilde{G}$. We show that if G contains two edge disjoint Hamilton cycles and has minimum degree at least 5, then asymptotically almost surely $\tilde{G}$ is Hamiltonian.

Streszczenie

Przedmiotem rozprawy doktorskiej są asymptotyczne własności losowych nakryć grafów zdefiniowanych przez Amita i Liniala w 2002 roku, jako nowy model grafu losowego. Dla zadanego grafu bazowego G losowe nakrycie stopnia n, oznaczane jako $\tilde{G}$, otrzymujemy poprzez zastąpienie każdego wierzchołka v przez n-elementowy zbiór $\tilde{G}_v$ oraz wybór, dla każdej krawędzi $\{x, y\} \in E(G)$, z jednostajnym prawdopodobieństwem, losowego skojarzenia pomiędzy zbiorami $\tilde{G}_x$ i $\tilde{G}_y$.

Pierwszym zagadnieniem poruszanym w pracy jest oszacowanie wielkości największej topologicznej kliki zawartej (jako podgraf) w losowym nakryciu danego grafu G. Udało się pokazać, że asymptotycznie prawie na pewno losowe nakrycie grafu G zawiera największą z możliwych topologiczną klikę.

Drugim badanym zagadnieniem jest pytanie o istnienie w podniesieniu grafu cyklu Hamiltona. W pracy pokazujemy, że jeżeli graf G zawiera dwa krawędziowo rozłączne cykle Hamiltona i ma minimalny stopień co najmniej 5, to asymptotycznie prawie na pewno nakrycie $\tilde{G}$ jest grafem hamiltonowskim.

Contents

> *What is there that confers the noblest delight? What is that which swells a man's breast with pride above that which any other experience can bring to him? Discovery! (...) To give birth to an idea, to discovery a great thought-an intellectual nugget, right under the dust of a field that many a brain-plough had gone over before.*

Mark Twain, *The Innocents Abroad*, 1869

1 Introduction

The main object of this thesis is to study selected properties of random coverings of graphs. The idea behind this model is to transfer the topological notion of covering maps to the case of graphs. Then, we introduce a probabilistic structure on the set of all graphs that cover a fixed base graph.

One of the simplest and most frequently used model of random graphs is the binomial random graph $G(n, p)$, studied already by Erdős and Rényi [14]. In this model, a graph is generated by drawing n vertices and adding edges between them with probability p, independently for each pair of vertices. $G(n, p)$ has been proved useful in many constructions of graphs with certain unusual properties, such as graphs with large chromatic number and large girth, graphs with some special extremal properties as well as in modelling various processes in statistical physics [19].

Nevertheless, the binomial model has some serious limitations. For instance, it poorly reflects the properties of so called Internet graphs. Moreover, since, roughly speaking, we cannot force $G(n, p)$ to have some local or global properties of certain types, there are some problems when it is applied to constructing error correcting codes, random maps, or to provide tight estimates for Ramsey numbers.

Random coverings of graphs were proposed to meet some of these needs. The model of random covering we are interested in was introduced by Amit and Linial [2]. The concept comes from the topological notion of covering maps. A graph is a topological object (e.g. it can be viewed as a one dimensional simplicial complex), so covering maps can be defined and studied for graphs. Later however to distinguish this model from the other existing concepts of coverings in graph theory, as edge coverings or vertex coverings, it has been proposed [3] to use the name “lift” instead of “covering”. From this point on we will be mainly using the second name.

For graphs G and H, a map $\pi : V(H) \to V(G)$ is a *covering map* from H to G if for every $v \in V(H)$ the restriction of π to the neighborhood of v is a bijection onto the neighborhood of $\pi(v) \in V(G)$. If such a mapping exists, we say that H is a lift of G and G is the base graph for H. It is easy to see that for connected graphs the number of vertices which are mapped to one vertex of the base graph is the same for all vertices $v \in G$. We denote this common value by n and call it the degree of covering. The set of all graphs that are n-lifts of G is called $L_n(G)$. The random n-lift of a graph G is obtained by choosing uniformly at random one graph from the set $L_n(G)$. More formal definition of the model can be found in Chapter 2.

Our interest lies in the asymptotic properties of lifts of graphs, when the parameter n goes to infinity. In particular, we say that a property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity. Sometimes, instead of saying that the random lift of G has aas a property $\mathcal{A}$, we write that almost every random lift of a graph G has $\mathcal{A}$, or, briefly, just that the random lift of G has property $\mathcal{A}$.

Random lifts of graphs are interesting mathematical objects by their own and there are several papers which study how typical properties of random lifts reflect properties of the their base graphs [2, 3, 4, 26, 28]. Nonetheless, the main motivation to introduce this model has been its applications, so let us mention some of them. The first one is to solve problems in extremal graph theory and construct graphs with good expanding properties [1, 12, 27]. Amit and Linial also suggested that random lifts can be found useful in some algorithmic problems, in particular, they were able to reformulate the Unique Game Conjecture in terms of random lifts [25]. Recently the idea of random coverings has been pushed further to study a random higher-dimensional complexes [5]. One can notice that every covering map is also a homomorphism of graphs, but the converse is not true. Thus one can consider coverings as the special class of homomorphisms of graphs, and study whether conjectures concerning homomorphism of graphs holds for the subclass of coverings.

For the applications the main challenge is to turn lifts and random lifts into tools in the study important questions in computational complexity and discrete mathematics. The most spectacular result obtained with random lifts of graphs concern spectral properties of graphs. Lifts can be used to construct regular graphs with large spectral gap. Currently we know how to construct Ramanujan graphs (i.e. d-regular graphs with second eigenvalue $\lambda_2 \leq 2\sqrt{d-1}$) only for $d = p^\alpha + 1$, with p being a prime number [30]. Bilu and Linial [6] presented a new explicit construction for expander graphs with nearly optimal spectral gap, namely having second eigenvalue of order $O(\sqrt{d\log^3 d})$. The construction is based on a series of 2-lift operations. Recently Lubetzky, Sudakov and Vu [29] proved that a typical n-lift of a Ramanujan graph is "nearly" Ramanujan.

As we have already mentioned there are only a handful of papers concerning asymptotic properties of random lifts. In the paper in which they introduced the model Amit and Linial [2] proved that random lifts are highly connected. In the second paper on random lifts the authors proved that random lifts have good expanding properties [3]. The infinite d-regular tree is an ideal expander. The main challenge is to find a finite graph with similar combinatorial and spectral properties. One idea is to look at the minors of a graph. An infinite tree has no nontrivial minors. The question is which of the minors M of a graph G are persistent, meaning they are minors of almost every lift of G. Drier and Linial [13] studied existence of minors in random lifts of complete graphs, proving existence of topological cliques of certain sizes in lifts of small degree. We continue the study of existence of topological cliques in random lifts of graphs [34], showing that almost every random lift of a given graph contains the largest possible topological clique.

In Chapter 3 of this thesis we discuss the basic properties of random lifts focusing especially on their connectivity properties. In the next part of the thesis, we prove the existence of large topological cliques in random lifts. Using basically the same argument we will argue that asymptotically almost surely a random lift of a graph G with minimum degree $\delta \geq 2k-1$ is k-linked. This result is a substantial strengthening of a theorem by Amit and Linial [2] from their first paper on random coverings.

We say that a theorem is a *zero-one law* if it specifies that an event of a certain type either happens asymptotically almost surely or asymptotically almost surely does not happen. This will mean to us that the probability of an occurrence of such event tends either to zero, or to one, as $n \to \infty$. Some part of research in the area of random lifts is connected with such theorems. Linial and Rozenman [28] showed that for any graph G its random lifts either almost surely has a perfect matching or almost surely does not have such a matching. Similar question has been raised regarding existence of Hamiltonian cycles [26]:

Problem 1. Is it true that asymptotically almost surely for every G every or none of the graphs in $L_n(G)$ have a Hamilton cycle?

Problem 2. Let G be a d-regular graph with $d \geq 3$. Is it true that random n-lift of G is asymptotically almost surely hamiltonian?

In fact, the question about existence of Hamiltonian cycle is one of the most studied in the topic of random lifts. Chebolu and Frieze [9] proved that random lifts of appropriately large complete directed graphs asymptotically almost surely contains a Hamiltonian cycle. Burgin, Chebolu, Cooper and Frieze [8] proved that there exists a constant c such that almost every lift of complete graphs on more than c vertices contains a Hamiltonian cycle. Together with Łuczak and Ł.Witkowski we were able to show that almost every random lift of a graph G with minimum degree at least 5 and two edge disjoint Hamiltonian cycles is Hamiltonian [32]. Proof of this fact can be found in Chapter 5.

Let us also mention similar concentration questions raised for the chromatic number of lifts of graphs. We do not know whether for every graph G the chromatic number of almost every lift of G tends to concentrate in one value [4]. The simplest case for which we can not determine a result is the complete graph on five vertices K_5. We know that chromatic number of random lift of G may be either 3 or 4, but we do not know whether both these values are obtained with probability bounded away from zero, or the chromatic number of a random lift of K_5 almost surely takes only one of them. Farzad and Theis tried to solve this problem but they were able to prove only that random lifts of K_5 minus one edge are almost surely 3-colourable [15].

In the Chapter 2 we will recall the definitions and notions necessary in this thesis. Here we also define the model of random coverings of graph we shall be dealt with. In Chapter 3 we presents known properties of random lifts in a more thorough way. Moreover some useful facts concerning asymptotic properties of lifts, which are used in next chapters, are proven in this part of the thesis.

> *I do not carry such information in my mind since it is readily available in books... The value of a college education is not the learning of many facts but the training of the mind to think.*

Albert Einstein, *In response to question about the speed of sound*, *NYT* 1921

2 Preliminaries

We start with basic definitions of terms and notions that are used throughout the thesis. The most fundamental notions of this thesis is the concept of a graph. A *simple graph* or just a *graph*, is a pair $G = (V, E)$, where $E \subset V^{(2)} = \{\{x, y\} \subset V : x \neq y\}$. The set V, also denoted $V(G)$, is called the set of *vertices* of G. The set E (sometimes denoted $E(G)$) is called the set of *edges* of G. The number of vertices $|G| = |V(G)|$ is called the *order* of G and $e(G) = |E(G)|$ is called the size of G. If H is a graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then we say that H is a *subgraph* of G.

Typically the first questions that are asked about special subclasses of graphs are their connectivity properties. The set of *neighbours* of a vertex v is denoted

$$N(v) = \{w \in V(G) : \{v, w\} \in E(G)\}.$$

For $\{v, w\} \in E(G)$ we say that a vertex $w \in N(v)$ is *adjacent* to v and an edge $\{v, w\}$ is *incidence* to v and w. The number of neighbours of a given vertex $d(v) = |N(v)|$ is called the *degree* of the vertex. The *minimum degree* over all vertices of G is denoted

$$\delta(G) = \min_{v \in V(G)} d(v),$$

while for the *maximum degree* over all vertices of G we write

$$\Delta(G) = \max_{v \in V(G)} d(v).$$

A *walk* is an alternating sequence of vertices and distinct edges, beginning and ending at vertices, where each vertex is incident to the edges that precede and follow it in the sequence.

If all the vertices in a walk are different we call it a *path*. The length of a path is the number of edges which belong to it. The path together with the edge joining its ends forms a *cycle*. A cycle containing all the vertices of a graph is called *Hamiltonian* or a *Hamilton cycle*. A graph which contains a Hamiltonian cycle as a subgraph is called *Hamiltonian*. Finding a Hamilton cycle is one of the most important problems in graph theory and has many applications in clustering of data arrays, route assignments, analysis of the structure of crystals and others [24].

For vertices u and v the *distance* $dist(u, v)$ is the length of the shortest path connecting u to v. The set of vertices at distance at most d to vertex v is called a *d-neighbourhood* and denoted

$$N_d(v) = \{u \in V(G) : dist(u, v) \leq d\}.$$

A graph is *connected* if for every pair of vertices $u, v \in V(G)$ there is a path in G from u to v (called uv path). A graph is k*-connected* if for every pair of vertices $u, v \in V(G)$ there are k vertex-disjoint paths in G from u to v. Equivalently, by Menger's theorem [11], graph is k-connected if and only if it stays connected after removing any set of k vertices.

A graph H is called a *minor* of a graph G if it can be obtained from G by a series of edge contradiction and deletions, and possibly omitting some vertices and edges. A graph that is obtained by replacing the edges of H with vertex disjoint paths is called a *subdivision* of H. If X is isomorphic to a subgraph of G, and X is a subdivision of a graph H, we say that H is a *topological minor* of G. Clearly, each topological minor is a minor as well, but it is easy to see that converse is not true.

We distinguish several special classes of graphs. By K_n we denote a graph with $E(K_n) = V^{(2)}$ and called it the *complete graph*, or *clique* of order n. A graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to a vertex in V is called a *bipartite graph*. If all vertices in G have the same degree equal d, then G is called d*-regular*. A set of disjoint edges of a graph is called a *matching*; a matching covering all vertices from $V(G)$ is called a *perfect matching*. A connected graph with no cycles is called *a tree*. The vertices of degree one in a tree are called *leaves*.

2.1 Coverings

The notion of covering maps between graphs is a restriction of more general topological notion of covering maps to the case of graphs (notice that graph can be viewed as one dimensional simplicial complexes). A *covering map* of topological spaces $f : X \to Y$ is an open surjective map that is locally homeomorphism, meaning that the neighbourhood on each point x in X looks the same after mapping $f(x)$ in Y. We will define a covering map of graphs in terms of homomorphism of graphs.

Definition. Let G and H be graphs. A *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that

$$\{x, y\} \in E(G) \Rightarrow \{f(x), f(y)\} \in E(H).$$

By $H \to G$ we denote the existence of a homomorphism of H onto G. Notice that the smallest k for which there is a homomorphism of G onto K_k is the chromatic number of G. A covering map between graphs is a "locally bijective" homomorphism.

Definition. For graphs G and H a homomorphism $\Gamma : V(H) \to V(G)$ is a *covering map* if for every $x \in V(H)$, the neighbor set $N(x)$ is mapped 1-to-1 onto $N(\Gamma(x))$.

We denote the covering of a graph G as $\tilde{G}$, and call the graph G the *base graph* of the covering while $\tilde{G}$ is called a *lift* of G. For each vertex $v \in G$ the inverse image $\Gamma^{-1}(v)$ is called the *fiber above* v and denoted $\tilde{G}_v$. For simplicity we sometimes say that u lies above v when $\Gamma(u) = v$.

The best way to visualize a covering is to put vertices of fibers as vertical stacks above the vertices of the base graph G as in Figure 2.1. It is easy to see that the condition of covering map being locally homomorphic forces all fibers to have the same size, provided G is connected. This common cardinality is called the *degree* of covering. If degree of a covering Γ equals to n we call it an n-covering, and call $\tilde{G}$ an n-lift of G.

We will mostly use the term *lift* rather than covering, to distinguished it from other concept of coverings in graph theory e.g. vertex covering, edge covering. That is why $\tilde{G}$ will often be called an n-lift of G, or simple a lift of G.

2.2 Random coverings

Let $\mathcal{G}$ be a family of graphs. A graph chosen from G according to some random experiment is called a *random graph*. A *random n-covering* of a graph G will be obtained by choosing a graph $\tilde{G}$ at random from the set $L_n(G)$ of all n-lifts of G. Notice that an edge $\{u, v\} \in E(G)$ results in a matching between vertices from fibers $\tilde{G}_u$ and $\tilde{G}_v$. Thus equivalently a random covering of a graph G can be generated by choosing independently and uniformly at random for every edge $\{u, v\} \in E(G)$ a perfect matching between $\tilde{G}_u$ and $\tilde{G}_v$.

Nevertheless most of the time we would use yet another approach to choose a random lift. Let G be a base graph and $\tilde{G}$ be its lift. For every edge $\{u, v\} \in E(G)$ we choose its

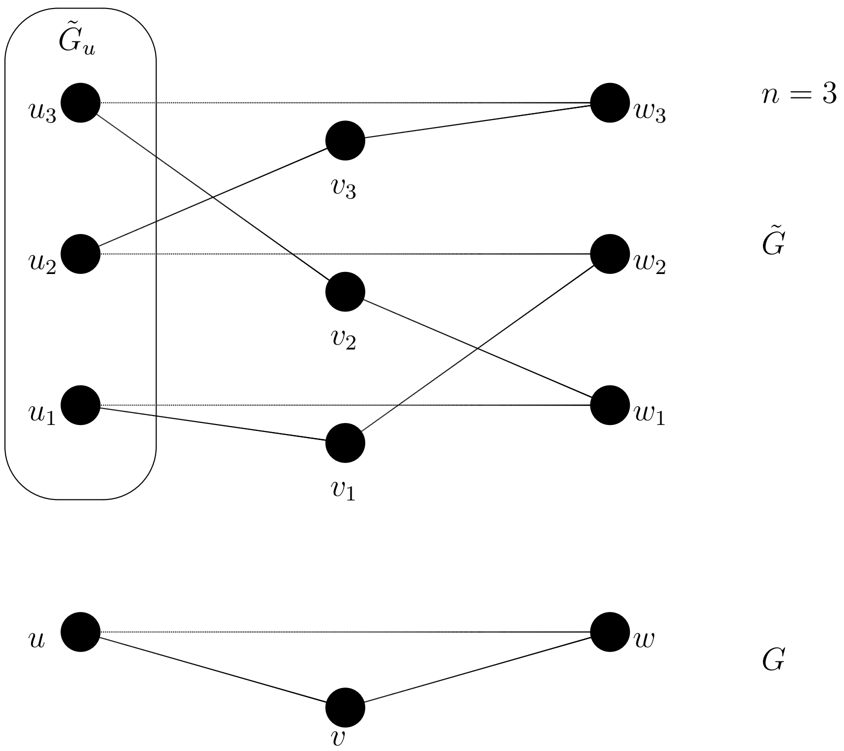

Figure 2.1: Example of a 3-covering (3-lift) $\tilde{G}$ of the graph $G = K_3$. Covering assigns u_i's to u. Vertices $\{u_1, u_2, u_3\}$ creates a fiber above vertex u.

orientation and enumerate all vertices in every fiber in $\tilde{G}$ from 1 to n. Then the random matching between two fibers is determined by a single permutation on n elements. Whenever $u_i \in \tilde{G}_u$ is connected with $v_j \in \tilde{G}_v$, we put j on i-th position of the permutation. For example in the Figure 2.1 the permutation for edge $\{v_1, v_2\}$ equals (231). Changing the permutation results in obtaining different covering.

Notice that a chosen orientation of the edge has no real effect on possible outcome, since reversing the edge and inverting the permutation yield the same covering. Nevertheless if we want to precisely describe a covering we need to orient each edge e in order to know how to attach to it a single permutation. It is also easy to see that indeed all coverings of G can be obtained in this manner.

Thus, formally we can define a *random n-covering* in a following way: choose a permutation $\sigma_e \in S_n$ uniformly and independently for every edge $e = \{u, v\}$ in G and connect u_i to $v_{\sigma_e(i)}$. One can also think about choosing the permutations non-uniformly or not independently, but none of those variations is a subject of this thesis. The following definition by Linial and Amit gives formal description of the model.

Definition ([2]). Given a graph G, a *random labeled n-covering of G* is obtained by arbitrarily orienting the edges of G, choosing permutations σ_e in S_n for each edge e uniformly and independently, and constructing the graph $\tilde{G}$ with n vertices $u_1, ..., u_n$ for each vertex u of G and edges $e_i = \{u_i, v_{\sigma_e(i)}\}$ whenever $e = \{u, v\}$ is an oriented edge. A covering $\Gamma : \tilde{G} \to G$ is defined by $\Gamma(u_i) = u$.

Note that analogously to the case of the random graph model of Erdős and Rényi, the standard model is defined for labelled graphs, where vertices of each fiber are equipped with a labelling $\{1, ..., n\}$. It can be proved [2] that asymptotic properties of coverings are the same in the labelled and unlabelled models. We may therefore prove results in the labelled model, and view them as valid statements in the unlabelled case.

2.3 Probability

In this work we shall deal only with finite probability spaces. Typically our probability space would be the set of all random n-lift of a given graph G. Each graph have the same probability to be drawn. Thus the properties of graphs become the events in this probability space and usually the random variables will count the number of specific structures in such random graph.

Our interest lies in the asymptotic properties of random lifts, that is when $n \to \infty$. In particular, we say that a graph property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity. In other words a graph H drawn at random from $L_n(G)$ has this property with probability $1 - \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$.

Throughout the paper we will use standard probabilistic inequalities to estimates the probabilities of events. The first one, the union bound says that for any set of events $A_1, ..., A_n$, we have

$$\mathbf{Pr} \bigcup_{i=1}^{n} X_i \leq \sum_{i=1}^{n} \mathbf{Pr}[X_i].$$

The second one is the Markov inequality, which states that for any random variable $X \geq 0$,

$$\mathbf{Pr}[X \geq \lambda] \geq \frac{\mathbf{E}[X]}{\lambda}$$

Note that if X is a random variable with non-negative integer values, then Markov inequality with $\lambda = 1$ implies that

$$\mathbf{Pr}[X > 0] \leq \mathbf{E}[X],$$

In particular if $\mathbf{E}[X] \to 0$, then $\mathbf{Pr}[X = 0] \to 1$.

The last inequality is particularly useful if X counts the occurrence of some structure we want to avoid. In the setup of lifts we look at the behaviour of expected value of X as degree n tends to infinity, arguing that almost every random lift does not have the desired structure or property.

Another frequently used tool in the theory of random structures is the Chebyshev inequality. It says that for any random variable X with finite expected value $\mathbf{E}[X]$ and finite non-zero variance $\mathbf{Var}[X]^2$ then for any $t > 0$ we have

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq t] \leq \frac{\mathbf{Var}[X]}{t^2}.$$

A common feature in many probabilistic arguments is the need to show that a random variable with large probability is not too far from its mean. More complex result than the one given by Chebyshev bound is the result of Chernoff. Chernoff inequality states that if $X \in B(r, p)$ (i.e. X has the binomial distribution with parameters r and p), for every ϵ, $0 < \epsilon \leq 3/2$,

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq \epsilon\, \mathbf{E}[X]] \leq 2 \exp\left(-\frac{\epsilon^2}{3}\, \mathbf{E}[X]\right).$$

In the thesis we also use some results from the theory of branching processes. Let X be an integer-valued non-negative random variable with probability mass function for each $k = 0, 1, ...$ given by $p_k = \mathbf{Pr}[X = k]$. We say that a sequence of random variable Y_n, $n = 0, 1, 2, ...$, is a *branching process* if

1. $Y_0 = 1$
2. $Y_{n+1} = X_1^{(n)} + X_2^{(n)} + ... + X_{Y_n}^{(n)}$,

Where $X_j^{(n)}$ is the number of descendants produces by the j^{th} ancestor of the n^{th} generation and the $X_j^{(n)}$ are i.i.d. random variables with the same distribution as X. We say that distribution of X is the generating distribution of the branching process.

This definition describes one of the simplest models for population growth. The process starts at time 0 with one ancestor: $Y_0 = 1$. At time $n = 1$ this ancestor dies producing a random number of descendants $Y_1 = X_1^{(0)}$. Each descendant behaves independently of the others living only until $n = 2$ and being then replaced by his own descendants. This process continues while $Y_n > 0$. If $Y_n = 0$, for some n, the branching process stops and we say that it dies out. Thus, Y_{n+1} is the number of descendants in the $(n+1)^{th}$ generation produced by Y_n individuals of generation n.

The random variable X defined above specifies the probability distribution on the number of offspring. We denote $\mathbf{E}[X] = \mu$ and $\mathbf{Var}[X] = \sigma^2$. Let $f : [0, 1] \to \mathbb{R}$ denote the probability generating function of X, defined as

$$f_X(x) = f(x) = \sum_{i \geq 0} x^i\, \mathbf{Pr}[X = i]$$

Let

$$\rho_n = \mathbf{Pr}[Y_n = 0]$$

be the probability that the population is extinct by generation n. The probability π_0 that the branching process dies out is then the limit of those probabilities.

$$\pi_0 = \mathbf{Pr}[\text{the process dies out}] = \mathbf{Pr}[Y_n = 0 \text{ for some } n] = \lim_{n\to\infty} \mathbf{Pr}[Y_n = 0] = \lim_{n\to\infty} \rho_n.$$

The basic result in the theory of branching processes is the following (see e.g. [17]).

Theorem 1. *If $\mu > 1$ and $\mathbf{Pr}[X = 0] > 0$, then π_0 is the smallest solution of the equation $f(x) = x$ which belongs to the interval $(0, 1)$.*

Note that this mean that whenever $\mu > 1$ the probability the process survives is some positive constant. We are particularly interested in branching processes where the number of descendants is given by a binomially distributed random variable. Let $X \in B(r, p)$. Then the probability generating function of X is

$$f_X(x) = \sum_{i=0}^{n} \binom{r}{i} x^i p^i (1-p)^{r-i} = (1 - p + xp)^r.$$

Thus the probability of extinction ρ_n of the branching process defined by X is uniquely determined by the solution of the equation

$$(1 - p + xp)^r = x \tag{2.1}$$

In the Chapter 5 of this thesis we construct a branching process with generating distribution given by $X \in B(3, 0.49)$. As $\mathbf{E}[X] = 1.47 > 1$, from above paragraph we know that with probability greater than 0.61 such a process will never die out. At this point we will need to estimate the grow of such a process, namely we want to know what is the expected number of individuals in the n-th generation.

Lemma 2. *Let X be a random variable with binomial distribution $B(r, p)$, where $rp > 1$. Let Y_i denote the number of individuals in n-th generation of the branching process with generating distribution given by distribution of X. For a given m let us choose the smallest n such that $\sum_{i=0}^{n-1} Y_i \geq 2m/(rp - 1)$. Then with probability at least $2\exp(-m\frac{rp-1}{3rp})$ we have $Y_n \geq m$.*

Proof. Note that the probability that there are fewer than m ancestors in the last generation is bounded from above by the probability that random variable $Z = \sum_{i=1}^{t} X_i$ defined as the sum of t, $t \geq 2m/(rp - 1)$ random variables $X_i \in B(r, p)$ has value less than $t + m - 1$. Observe that Z has the binomial distribution $B(tr, p)$; in particular $\mathbf{E}\, Z = trp$. Thus, from Chernoff inequality, we get

$$
\begin{aligned}
\mathbf{Pr}[Y_n \le m] &\le \mathbf{Pr}[Z \le m + t] \\
&\le \mathbf{Pr}\left[Z - \mathbf{E}[Z] \le \frac{m + t - trp}{trp}\,\mathbf{E}[Z]\right] \\
&\le 2\exp\left(-\frac{(trp - t - m)^2}{3trp}\right) \\
&\le 2\exp\left(-\frac{\left(t(rp-1) - \frac{t(rp-1)}{2}\right)^2}{3trp}\right) \\
&\le 2\exp\left(-\frac{t}{6}\frac{(rp-1)^2}{rp}\right) \\
&\le 2\exp\left(-\frac{m(rp-1)}{3rp}\right).
\end{aligned} \tag{2.2}
$$

This concludes the proof of Lemma 2. $\square$

> *Young man, in mathematics you don't understand things. You just get used to them.*
>
> Johann von Neumann, 1921

3 Properties of random lifts of graphs

In this chapter we survey the results concerning properties of random lifts and show some of its properties which will be useful in the upcoming chapters. In order to present wider picture we mention results on matchings and chromatic number of random lifts which strictly speaking are not related to the issues we are concerned in this thesis, but since, in general, not much is known about the properties of random coverings we like to present current picture of the whole area.

Let us recall that we shall be only interested in the asymptotic properties of random lifts, when $n \to \infty$. Thus in every proof in this and following chapters we claim that all inequalities we state holds only for sufficiently large n.

It is easy to see that some properties of the base graph are in a way preserved by the covering graph. For example the degrees of the vertices in the fibers are the same as the degree of a vertex they are mapped to, and so the lift of a d-regular graph is d-regular. Since the covering map is a homomorphism, the chromatic number of the lift is not greater than the chromatic number of the base graph. On the other hand lifts of graphs can have much better connectivity properties than base graphs. Our main interest lies in a question how the family of lifts preserves and reflects the local and global structure of the base graph. The simplest case is when the base graph is a tree. An easy argument proves that a lift of a tree T is a collection of disjoint trees isomorphic to T.

Fact 1. Let $\Gamma : \tilde{G} \to G$ be an n-covering. Every tree T in G is covered in $\tilde{G}$ by n disjoint trees isomorphic to T.

Proof. We will prove this fact by induction on the size of a tree. The base case is a single vertex t. A covering of one vertex is simply a sum of n disjoint vertices. For the induction hypothesis, suppose that the statement of the fact is true for every connected tree on $n-1$ vertices. Now consider a tree T on n vertices with a vertex u of degree one adjacent to vertex v in T. A covering of $T \backslash u$ is a sum of n disjoint trees $T_1, ..., T_n$ isomorphic to T. Consider an edge $e = \{u, v\}$, its lift match trees $T_1, ..., T_n$ with n vertices that covers u. □

If T is a path the above property is sometimes called the unique path-lifting property of random lifts [2]. In the previous chapter we stated that random coverings model has the same properties for labelled and unlabelled graphs. It is worth noting that imposing certain restrictions on the random permutation yields an equivalent model. Specifically, it is not difficult to prove that if E is a set of edges that does not contain a cycle, then the probability of any graphical property of the covering is unchanged if we condition on all the permutations assigned to edges in E being the identity.

3.1 General properties of random lifts

Adding one edge to a tree results in creating a cycle in the graph. A random lift of a cycle is the first non-trivial case we have to review. One can easily check that the lift of a cycle is a set of disjoint cycles, but in this case lengths of those cycles varies.

Lemma 3. *Let $h \geq 3$. Asymptotically almost surely a random lift of a cycle C_h on h vertices consists of a collection of at most $2 \log n$ disjoint cycles.*

Proof. If we remove one edge e from a cycle, then a lift of the path obtained in this way is a collection of n disjoint paths (see Fact 1 above). Lifting the missing edge e is the same as matching at random the two sets of ends of those paths or connecting those ends according to some random permutation. The number of cycles created after joining those paths is then the same as the number of cycles in a random permutation on set $[n] = \{1, 2, \ldots, n\}$. The precise distribution of the number of cycles in random permutation is well known [16], but here we estimate it for the completeness of the argument.

Let $X_d = X_d(n)$ denote the number of d-cycles in the random permutation on $[n]$. There are $(d-1)!$ ways of arranging d given symbols in a cycle, $(n-d)!$ permutations of the remaining symbols and $n!$ permutations in total, so the probability for d given symbols to form a cycle in a permutation chosen uniformly at random from the set of all permutations of n symbols is

$$\frac{(d-1)!(n-d)!}{n!}$$

There are $\binom{n}{d}$ selections of d out of n symbols, so for expected number of d-cycles $\mathbf{E}\,X_d$ we have

$$\mathbf{E}\,X_d = \binom{n}{d}(d-1)!\frac{(n-d)!}{n!} = \frac{1}{d} \tag{3.1}$$

Thus, if $X = X(n) = \sum_{d=1}^{n} X_d$ denotes the total number of cycles, then

$$\mathbf{E}\,X = \sum_{d=1}^{n} \mathbf{E}\,X_d = \sum_{d=1}^{n} \frac{1}{d} = \log n + O(1). \tag{3.2}$$

In order to compute the variance note that if we fix a cycle in a random permutation, then each permutation on the remaining vertices is equally likely. Hence

$$\mathbf{E}\,X(n)[(X(n)-1)] = \sum_{d=1}^{n}\sum_{\ell=1}^{n-d} \mathbf{E}\,X_d(n)\,\mathbf{E}\,X_\ell(n-d) = \sum_{d=1}^{n}\sum_{\ell=1}^{n-d} \frac{1}{d\ell}\,. \tag{3.3}$$

Let $s = n\exp(-\sqrt{\log n})$. Then,

$$\begin{aligned}
\mathbf{E}\,X[(X-1)] &= \sum_{d=1}^{s}\sum_{\ell=1}^{n-d} \frac{1}{d\ell} + \sum_{d=s+1}^{n}\sum_{\ell=1}^{n-d} \frac{1}{d\ell} \\
&= (\log s + O(1))(\log n + O(1)) + (\log(n/s) + O(1))O(\log n) \\
&= (\log n)^2 + O\big(\log^{3/2} n\big)\,.
\end{aligned} \tag{3.4}$$

and

$$\mathbf{Var}\,X = \mathbf{E}\,X(X-1) + \mathbf{E}\,X - (\mathbf{E}\,X)^2 = O\big(\log^{3/2} n\big)\,.$$

Hence, from Chebyshev's inequality we get

$$\mathbf{Pr}[X \geq 2\log n] \leq \mathbf{Pr}[|X - \mathbf{E}\,X| \geq 0.5\log n] \leq \frac{4\,\mathbf{Var}\,X}{\log^2 n} = o(1)\,.$$

□

The lifts of more complex graphs is much harder to describe. That is why from this point on we focus on selected graph properties that are preserved in lifts. In the case of general graphs it can be proven that all short cycles are typically "sparsely distributed" in the lifts.

Lemma 4. *Let G be a simple graph. Then asymptotically almost surely in $\tilde{G}$ no two cycles of length smaller than $(\log\log n)^2$ lie within distance less than $(\log\log n)^2$ from each other.*

Proof. Let G be a simple graph of order k. Let Z be a random variable which counts the number of pairs of cycles in $\tilde{G} \in L_n(G)$, which are shorter than $(\log\log n)^2$, that intersect each other or are connected by path of length at most $(\log\log n)^2$. We can bound from above the expected value of Z counting the number of paths P of length at most $3(\log\log n)^2$ such that ends of P are adjacent to at least two elements of P (denote this new random variable as Z'). Note that for a given ordered set of m vertices $\{u_1, ..., u_m\}$ and two selected vertices u_i and u_j, the probability that there is a path $u_1...u_m$ with additional edges between u_1 and u_i, and u_m and u_j in $\tilde{G}$ is less than $\left(\frac{1}{n-m}\right)^{m+1}$. Thus

$$\begin{aligned}
\mathbf{E}\, Z' &\leq \sum_{m=1}^{3(\log\log n)^2} \binom{kn}{m} m!\, m^2 \left(\frac{1}{n-m}\right)^{m+1} \\
&\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(kn)^m}{m!} \frac{m!\, m^2}{(n-m)^{m+1}} \\
&\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(kn)^m m^2}{(n-m)^{m+1}}
\end{aligned}$$

Since $m < n/2$, we have

$$\begin{aligned}
\mathbf{E}\, Z' &\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(2k)^{m+1} m^2}{n} \\
&\leq \frac{9(2k)^{3(\log\log n)^2}(\log\log n)^6}{n} \leq \frac{\exp\left((\log\log n)^3\right)}{n}.
\end{aligned}$$

Consequently, from Markov's inequality,

$$\mathbf{Pr}[Z > 0] \leq \mathbf{E}\, Z \leq \mathbf{E}\, Z' = o(1)\,,$$

and the assertion follows.

$\square$

Our next result states that for almost every lift of a graph G, $d = d(n)$ and any constant C, the d-neighbourhoods of the C lexicographically first vertices of every fiber are mutually disjoint and have a structure of a tree. Moreover d can be chosen to be of order $O(\log\log n)$, so each neighbourhood can have, say $\log^{10} n$ vertices.

Lemma 5. *Let G be a simple graph, with $\delta(G) \geq 2$, and $C > 0$ be a constant. Asymptotically almost surely $\tilde{G}$ has the following property. For any vertex $v \in G$, the C lexicographically first vertices from the fiber above vertex v are at distance at least $11 \log\log n$ from each other and each such vertex is at distance at least $11 \log\log n$ from any cycle shorter than $10 \log\log n$.*

Proof. Let G be a simple graph and set $k = |G|$. For every fiber in $\tilde{G}$ enumerate its vertices from 1 to n. Let $x, y \in \tilde{G}_v$ be any two out of the first C vertices from fiber above vertex v. Let $Z_{x,y}$ counts the number of paths shorter than $11 \log n \log n$ connecting x with y. For a given ordered set of m vertices $\{u_1, ..., u_m\}$, the probability that there is a path $xu_1...u_my$ in $\tilde{G}$ is less than $\left(\frac{1}{n-m}\right)^{m+1}$. Hence, similarly as in the proof of Lemma 4 we have

$$\begin{aligned}\mathbf{E}\, Z_{x,y} &\leq \sum_{m=1}^{11 \log \log n} \binom{kn}{m} (m!) \left(\frac{1}{n-m}\right)^{m+1} \\ &\leq \sum_{m=1}^{11 \log \log n} \frac{(kn)^m}{m!} \frac{m!}{(n-m)^{m+1}} \\ &\leq \sum_{m=1}^{11 \log \log n} \frac{(kn)^m}{(n-m)^{m+1}}\end{aligned}$$

Since $m < n/2$, we have

$$\begin{aligned}\mathbf{E}\, Z_{x,y} &\leq \sum_{m=1}^{11 \log \log n} \frac{(2k)^{m+1}}{n} \\ &\leq 11 \log \log n \frac{(2k)^{11 \log \log n}}{n} \leq \frac{\exp\left((\log \log n)^2\right)}{n}.\end{aligned}$$

Let Z counts, for every fiber $\tilde{G}_v$, the expected number of paths shorter than $11 \log \log n$ connecting any pair of its C lexicographically first vertices. Then, using union bound, we get

$$\mathbf{E}\, Z \leq k\binom{C}{2} \mathbf{E}\, Z_{a,b} \leq k\binom{C}{2} \frac{\exp\left((\log \log n)^2\right)}{n} \longrightarrow 0.$$

Thus $\mathbf{Pr}[Z > 0] = o(1)$ which proves the first part of the statement.

Now we would like to count the expected number of cycles shorter than $10 \log \log n$ which are closer than $11 \log \log n$, to some of C firsts vertices of any fiber. Let X count the number of paths starting at one of those vertices which are shorter than $22 \log \log n$ and for which the last vertex has an edge connecting it to any from $n-2$ firsts vertices of this path. Then

$$\begin{aligned}
\mathbf{E}\,X &\le \sum_{m=1}^{22\log\log n} (Ck)\binom{kn}{m}\, m!\, m \left(\frac{1}{n-m}\right)^{m+1} \\
&\le \sum_{m=1}^{22\log\log n} (Ck)\frac{(kn)^m m}{(n-m)^{m+1}} \\
&\le \sum_{m=1}^{22\log\log n} C\frac{(2k)^{m+1} m}{n} \\
&\le C\frac{(2k)^{22\log\log n} 22(\log\log n)}{n} \longrightarrow 0.
\end{aligned}$$

Hence, asymptotically almost surely such a cycle does not appear in $\tilde{G}$ and the assertion follows. □

3.2 Connectivity

In this section we focus on the connectivity properties of random lifts. More precisely, we study the expected number of vertex disjoint paths connecting any two vertices in a random lift. Let δ denote the minimum degree of a graph G so, if it is ℓ-connected, we have $\ell \le \delta$. Notice that every lift of G contains vertices of degree δ and therefore is at most δ-connected. We already know that there exists examples of graphs (e.g. cycles) with $\delta \le 2$ of which random lifts are aas not connected. Amit and Linial [2] proved that if $\delta \ge 3$, then almost every random lift is in fact δ-connected. We present here a short proof of this fact (much simpler than the original argument of Amit and Linial).

Theorem 6 ([2]). *Let G be a connected simple graph with minimal degree $\delta \ge 3$. Then asymptotically almost surely an n-lift $\tilde{G}$ is δ-connected.*

Proof. Let G be a connected graph with $\delta(G) \ge 3$. To show that a covering $\tilde{G}$ is δ-connected, we need to show that for every set X of vertices with $|X| < |\tilde{G}|/2$, we have $|N(X)| \ge \delta$, where $N(X)$ is the set of vertices from $V(\tilde{G})\backslash X$ that are adjacent to some vertex in X. Notice that it is enough to show that this property is true for connected subsets X of G.

By Lemma 4 we know that whenever we take a set $X \subseteq V(\tilde{G})$ of a size $x = |X| \le \log\log n$ then aas there is at most one cycle in a subgraph of $\tilde{G}$ induced on X. Therefore there are no more than x edges connecting the vertices inside X. It implies that $N(X)$ equals at least $(\delta-2)|X|$. However the inequality $(\delta-2)|X| \ge \delta$ holds for $|X| \ge \frac{\delta}{\delta-2}$, while for $|X| \le 2$ the statement is trivial. Thus the assertion holds for $|X| \le \log\log n$.

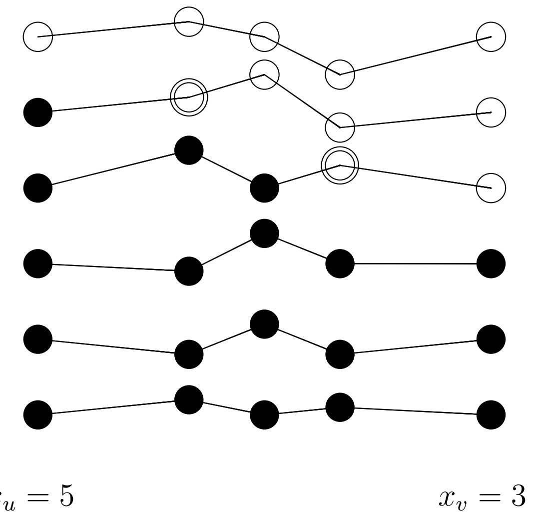

Figure 3.1: An example of the set X which is not 2-outside with a lift of path. The black vertices are in X, the white ones are outside X. Double-circled points are elements from the set $N(X)$.

In order to deal with the case when $|X| = x > \log\log n$ we lift G in two stages. Let T be a spanning tree of a graph G. First we lift edges of T; then we lift the rest of the graph. Let us recall that, by the Fact 1, the lift $\tilde{T}$ of T consists of n disjoint copies of T; we denote them by $T_1, \dots, T_n$.

We say that a set of vertices $X \subset V(\tilde{G})$ is *δ-outside of $\tilde{T}$* if all except at most $\delta - 1$ trees from the family $T_1, \dots, T_n$ which intersect X are entirely contained in X. We show first that the edges of $\tilde{T} \subset \tilde{G}$ already guarantee that each subset X which is not δ-outside of $\tilde{T}$ has neighbourhood at least δ. Indeed, it is enough to note that if X properly intersects some tree T_i, then T_i contributes at least one vertex to $N(X)$ (see Figure 3.1, where we illustrate it for the case when T is a path). Consequently, whenever X intersects properly at least δ trees from $\tilde{T}$, we have $|N(X)| \geq \delta$.

Thus, to conclude the proof, we can restrict our attention to the sets X of size $\log\log n \leq |X| = x \leq nk/2$ which are δ-outside of $\tilde{T}$ and show that aas for all of them we have $|N(X)| \geq \delta$. Let choose a set X with the above property. Then, about $x/|G|$ trees from $\tilde{T}$ are entirely contained in X. Let v be a vertex of degree one in the tree T. There are $\delta - 1$ edges connecting v with other vertices $u_1, ..., u_{\delta-1}$ from T, where, let us recall, $\delta \geq 3$. We prove that the probability that vertices from X_v are connected to fewer than δ vertices from $\tilde{G}_{u_1} \backslash X_{u_1} \cup \dots \cup \tilde{G}_{u_{\delta-1}} \backslash X_{u_{\delta-1}}$ tends to zero as n goes to infinity. More specifically, let $B(y)$ be the expected number of sets X of size y such that $|N(X_v) \cap (V(\tilde{G}) \backslash X)| \leq \delta$. We shall show that

$$\sum_{y=\log\log n}^{nk/2} B(y) = o(1). \tag{3.5}$$

Let us divide X into two parts: X_1 that contains trees from $\tilde{T}$ which are entirely contained in X, and X_2 containing trees from $\tilde{T}$ which intersect properly with X. Let $y_1 = |X_1|$ and $y_2 = |X_2|$, where $y_1 + y_2 = y$. To choose X we have to choose y_1/k trees that are contained in X, and then select possible additional $z \leq \delta - 1$ trees that are not entirely contained in X. Finally we have to decide which $y_2 \leq z(k-1)$ vertices of the second type trees we want to include in X_2. For every edge $e = \{v, u\}$ there is a matching between sets $\tilde{G}_v$ and $\tilde{G}_u$, so the probability that $|N(X_v) \cap (\tilde{G}_u \backslash X_u)| \leq \delta$ is bounded from above by the probability that the chosen random set of y_1 elements would be a subset of admissible $y_1 + \delta$ vertices. Thus we have

$$B(y) = \sum_{y_2=0}^{(\delta-1)(k-1)} \sum_{y_1=(\log\log n - y_2)/k}^{(nk/2-y_2)/k} B'(y_1, y_2),$$

where

$$\begin{aligned} B'(y_1, y_2) &\leq \binom{n}{y_1}\binom{n}{z}\binom{(k-1)z}{y_2}\left(\frac{\binom{y_1+\delta}{y_1}}{\binom{n}{y_1}}\right)^{\delta-1} \\ &\leq \binom{n}{y_1}\binom{n}{\delta-1}\binom{k(\delta-1)}{y_2}\left(\frac{(y+\delta)\cdot\ldots\cdot(\delta+1)}{n\cdot\ldots\cdot(n-y+1)}\right)^{(\delta-1)}. \end{aligned}$$

Note that $c = \binom{k(\delta-1)}{y_2}$ is a constant that does not depend on n. Moreover,

$$\begin{aligned} B'(y_1, y_2) &\leq cn^{\delta}\frac{n!}{(y_1)!(n-y_1)!}\left(\frac{(y+\delta)\cdot\ldots\cdot(\delta+1)}{n\cdot\ldots\cdot(n-y+1)}\right)^{(\delta-1)} \\ &\leq cn^{\delta}\left(\frac{n\cdot\ldots\cdot(n-y+1)}{(y_1)!}\right)\left(\frac{(y+\delta)\cdot\ldots\cdot(\delta+1)}{n\cdot\ldots\cdot(n-y+1)}\right)^{(\delta-1)} \\ &\leq cn^{\delta}(2y)^{\delta-1}\left(\frac{(y-1)\cdot\ldots\cdot(\delta+1)}{n\cdot\ldots\cdot(n-y+1)}\right)^{(\delta-2)}. \end{aligned}$$

Now for $\log\log n \leq y \leq \log^2 n$, we have

$$B'(y_1, y_2) \leq cn^{2\delta}\left(\frac{\log^2 n}{n}\right)^{(\delta-2)\log\log n} = o(1/n)$$

while for $\log^2 n \leq y \leq n/2$, we get

$$B'(y_1, y_2) \leq cn^{2\delta} \left(\frac{1}{2}\right)^{(\delta-2)\log^2 n} = o(1/n)$$

Hence, (3.5) holds and so the assertion follows. $\square$

In the next Chapter 4 we prove that almost every random lift of minimal degree at least $2k-1$ has much stronger connectivity property, namely it is k-linked (see Chapter 4 for definition). Furthermore the lengths of the paths connecting every pair of vertices has length of order $O(\log n)$.

As we have seen in order for a graph to be α-connected, for any subset S of vertices of a graph, we need it neighbourhood to be greater than α. A natural question to ask is whether it is possible to obtain a stronger property, i.e. the size of neighbourhood of S to be some function of the size of S. This question may be asked in terms of number of edges connecting set S with the rest of the graph or the number of vertices adjacent to some vertex from S. A parameter that measure this property in the case of the size of edge-cut between S and $G \backslash S$ is the edge expansion.

Definition. Let G be a graph with v vertices. For $S \subset V(G)$, let ∂S be the set of edges with one vertex in S and one outside S. The *edge expansion* $\xi(S)$ is defined to be $|\partial S|/|S|$, and the edge expansion of G is

$$\xi(G) = \min\{\xi(S) : S \subset V(G), |S| \leq v/2\}.$$

Graphs which have a large edge expansion are called *expanders*. These graphs have the property that it is easy to get from one point to any other in the graph. Notice that a lift $\tilde{G}$ cannot have higher edge expansion than G. Given $S \subset V(G)$ with some small $\xi(S)$, take $\tilde{S}$ to be union of the fibers $\tilde{G}_u = \{u\} \times [n]$, for $u \in S$. Then $\xi(\tilde{S}) = \xi(S)$ and $|\tilde{S}| \leq |V(\tilde{G})|/2$ iff $|S| \leq |V(G)|/2$. Amit and Linial proved that edge expansions of lifts are asymptotically almost surely bounded away from 0.

Theorem 7 ([3]). *Let $G = (V, E)$ be a connected graph with $|E| > |V|$. Then there is a positive constant $\xi_0 = \xi_0(G)$ such that aas lift $\tilde{G}$ has edge expansion at least ξ_0.*

Note that the constant in the theorem of Amit and Linial is a function of the order of the graph G. If we put a restriction on the size of the set S, we can show a bound for the size of of the set $N(S)$ (where $N(S)$ is the set of vertices from $V(\tilde{G}) \backslash S$ that are adjacent to some vertex in S) as a function of the minimum degree of a graph G.

Lemma 8. *Let $\delta \geq 12$, for every simple graph G of order k with minimum degree δ aas every subset $|S|$ of vertices in $\tilde{G}$ with*

$$|S| \leq \frac{n}{1000k^4\delta}$$

satisfy

$$|S \cup N(S)| > \frac{\delta}{3}|S|.$$

Proof. Let G be a graph of order k. Let S be any subset of vertices of $\tilde{G}$ and denote its size by s. We estimate the probability of an event $B(s)$ that any of the sets of size $s \leq \alpha n$ has a neighborhood larger than $\frac{\delta}{3}|S|$ and show that this probability tends to zero as n tends to infinity. For a given set of vertices $T \in \tilde{G}$, $|T| \leq (\delta s/3 - 1)$ the probability that $N(S) \subset S \cup T$ is bounded from above by

$$\left(\frac{|S| \cup |T|}{n}\right)^{\delta s/2},$$

since for each vertex $v \in S$ we have to choose all its neighbors in $S \cup T$. There are at least $\delta s/2$ neighbors to be chosen, where each neighbor can be chosen from all vertices of appropriate fibers. There are $\binom{nk}{s}$ sets S with $|S| = s$ and $\binom{nk}{(\delta s/3-1)}$ choices for T, so we need to show that

$$\sum_{s=1}^{\alpha n} B(s) = o(1), \tag{3.6}$$

where

$$B(s) = \binom{nk}{s}\binom{nk}{\frac{\delta s}{3} - 1}\left(\frac{\delta s}{3n}\right)^{\delta s/2}.$$

Using the fact that

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

we get

$$\begin{aligned}
B(s) &\leq \left(\frac{ekn}{s}\right)^s \left(\frac{ekn}{(\delta s/3 - 1)}\right)^{(\delta s/3-1)} \left(\frac{e\delta s}{3n}\right)^{\delta s/2} \\
&\leq \left(\left(\frac{e\delta}{3}\right)^{\delta/2} \left(\frac{3ek}{\delta - 3}\right)^{\delta/3} \left(\frac{s}{n}\right)^{\delta/6}\right)^s \\
&\leq \left(\left(2ek\right)^{2\delta/3} \left(\frac{\delta s}{n}\right)^{\delta/6}\right)^s.
\end{aligned}$$

Hence, if $s \leq \alpha n$, where $\alpha = \frac{1}{1000k^4\delta}$, we have

$$\left((2ek)^4\delta\alpha\right)^{\delta/6} < 0.99,$$

therefore $B(s) = o(1/n)$ and the assertion follows. $\square$

There have been extensive studies of lifts in terms of their expanding features, and this topic has brought a lot of attention because of their possible important applications. However, most of them concentrate on lift of special classes of graphs (so called Ramanujan graphs) or a construction of finite lifts, and so have different flavour than other results presented in this chapter. Since covering of this topic would require a commodious introduction and analysis we do not present those results in this thesis, more details on this topic can be found in [1, 6, 29].

3.3 Minors

Drier and Linial [13] discussed the existence of minors and topological minors in lifts of graphs. They used slightly different approach and consider the asymptotic behaviour of the n-lifts of complete graph of order ℓ, when $\ell = \ell(n)$. They proved that for $n \leq O(\log \ell)$ almost every n-lift of the complete graph K_ℓ contains a clique minor of size $\Theta(\ell)$, and for $n > \log \ell$ it contains a clique minor of size at least $\Omega\left(\frac{\ell\sqrt{n}}{\sqrt{\log \ell}}\right)$. The last result was shown to be tight as long as $\log \ell < n < \ell^{1/3-\epsilon}$.

Denote by $\sigma(L)$ the size of the largest clique which topological minor can be found in a lift L. The following bound is true for every lift of complete graph K_ℓ.

Lemma 9 ([13]). *Let $\tilde{K}_\ell$ be a lift of K_ℓ, then*

$$\Omega(\sqrt{n}) \leq \sigma(\tilde{K}_\ell) \leq n$$

Indeed since every vertex in $L \in \tilde{K}_\ell$ has only $n-1$ neighbours it is easy to notice that $\sigma(L) \leq n$. Lower bounds comes from theorem of Komlós and Szemerédi [22] that says that every graph of average degree d contains a subdivision of $K^{\Omega(\sqrt{d})}$. For n sufficiently large Drier and Linial proved the following results for random lifts.

Theorem 10 ([13]). *Aas for $L \in L_n(K_\ell)$ we have $\sigma(L) \leq O(\sqrt{\ell n})$.*

Theorem 11 ([13]). *If $\ell \geq \Omega(n)$, then aas for $L \in L_n(K_\ell)$, we have $\sigma(L) \geq \Omega(n)$.*

Authors left the problem of finding topological minors in lifts of complete graphs when $n \geq \Omega(\ell)$ and for lifts of general base graphs as an open question. The main question in this area is to understand, for a given graph, which of its minors M is persistent, i.e. M is a minor of every lift of G; and which are not.

In the Chapter 4 we show that in almost every lift of any graph G we can find a topological clique of size equal to the maximal degree in the $core(G)$ plus one. This results is best possible.

3.4 Other properties

There are only a handful papers on random lifts, therefore only few properties of those graphs has been studied. Thus for the completion of the picture we briefly present here also the results on matching and chromatic number of random coverings, even though they are not the topic of research presented in upcoming chapters.

3.4.1 Matchings in random lifts

Some part of the research in the area of random lifts is dedicated to analyse which properties of random lifts are aas preserved by almost all or almost none of the lifts regardless of the choice of the base graph. Let us consider the property that a graph contains a perfect matching. It is easy to see that a lift of the perfect matching in G is a perfect matching in $\tilde{G}$. However, it is possible that G does not have a perfect matching while aas every lift does. The main role in determining whether the lift of a graph contains a perfect matching plays a concept of *fractional matching*.

Definition. A *fractional matching* in a graph $G = (V, E)$ is mapping $f : E \to \mathbb{R}^+$ such that $\sum_{e=\{v,x\}} f(e) \leq 1$ for every vertex $v \in V$. If the equality holds at every vertex, f is called a *perfect fractional matching*.

Since a covering graph can have an odd number of vertices we define an *almost-perfect matching*, as a matching that misses at most one vertex. A perfect matching in $\tilde{G}$ determines a fractional perfect matching in G. We just set for every edge of G the $f(e)$ to be the proportion of edges in the matching in lift. It turned out that this condition is also sufficient for lift to admit a perfect matching.

Theorem 12 ([28]). *Let G be a graph that satisfies the following conditions:*

1 G is connected.

2 |E(G)| > |V(G)|.

3 G has a perfect fractional matching.

Then asymptotically almost surely a lift $\tilde{G}$ has an almost-perfect matching.

Linial and Rozenman were able to prove even more tight classification result.

Theorem 13 ([28]). *Let G be finite connected graph. Exactly one of the following situations occurs:*

1 *Every lift $\tilde{G}$ of G has a perfect matching. This occurs when G has a perfect matching.*

2 *Asymptotically almost surely a lift $\tilde{G}$ of G has an almost-perfect matching.*

3 *Asymptotically almost surely in a lift $\tilde{G}$, the largest matching misses $\Theta(\log n)$ vertices. This happens e.g. when G is an odd cycle.*

4 *Asymptotically almost surely every matching in an n-lift $\tilde{G}$ misses $\Omega(n)$ vertices. This happens if $\sum f(e) \leq (1/2 - \epsilon)|V|$ for every fractional matching in G.*

The implicit constants on the Θ and Ω terms depend only on G.

3.4.2 Chromatic number

We say that G is k-*colorable* if one can assign the colors $\{1, .., k\}$ to the vertices in $V(G)$, in such a way that every vertex gets exactly one color and no edge in $E(G)$ has both of its endpoints colored the same color. The smallest k such that G is k-colorable is called the *chromatic number of* G. It turns out that finding the distribution of the chromatic number $\chi(\tilde{G})$ of random lifts of G is an interesting and challenging problem. We will focused on two parameters which are in a sense upper and lower bound on the chromatic number of lifts.

Definition.

$$\tilde{\chi}_h(G) = \min\{k \mid \chi(\tilde{G}) \leq k \text{ for almost every lift } \tilde{G} \text{ of } G\}$$

$$\tilde{\chi}_l(G) = \max\{k \mid \chi(\tilde{G}) \geq k \text{ for almost every lift } \tilde{G} \text{ of } G\}$$

Obviously $\tilde{\chi}_l(G) \leq \tilde{\chi}_h(G) \leq \chi(G)$. Linial, Amit and Matousek conjecture that the chromatic number of random lifts concentrates essentially in a single value.

Conjecture 1. *For every graph G, $\tilde{\chi}_l(G) = \tilde{\chi}_h(G)$.*

Conjecture has been settled in the affirmative for bipartite graphs, cubic graphs and certain "blow-up" graphs defined below. For paths and trees the chromatic number of their lift is aas equal 2. A lift of a graph with at least one odd cycle has chromatic number at least 3, since with high probability such lift contain an odd cycle. The smallest graph for which we do not know if this conjecture is true is K_5, the complete graph on 5 vertices. The chromatic number of its n-lift is either 3 or 4, but so far we do not know the probability distribution of $\chi(\tilde{K}_5)$. For the complete graph on 5 vertices minus one edge the chromatic number of the random lift was found by Farzad and Theis.

Theorem 14 ([15]). *Asymptotically almost surely a random lift of K_5/e, (i.e. the complete graph of order 5, minus one edge) is 3-colorable.*

As it comes to determining the values of $\tilde{\chi}_l(G)$ and $\tilde{\chi}_h(G)$ in general case the following was proven by Amit, Linial and Matousek.

Theorem 15 ([4]). *For every graph G,*

$$\tilde{\chi}_l(G) \geq \sqrt{\frac{\chi(G)}{3 \log \chi(G)}}$$

As a matter of fact, the authors of this result conjectured that it can be substantially improved.

Conjecture 2 ([4]). *For each graph G,*

$$\tilde{\chi}_l(G) \geq C \frac{\chi(G)}{\log \chi(G)}$$

A better estimate can be obtained if instead of the chromatic number $\chi(G)$ we use the fractional chromatic number $\chi_f(G)$, defined as the minimum total weight of linear combination of independent sets, such that the weight at each vertex is at least 1.

Theorem 16 ([4]). *For each graph G,*

$$\tilde{\chi}_l(G) \geq \Omega\left(\frac{\chi_f(G)}{\log^2 \chi_f(G)}\right)$$

On the other hand, theorem of Kim [21], about chromatic number of graphs with high girth (the length of shortest cycle in a graph), yields an upper bound on $\tilde{\chi}_h(G)$. This bound can be proven to be tight for some classes of graphs.

Theorem 17 ([21]). *Let G be a graph with minimal degree $\Delta = \Delta(G)$. Then*

$$\tilde{\chi}_h(G) \leq \frac{\Delta}{\ln \Delta}(1 + o(1))$$

For complete graphs, we have then the following estimates

Corollary 1. *There exist constants $A > B > 0$ such that*

$$A \frac{r}{\log r} \geq \tilde{\chi}_h(K_r) \geq \tilde{\chi}_l(K_r) \geq B \frac{r}{\log r}$$

The above means that if we randomly lift complete graphs, its chromatic number drops from r to $r/\log r$. On the other hand there exist graphs whose chromatic numbers are preserved for all theirs lifts.

Proposition 1 ([4]). *For any graph G with $\chi(G) \geq 2$, put $r = 3\chi(G) \log \chi(G)$, and let H be constructed from G by replacing each vertex by an independent set of size r and every edge by a complete bipartite graph $K_{r,r}$. Then aas a lift $\tilde{H}$ of H has chromatic number $\chi(\tilde{H}) = \chi(H) = \chi(G)$.*

> *Oh, he seems like an okay person, except for being a little strange in some ways. All day he sits at his desk and scribbles, scribbles, scribbles. Then, at the end of the day, he takes the sheets of paper he's scribbled on, scrunches them all up, and throws them in the trash can.*
>
> John von Neumann's housekeeper, describing her employer.

4 Topological cliques in random lifts

In this part of the work we give a more detailed insight into the size of the largest topological clique in random lifts and some other related properties.

Let us recall that a graph obtained by replacing the edges of H with vertex disjoint paths is called a *subdivision* of H. If X is isomorphic to a subgraph of G, and X is a subdivision of a clique K_ℓ, we say that there is a *topological clique* of order ℓ in G. The vertices in G corresponding to the vertices in $K\ell$ are then called branch vertices.

Observe that a vertex v of degree d can be a branch vertex in a topological clique of size at most $d+1$. Moreover no vertex of degree one can be a vertex connecting two branch vertices. That is why the concept of the core of a graph takes crucial place in the analysis of topological cliques.

Definition. The *core* of a connected graph G, denoted as $core(G)$, is the unique maximal subgraph of G with minimum degree at least two.

Notice that the core of the lift $\tilde{G}$ is the same as the lift of the $core(G)$. Consequently the maximum size of the topological clique contained in the lift of the graph G is bounded from above by $\Delta(core(G))+1$.

The main theorem of this chapter is that this bound is tight. That is, for any graph G, a random lift of G will aas contain a topological clique of size $\Delta(core(G))+1$.

Theorem 18. *For a given graph G asymptotically almost surely $\tilde{G}$ contains a topological clique of size $\Delta(core(G))+1$. Moreover, the clique can be chosen in such a way that each path joining two branch vertices is shorter than $c\log n$, for some constant $c = c(|G|)$.*

4.1 Idea of the proof

In this section we present the main idea and describe some obstacles which we shall have to overcome in the proof of Theorem 18. We also introduce some of the notation that is used throughout this chapter.

The idea behind the proof is roughly the following. Let G be a simple, connected graph and denote by H the core of the graph $\tilde{G}$ chosen randomly from the set $L_n(G)$. Our goal is to find a topological clique of size $\Delta(H) + 1$ in H. Therefore the branch vertices of such clique must have degree at least $\Delta(H)$. Let v be a vertex of the maximum degree in the graph H. Since vertex v could be the only one having the required degree in H we focus on vertices from the fiber H_v. We show that asymptotically almost surely lexicographically first $\Delta(H) + 1$ vertices from this fiber are branch vertices of a topological clique.

Denote the set of $\Delta(H)+1$ lexicographically first vertices from H_v as $U = \{u_1, u_2, \ldots,$ $\ldots, u_{\Delta(H)+1}\}$. In order to prove that they form a topological clique in H we perform a breadth-first search type procedure on the graph H. Let $\mathcal{W}$ be a family of directed closed walks which start and end in v. Starting from each u_i we follow the lifts of walks from the family $\mathcal{W}$ and end up at another vertex from the fiber H_v. The set of vertices reached in that way after z iterations is denoted by $R_z(u_i)$. We proceed in two phases. First, using general structural properties of random lifts we show that aas we can find sets $R_\ell(u_i)$ of the size $O(\log^4 n)$. Next, we extend sizes of $R_\ell(u_i)$ from $O(\log^4 n)$ to the size of $O(\sqrt{n}\log n)$ by the use of coverings of walks from the set $\mathcal{W}$. Finally we show that with probability tending to one, for every pair $u_i, u_j \in U$ there would be a common vertex $x \in R_\ell(u_i) \cap R_\ell(u_j)$. Thus along the path we used to get to x we can find a path connecting u_i to u_j. We repeat this reasoning for every pair of vertices in U which are the branch vertices of our topological clique.

The main technical obstacle in the argument is that paths which connect the branch vertices should be vertex disjoint. Thus in the process of generating $R_\ell(u_i)$ we want to avoid the vertices which have been added to the sets $R_\ell(u_j)$ generated earlier. Hence, whenever we reach already "visited" vertex we will not use this vertex to expand $R_\ell(u_i)$. Consequently sets $\hat{R}_\ell(u_i)$ modified in such a way will be slightly smaller than in the case in which they would be generated independently from each other. We argue that this difference is not substantial and would not affect the probability that the random sets $\hat{R}_\ell(u_j)$ and $\hat{R}_\ell(u_i)$ have a non-empty intersection.

4.2 Preliminaries

As mentioned above in our argument we use a family $\mathcal{W} = \{W_1, W_2, ..., W_{\Delta(H)}\}$ of directed closed walks which start and end at v. We choose those walks in such a way that their first edges are different. It is easy to see that such a family always exists. Indeed, assume we start a walk choosing an edge $e = \{v, x\}$. There are two cases, either e lies on a cycle and we choose this cycle as our walk, or we choose the next adjacent edge $e' = \{x, y\}$ and so on. Since H has minimum degree greater than 2 at some point of this procedure we will reach an edge which lies on a cycle. A path from v to this edge together with this cycle and way back to vertex v will be our closed walk in $\mathcal{W}$.

The probability that the lift of a walk from $\mathcal{W}$ start and end at the same vertex from H_v equals $\frac{1}{n}$. The fact that every walk contains a directed edge separate from other walks allows us to treat them as almost independent from each other (although the lifts of different walks can intersect in H, as we will see shortly, this fact does not affect much the whole analysis).

We will use walks $\{W_1, W_2, ..., W_{\Delta(H)}\}$ to recursively build sets of vertices of the graph H which can be reached from u_i. Let $T_0(u_i) = R_0(u_i) = u_i$. By $T_1(u_i)$ we denote the set of vertices of the lifts of walks from the set $\mathcal{W}$ which starts at vertex u_i. Let $R_1(u_i) = T_1(u_i) \cap H_v$ denote the set of all vertices of the fiber above v in which those walks ends. Next, $T_2(u_i)$ would be the sum of $T_1(u')$'s, for all $u' \in R_1(u_i)$. Thus, let us recall, we start with a vertex u_i, use the lifts of each W_i, for $i \in \{1, 2, \dots, \Delta(H)\}$, to travel from u_i back to the fiber H_v and then use all the walks from $\mathcal{W}$, whose first edge is different than the last edge of the previous path, again to reach successive vertices. In general we set $R_\ell(u) = T_\ell(u) \cap H_v$ and call it *ℓ-vicinity* of u. The set of vertices $T_\ell(u_i)$ is defined recursively, we take all cycles which cover closed walks from $\mathcal{W}$ and start at vertices from $R_{\ell-1}(u_i)$.

As mentioned before in each step of the branching through graph H we are avoiding vertices visited in previous steps. The reason is that we do not want to have an intersection between generated paths, moreover we want the neighbourhoods to be generated randomly and (roughly) independently. To this end, during our procedure we will generate the random lift $\tilde{G}$ on the way, i.e. if we visit a vertex from the lift we reveal its incident edges as a result of the random experiment by choosing one out of the m possible edges. Let us call a vertex $v \in H$ as *active* if we did not generate any edge incidence to it, and call all vertices that are not active as *inactive*. Our object is to avoid inactive vertices since if at any point of the procedure we reach an inactive vertex, then at least some of edges incident to it are already chosen, which interfere with our probabilistic analysis. Let D be the set of inactive vertices in the graph H. Let $D_v = D \cap \tilde{H}_v$ and $D_{W_k} = D_v \cap \{\tilde{W}_k : |\tilde{W}_k \cap D| \geq 1\}$ be the set of ends of the walks from the set of all covers of walks W_k which contains an inactive vertex.

Notice that two walks W_i and W_j can also intersect in G at vertices other than v. For each common point $c \in W_i \cap W_j$, every cover of W_i intersect with at most one of W_j. Hence, whenever we use a cover of the cycle W_i to expand $R_\ell(u_i)$ it prevents us from using exactly one cover of W_j it intersects with. Thus, in this case, in order to prevent W_j from being a part of $R_\ell(u_i)$, for any prospective vertex, we add the ends of the second walk to the set of inactive vertices. This implies that we would never branch from this vertex in the future. Let c denote the total number of intersections between walks $W_1, W_2, ..., W_{\Delta(H)}$ apart from at vertex v. Note that c is bounded from above by the square of the number of vertices of G which, let us recall, is a constant which does not depend on n.

Note that whenever we expand the $T_\ell(u_i)$ there is no point to use edges by which we arrived to the points of $R_{\ell-1}(u_i)$ from $R_{\ell-2}(u_i)$. Otherwise it would contradict the assumption that we want to avoid branching from vertices that we have visited in previous steps. Moreover, for any set $T_\ell(u_i)$ we exclude the vertices which were elements of $T_k(u_i)$, for every $k < \ell$. We treat in the same way possible intersections between $T_k(u_j)$ and $T_\ell(u_i)$ for any $u_i, u_j \in U$ and respectively $i < j$ and $k < \ell$. The modified sets obtained by applying this rule are denoted as $\hat{R}_\ell(u_i)$ and $\hat{T}_\ell(u_i)$. Note that the set $\hat{R}_\ell(u)$ has a structure of a tree T rooted at u_i, which has all leaves placed on the fiber H_v. We can order the vertices of this tree from the root to the leaves. Observe that because $\delta(T) \geq 3$ the sizes of $\hat{R}_\ell(u)$ are expected to grow exponentially with ℓ, at least for small ℓ. Note also that for a given closed walk $W_i \in \mathcal{W}$ the mapping assigning in H to each vertex of H_v its closest successor on H_v is a random matching, which can be viewed as a random permutation.

4.3 Proof

Proof of Theorem 18. Let H be the core of the graph $\tilde{G}$ and v be a vertex of maximal degree in H. If $\Delta(H) = 2$, then H is a cycle. The lift of a cycle is a sum of disjoint cycles, so the lift of G contains a topological clique of size 3. Therefore we may assume $\Delta(H) > 2$ and, since we are considering the core of G, we have also $\delta(H) \geq 2$. For the remainder of this section, we condition on the event that a graph H satisfies conditions of Lemma 5 (i.e. that the $\Delta(H) + 1$ lexicographically first vertices of the fiber H_v are at distance $11 \log\log n$ from each other and every short cycle in H).

Let $U = \{u_1, u_2, \ldots, \ldots, u_{\Delta(H)+1}\}$ be the set of the $\Delta(H) + 1$ lexicographically first vertices from H_v. Let $q < 5 \log\log n$ be the smallest number such that for each $i = 1, ..., \Delta(H) + 1$ the size of the q-neighbourhood $N_q(u_i)$, in the graph H is at least $\log^4 n$.

We may assume, due to Lemma 5, that the vertices in U are at distance at least $11 \log \log n$ from each other and any cycle of length at most $10 \log \log n$. Therefore for all the i's we can choose neighborhoods $N_q(u_i)$ which form a tree and are disjoint from other neighborhoods. Note that in these neighborhoods the distance between two vertices from the fiber H_v is bounded by $|G|$, which is a constant and does not grow with n.

For all trees $N_q(u_i)$ we restrict our attention only to vertices from H_v. Let $M(u_i)$ be a graph whose set of vertices is $N_q(u_i) \cap H_v$. Two vertices x, y are connected in $M(u_i)$ if and only if they are the closest neighbours in the $N_q(u_i)$, i.e. there is no $z \in N_q(u_i)$ such that $d(x,y) = d(x,z) + d(z,y)$. Notice that $M(u_i)$ is a topological minor of $N_q(u_i)$ and the number of vertices of each $M(u_i)$ is of order $\Theta(\log^4 n)$. We subdivide those trees into disjoint subtrees. For each u_i, we choose a subset of vertices $U_i = \{u_i^1, ..., u_i^{\Delta(H)}\} \in M(u_i)$ and divide $M(u_i)$ into disjoint connected subtrees $M(u_i^1) \cup ... \cup M(u_i^{\Delta(H)})$ rooted at u_i^j's and of order $\Theta(\log^4 n)$.

After choosing the u_i's and generating the M_{u_i}'s the set D of inactive vertices is the sum of $\{u_1, u_2, \ldots, u_{\Delta(H)+1}\}$ together with vertices of M_{u_i}'s and ends of walks which cross those neighbourhoods. Our ultimate goal is to expand the vicinity $\hat{R}_\ell(u_1^2)$ to the size of $\sqrt{n} \log n$. We show that we can obtain it aas deactivating at most $O(\sqrt{n} \log n)$. Moreover it will imply that the number of deactivated vertices in the whole process is of order $O(\sqrt{n} \log n)$.

Consider the first pair of vertices (u_1^2, u_2^1). Our first goal is to further expand the set $R_q(u_1^2)$. To this end, we take all leaves of the tree $M(u_1^2)$ and expand their vicinities in the following manner: Let $w \in M(u_1^2)$ be the currently processed vertex. We generate consecutively $\hat{R}_1(w), \hat{R}_2(w), ..., \hat{R}_j(w)$. This is equivalent to consecutively choosing, for each vertex $w' \in \hat{R}_{j-1}(w)$, and for all walks $W_k, k \in \{1, 2, ..., \Delta(H)\}$, an element from corresponding set $H_v - D_v$ at random with uniform distribution. The selected vertices are then deactivated. Furthermore if a walk W_k crosses any other walk W_q then we deactivate ends of the walk $\tilde{W}_q$. We continue expanding until for some ℓ we have

$$\sum_{i=0}^{|M(u_1^2)|} |\hat{R}_\ell(w_i)| = \Theta(\sqrt{n}/\log^3 n).$$

Let $\mathcal{A}$ denote the event that at some point of expanding the vicinity of the vertex $w \in M(u_1^2)$ we choose an inactive vertex. The probability of event $\mathcal{A}$ to occur is bounded by

$$\mathbf{Pr}[\mathcal{A}] \leq c \frac{\sqrt{n}}{\log^3 n} \cdot \frac{|D|}{n - |D|} \leq c \frac{\sqrt{n}}{\log^3 n} \cdot \frac{O(\sqrt{n} \log n)}{n - O(\sqrt{n} \log n)} \leq \frac{O(1)}{\log^2 n} \longrightarrow 0. \tag{4.1}$$

We repeat this action for all $c\log^4 n$ leaves in $M(u_1^2)$. As we just showed the probability of failure in expanding the vicinity for a single vertex is bounded by $\frac{O(1)}{\log^2 n}$. Thus the probability of the event $\mathcal{B}$ that we fail to expand one half of the vertices from $M(u_1^2)$, is bounded by

$$\mathbf{Pr}[\mathcal{B}] \le 2^{c\log^4 n} \left(\frac{O(1)}{\log^2 n}\right)^{c\log^4 n} = o(n^{-3\Delta(H)}) \longrightarrow 0. \tag{4.2}$$

Thus asymptotically almost surely we can expand the vicinities of leaves of $M(u_1^2)$ to the size of $\sqrt{n}/\log^3 n$, avoiding all inactive vertices. Therefore in total we expand the vicinity $\hat{R}_\ell(u_1^2)$ to the size $\Theta(\sqrt{n}\log n)$. Notice that the number of vertices we deactivated during the process is also of order $\Theta(\sqrt{n}\log n)$.

Since we would like to find a path between u_1^2 and u_2^1 in the next step we repeat the same reasoning in respect to the vertex u_2^1. We proceed in exactly the same manner as with the vertex u_1^2, trying to expand $\hat{R}_\ell(u_2^1)$ step by step. The only difference is that the size of the set of inactive vertices grow and whenever there is a connection between any vertex $w_2^1 \in \hat{R}_\ell(u_2^1)$ and any vertex $w_1^2 \in \hat{R}_\ell(u_1^2)$ we stop the procedure. Thus, as before, the probability of the event $\mathcal{A}'$, that at some point of expanding the vicinity of the leaf $w \in M(u_2^1)$ we choose a vertex which is inactive is bounded by

$$\mathbf{Pr}[\mathcal{A}'] \le c\frac{\sqrt{n}}{\log^3 n} \cdot \frac{|S|}{n-|S|} \le \frac{O(1)}{\log^2 n} \longrightarrow 0.$$

Likewise in previous case we repeat this action for all $c\log^4 n$ leaves in $M(u_2^1)$. Again, the probability of failure in expanding the vicinity for a single vertex is bounded by $\frac{O(1)}{\log^2 n}$. Thus, again the probability of the event $\mathcal{B}'$ that we fail to expand one half of the leaves from $M(u_2^1)$, is bounded by $o(n^{-3\Delta_H})$.

Finally we can expand both sets $\hat{R}(u_1^2)$ and $\hat{R}(u_1^2)$ to the size of $\Theta(\sqrt{n}\log n)$. In order to connect vertices u_1^2 and u_2^1 by a path we need to find some vertex $w \in \hat{R}(u_1^2) \cap \hat{R}(u_1^2)$, then the path $u_1^2 ... w ... u_2^1$ would connect u_1 with u_2. The probability that such a vertex does not exist can be bounded above by the probability that we can choose a random set $\hat{R}(u_2^1) \subseteq H_v$ of size $\sqrt{n}\log n$ which avoids $\hat{R}(u_1^2)$. This probability is smaller than

$$\begin{aligned}
\frac{\binom{n-|D|-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{n-|D|}{\sqrt{n}\log n}} &\le \frac{(n-|D|-\sqrt{n}\log n)!}{(\sqrt{n}\log n)!(n-|D|-2\sqrt{n}\log n)!} \cdot \frac{(\sqrt{n}\log n)!(n-|D|-\sqrt{n}\log n)!}{(n-|D|)!} \\
&= \frac{(n-|D|-2\sqrt{n}\log n)(n-|D|-2\sqrt{n}\log n+1)\cdot \ldots \cdot (n-|D|-\sqrt{n}\log n)}{(n-|D|-\sqrt{n}\log n)(n-|D|-\sqrt{n}\log n+1)\cdot \ldots \cdot (n-|D|)} \\
&\le \left(1-\frac{\log n}{\sqrt{n}-O(\log n)}\right)^{\sqrt{n}\log n} = o(n^{-3\Delta(H)}) \longrightarrow 0. \qquad (4.3)
\end{aligned}$$

Our aim is to connect u_i's by disjoint paths so that they create a topological clique. Therefore we will take pairs of vertices (u_i^j, u_j^i) for $i \neq j$ and $(u_i^j, u_{\Delta(H)+1}^i)$ for $i = j$ and try to build a set of disjoint paths between them.

The argument for all the pairs of vertices u_i^j, u_j^i is similar to the one above. Again the only thing that changes is the size of the set of inactive vertices D we have to avoid, but since there are only $(\Delta(H) + 1)^2$ pairs it will never grow beyond $O(\sqrt{n} \log n)$. Consequently all the previous calculations carry over to this case. Thus, the probability of choosing some previously visited vertex while expanding the vicinity of any leaf of $M(u_i^j)$ or $M(u_i^j)$ to the size of $\sqrt{n}/\log^3 n$, is bounded by $\frac{O(1)}{\log^2 n}$ as in (4.1). Hence, as before, the probability of the event $\mathcal{B}''$, that we fail to expand one half of the vertices from $M(u_j^i)$ and half from $M(u_i^j)$ is $o(n^{-3\Delta(H)})$ (see (4.2)). This implies that in $O(\log n)$ stages we can expand vicinities of leaves from $M(u_j^i)$ and $M(u_i^j)$ to the size of $\sqrt{n} \log n$. Finally, as in (4.3), the probability that we do not find a vertex which connects these two vicinities can be bounded from above by

$$\frac{\binom{n-|D|-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{n-|D|}{\sqrt{n}\log n}} = o(n^{-3\Delta(H)}) \longrightarrow 0, \tag{4.4}$$

Thus, we have showed that the probability of failure in connecting any pair is of order $o(n^{-3\Delta(H)}) = o(1)$. Because there are only finite number of pairs, the probability that we do not find a topological clique of size $\Delta_H + 1$ can also be bounded by $o(1)$. Note that for each vertex u_i^j we chose some vertex at distance at most $5 \log \log n$ from u_i^j and in $O(\log n)$ steps we connected it with some other u_j^i. Thus the generated paths connecting u_i^j's with u_i^j's are of length $O(\log n)$. $\square$

4.4 Links

In the Section 3.2 we reviewed the results on connectivity properties of random lifts in terms of the number of vertices or edges you have to delete from a graph to separate given subset of vertices from the rest of the graph. Now we consider a related yet slightly different problem.

Definition. A graph G with at least $2k$ vertices is said to be k*-linked* if for every $2k$ distinct vertices $s_1, s_2, ..., s_k, t_1, t_2, ..., t_k$ it contains k vertex-disjoint paths $P_1, P_2, ..., P_k$ such that P_i connects s_i to t_i, $1 \leq i \leq k$.

Notice that, from Menger's theorem [11], each k-linked graph is k-connected, but the converse is far from being true (for example a cycle is 2-connected but it is not 2-linked).

Now we try to answer the question about maximal k, for which almost every random lift of a given graph is k-linked. Jung [20] and, independently, Larman and Mani [23] proved that every $2k$-connected graph that contains a K_{3k} as a topological minor is k-linked. Combining their result with Theorem 18 and Theorem 6 we get the following corollary.

Corollary 2. *If G is a connected graph with minimum degree δ, then aas $\tilde{G} \in L_n(G)$ is $\min\{\Delta(\mathit{core}(G))/3, \delta/2\}$-linked.*

A slight modification of the argument used in the proof of Theorem 18 together with result from Lemma 4 gives us better result.

Theorem 19. *For a given graph G with $\delta(G) = 2k - 1 \geq 3$ asymptotically almost surely $\tilde{G} \in L_n(G)$ is k-linked.*

Proof. In the proof we condition on the random lift of G to fulfill Lemma 4. Our plan is to mimic the proof of Theorem 18. If vertices from the set S are at distance $11 \log\log n$ from each other and all short cycle in $\tilde{G}$, then we can choose one fiber $\tilde{G}_u$ and connect them, by paths of length smaller than $|G|$, to vertices $u_1, ..., u_{2k}$ from this fiber. Those vertices will have the same properties as vertices from the statement of Lemma 5. Thus from this point we proceed in the same way as in the proof of Theorem 18. Asymptotically almost surely designated vertices $u_1, ..., u_{2k}$ will be the branch vertices of a topological clique in $\tilde{G}$. Hence we can find a vertex-disjoint paths connecting s_i to t_i.

If any two vertices $x, y \in S$ are at smaller distance to each other than $11 \log\log n$, then we would like to switch them with to ones which are far from all the others vertices in S. By Lemma 4 the $(\log\log n)^2$-neighbourhoods of vertices in S have at most one cycle. It means we can find a path connecting x with vertex $\bar{x} \in N_q(x)$, $q = (\log\log n)^2$ which is at distance greater than $11 \log\log n$ from any other vertex in S. We can repeat this operation for any other vertex $x \in S$. Notice that branching through the tree we either find a short path connecting particular s_i with t_i inside $N_q(s_i) \cap N_q(t_i)$ or vertex-disjoint paths connecting x's with $\bar{x}$'s.

In this way we can create some short paths connecting some of pairs s_i and t_i inside $N_q(s_i) \cap N_q(t_i)$ and for those among vertices $s_1, s_2, ..., s_k$ and $t_1, t_2, ..., t_k$ which remain unmatched we can find a set of disjoint paths connecting them with vertices $u_1, ..., u_k$ on the fiber of $\tilde{G}_u$. Furthermore, we can assume that u_i's are at distance at least $11 \log\log n$ from each other and all, apart from at most one, short cycles in $\tilde{G}$. It easy easy to notice that such a single cycle do not influence the analysis made in the proof of Theorem 18. Then we can mark all vertices of constructed paths and their neighbours as inactive and mimic the argument from the proof of Theorem 18 to construct the topological clique on set

$u_1, u_2, ..., u_k$. Then, to find a path from s_i to t_i one needs to go from s_i to the branch vertex u_i, next use edges of the clique to reach the branch vertex u_{k+i} matched to t_i and finally go to t_i.

The probability that we fail in any step of the proof is less than $o(n^{-3\Delta(H)})$ (see the estimates in (4.2)-(4.4)). Since there are at most $\binom{n|G|}{2k} \leq (n|G|)^{2k}$ possibilities to choose $2k$ vertices out of $n|G|$ vertices of the lift of G the probability of failure in connecting any of them tends to 0 as $n \to \infty$.

$\square$

Let us remark that the above statement does not hold for $k = 1$ even if $\delta(G) = 2$, since asymptotically almost surely random lift of a cycle is not connected. On the other hand, for $k \geq 2$ it is clearly best possible, since k-linked graph contains a vertex of degree at most $2k - 2$. Indeed, in this case we can put v as s_1, as t_1 take any vertex outside $N(v)$, and separate s_1 from t_1 by $N(v) \subseteq \{s_2, ..., s_k, t_2, ..., t_k\}$.

4.5 k-diameter

In the previous section we showed that for any two sets of k vertices we can connect pairs of vertices from those sets by mutually disjoint paths. In addition the proof of this fact gives us an insight into the length of the paths connecting those vertices. A parameter which is focused on the length of different paths connecting any pair of two vertices in a graph is the *k-diameter of a graph.*

Let G be a k-connected graph and u, v, $u \neq v$, be any pair of vertices of G. Let $P_k(u, v)$ be a family of k vertex disjoint paths between u and v, i.e.

$$P_k(u, v) = \{p_1, p_2, ..., p_k\}, \text{ where } |p_1| \leq |p_2| \leq ... \leq |p_k|$$

and $|p_i|$ denotes the number of edges in path p_i. The *k-distance* $d_k(u, v)$ between vertices u and v is the minimum $|p_k|$ among all $P_k(u, v)$ and the *k-diameter* $d_k(G)$ of G is defined as the maximum k-distance $d_k(u, v)$ over all pairs u, v of vertices of G.

The concept of k-diameter comes from analysis of the performance of routing algorithms [10] but has also drawn some attention as a graph parameter [18]. In the case of lifts of a given graph G, for all vertices $u, v \in V(G)$, by the proof of Theorem 18, we know that for almost every random lift whenever we choose nearest neighbors of u and v we find a set of disjoint paths connecting vertices from these two sets. Thus, as an immediate consequence of Theorem 18 we get the following result.

Corollary 3. *If G is a connected graph with minimum degree $\delta \geq 3$, then aas δ-diameter of $\tilde{G} \in L_n(G)$ is $O(\log n)$.*

> *Do not keep saying to yourself, if you can possibly avoid it, "But how can it be like that?" because you will get "down the drain," into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that.*
>
> Richard Feynman, *The Character of Physical Law, 1965*.

5 Hamilton cycles in random lifts

Finding a Hamiltonian cycle is one of the most celebrated problems in graph theory and theory of random graphs (see [7] for many results in this area). It is no surprise that it also caught the attention of the researchers for the case of random lifts. The main question is whether it is true that for every G either almost all or almost none of the random lifts of G contain a Hamilton cycle as in the case for perfect matching (see Theorem 13). In a weaker version of the problem, posed by Linial [26], we ask whether this property is true for a subclass of d-regular graphs.

Problem 1. *Let G be a d-regular connected graph with $d \geq 3$. Is it true that almost every random lift of G is Hamiltonian?*

Burgin, Chebolu, Cooper and Frieze has proven that for sufficiently large complete graphs and complete bipartite graphs almost every lift is Hamiltonian.

Theorem 20 ([8])**.** *There exists a constant t_0, such that if $t \geq t_0$, then asymptotically almost surely $\tilde{K}_t$ is Hamiltonian.*

Theorem 21 ([8])**.** *There exists a constant t_1, such that if $t \geq t_1$, then asymptotically almost surely $\tilde{K}_{t,t}$ is Hamiltonian.*

It can be shown [33] that the constant t_0 is less than 30. Chebolu and Frieze [9] were able to expand this result to the random lifts of complete directed graphs (where lifted edges preserve orientation of edges from the base graph). Here we prove the following statement.

Theorem 22. *Let G be a graph with minimum degree at least five which contains at least two edge-disjoint Hamilton cycles, then aas $\tilde{G}$ is Hamiltonian.*

The structure of the proof is the following. First we describe the algorithm which finds the Hamilton cycle in $\tilde{G}$. Then we show that asymptotically almost surely it succeeds in finding Hamilton cycle in $\tilde{G}$.

5.1 Preliminaries

Let G be a connected graph on k vertices with $\delta(G) \geq 5$ which contains two edge disjoint Hamilton cycles H_1 and H_2. Choose any vertex h_1 and label each vertex twice according to its appearance in Hamilton cycles i.e. $H_1 = h_1 h_2 \dots h_k h_1$ and $H_2 = h'_1 h'_2 \dots h'_k h'_1$, where $h'_1 = h_1$. Let $G_1 = G - H_1$ and note that $\delta(G_1) \geq 3$.

Due to Lemma 3 aas the random lift of H_1 consists of disjoint cycles $C_1, C_2, ..., C_\ell$, where $\ell \leq 2 \log n$. We refer to these as *basic cycles*. We will use the property that cycles in the lift preserve the order of vertices from the cycles in the base graph, i.e. they can be written as

$$h_1^1 h_2^1 \dots h_k^1 h_1^2 h_2^2 \dots h_k^2 \dots h_1^r h_2^r \dots h_k^r h_1^1, \tag{5.1}$$

where h_j^i is an element of the fiber above h_j. In other words when we project them on the base graph G we get r copies of the cycle H_1 glued together at vertex h_1.

The algorithm uses a path reversal technique of Pósa [31]. Let G be any connected graph and $P = v_0 v_1 ... v_m$ be a path in G. If $1 \leq i \leq m-2$ and $\{v_m, v_i\}$ is an edge of G, then $P' = v_0 v_1 ... v_i v_m v_{m-1} ... v_{i+1}$ is a path with the same vertex set as P. We call P' a Pósa rotation of P with preserved *starting point* v_0 and *pivot* v_i. Note that new edge $\{v_m, v_i\}$ in path P' is not incidence to the current ends of P'. By $\mathcal{P}_b(v_0)$ we denote the set of all paths of G which can be obtained from P by b rotations preserving the starting point v_0.

Our strategy will be rather natural. Denote the longest cycle in $\tilde{H}_1$ by C. We shall try to connect C to any remaining basic cycle in the lift using the edges of $\tilde{G}_1$. Thus we want to increase the length of C by “absorbing” one basic cycle at a time. Once we succeed we break the cycle C and connect it to other basic cycle, a path created in this way will be denoted by P. Next we continue to expand the path. We shall do it by generating edges of $\tilde{G}_1$ which are incident to one of the ends of the path P. If we connect it to some of the basic cycles, say C'_s, then we replace P by a longer path adding all vertices from C'_s, otherwise, either we try to connect the ends of P by creating a new cycle C or try to replace P by another path using the Pósa transformation. If the obtained cycle C is still not a Hamilton cycle, then we merge C with some of the the remaining basic cycles and repeat the procedure.

In the analysis of the algorithm we shall show that asymptotically almost surely after fewer than $5n^{4/5}$ rotations we can always merge P with one of the remaining basic cycles. At each iteration we connect one additional basic cycle to the cycle C. Thus in order to perform the whole procedure, we need to generate edges of $\tilde{G}_1$ incident to not more than $10n^{4/5}\log n \leq n^{5/6}$ vertices. We generate a graph $\tilde{G}$ in each step of the algorithm edge by edge, at each point choosing for a given vertex v its neighbour at random from all available candidates. Whenever we have already generated an edge adjacent to vertex v we call such a vertex *inactive*, vertices that are not inactive are called *active*.

5.2 The algorithm

The algorithm consist of seven phases.

Phase 1 – Cycle lift

Generate a lift $\tilde{H}_1$. Assign C to be the longest cycle in $\tilde{H}_1$.

Phase 2 – Cycle Merge

Given a cycle C and a set of basic cycles $C'_1, \ldots, C'_s$ disjoint with C do the following:

A. If $0 < \sum_{i=1}^{s} |V(C'_i)| < n^{9/10}$ take any active vertex v which belongs to a basic cycle C'_1 and generate edges of $\tilde{G}_1$ incident to it. If one of these edges e connects C'_1 to C assign to P a path whose vertex set is $V(C) \cup V(C'_1)$ and those two parts are joined by e.

B. If $\sum_{i=1}^{s} |V(C'_i)| \geq n^{9/10}$ choose any out of $n^{1/3}$ vertices of C which are at distance at least 2 from any inactive vertex and generate edges of $\tilde{G}_1$ incident to them. If one of these edges e connects C'_i to C assign to P a path whose vertex set is $V(C) \cup V(C'_i)$ and those two parts are joined by e.

Phase 3 – Path Merge

Given a path P and some basic cycles $C'_1, \ldots, C'_s$, if any end of P is connected to a basic cycle C'_i replace P by a new path with vertex set $V(P) \cup V(C'_i)$.

Phase 4 – Cloning Path

Let us suppose we are given a path P whose ends are both active and a set of basic cycles $C'_1, \ldots, C'_s$.

Repeat following actions :

Take $P = w_1 w_2 \ldots w_t$ and apply to it repeated Pósa transformation preserving starting point w_t. Continue until you find $\log^2 n$ different paths starting at w_1 and ending at w_{i_j}, $j = 1, 2, \ldots, \log^2 n$. Now reverse each of these paths and apply to each of them the *transformation preserving point w_{i_j}. Continue to perform the operations until one of the conditions is true:*

— there is a connection between a path P', $V(P') = V(P)$ and some basic cycle C_{i_0},

— we find $\log^2 n$ paths $P_1, \ldots, P_r$ such that each of them has the same vertex set as P, and all $2r$ vertices which are ends of these paths are pairwise different and active.

In the case we fulfil the first condition go back to Phase 3, in the case we fulfil the second condition continue to Phase 5.

Phase 5 – Multiplying Ends

For every path $P_1, \ldots, P_r$ constructed in Phase 4 split the vertex set $V(P_j)$ of P_j into two roughly equal disjoint sets $V_1, V_2 \subset V$, $|V_1|, |V_2| \geq (|V| - 1)/2$. Thus every path $P_j = w_1 \ldots w_t$ splits into two paths $P'_j = w_1 w_2 \ldots w_{i-1} w_i$ and $P''_j = w_{i+1} w_{i+2} \ldots w_t$, where $i = \lceil t/2 \rceil$.

At any point of the phase if there is:

— an edge closing some path P_j to form a cycle, then go to Phase 2,

— an edge connecting P_j with some basic cycle, then go to Phase 3.

Repeat simultaneously for each path $P_1, \ldots, P_r$:

Apply a series of Pósa transformations to the path P'_j which preserve the starting point w_i and a series of Pósa transformations to the path P''_j which preserve starting point w_{i+1}. (We apply a single Pósa transformation to each of the paths in turn before we apply the next Pósa transformation).

Stop if for any path you find two sets $S_1 \subset V_1$, $S_2 \subset V_2$, such that $|S_1|, |S_2| \geq n^{3/5} \log^2 n$ with the following property:

> *For every $x \in S_1$ and $y \in S_2$ there is a path P_{xy} of length $|P_j|$ which starts at x ends at y whose first $|V_1|$ vertices are those from V_1 and last $|V_2|$ vertices are those from V_2.*

Phase 6 – Adjusting

Choose any edge $\{x, y\}$ from $G \setminus (H_1 \cup H_2)$. Use at most $2|G|$ Pósa transformations to switch the end w_1 of the path P' and the end w_t of the path P'' to replace the sets S_1, S_2 generated in the previous stage by slightly smaller sets $S'_1 \subset V_1$, $S'_2 \subset V_2$, $|S'_1|, |S'_2| \geq n^{3/5}$, such that S_1 is contained in the fiber G_x and $S_2 \subset G_y$.

Phase 7 – Closing a cycle

Generate all edges of $\tilde{G}_1$ incident to vertices from S_1'. If one of them has an end in S_2' then STOP if the resulted cycle is a Hamilton cycle, or otherwise go to Phase 2.

5.3 The analysis of the algorithm

In this section we show that aas the algorithm returns a Hamiltonian cycle and, consequently, Theorem 22 follows.

Phase 1. We start the analysis of the algorithm with Phase 1. As already mentioned, Lemma 3 states that the random lift of H_1 asymptotically almost surely consists of disjoint cycles $C_1, C_2, ..., C_\ell$, where $\ell \leq 2\log n$. Note that this means that the length of the longest cycle $C \in \tilde{H}_1$ is at least $n/(2\log n)$.

Observe that since the number of basic cycles is bounded from above by $2\log n$, Phase 2 and Phase 3 can be invoked only at most $2\log n$ times. Our aim is to show that with probability at least $1 - o(1/\log n)$ we enlarge the path P during Phases 2-7 each time deactivating fewer that $5n^{4/5}$ vertices. Thus, the total number vertices deactived during the Algorithm is bounded from above by $n^{5/6}$. Note that in any step in which we deactivate a vertex either it is already in P or we have just added it to P. Im implies that all vertices outside P are active.

Phase 2. In this step we want to connect cycle C with a basic cycle disjoint with it, creating a long path P. Since we want the ends of path P to be active vertices, we require that the vertices which connect those two cycles are not adjacent to any inactive vertices.

In case A the total number of vertices in the remaining basic cycles which are yet to be joined to C is smaller than $n^{0.9}$. The probability that a vertex from the basic cycle C_1' will have a neighbour in C which is at distance at least 2 from any inactive vertex is larger than $1 - 2\Delta^2(G)n^{-0.9} = 1 - o(1/\log n)$, since we need to exclude vertices outside C together with all inactive vertices and their neighbours. Hence, aas merging will deactivate only one vertex.

For case B note that since $|C| \geq n/(2\log n)$ and fewer than $n^{5/6}$ vertices have been deactivated in the procedure, one can greedily select $n^{1/3}$ vertices which are at distance at least 2 from any inactive vertex and from each other. Clearly, the probability that none of these vertices is adjacent in $\tilde{G}_1$ to one of the basic cycles is bounded from above by

$$1 - \left(\frac{n - n^{0.9}}{n}\right)^{n^{1/3}} \leq 1 - (1 - n^{-1/10})^{n^{1/3}} = 1 - o(1/\log n).$$

In this way each time we invoke this phase aas we deactivate at most $n^{1/3}$ vertices.

Phase 3.

We do not generate any edges in this step, and so we do not deactivate any vertices.

Phase 4. Let $P = w_1, ..., w_t$. Our aim is either to find an edge of $\tilde{G}_1$ joining one end of a path P', $V(P') = V(P)$, to one of the cycles outside P and go to Phase 3, or to find for $r = \log^2 n$ a set of paths $P_1, \ldots, P_r$ such that each of them has the same vertex set as P, and all $2r$ vertices which are ends of these paths are pairwise different and active.

There are two stages in this phase. First we take path P and find a set of r paths which start at w_1 and whose $2r$ ends are distint and active. Notice that after any Pósa transformation we want our new ends to be active so we require that our pivot w_i has no inactive neighbours. Thus we estimate the probability that in any of the $\log^2 n$ possibly required Pósa transformations the new end of our transformed path is connected to a vertex which is at distance at least 2 to any inactive vertex or has been the end of one of the previously generated paths. Since there are fewer than $n^{5/6}$ inactive vertices this probability can be crudely bounded above by

$$\log^2 n \, \frac{2\Delta^2(G)n^{5/6}}{n - 10\Delta(G)n^{5/6}} = o(1/\log n) \, .$$

In the second stage we take all paths $P_1, ..., P_r$ and apply to them the Pósa transformations preserving the ends chosen in the first stage. At this time the structure of each path is distinct, so in the process of applying consecutive transformations we might get different results for each path. Moreover we want those new ends to be different from ends generated in previous stage. Thus we take the first path P_1 and apply transformations in order to generate a set of $2\log^2 n$ active ends for it and choose one of them as the end of P_1. Then we take path P_2; if it admits the same transformations as P_1, then we select one of the vertices generated for P_1, which has not already been taken, as the end for P_2. In the opposite case we apply Pósa transformations for P_2 and generate a new set of $2\log^2 n$ ends for it. We repeat the same operations for other paths. Notice that in the worst case scenario we need to make at most $2\log^4 n$ single transformations in total. Similarly to before the probability that in any of $2\log^4 n$ required Pósa transformations the new end of our transformed path is connected to a vertex which is at distance at least 2 to any inactive vertex or has been the end of one of the previously generated paths is bounded from above by

$$2\log^4 n \, \frac{2\Delta^2(G)n^{5/6}}{n - 10\Delta(G)n^{5/6}} = o(1/\log n) \, .$$

Note also that we have deactivated at most $2\log^2 n + 2\log^4 n \leq 3\log^4 n$ vertices in this stage.

Phase 5. Let us recall that, roughly speaking, in this phase we want to take any of the paths $P_i = w_1 w_2 \dots w_t$ constructed in the previous case, split it into two halves $P' = w_1 w_2 \dots w_{i-1} w_i$ and $P'' = w_{i+1} w_{i+2} \dots w_t$, where $i = \lceil t/2 \rceil$, and apply to them transformations preserving respectively w_i and w_{i+1} in order to find at least $n^{3/5} \log^2 n$ new feasible ends for each of them.

We show that the probability that we succeed in doing it for one path is bounded away from zero, by some constant $\alpha > 0$. Thus if we repeat this for $\log^2 n$ paths, then with probability $1 - o(1/\log n)$ for at least one of them we expand the set of feasible ends to the required size.

The existence of a constant $\alpha > 0$ follows easily from the theory of branching process (see Section 2.3). Indeed, take one path, say P', and first generate all its possible ends using the transformation preserving the end w_i (this will be the first generations of ends), then apply consecutive transformation to obtained ends in order to get the second generations of ends, and so on. In each step we generate at least three new vertices (since the minimum degree of G is three) and we fail if we choose in such a trial either a vertex from the other path P'', or a vertex which is adjacent to inactive vertex or one of the ends chosen so far. Hence, the probability of making a bad choice is in each step bounded from above by

$$\frac{n/2 + \Delta^2(G)(n^{5/6} + 1)}{n - \Delta(G) n^{5/6}} \leq 0.51.$$

Consequently, the number of successful choices (i.e. the ones which either lead to a new end or allow us to go to Phase 3) in one round is stochastically bounded from below by the binominally distributed random variable $B(3, 0.49)$.

Thus, let us recall, we treat the process of applying consecutive Pósa transformations as a branching process. Since every active vertex v has at least 3 edges in G_1 which are still to be revealed, then the possible number of descendants for each ancestor is bounded from below by 3. The probability of producing new individual in the next generation equals the probability that generated edge connects v with a vertex of P' which is not adjacent to an inactive vertex or vertex generated in previous steps. Thus while the number of individuals is of order $O(n^{3/5})$, and do not affect much the number of inactive vertices, the process of generating feasible ends for the path P can be closely approximated from below by the branching process defined by a variable with binomial distribution $B(3, 0.49)$.

Since $3 * 0.49 > 1$, by Theorem 1 with probability $\beta > 0.61$ the branching process will not die out. Furthermore, in the Section 2.3, we showed that with probability at least $1 - 2\exp(-n^{3/5})$ the first time we get $n^{3/5} \log^2 n$ vertices in one generation the total number of offspring is bounded from above by $5n^{3/5} \log^2 n$ (see Lemma 2). Consequently, with probability at least $\beta/2$, after using at most $5n^{3/5} \log^2 n$ vertices we either merge the end of

P' with one of basic cycles (and so go to Phase 3) or generate at least $n^{3/5}\log^2 n$ different active ends for this path. Hence, the probability that it happens at the same time for P' and P'' is bounded from below by $\alpha = (\beta/2)^2$.

As mentioned at the beginning the previous phase of the algorithm provided us not one, but $\log^2 n$ paths with different ends. Consequently, with probability

$$1 - (1-\alpha)^{\log^2 n} = 1 - o(1/\log n)$$

we succeed in expanding the set of feasible ends for at least one of the paths. Hence, with probability at least $1 - o(1/\log n)$ this phase of the algorithm can be completed with the total number of deactivated vertices bounded from above by $5\Delta(G)n^{3/5}\log^4 n \leq n^{4/5}$.

Phase 6. The sets S_1 and S_2 found in the previous phase are such that each edge connecting them creates a cycle on vertex set $V(P)$. Such a cycle is either a Hamilton cycle or can be merged to some remaining basic cycles. However, it might happen that sets S_1 and S_2 are placed in two fibers which correspond to non-adjacent vertices of G and so we cannot expect them to be connected by an edge. Hence, in this phase we want to use the second Hamilton cycle H_2, to "switch" elements of the sets S_1 and S_2 (or at least a large portion of it) to the chosen fibers $\tilde{G}_x$ and $\tilde{G}_y$.

Let $P' = w_1 w_2 \dots w_i$ be defined as "the half" of path we have dealt with in the previous phase, and let $w_1 \in S_1$. We would like to argue that, with probability bounded away from zero by some constant $\gamma > 0$, we can deactivate at most $|G|$ vertices in order to either connect P' to some remaining basic cycle, or turn P' by a sequence of transformations preserving the end w_i into a path with an end in the fiber above a given vertex x.

Let us recall first that P' has been obtained in the process of merging and transforming basic cycles obtained in the first phase. Each of the basic cycles has a periodic structure (see (5.1)) which implies that they are evenly distributed across the fibers of the lift. Let $k = |G|$ where, let us recall, k is a constant which does not grow with n. In the case when the length of P' is smaller than $n/3$ the total length of basic cycles outside P' and P'' is $m \geq n/3$ and furthermore each fiber contains precisely m/k vertices which belong to basic cycles outside $V(P') \cup V(P'')$. Consequently, with positive probability (at least $m/(nk) \geq 1/(3k)$) we merge the end of P' with a basic cycle deactivating just one vertex.

Let us consider now the more challenging case, when P is very long and the length of P' is at least $n/3$. We are interested in the structure of path P', namely to what extent it preserves the structure of basic cycles. Whenever we join two cycles or perform a Pósa transformation we perturb the cyclic distribution of vertices. More precisely, one merge or transformation can spoil at most three of sequences $h_1^i h_2^i \dots h_{k-1}^i h_k^i h_1^i$ which occur in the path P. See Figure 5.1 for an example of transformation, note that after transformation in part of the path the order of the vertices in the sequence is reversed.

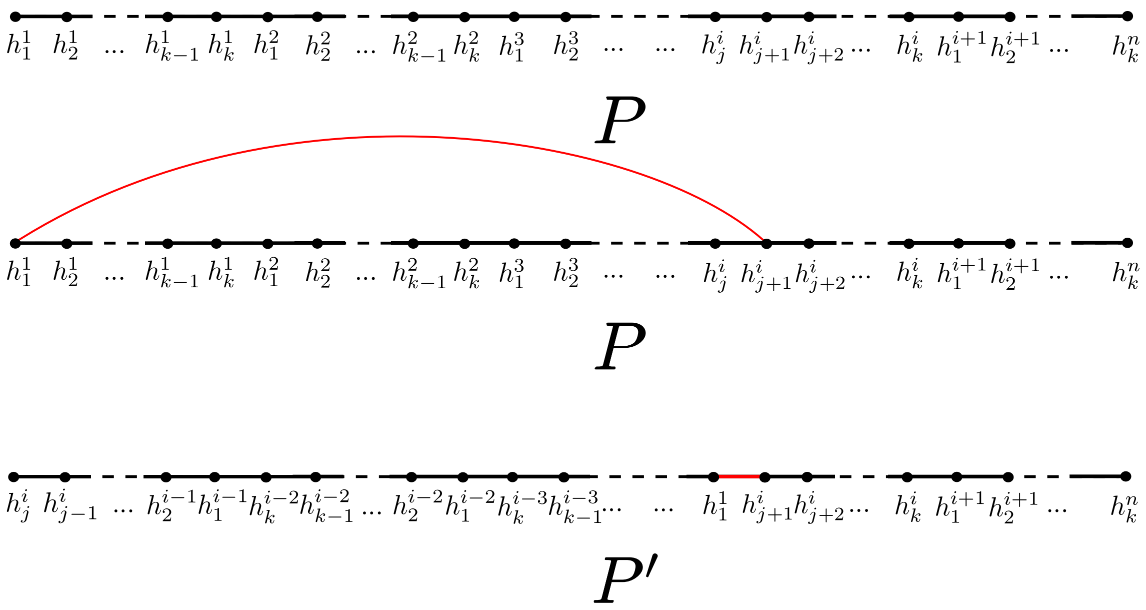

Figure 5.1: Let $H_1 = h_1...h_k$ be a Hamilton cycle, by h_i^j we denote a vertex from the fiber above h_i. Path P consist of sequences of vertices from fibers above consecutive vertices in H_1. After Pósa transformation with pivot h_{j+1}^i we get a new path P'. Notice that $h_1^i...h_k^i h_1^{i+1}$ is not a consecutive sequence of vertices on the path P'. Moreover sequences $h_1^1...h_1^{i-1}$ are reversed in the path P'.

Observe that the number of joins and transformations made to a path P' is bounded by the number of inactive vertices. Since during the algorithm we deactivate at most $n^{5/6}$ vertices, there are at least $(n/3k) - 3n^{5/6} > 2n/(7k)$ sequences of consecutive vertices which belong to fibers given by the order of vertices in H_1. Some of the sequences could get reversed in the transformations (see Figure 5.1), but at least half of them, i.e. at least $n/(7k)$, are sequences of consecutive vertices appearing in the order $h_1 \dots h_{k-1}h_kh_1$ or $h_1h_kh_{k-1} \dots h_2h_1$. In the first case we will say that a sequence have a positive orientation, in the second case we will say that is has a negative orientation.

Let us assume that we can choose $n/(7k)$ sequences with the same orientation. We subdivide P' into $k-1$ connected sections, such that each of them contain at least $z = n/(8k^2)$ sequences and denote those sections as $Q_1, ..., Q_{k-1}$. See Figure 5.2 for an example.

Let $H_2 = h'_1h'_2 \dots h'_kh'_1$ be the second Hamiltonian cycle in the graph G, that is edge-disjoint from the cycle H_1. Without loss of generality we may assume that the end of P' belongs to the fiber above h'_1 and denote it by u'_1. Notice that H_1 is just a permutation of

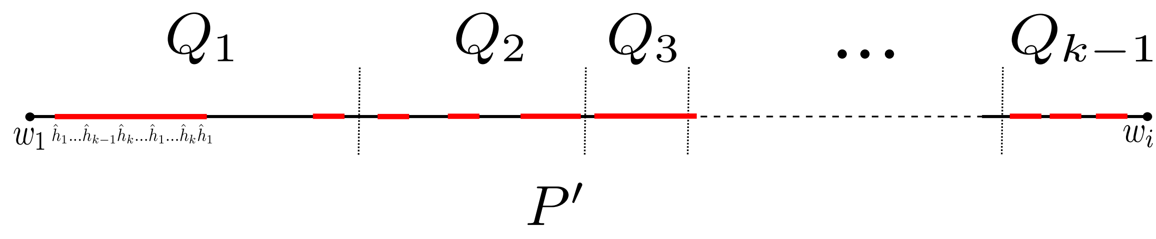

Figure 5.2: The path P' divided into sections $Q_1, ..., Q_{k-1}$. By $\hat{h}_i$ we denote that vertex is an element of the fiber above h_i. Red fragments indicates segments with sequences of vertices which belong to fibers given by order of vertices in H_1 (there could be more than one sequence in one segment).

the cycle H_2. Associate with every vertex h'_i of H_2 a vertex which precedes it on the cycle H_1, and denote them as μ_i (notice that such an assignment is a surjective function). What we like to do now is to generate an edge from u'_1 to a vertex u'_2 which lies above h'_2. Then use u'_2 as a pivot in Pósa transformation that would change P' into a path P'_1 which ends at a vertex u'_i from fiber above $\mu_1 = h'_i$. As the vertex μ_1 corresponds to a vertex h'_i in the cycle H_2, we continue the transformations in the same manner as previously. We generate an edge from u'_i to a vertex u'_{i+1} which lies above h'_{i+1}. Next use u'_{i+1} as a pivot in Pósa transformation that would change P'_1 into a path P'_2 which ends at a vertex u'_j from fiber above $\mu_{i+1} = h'_j$. We apply the same operations until, for some i, the path P_i end in a vertex from the fiber $\tilde{G}_x$. See Figure 5.3 for example.

Note that since we switch between fibers according to the order of vertices in the Hamilton cycle H_1 the paths $P'_1, P'_2, \ldots, P'_k$ have ends on different fibers of $\tilde{G}$. Thus one of them has to belong to the fiber $\tilde{G}_x$.

To perform described switching we have to make sure that prospective path ends are elements of sequences of the same orientation (which changes due to transformations). Otherwise the path end after the transformation could not be an element of the respective fiber $\tilde{G}_{\mu_i}$. We can easily preserve this property if every vertex chosen as a pivot for path P_i will be closer on a path P_i to the vertex w_i than the pivot used for path P_{i-1} (see the Figure 5.3 again). That is why for all generated ends we put a condition that the end of path P_i has to be an element of Q_i. The probability that chosen end belongs to Q_i equals $1/9k^2$. Hence, with probability at least $(9k^2)^{-k}$ we can do at most k switches to move a given vertex from S_1, from any fiber to the designated fiber above vertex x. The same analysis can be repeated in respect to the second path P'' and vertices from S_2 which we would like to place on fiber $\tilde{G}_y$. Since $|S_1|, |S_2| \geq n^{3/5} \log^2 n$, with probability at least

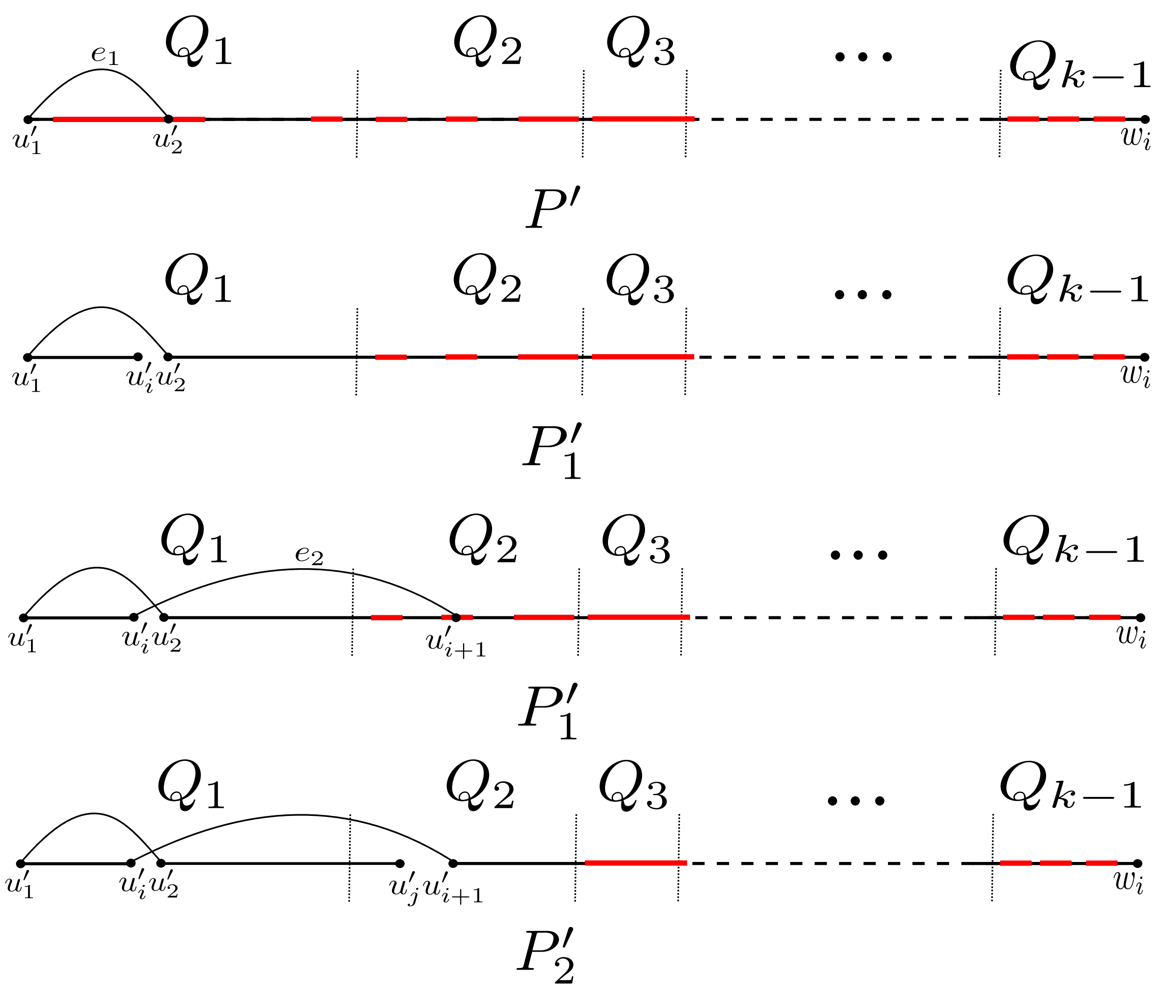

Figure 5.3: Two steps of the process of switching the end of path P' onto desired fiber. Red sections indicates positively oriented sequences of vertices from fibers above consecutive vertices of the cycle H_1. Vertex u'_i belong to the fiber above vertex h'_i, where h'_i is the i-th element of the cycle H_2. Edges e_1 and e_2 connects vertex from fiber above h'_ℓ with vertex from fiber above vertex succeeding h'_ℓ in Hamilton cycle H_2. In the example vertex h'_i precedes vertex h'_2, and vertex h'_j precedes vertex h'_{i+1} in the Hamilton cycle H_1.

$1 - \exp(-n^{3/5}) = 1 - o(1/\log n)$ we can successfully switch at least $n^{3/5}$ of them. Note that in this process we deactivated at most $2|G|n^{3/5}\log^2 n < n^{4/5}$ new vertices.

Phase 7 Since S'_1 and S'_2 belong to different fibers which correspond to adjacent vertices from G the probability that we shall not close the cycle is bounded from above by

$$\left(\frac{n - |S'_1| - |D|}{n - |D|}\right)^{|S'_2|} \leq \left(\frac{n - |S'_1|}{n}\right)^{|S'_2|} \leq \exp\left(-\frac{|S'_1||S'_2|}{2n}\right) \leq \exp(-n^{1/6}/2) = o(1/\log n)\,.$$

Where by $|D|$ we denoted the set of inactive vertices. Clearly, in this process we deactivated at most $2|S_1'| \leq n^{4/5}$ vertices.

This completes the analysis of the algorithm and the proof of Theorem 22. $\square$

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