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On a topological relaxation of a conjecture of Erdős and Nešetřil

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Abstract

The strong chromatic index of a graph G, denoted by $s'(G)$, is the minimum number of colors in a coloring of edges of G such that each color class is an induced matching. Erdős and Nešetřil conjectured that $s'(G) \leq \frac{5}{4}\Delta^2$ for all graphs G with maximum degree Δ. The problem is far from being solved and the best known upper bound on $s'(G)$ is $1.98\Delta^2$, even in the case when G is bipartite.

In this note we study the topological strong chromatic index, denoted by $s'_t(G)$, defined as the $\mathbb{Z}_2$-index of a topological space obtained from the graph. It is known that $s'(G) \geq s'_t(G)$. We show that for bipartite graphs G we have $s'_t(G) \leq 1.703\Delta^2$.

1 Introduction

A *strong edge-coloring* of a graph G is a coloring of edges of G where each color class forms an induced matching (that is, if edges uv and wx have the same color, then G does not contain any of $\{uw, ux, vw, vx\}$ as an edge). The *strong chromatic index* of G, denoted by $s'(G)$, is the minimum possible number of colors in a strong edge-coloring of G.

One of the most intriguing questions concerning strong chromatic index of graphs was stated by Erdős and Nešetřil in 1985. They asked about the maximum possible value of strong chromatic index of a graph with given maximum degree Δ. The best known construction is obtained from the cycle C_5, by expanding each vertex to an independent set of order $\Delta/2$. The following conjecture states that this is best possible.

Conjecture 1 (Erdős and Nešetřil, 1985)
For any graph G of maximum degree Δ we have $s'(G) \leq \frac{5}{4}\Delta^2$.

Only partial results are known. The best upper bound on $s'(G)$ that applies for the general case is due to Molly and Reed [6]. They showed that $s'(G) \leq 1.98\Delta^2$, provided that Δ is large enough. Stronger bounds are known for restricted classes of graphs, e.g. if G is planar then $s'(G) \leq 4\Delta + 4$ (Faudree *et. al* [3]) and for any C_4-free graph G we have $s'(G) = O(\frac{\Delta^2}{\ln \Delta})$ (Mahdian [4]).

Faudree *et. al* [2] conjectured that for bipartite graphs the upper bound for $s'(G)$ anticipated by Erdős and Nešetřil can be replaced by Δ^2. The complete bipartite graph $K_{\Delta,\Delta}$ shows that this bound, if true, is sharp. Although at first sight this version looks easier, it still remains unsolved. In fact, there are no general upper bounds on s' in the bipartite case that improve on the result by Molloy and Reed, mentioned above.

In this paper we consider the relaxation of Conjecture 1, where the strong chromatic index is replaced by its topological variant $s'_t(G)$, defined in the next section. We show that for a bipartite graph G of maximum degree Δ, we have $s'_t(G) \leq 1.703\Delta^2$.

2 Topological chromatic number

In this section we define the topological equivalent of the chromatic number and explain the meaning of the topological strong chromatic index. There is a number of parameters that may be considered a topological counterpart of the chromatic number, and among those we focus only on the largest one (being an upper bound for all other). We refer to the paper by Simonyi and Zsbán [7] for the discussion of other similar notions.

A $\mathbb{Z}_2$-*space* is a pair (X, v), where X is a topological space and $v : T \to T$ is a continuous function satisfying $v^2 = id_X$ (v is called a $\mathbb{Z}_2$-*action*). We say that a $\mathbb{Z}_2$-space (X, v) is *free* if $v(x) \neq x$ for all $x \in X$. We apply this notion to topological spaces arising from simplicial complexes. We say that a simplicial complex F equipped with a simplicial map $f : V(F) \to V(F)$ is a *free* $\mathbb{Z}_2$-*complex* if $(||F||, ||f||)$ is a free $\mathbb{Z}_2$-space, where $||F||$ is a geometric realization of F and $||f||$ is a natural extension of f to a continuous function on $||F||$.

A $\mathbb{Z}_2$-*map* between two $\mathbb{Z}_2$-spaces (X, v) and (Y, u) is a continous map $m : (X, v) \to (Y, u)$, such that $m(v(x)) = u(m(x))$ for any $x \in X$. The $\mathbb{Z}_2$-*index* of a free $\mathbb{Z}_2$-space (X, v) is the minimum d such that there exists a $\mathbb{Z}_2$-map $m : (X, v) \to (\mathbb{S}^d, -)$, where $\mathbb{S}^d$ is a d-dimensional sphere and $-$ is a natural antipodal operation. We define the $\mathbb{Z}_2$-index of a free $\mathbb{Z}_2$-complex (F, f) to be the $\mathbb{Z}_2$-index of the underlying $\mathbb{Z}_2$-space and denote it by $ind(F)$.

We define the *box complex* of a graph G (denoted by $B(G)$) to be a free $\mathbb{Z}_2$-complex on two copies of vertices of G, $V(G)$ and $\overline{V(G)}$, where $A \cup \overline{B}$ is a face if and only if either G contains a complete bipartite subgraph with partition classes A and B (for A, B being nonempty) or all vertices in A and B have at least one common neighbor (for A or B being an empty set). An $\mathbb{Z}_2$-action v is defined by $v(x) = \overline{x}$ and $v(\overline{x}) = x$ for $x \in V(G)$.

The $\mathbb{Z}_2$-index of the box complex of G (plus 2) can be thought of as a topological analog of the chromatic number of G. We have $\chi(G) \geq ind(B(G)) + 2$ and in many cases this lower bound turns out to be sharp. In particular, we have equality for Kneser graphs, which shows that $ind(B(G)) + 2$ can be greater than the fractional chromatic number of G. Results concerning this chromatic parameter provide supporting evidences for a number of conjectures on chromatic number of specific graphs [7].

Note that a strong edge-coloring of G is in fact a coloring of a graph $L(G)^2$ (that is, the square of a line graph of G). Using this interpretation, we define the *topological strong chromatic index* of G, denoted $s'_t(G)$, as $s'_t(G) = ind(B(L(G)^2)) + 2$. As $s'_t(G) \leq s'(G)$, the following constitutes a relaxation of the original Erdős and Nešetřil conjecture, in which we are interested.

Conjecture 2 (Erdős and Nešetřil, topological variant)
For any graph G of maximum degree Δ we have $s'_t(G) \leq \frac{5}{4}\Delta^2$.

In the same way we can relax the conjecture of Faudree *et. al*, concerning the strong chromatic index of bipartite graphs.

Conjecture 3 (Faudree, Gyárfás, Schelp, Tuza, topological variant)
For any bipartite graph G of maximum degree Δ we have $s'_t(G) \leq \Delta^2$.

Both conjectured bounds would be sharp, as the discussed extremal examples (blowup of C_5 and $K_{\Delta,\Delta}$) admit the same strong chromatic index and topological strong chromatic index. The best general upper bound $s'_t(G) \leq 1.98\Delta^2$ follows directly from result of Molloy and Reed [6].

We need two properties of the $\mathbb{Z}_2$-index. The first one is a topological counterpart of the observation that adding a new vertex of degree d to a graph cannot push its chromatic number above $d+1$. It is implicitly proved in book by Matoušek [5] in (see Proposition 5.3.2).

Lemma 4 *Let G be a graph and take $G' = G - v$, where v is a vertex of G of degree d. We have*

$$ind(B(G)) + 2 \leq \max(ind(B(G')) + 2, d + 1).$$

The second tool is $K_{l,m}$-theorem of Csorba *et. al* [1] stating that a graph of large $\mathbb{Z}_2$-index must contain large complete bipartite subgraphs.

Theorem 5 ([1]) *If G is a graph satisfying $ind(B(G)) + 2 \geq t$, then for every possible $l, m \in \mathbb{N}$ with $l + m = t$, the complete bipartite graph $K_{l,m}$ appears as a subgraph of G.*

3 Bipartite graphs

The main idea of our argument goes as follows. First we consider the case when G contains no complete bipartite subgraph with $z\Delta^2$ edges, where z is some constant, to be revealed later. Once we establish Corollary 8, the rest of the proof goes by a simple greedy coloring argument.

A complete bipartite subgraph of $L(G)^2$ can be viewed as two disjoint sets of edges of G, such that each two edges e and f, where e belongs to the first set and f to the second one, are joined by a link (i.e. there exist vertices $u \in e, v \in f$ such that $uv \in E(G)$). We refer to those sets as red edges (denoted R) and blue edges (denoted B). We use the term *red degree* (respectively, *blue degree*) of $v \in V(G)$, denoted $d_r(v)$ (resp. $d_b(v)$), which is defined to be the number of red (blue) edges that are incident to v (in G). Finally, by the *second* *red degree* (the *second blue degree*) of $v \in V(G)$ we mean the number of red (blue) edges incident to at least one neighbor v.

We refer to a pair (R, B) of subsets of $L(G)^2$ as a *selection* (in G), if $|R| = |B|$, $R \cap B = \emptyset$ and each edge in R is within distance 1 from each edge in B. The *order* of a selection (R, B) is defined as$|R| + |B|$.

Lemma 6 *Let G be a graph of maximum degree Δ which contains no complete bipartite subgraph with at least $z\Delta^2$ edges, and let (R, B) be a selection in G. If $u, v \in V(G)$ are in different partition classes of G, then*

$$d_r^2(u) + d_b^2(v) < (1 + 2z - z^2)\Delta^2.$$

Proof. Note that in bipartite graph $d_c^2(w)$ (where c equals r or b) is equal to the sum $\sum_{i=1}^{\Delta} d_c(w_i)$, where $w_1, \ldots, w_\Delta$ are neighbors of w. For $i = 1, 2, \ldots, \Delta$, define β_i to be a red degree of the i-th neighbor of v and let γ_i be the blue degree of i-th neighbor of u, setting $\beta_i = 0$ ($\gamma_i = 0$) if there are fewer than i neighbors. We denote the neighbors of v by $v_1, v_2, \ldots$ and neighbors of u by $u_1, u_2, \ldots$.

Note that if we have $\beta_i \gamma_j \geq z\Delta^2$ for some i and j, then $v_i u_j \in E(G)$. Indeed, if $v_i u_j \notin E(G)$, then every neighbor x of v_i connected to it by a red edge must be adjacent to each neighbor y of u_i connected to it by a blue edge (since G is bipartite we have $x \neq y$, so xy remains the only possible link between those two edges), so G contains a complete bipartite subgraph with $\beta_i \gamma_j$ edges and the claim follows by our assumption on G.

Clearly, the sum $d_r^2(u) + d_b^2(v)$ is equal to $\sum_i \beta_i + \sum_i \gamma_i$. For any $\alpha \in (0, 1]$, let $n_\alpha = |\{i : \beta_i \geq \alpha\Delta\}| \left|\{i : \gamma_i \geq \frac{z}{\alpha}\Delta\}\right|$ (that is, the number of pairs (i, j) such that $\beta_i \gamma_j \geq z\Delta^2$ and $\beta_i \geq \alpha\Delta$). By the above claim, for each such pair, there is an edge $v_i u_i$, so G contains a complete bipartite subgraph with at least n_α edges. Therefore, we must have

$$n_\alpha \leq z\Delta^2. \tag{1}$$

Our aim is to show, that the condition (1) implies the upper bound of $(1 + 2z - z^2)\Delta^2$ on $s = \sum_i \beta_i + \sum_i \gamma_i$ for any real numbers $\beta_1, \ldots, \beta_\Delta, \gamma_1, \ldots, \gamma_\Delta$ that are nonnegative and at most Δ. Note that this claim immediately completes the proof.

Our first step is to prove that for any configuration (i.e. the choice of values β_i and γ_i satisfying (1) for all α) there is a configuration at least as good (with not lower value of s), in which there is no β_i nor γ_i in the interval $[z\Delta, \Delta)$. Indeed, suppose that there is some $\beta_k \in [z\Delta, \Delta)$ and there exists γ_m strictly smaller than Δ. Without loss of generality we may assume that β_k is the lowest among such β_i, γ_m is the highest among such γ_i and $\beta_k \leq \gamma_m$. Note that if we decrease β_k to $\frac{\beta_k \gamma_m}{\Delta}$ and increase γ_m to Δ, s will increase. Since no β_i lies in the interval $[z\Delta, \beta_k)$, the increase of γ_m will not result in increasing any n_α, so the condition (1) would hold. If for every i we have $\gamma_i < \beta_i$ then replacing β_k by Δ does not change n_α. Note that the same argument applies if we exchange β with γ. By repeating this process we obtain the desired configuration.

Hence, we may and shall assume that all β_i's and γ_j's are either equal to Δ or smaller than $z\Delta$. Let b_r be the number of β_i that are equal to Δ ($b_r = |\{i : \beta_i = \Delta\}|$) and let b_b

stand for the number of γ_i equal to Δ. Then

$$s \leq \Delta(b_r + b_b) + z\Delta((1 - b_r) + (1 - b_b)) = \Delta(1 - z)(b_r + b_b) + 2z\Delta.$$

Note that $0 < b_r, b_b \leq \Delta$ and, by (1), we have $b_r b_b < z\Delta^2$, so we obtain $s < (1 + 2z - z^2)\Delta^2$, as desired. ■

Lemma 7 *Let G be a graph of maximum degree Δ which contains no complete bipartite subgraph with at least $z\Delta^2$ edges. Then, the order $|R| + |B|$ of each selection (R, B) is at most*

$$\max((2 - z), 2f(1 + 2z - z^2, z), 2f(1 - z/2, z))\Delta^2,$$

where $f(\alpha, z) = \alpha/2 + 2z/\alpha - z$.

Proof. Let w, t be a pair of vertices from different partition classes of G maximizing $d_b^2(t) + d_r^2(w)$. Take $s = d_b^2(t) + d_r^2(w)$ and without loss of generality suppose that $d_r^2(w) \geq \frac{s}{2}$. Let v be a neighbor of w of highest red degree. Clearly, $d_r(v) \geq \frac{d_r^2(w)}{\Delta}$. Consider $d_r(v)$ neighbors of v that are connected to v by a red edge, denoted $u_1, u_2, \ldots, u_{d_r(v)}$. We start by proving the following claim.

Claim 1. The number of blue edges is at most $s/2 + 2z\Delta^4/s - z\Delta^2 = f(s/(2\Delta^2), z)$

Let us count the number of edges incident to a neighbor of some u_i and not incident with any neighbor of v. Consider a vertex x at distance 2 from v and distance 1 from u_1. If there is a blue edge xy, where y is different than all u_i, then either y is at distance 1 from v (so we do not count xy) or all u_i must be adjacent to x (because xu_i is the only possible link between xy and vu_i by G being bipartite). There are clearly at most $\Delta - d_b(v)$ blue edges incident with x and not incident with any neighbor of v. As G has no complete bipartite subgraph on $z\Delta^2$ edges, there are at most $z\Delta^2/d_r(v)$ such vertices x (incident with at least one blue edge not incident to a neighbor of v), so the number of edges in question is at most $(\Delta - d_r(v))z\Delta^2/d_r(v)$.

To bound the number of all blue edges, we need to add the number of blue edges incident with a neighbor of v. Consequently, there are at most

$$d_b^2(v) + (\Delta - d_r(v))\frac{z\Delta^2}{d_r(v)} \leq d_b^2(v) + \frac{z\Delta^4}{d_r^2(w)} - z\Delta^2$$

blue edges. Since v and w belong to different partition classes, $d_r^2(w) \geq s/2$ and $d_b^2(v) \leq s/2$ which concludes the proof of the claim.

Claim 2. If $s \leq (1 - z/2)\Delta^2$, then the order of selection (R, B) is at most $(2 - z)\Delta^2$.

Let $m_{i,c}$, where $i \in \{1, 2\}$ and $c \in \{r, b\}$ denote the maximum value of $d_c^2(u)$ for a vertex u in i-th partition class of G. Without loss of generality we may assume that $m_{2,b} \leq m_{2,r}$. For any red edge $ab \in R$ (where a is in the first partition class of G) the number of blue edges is at most

$$|B| \leq d_b^2(a) + d_b^2(b) \leq m_{1,b} + m_{2,b} \leq m_{1,b} + m_{2,r} \leq s.$$

Consequently, since $|R| = |B|$, the order of selection (R, B) is at most $2s$, which proves Claim 2.

Now note that as a function of s, when z and Δ are fixed, the function $s/2+2z\Delta^4/s-z\Delta^2$ is unimodal, so if we could bound s both from above and below, it would result in an upper bound on the number of blue edges by Claim 1.

If s is smaller than $(1-z/2)\Delta^2$, then the result follows by Claim 2. Therefore, we may assume that $s/\Delta^2 \geq 1-z/2$. Moreover, by Lemma 6 we have $s/\Delta^2 < 1+2z-z^2$. Using the observation from the above paragraph and Claim 1 we have that the number of blue edges is at most $\max(f(1-z/2, z), f(1+2z-z^2, z))\Delta^2$, which completes the proof. ■

After plugging in the value $z = 0.298$, we immediately get the following Corollary.

Corollary 8 *Let G be a graph of maximum degree Δ which contains no complete bipartite subgraph with at least $0.298\Delta^2$ edges. Then, the order of each selection (R, B) is less than $1.703\Delta^2$.*

Now, we are ready to prove our main result.

Theorem 9 *Let G be a bipartite graph of maximum degree Δ. We have*

$$s'_t(G) \leq 1.703\Delta^2.$$

Proof. Suppose that G is a minimal counterexample to our conjecture. If G has no complete bipartite subgraphs with at least $0.298\Delta^2$ edges, then by Corollary 8 the maximum order of a selection in G (equal to maximum order of a bipartite subgraph of $L(G)^2$ with partition classes of equal order) is smaller than $1.703\Delta^2$. By Theorem 5 we get that $s'_t(G) \leq 1.703\Delta^2$, a contradiction.

In the remaining case, let H be a complete bipartite subgraph of G with at least $0.298\Delta^2$ edges and consider the graph $G' = G \setminus V(H)$. Note that, by our choice of G, vertices of $L(G')^2$ can be colored using $1.703\Delta^2$ colors. Thus, to complete the proof it is enough to verify that $L(G)^2$ can be obtained from $L(G')^2$ by adding to it vertices one by one in such a way that each vertex has degree less than $1.702\Delta^2$ in the existing graph.

Let $e = uv \in E(H)$ be an edge of $L(G)^2$. There are $2\Delta - 2$ vertices adjacent to u or v (and not equal neither to u nor to v), each incident with at most Δ edges, so the degree of e (in $L(G)^2$) is equal to $2\Delta^2 - 2\Delta$ minus the number of edges (other than e) incident to both a neighbor of u and a neighbor of v. The latter number is at least the number of edges of H that are not incident with neither u nor v, so it is strictly greater than $0.298\Delta^2 - 2\Delta$. consequently, the degree of any edge $e = uv \in E(H)$ in $L(G)^2$ is less than $1.702\Delta^2$.

This completes the proof of Theorem 9. ■

4 Concluding remarks

It seems that further exploration of our ideas may lead to results stronger than Theorem 9. The first possibility to strengthen Lemma 6, but even if we could replace the constant

$(1 + 2z - z^2)$ by 1, it would result in strengthening of Theorem 9 by only $0.014\Delta^2$. The obstacle to greater improvement is hidden in the proof of Lemma 7, where we must carefully consider the case when s is small: if the lower bound on s provided by Claim 2 would be a bit weaker (where *a bit* stands for at least $0.061\Delta^2$), then it would lead to worse constants in Corollary 8.

Other possible way of strengthening the result is to directly bound the number of red and blue edges, like in the proof of Lemma 6, getting rid of the weaknesses of Lemma 7 mentioned above. There is also some hope that this approach will let us remove the assumption that G is bipartite.

Note that our proof relies on Theorem 5. As useful as it is, it is not strong enough to confirm Conjectures 2 and 3. Indeed, if we take the graph $K_{\Delta,\Delta}$ and subdivide each edge of some small complete bipartite subgraph, we can produce a graph G' such that $L(G')^2$ contains the bipartite subgraph $K_{l,m}$, for any given l, m such that $l + m = (1 + \epsilon)\Delta^2$ (where ϵ is some constant around 0.05). Similar statement holds for the blowup of C_5 (we can achieve $l + m = (\frac{5}{4} + \epsilon)\Delta^2$ for ϵ around 0.02).

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