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Tight Hamiltonian cycles in hypergraphs of matroid bases

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1. Introduction

Our aim is to give a survey on the conjecture on existence of tight Hamiltonian cycles in hypergraphs of matroid bases. We will begin with recalling a conjecture called the "opposing chevron problem". It is a simple, yet still unsolved, conjecture regarding nonsingular matrices over a field. We will also state an equivalent conjecture using bases of vector spaces. Afterwards we will give a brief introduction to matroids and show, how the conjecture on the existence of tight Hamiltonian cycles in hypergraphs of matroid bases is related to the "inverse chevron problem". Then we will state other related conjectures and conclude with describing the so far known results.

2. Linear algebra setting

In this section we will work with matrices and vector spaces over some fixed field $\mathbb{K}$. Unless stated otherwise, matrices will be assumed to be square of size $n \times n$ and vector fields will be assumed to be of dimension n. We will use the following terminology:

Definition 2.1. Let $A = [a_{i,j}]$ be a square matrix. For $1 \leqslant k \leqslant n$ we will call the square submatrix of A consisting of $a_{i,j}$ with $1 \leqslant i, j \leqslant k$ an upper corner minor of A. Similarly a submatrix of A consisting of $a_{i,j}$ with $n - k + 1 \leqslant i, j \leqslant n$ will be called a lower corner minor of A. A submatrix of A will be called a corner minor of A if it is an upper corner minor of A or a lower corner minor of A.

To avoid confusion let us note that in literature a minor sometimes means the determinant of what we call a minor. What we call an upper corner minor is sometimes called in literature a leading principal minor, or a matrix corresponding to a leading principal minor.

We call a matrix B a permutation of a matrix A if there are some permutations of the rows and columns of A that yield B, i.e. $A = [a_{i,j}]$, $B = [b_{i,j}]$ and $b_{i,j} = a_{\sigma(i),\delta(j)}$ for some $\sigma, \delta \in \mathrm{S}_n$. Since permutations form a group, the relation of "being a permutation of" is an equivalence. Note that B is a permutation of A iff there exist permutation matrices, i.e. matrices P and Q with entries 0 and 1 having exactly one 1 in each row and column, such that $B = PAQ$. Permutation matrices have determinant either 1 or -1 thus the permutation of a nonsingular matrix is nonsingular.

We may now state the following:

Conjecture 2.2 (opposing chevron problem)**.** *If A is a nonsingular square matrix then there exists a permutation B of A such that all corner minors of B are nonsingular.*

The name of this conjecture was taken from [5]. Note that we can easily solve a weaker version of this problem – we have the following:

Theorem 2.3. *If A is a nonsingular square matrix then there exists a permutation B of A such that all upper corner minors of B are nonsingular.*

Before we give the proof we would like to recall a standard algorithm of transforming a square matrix $A = [a_{i,j}]$ with rows r_i into upper triangular form:

(1) for $i := 1$ to n do
(2) $k := i + 1$; while $a_{i,i} = 0$ and $k \leqslant n$ do
(3) if $a_{k,i} \neq 0$ then
(4) exchange r_i and r_k; $k := n + 1$
(5) else $k := k + 1$
(6) if $a_{i,i} \neq 0$ then for $k := i + 1$ to n subtract $r_k := r_k - a_{k,i}/a_{i,i} \cdot r_i$

The matrix A is modified only in steps (4) and (6). Each execution of step (4) gives a transposition of the rows of A and changes $\det(A)$ to $-\det(A)$. Thus all executions of step (4) give a permutation σ of the rows A and change $\det(A)$ to $\operatorname{sgn}(\delta)\det(A)$. Step (6) does not change the determinant, but ensures that the resulting matrix is upper triangular. We can now proceed with the proof of theorem 2.3:

Proof of theorem 2.3. Let C be the matrix obtained from A by executing the above algorithm. Let σ be the permutation of rows obtained during the execution and $B = [b_{i,j}]$, where $b_{i,j} = a_{\sigma i,j}$. Note that executing the algorithm on B will also give C. Furthermore step (4) of the algorithm will never be executed.

The matrix A is nonsingular and $\det(C) = \operatorname{sgn}(\delta)\det(A)$ thus $\det(C) \neq 0$. Furthermore C is upper triangular so $\det(C) = \prod_{i=1}^{n} c_{i,i}$ and all corner minors of C are nonsingular. Note that step (6) of the algorithm modifies the i-th row of B only by adding some multiples of the j-th rows with $j \leqslant i$. This means that the corresponding upper corner minors of B and C have equal determinants. Thus B is a permutation of A with nonsingular upper corner minors. $\square$

Note that to construct a permutation of a nonsingular matrix with nonsingular upper corner minors we only used a permutation of the rows, i.e. $B = PA$ for some permutation matrix P. The same result can be obtained by permutating only columns because $(PA)^T = A^T P^T$ and transposition does not change corner minors. One can ask, whether permutating rows is enough to obtain a matrix with nonsingular corner minors. The following example shows, that the answer is no:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Let us now state the following conjecture:

Conjecture 2.4. *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases of $\mathbb{K}^n$ then there are permutations $\sigma, \delta \in \mathrm{S}_n$ such that every n cyclically consecutive elements of $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$, $w_{\delta(1)}, \ldots, w_{\delta(n)}$ are independent.*

By cyclically consecutive elements of a sequence $a_1, \ldots, a_k$ we mean consecutive elements of the cycle $a_1, \ldots, a_k, a_1$.

We will now prove the following theorem:

Theorem 2.5. *Conjectures 2.2 and 2.4 are equivalent.*

Proof. Let us first assume that conjecture 2.2 is true. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two bases of $\mathbb{K}^n$. Let $w_i = \sum_{j=1}^n a_{i,j} v_j$, such expressions exist and are unique, because $v_1, \ldots, v_n$ is a base. Let $A = [a_{i,j}]$ be the matrix with $a_{i,j}$ in i-th row and j-th column. Since $w_1, \ldots, w_n$ are independent, A is nonsingular. Assuming conjecture 2.2 we obtain two permutations $\sigma, \delta \in \mathrm{S}_n$ such that $B = [a_{\sigma(i),\delta(j)}]$ has nonsingular corner minors. We claim that every n cyclically consecutive vectors in the sequence $v_{\delta(1)}, \ldots, v_{\delta(n)},\ w_{\sigma(1)}, \ldots, w_{\sigma(n)}$ are independent. Indeed, since v_i are independent, the vectors $w_{\sigma(k+1)}, \ldots, w_{\sigma(n)},\ v_{\delta(1)}, \ldots, v_{\delta(k)}$ are independent iff the vectors $\sum_{j=k+1}^n a_{\sigma(k+1),\delta(j)} v_{\delta(j)}$, ..., $\sum_{j=k+1}^n a_{\sigma(n),\delta(j)}$ are independent, which happens iff the lower $k \times k$ minor of B is nonsingular. Similarly the vectors $v_{\delta(k+1)}, \ldots, v_{\delta(n)}, w_{\sigma(1)}, \ldots, w_{\sigma(k)}$ are independent iff the upper $k \times k$ minor of B is nonsingular.

For the other implication let us assume that the conjecture 2.4 is true. Let $A = [a_{i,j}]$ be a nonsingular matrix and $v_1, \ldots, v_n$ a basis of $\mathbb{K}^n$. Then $w_i = \sum_{j=1}^n a_{i,j} v_j$ will also be a basis of $\mathbb{K}^n$. Then there are permutations $\sigma, \delta \in \mathrm{S}_n$ such that every n cyclically consecutive vectors in the sequence $v_{\delta(1)}, \ldots, v_{\delta(n)}, w_{\sigma(1)}, \ldots, w_{\sigma(n)}$ are independent. As we have shown above this means that $B = [a_{\sigma(i),\delta(j)}]$ is a permutation of A with nonsingular corner minors. □

Following the proof of theorem 2.5 we see that theorem 2.3 gives us the following result:

Remark 2.6. *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases of $\mathbb{K}^n$ then there is a permutation $\sigma \in \mathrm{S}_n$ such that every n consecutive elements of $v_1, \ldots, v_n, w_{\sigma(1)}, \ldots, w_{\sigma(n)}$ are independent.*

3. Matroid setting

We will now recall the definition and some basic facts about matroids. A matroid is a pair $M = (E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a subset of $\mathcal{P}(E)$ satisfying the following three conditions:

(I1) $\mathcal{I} \neq \emptyset$
(I2) If $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$
(I3) If $A, B \in \mathcal{I}$ and $|A| = |B| + 1$ then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathcal{I}$

We call elements of $\mathcal{I}$ independent subsets of the matroid M. Note that using condition (I2) one can show that (I1) is equivalent to $\emptyset \in \mathcal{I}$ and (I3) can be expressed as (I3'): If $A, B \in \mathcal{I}$ and $|A| > |B| + 1$ then there exists $C \subset A \setminus B$ such that $B \cup C$ is in $\mathcal{I}$ and $|A| = |B \cup C|$. We call a maximal independent subset a base of the matroid. Condition $(I3')$ shows that all bases have equal cardinality.

Note that if E is a finite subset of a vector space and $\mathcal{I} = \{A \subset E \mid A$ is linearly independent$\}$ then $(E, \mathcal{I})$ is a matroid. Hence we have a natural generalization of conjecture 2.4:

Conjecture 3.1. *If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are two bases of a matroid then there are permutations $\sigma, \delta \in \mathrm{S}_n$ such that every n cyclically consecutive elements of $v_{\sigma(1)}, \ldots, v_{\sigma(n)}, w_{\delta(1)}, \ldots, w_{\delta(n)}$ form an independent set.*

The question whether two bases can be ordered in such way, that their concatenation gives a bases cyclic ordering, i.e. a cyclic ordering in which every n cyclically consecutive elements form a base, was asked in [2] and again in [5]. It was given as a conjecture in [1] and [3].

Obviously we can also generalize remark 2.6 to obtain:

Remark 3.2. *If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are two bases of a matroid then there is a permutation $\sigma \in \mathrm{S}_n$ such that every n consecutive elements of $v_1, \ldots, v_n, w_{\sigma(1)}, \ldots, w_{\sigma(n)}$ form an independent set.*

This remark is easy to prove using induction and conditions $(I2)$ and $(I3)$. For $k = 0, \ldots, n$ let $A_k = \{v_{k+1}, \ldots, v_n\}$, where $A_n = \emptyset$. We construct $B_k \subset \{w_1, \ldots, w_n\}$ such that $B_0 = \emptyset$, $B_{k+1} = B_k \cup \{w_{\sigma(k+1)}\}$ for some $w_{\sigma(k+1)} \in B \setminus B_k$ and $A_k \cup B_k$ is independent. Obviously $A_0 \cup B_0$ is independent. If $A_k \cup B_k$ is independent then so is $A_{k+1} \cup B_k$. Then from $(I3)$ there is a suitable $w_{\sigma(k+1)} \in B \setminus B_k$.

Remark 3.2 obviously implies a more general remark:

Remark 3.3. *If for $i = 1, \ldots, k$ the sets $\{v_1^i, \ldots, v_n^i\}$ are bases of a matroid then there are permutations $\sigma_i \in \mathrm{S}_n$ such that every n consecutive elements of $v_1^1, \ldots, v_n^1$, $v_{\sigma_2(1)}^2, \ldots, v_{\sigma_1(n)}^2 \ldots, v_{\sigma_k(1)}^k, \ldots, v_{\sigma_k(n)}^k$ form an independent set.*

Conjecture 3.1 can also be generalized to the case of an arbitrary number of bases. However in this case there is no obvious way to deduct the more general result from the case of two bases.

One can also formulate a weaker version of the conjecture 3.1 allowing a permutation of all the elements, not only individual bases. We formulate it in the case of multiple bases:

Conjecture 3.4. *If for $i = 0, \ldots, k-1$ the sets $\{v_{1+ni}, \ldots, v_{n+ni}\}$ are bases of a matroid then there is a permutation $\sigma \in \mathrm{S}_{nk}$ such that every n cyclically consecutive elements of $v_{\sigma(1)}, \ldots, v_{\sigma(nk)}$ form an independent set.*

Recall that $r(A)$ denotes the rank of a set $A \in E$ which is equal to the maximum of cardinalities of independent subsets of A. We call a matroid loopless if $r(\{x\}) = 1$ for every $x \in E$. We have the following conjecture formulated in [4]:

Conjecture 3.5. *If M is a loopless matroid then there is a cyclic ordering of E such that every $r(E)$ cyclically consecutive elements form a base of M iff for all nonempty $A \subset E$ we have $|A|r(E) \leqslant |E|r(A)$.*

We will show that conjecture 3.5 implies conjecture 3.4. First observe that we may assume that the bases in conjecture 3.4 are disjoint. Indeed, if some $x \in E$ is an element of $t > 1$ bases then we may "split it" into t copies, i.e. replace $M = (E, \mathcal{I})$ with $M' = (E', \mathcal{I}')$ where $E' = E \backslash \{x\} \cup \{x_1, \ldots, x_t\}$ and $\mathcal{I}' = \{I \mid I \in \mathcal{I}' \text{ and } x \notin I\} \cup \bigcup_{i=1}^{t} \{I \backslash \{x\} \cup \{x_i\} \mid I \in \mathcal{I}' \text{ and } x \in I\}$. We leave the bases unchanged, with the exception, that every occurrence of x is replaced by a unique x_i. We may consecutively split all elements occurring in more then one base. A permutation obtained for the matroid with split elements will also be suitable for the original matroid.

Now let $M'' = (E'', \mathcal{I}'')$ be the matroid restricted to the nk elements of the bases. If $A \subset E$ then from the pigeon hole principle one of the bases contains at least $\lceil |A|/k \rceil$ of elements of A. It follows that $|E|r(A) \geqslant nk\lceil |A|/k \rceil \geqslant |A|r(E)$. Thus from conjecture 3.5 we find a suitable permutation.

We would like to emphasise that conjecture 3.4 has a particularly nice expression in terms of hypergraphs. Recall that a k-uniform hypergraph is a pair $(V, \mathcal{E})$ where $\mathcal{E} \subset \binom{V}{k}$. Let $1 \leqslant l \leqslant k$. An l-overlapping Hamiltonian cycle is a cycle of $|V|/(k-l)$ edges E_i such that for some cyclic order of V each E_i consist of k consecutive vertices and $|E_{i-1} \cap E_i| = l$. A tight Hamiltonian cycle is an l-overlapping Hamiltonian cycle where $l = k-1$. Now if $M = (E, \mathcal{I})$ is a matroid and $\mathcal{B}$ is the set of its bases then we can associate with M the hypergraph of bases of M, i.e. the $r(E)$-uniform hypergraph $H_M = (E, \mathcal{B})$. We have the following:

Conjecture 3.6 (existence of tight Hamiltonian cycles in hypergraphs of matroid bases)**.** *If $M = (E, \mathcal{I})$ is a matroid such that E can be expressed as a disjoint union of bases then the associated hypergraph H_M contains a tight Hamiltonian cycle.*

Since we have shown that in conjecture 3.4 we can assume that the bases are disjoint it follows that conjecture 3.6 is just conjecture 3.4 restated in terms of hypergraphs.

4. Known results

In [3] it was shown that conjecture 3.5 holds for matroids satisfying $\gcd(|E|, r(E)) = 1$. This does not help in proving conjecture 3.4 since we have to order nk elements of a matroid having rank n. However we can deduct a weaker result:

Theorem 4.1. *If for $i = 0, \ldots, k-1$ the sets $\{v_{1+ni}, \ldots, v_{n+ni}\}$ are bases of a matroid then there is a permutation $\sigma \in \mathrm{S}_{nk}$ such that every $n-1$ cyclically consecutive elements of $v_{\sigma(1)}, \ldots, v_{\sigma(nk)}$ form an independent set.*

Proof. As shown after stating conjecture 3.5 we can assume that $v_1, \ldots, v_{kn}$ are distinct elements of a matroid $M = (E, \mathcal{I})$ with $|E| = kn$ and $r(E) = n$. Let $M' = (E', \mathcal{I}')$, where

$E' = E \cup \{x\}$ and $\mathcal{I}' = \mathcal{I} \cup \{I \cup \{x\} \mid I \in \mathcal{I}$ and $r(I) \leqslant n-1\}$. Obviously $|E'| = kn+1$ and $r(E') = n$ thus we may apply conjecture 3.5. Let A be a nonempty subset of E'. If $x \notin A$ then from the pigeon hole principle $r(A) \geqslant \lceil |A|/k \rceil$. If $x \in A$ then either $r(A) = r(A \setminus \{x\}) = n$ or $r(A) \geqslant \lceil |A-1|/k \rceil + 1 \geqslant \lceil |A|/k \rceil$. Hence $|E'|r(A) \geqslant |A|r(E')$ and we obtain a cyclic ordering of E' such that every n cyclically consecutive elements form an independent set. This ordering induces a permutation of v_i such that every $n-1$ cyclically consecutive elements form an independent set. □

Another result worth mentioning is that conjecture 3.1 holds for graphic matroids. Recall that a matroid $M = (E_1, \mathcal{I})$ is called graphic iff there exists a graph $G = (V, E_2)$ and a bijection $f : E_1 \to E_2$ such that for every $A \subset E_1$ we have: $A \in \mathcal{I}$ iff the graph $(V, f(A))$ does not contain a cycle.

In terms of graph theory conjecture 3.1 for graphic matroids can be expressed in the following form:

Theorem 4.2. *Let $T_1 = \{e_1^1, \ldots, e_n^1\}$ and $T_2 = \{e_1^2, \ldots, e_n^2\}$ be two spanning trees of a connected graph G. Then there are permutations $\sigma, \delta \in \mathrm{S}_n$ such that every n cyclically consecutive elements of $e_{\sigma(1)}^1, \ldots, e_{\sigma(n)}^1, e_{\delta(1)}^2, \ldots, e_{\delta(n)}^2$ form a spanning fores of G.*

The theorem above was proved in [5]. Obviously, one can drop the assumption on connectedness of G and state the theorem about spanning forests.

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