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Beyond Shannon's bound for 4 and 5 colors

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Beyond Shannon's bound for 4 and 5 colors

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Abstract

Let $G = (V, E)$ be a multigraph of maximum degree Δ. The edges of G can be colored with at most $\frac{3}{2}\Delta$ colors by Shannon's theorem. We study lower bounds on the size of subgraphs of G that can be colored with Δ colors.

Shannon's Theorem gives a bound of $\frac{\Delta}{\lfloor\frac{3}{2}\Delta\rfloor}|E|$. However, for $\Delta = 3$, Kamiński and Kowalik [8] showed that there is a 3-edge-colorable subgraph of size at least $\frac{7}{9}|E|$, unless G is isomorphic to $K_3 + e$ (a K_3 with an arbitrary edge doubled). Here we extend this line of research by showing that

- every multigraph of maximum degree 4 has a 4-edge colorable subgraph with at least $\frac{4}{5}|E|$ edges, unless G is isomorphic to $2K_3$ (a K_3 with every edge doubled),
- every multigraph of maximum degree 5 has a 5-edge colorable subgraph with at least $\frac{5}{6}|E|$ edges, unless G is isomorphic to $2K_3 + e$ (a K_3 with two edges doubled and one edge tripled).

Our results have immediate applications in approximation algorithms for the Maximum k-Edge-Colorable Subgraph problem, where given a graph G (without any bound on its maximum degree or other restrictions) one has to find a k-edge-colorable subgraph with maximum number of edges. In particular, when G is a multigraph and for $k = 4, 5$ we obtain approximation ratios of $\frac{5}{7}$ and $\frac{11}{15}$, respectively. This improves earlier ratios of $1 - \left(\frac{3}{4}\right)^4$ and $\frac{5}{7}$ due to Feige et al. [6].

1 Introduction

A graph is k-edge-colorable if there exists an assignment of k colors to the edges of the graph, such that every two incident edges receive different colors. By Shannon's theorem [13], $\lfloor\frac{3}{2}\Delta\rfloor$ colors suffice to color any multigraph, where Δ denotes the maximum degree. This bound is tight, e.g. for every even Δ consider the graph $(\Delta/2)K_3$ (K_3 with every edge of multiplicity $\Delta/2$) and for odd Δ consider the graph $\lfloor\Delta/2\rfloor K_3 + e$ (K_3 with two edges of multiplicity $\lfloor\Delta/2\rfloor$ and one edge of multiplicity $\lfloor\Delta/2\rfloor + 1$).

It is natural to ask *how many* edges of a graph of maximum degree Δ can be colored with *less* than $\lfloor\frac{3}{2}\Delta\rfloor$ colors. The *maximum k-edge-colorable subgraph of G* (maximum k-ECS in short) is a k-edge-colorable subgraph H of G with maximum number of edges. Let $\gamma_k(G)$ denote the ratio $|E(H)|/|E(G)|$; when $|E(G)| = 0$ we define $\gamma_k(G) = 1$. If Δ is the maximum degree of G we write shortly $\gamma(G)$ for $\gamma_\Delta(G)$. Lower bounds on $\gamma(G)$ were studied first by Albertson and Haas [1]. They showed that $\gamma(G) \geq \frac{26}{31}$ for simple graphs of maximum degree 3. Today, the case of simple graphs is pretty well researched. Since by Vizing's theorem any simple graph of maximum degree Δ can be edge-colored with $\Delta + 1$ colors, by simply discarding the smallest color class we get $\gamma(G) \geq \Delta/(\Delta + 1)$. This ratio grows to 1 with Δ, and for $\Delta \leq 7$ much more precise bounds are known (see [8] for a discussion).

In this paper we study lower bounds on $\gamma(G)$ for *multigraphs*. Note that in this case we can apply Shannon's theorem similarly as Vizing's theorem above and we get the bound $\gamma(G) \geq \Delta / \lfloor \frac{3}{2}\Delta \rfloor$. As far as we know, so far better bounds are known only for subcubic graphs (i.e. of maximum degree three). The Shannon's bound gives $\gamma(G) \geq \frac{3}{4}$ then, which is tight by $K_3 + e$. Rizzi [10] showed that when G is a subcubic multigraph with no cycles of length 3, then $\gamma(G) \geq \frac{13}{15}$, which is tight by the Petersen graph. Kamiński and Kowalik [8] extended this result and proved that $\gamma(G) \geq \frac{7}{9}$ when G is a subcubic multigraph different from $K_3 + e$.

1.1 Main Result

In what follows $2K_3$ denotes the graph on three vertices with every pair of vertices connected by two edges, while $2K_3 + e$ denotes the graph that can be obtained from $2K_3$ by adding a third edge between one pair of vertices. Below we state our main result.

Theorem 1. *Let G be a connected graph of maximum degree $\Delta \in \{4, 5\}$. Then G has a Δ-edge-colorable subgraph with at least*

1. *$\frac{4}{5}|E|$ edges when $\Delta = 4$ and $G \neq 2K_3$,*
2. *$\frac{5}{6}|E|$ edges when $\Delta = 5$ and $G \neq 2K_3 + e$.*

Moreover, the subgraph and its coloring can be found in polynomial time.

Note that the bounds in Theorem 1 are tight. The smallest examples are K_3 with two edges added and K_3 with three edges added.

1.2 Applications

One may ask why we study $\gamma_\Delta(G)$ and not, say $\gamma_{\Delta+1}(G)$. Our main motivation is that finding large Δ-edge-colorable subgraphs has applications in approximation algorithms for the Maximum k-Edge-Colorable Subgraph problem (aka Maximum Edge k-coloring [6]). In this problem, we are given a graph G (without any restriction on its maximum degree) and the goal is to compute a maximum k-edge colorable subgraph of G. It is known to be APX-hard when $k \geq 2$ [4, 7, 5]. The research on approximation algorithms for max k-ECS problem was initiated by Feige, Ofek and Wieder [6]. (In the discussion below we consider only multigraphs, consult [8] for an overview for simple graphs.)

Feige et al. [6] suggested the following simple strategy. Begin with finding a maximum k-matching F of the input graph, i.e. a subgraph of maximum degree k

which has maximum number of edges. This can be done in polynomial time (see e.g. [12]). Since a k-ECS is a k-matching itself, F has at least as many edges as the maximum k-ECS. Hence, if we color $\rho|E(F)|$ edges of F we get a ρ-approximation. If we combine this algorithm with (the constructive version of) Shannon's Theorem, we get $k/\left\lfloor\frac{3}{2}k\right\rfloor$-approximation. By plugging in the Vizing's theorem for multigraphs, we get a $\frac{k}{k+\mu(G)}$-approximation, where $\mu(G)$ is the maximum edge multiplicity.

Feige et al. [6] show a polynomial-time algorithm which, for a given multigraph and an integer k, finds a subgraph H such that $|E(H)| \geq$ OPT, $\Delta(H) \leq k+1$ and $\Gamma(H) \leq k+\sqrt{k+1}+2$, where OPT is the number of edges in the maximum k-edge colorable sugraph of G, and $\Gamma(H)$ is the odd density of H, defined as $\Gamma(H) = \max_{S\subseteq V(H),|S|\geq 2} \frac{|E(S)|}{\lfloor |S|/2\rfloor}$. The subgraph H can be edge-colored with at most $\max\left\{\Delta+\sqrt{\Delta/2}, \lceil\Gamma(H)\rceil\right\} \leq \lceil k+\sqrt{k+1}+2\rceil$ colors in $n^{O(\sqrt{k})}$-time by an algorithm of Chen, Yu and Zang [2]. By choosing the k largest color classes as a solution this gives a $k/\lceil k+\sqrt{k+1}+2\rceil$-approximation. One can get a slightly worse $k/(k+(1+3/\sqrt{2})\sqrt{k}+o(\sqrt{k}))$-approximation by replacing the algorithm of Chen et al. by an algorithm of Sanders and Steurer [11] which takes only $O(nk(n+k))$-time. Note that in both cases the approximation ratio approaches 1 when k approaches ∞.

k	simple graphs	reference	multigraphs	reference
2	0.842	[3]	$\frac{10}{13}$	[6]
3	$\frac{13}{15}$	[8]	$\frac{7}{9}$	[8]
4	$\frac{9}{11}$	[8]	$\mathbf{\frac{5}{7}}$	**this work**
5	$\frac{23}{27}$	[8]	$\mathbf{\frac{11}{15}}$	**this work**
6	$\frac{19}{22}$	[8]	$\max\{\frac{2}{3}, \frac{6}{6+\mu}\}$	[13, 14, 6]
7	$\frac{22}{25}$	[8]	$\max\{\frac{7}{10}, \frac{7}{7+\mu}\}$	[13, 14, 6]
$8,\ldots,13$	$\frac{k}{k+1}$	[14, 6]	$\max\{\frac{k}{\lfloor 3k/2\rfloor}, \frac{k}{k+\mu}\}$	[13, 14, 6]
≥ 14	$\frac{k}{k+1}$	[14, 6]	$\max\{\frac{k}{\lceil k+\sqrt{k+1}+2\rceil}, \frac{k}{k+\mu}\}$	[2, 14, 6]

Table 1: Best approximation ratios for the Maximum k-Edge-Colorable Subgraph problem

The results above work for all values of k. However, for small values of k tailor-made algorithms are known, with much better approximation ratios. Feige et al. [6] proposed a $\frac{10}{13}$-approximation algorithm for $k=2$ based on an LP relaxation. They also analyzed a simple greedy algorithm and showed that it has approximation ratio $1-\left(1-\frac{1}{k}\right)^k$, which is still the best result for the case $k=4$ in multigraphs. For $k=3$ Shannon's bound gives a 3/4-approximation. However, Kamiński and Kowalik [8] showed that K_3+e is the only tight example for the Shannon's bound in subcubic graphs; otherwise $\gamma(G) \leq \frac{7}{9}$. One cannot combine this result directly with the k-matching technique, since the k-matching may contain components isomorphic to K_3+e. However, inspired by the paper of Kosowski [9], Kamiński and Kowalik [8] showed a general algorithmic technique which leads to improved approximation factors even if the bound on $\gamma(G)$ does not hold for a few special graphs. Using this technique they get a $\frac{7}{9}$-approximation for $k = 3$.

In this paper we also apply the constructive versions of our combinatorial bounds with the algorithmic technique from [8] and we obtain new approximation algorithms for four and five colors, with approximation ratios $\frac{5}{7}$ and $\frac{11}{15}$, respectively. The current state of art in approximating MAXIMUM k-EDGE-COLORABLE SUBGRAPH is given in Table 1.

1.3 Further Research

When considering Theorem 1 one might ask what about higher values of Δ, e.g. $\Delta = 6$. To prove Theorem 1 we analyze the possible free components of a coloring. In case of $\Delta \in \{4, 5\}$ there are 3 distinct free components and only a couple ways of embedding them in G to consider. In case of $\Delta = 6$ there are 6 distinct free components and quite a few cases that require dealing with. Analyzing all the cases for values of Δ higher then 6 without the aid of a computer would be very cumbersome. Thus we restrain ourselves to $\Delta \in \{4, 5\}$ but we hope the bounds for $\Delta = 6$ (both the bound on $\gamma(G)$ and the on the approximation ratio) will be improved some day.

1.4 Preliminaries

We will work with undirected multigraphs (though for simplicity we will call them graphs). Our notation is mostly consitent with the one used in [8], which we recall below.

Let $G = (V, E)$ be a graph. For a vertex $x \in V$ by $N(x)$ we denote the set of neighbors of x and $N[x] = N(x) \cup \{x\}$. For a set of vertices S we denote $N(S) = \bigcup_{x \in S} N(x) \setminus S$ and $N[S] = \bigcup_{x \in S} N[x]$. We also denote the subgraph of G whose set of vertices is $N[S]$ and set of edges is the set of edges of G incident with S by $I[S]$. For a subgraph H of G we denote $N[H] = N[V(H)]$ and $I[H] = I[V(H)]$.

A *partial k-coloring* of a graph $G = (V, E)$ is a function $\pi : E \to \{1, \ldots, k\} \cup \{\perp\}$ such that if two edges $e_1, e_2 \in E$ are incident then $\pi(e_1) \neq \pi(e_2)$, or $\pi(e_1) = \pi(e_2) = \perp$. From now on by a *coloring* of a graph we will mean a partial $\Delta(G)$-coloring. We say that an edge e is *uncolored* if $\pi(e) = \perp$; otherwise, we say that e is *colored.* For a vertex v, $\pi(v)$ is the set of colors of edges incident with v, i.e. $\pi(v) = \{\pi(e) : e \in I[v]\} \setminus \{\perp\}$, while $\overline{\pi}(v) = \{1, \ldots, k\} \setminus \pi(v)$ is the set of free colors at v.

Let $V_\perp = \{v \in V : \overline{\pi}(v) \neq \emptyset\}$. In what follows, $\perp(G, \pi) = (V_\perp, \pi^{-1}(\perp))$ is called *the graph of free edges.* Every connected component of the graph $\perp(G, \pi)$ is called a *free component.* If a free component has only one vertex, it is called *trivial.*

Below we state a few lemmas proved in [8] which will be useful in the present paper. Although the lemmas were formulated for simple graphs one can easily check that the proofs apply to multigraphs as well.

Lemma 2 ([8], Lemma 7)**.** *Let (G, π) be a colored graph that maximizes the number of colored edges. For any free component Q of (G, π) and for every two distinct vertices $v, w \in V(Q)$*

(a) $\overline{\pi}(v) \cap \overline{\pi}(w) = \emptyset$,

(b) for every $a \in \overline{\pi}(v)$, $b \in \overline{\pi}(w)$ there is an (ab, vw)-path.

For a free component Q, by $\overline{\pi}(Q)$ we denote the set of free colors at the vertices of Q, i.e. $\overline{\pi}(Q) = \bigcup_{v \in V(Q)} \overline{\pi}(v)$.

Corollary 3 ([8], Lemma 8)**.** *Let (G, π) be a colored graph that maximizes the number of colored edges. For any free component Q of (G, π) we have $|\overline{\pi}(Q)| \geq 2|E(Q)|$. In particular Q has at most $\lfloor \frac{\Delta}{2} \rfloor$ edges.*

Let Q_1, Q_2 be two distinct free components of (G, π). Assume that for some pair of vertices $x \in V(Q_1)$ and $y \in V(Q_2)$, there is an edge $xy \in E$ such that $\pi(xy) \in \overline{\pi}(Q_1)$. Then we say that Q_1 *sees* Q_2 *with* xy, or shortly Q_1 *sees* Q_2.

Lemma 4 ([8], Lemma 10)**.** *Let (G, π) be a colored graph that maximizes the number of colored edges. If Q_1, Q_2 are two distinct free components of (G, π) such that Q_1 sees Q_2 then $\overline{\pi}(Q_1) \cap \overline{\pi}(Q_2) = \emptyset$.*

We use the notion of the potential function Ψ introduced in [8]:

$$\Psi(G, \pi) = (c, n_{\lfloor \Delta/2 \rfloor}, n_{\lfloor \Delta/2 \rfloor - 1}, \ldots, n_1),$$

where c is the number of colored edges, i.e. $c = |\pi^{-1}(\{1, \ldots, \Delta\})|$ and n_i is the number of free components with i edges for every $i = 1, \ldots, \lfloor \Delta/2 \rfloor$.

1.5 Our Approach and Organization of the paper

Informally, our plan for proving Theorem 1 is to consider a coloring that maximimizes the potential Ψ and injectively assign many colored edges to every free component in the coloring. To this end we introduce edges controlled by a component (each of them will be assigned to the component which controls it) and edges influenced by a component (as we will see every edge is influenced by at most two components; if it is influenced by exactly two components, we will assign *half* of the edge to each of the components).

In Section 2 we develop structural results on colorings maximimizing Ψ. Informally, these results mean that in such a coloring every free component influences/controls many edges. Then, in Section 3 we prove lower bounds for the number of edges assigned to various types of components, using a convenient formalism of sending charge. The section concludes with the proof of Theorem 1.

2 The structure of colorings maximimizing Ψ

2.1 Moving free components

Note that if P and Q are distinct free components of a coloring (G, π) then $E(P) \cap E(I[Q]) = \emptyset$.

Definition 5. Let (G, π) be a colored graph and let P be a free component of (G, π). An *elementary move* of P in π is a coloring π' such that:

1. π' can be obtained from π by uncoloring k edges of $I(P)$ and coloring k edges of P for some $k \geq 0$,
2. $\pi'|_{I[P]}$ has exactly one nontrivial free component, denote it P'.

If the above holds we say that π' and P' have been obtained respectively from π and P by an elementary move. Sometimes we will write shortly *π' is a move of π*, meaning that π' is a move of a free component of π.

Note that π and π' have the same number of uncolored edges. Furthermore the component P is either replaced with a component P' or merged with a component Q into $P' \cup Q$. Either way an elementary move does not decrease the potential Ψ. Furthermore we have the following:

Remark 6. *If π maximizes the potential Ψ then moving a component P cannot cause a merge of components and hence P' is a free component of π'.*

We consider π to be a trivial elementary move of any of its components.

Lemma 7. *Let (G, π) be a coloring that maximizes the potential Ψ and let P and Q be two distinct free components. Suppose that P' and Q' can be obtained respectively by elementary moves π_1 and π_2 of P and Q. Then:*

(i) Q is a free component of π_1,

(ii) Q' can be obtained by an elementary move of Q in π_1,

(iii) the move of Q to Q' in π_1 is the same coloring π' as the move of P to P' in π_2.

Proof.

(i) Directly from Remark 6.

(ii) We have $E(P) \cap E(I[Q]) = \emptyset$ (for otherwise P and Q would form a single component) and $E(P') \cap E(I[Q]) = \emptyset$ (for otherwise Ψ can be increased by moving P to P'). Hence $I[Q] \subset G \setminus (E(P) \cup E(P'))$ and thus $\pi_1|_{I[Q]} = \pi|_{I[Q]}$. Note that in an elementary move of Q the only edges that obtain new colors are in Q, so whether a move is possible is determined only by colors of edges in $I[Q]$. Thus we may move Q to Q' in π_1.

(iii) The coloring π' can be given explicitly by setting $\pi'|_{E(G)\setminus(E(I[P])\cup E(I[Q]))} = \pi|_{E(G)\setminus(E(I[P])\cup E(I[Q]))}$, $\pi'|_{E(I[P])} = \pi_1|_{E(I[P])}$ and $\pi'|_{E(I[Q])} = \pi_2|_{E(I[Q])}$. ☐

We say that a coloring π' is a *move* of π if there is a sequence of colorings $\pi_0 = \pi, \pi_1, \ldots, \pi_k = \pi'$ such that π_i is an elementary move of a free component in π_{i-1}. We say that a free component P' of π' is obtained from a free component P in π if there are $P_0, \ldots, P_k$ such that for $i = 0, \ldots, k$ P_i is a free component in π_i and either $P_i = P_{i-1}$ or π_i is the elementary move of P_{i-1} to P_i. We denote the free component of a coloring π' obtained from a free component P by $P(\pi')$. This notation naturally extends to $P(\pi_i)$ for each $i = 1, \ldots, k$, since π_i can be treated as a move of P in π.

Theorem 8. *Let (G, π) be a coloring that maximizes the potential Ψ. Then:*

(i) if π' is a move of π then π is a move of π',

(*ii*) *if $P(\pi')$ is a free component in a move π' obtained from P then there is a sequence of elementary moves π_i of P_{i-1} where $\pi_0 = \pi$, $\pi_k = \pi'$ and $P_i = P(\pi_i)$ for $i = 1, \ldots, k$,*

(*iii*) *if $P_1, \ldots, P_k$ are distinct free components of π and $\pi_1, \ldots, \pi_k$ are moves of π then there is a move π' of π such that $P_i(\pi') = P_i(\pi_i)$ and $\pi'|_{E(I[P_i(\pi')])} = \pi_i|_{E(I[P_i(\pi_i)])}$ for $i = 1, \ldots, k$.*

Proof.

(*i*) By Lemma 6 if π' is an elementary move of a free component P in π then π is an elementary move of $P(\pi')$ in π'. Since a move is a sequence of elementary moves the claim follows.

(*ii*) Let $\pi_1, \ldots, \pi_k$ be a shortest sequence of elementary moves giving a move from P to P'. Let $P_0 = P$ and $P_i = P_{i-1}(\pi_i)$. We claim that π_i are elementary moves of P_{i-1}. Suppose this is not true and let π_j be the elementary move of $Q_{j-1} \neq P_{j-1}$ with the highest index. By using Lemma 7 multiple times we see that π_k can be obtained from π_{i-1} by first moving $P_j, \ldots, P_{k-1}$ and then Q_{j-1}. However reversing the moving of Q_{j-1} does not change P', thus we obtain a sequence of $k-1$ elementary moves giving a move from P to P' – a contradiction.

(*iii*) We will show the claim for $k = 2$. The proof for more components is a trivial generalization of this one but involves a multitude of indices. Using 2. we get sequences of elementary moves $\pi_{1,0}, \ldots, \pi_{t_1+1,0}$ and $\pi_{0,1}, \ldots, \pi_{0,t_2+1}$ of free components $P_1^0, \ldots, P_1^{t_1}$ and $P_2^1, \ldots, P_2^{t_2}$ giving respectively a move from P_1 to $P_1(\pi_1)$ and a move from P_2 to $P_2(\pi_2)$ such that all $P_1^{j_1}$ are free components obtained from P_1 and all $P_1^{j_1}$ are free components obtained from P_1. We claim that for every $j_1 = 1, \ldots, t_1 + 1$ and $j_2 = 1, \ldots, t_2 + 1$ there exists a move π_{j_1,j_2} of π such that $P_1(\pi_{j_1,j_2}) = P_1(\pi_{j_1,0})$ and $P_2(\pi_{j_1,j_2}) = P_2(\pi_{0,j_2})$. Furthermore we require π_{j_1,j_2} to simultaneously be an elementary move of $P_1(\pi_{j_1-1,j_2})$ in π_{j_1-1,j_2} and $P_2(\pi_{j_1,j_2-1})$ in π_{j_1,j_2}. By Lemma 7 the existence of π_{j_1,j_2} follows from the existence of π_{j_1-1,j_2} and π_{j_1,j_2-1} thus the claim follows by induction on $j_1 + j_2$. The fact that $\pi'|_{E(I[P_i(\pi')])} = \pi_i|_{E(I[P_i(\pi_i)])}$ follows from the explicit definition of coloring π' in the proof of Lemma 7. □

2.2 Controlling and influencing edges

Let P be a free component of a coloring (G, π) that maximizes the potential Ψ. We say that an edge e is *influenced* by P if $e \in E(I[P])$. We say that an edge is *controlled* by P if $e \in E(I[P])$ and one of the following conditions is satisfied:

(C1) for each vertex x of e there is a move, possibly trivial, π' of π such that $x \in V(P(\pi'))$,

(C2) $\pi(e) \in \overline{\pi}(P)$ and $|\overline{\pi}(P)| \geqslant \Delta(G) - 1$.

We denote the set of edges controlled by P by $\mathrm{con}(P)$. By $\mathcal{M}(P)$ we denote the set of moves of π that can be obtained by moving only P (and the components obtained from P). We define the *sphere of influence* of P, denoted by $\mathrm{I}^*(P)$, to be $\bigcup_{\pi' \in \mathcal{M}(P)} E(I[P(\pi')])$ and the *sphere of control* of P, denoted by $\mathrm{con}^*(P)$, to be $\bigcup_{\pi' \in \mathcal{M}(P)} \mathrm{con}(P(\pi'))$.

Lemma 9. *Let (G, π) be a coloring that maximizes the potential Ψ and let P and Q be two distinct free components of π. Then $\mathrm{con}^*(P)$ and $\mathrm{I}^*(Q)$ have no common edges.*

Proof. Suppose e is a common edge of $\mathrm{con}^*(P)$ and $\mathrm{I}^*(Q)$. Take $\pi_1 \in \mathcal{M}(P)$ and $\pi_2 \in \mathcal{M}(Q)$ such that $e \in \mathrm{con}(P(\pi_1)) \cap I[Q(\pi_2)]$. Consider two cases:

(a) e satisfies condition C1 in the definition of $\mathrm{con}(P)$. Since $e \in I[Q(\pi_2)]$, there is an endpoint of e, say x, contained in $Q(\pi_2)$. By C1 there is a move π_1' of π_1 such that $x \in V(P(\pi'))$. By Theorem 8 there is a move of π with free components $P(\pi_1')$ and $Q(\pi_2)$, however $P(\pi_1)$ and $Q(\pi_2)$ share the vertex x, thus they have merged into a single component in (G, π), a contradiction.

(b) e satisfies condition C2 in the definition of $\mathrm{con}(P)$. Let Q' be the free component of π_1 obtained from Q. By Theorem 8 we can obtain $Q(\pi_2)$ by a move of Q' in π_1 which is a sequence $\pi_1 = \pi_{1,1}, \ldots, \pi_{1,k} = \pi_2$ of elementary moves such that for every $i = 2, \ldots, k$ we have that $\pi_{1,i}$ is an elementary move of a free component Q'_{i-1} obtained from Q'. Let Q'_j be the free component satisfying $e \in I[Q'_j]$ with the smallest possible index. Thus $Q'_j = Q'$ or $e \notin I[Q'_{j-1}]$, either way $\pi_{1,j}(e) = \pi_1(e)$. So $P(\pi_1)$ sees Q_j through e in the colored graph $(G, \pi_{1,j})$ and by Lemma 4 they have disjoint sets of free colors, a contradiction. $\square$

Lemma 10. *Let (G, π) be a coloring that maximizes the potential Ψ. Then for each edge e of G there are at most two distinct free components of π such that their spheres of influence contain e.*

Proof. Suppose there are k components $P_1, \ldots, P_k$ and an edge e such that and for every $i = 1, \ldots, k$ we have $e \in \mathrm{I}^*(P_i)$. Then for every $i = 1, \ldots, k$ there is a move $\pi_i \in \mathcal{M}(P_i)$ such that $e \in I[P_i(\pi_i)]$. Hence, for every $i = 1, \ldots, k$ the component $P_i(\pi_i)$ contains a vertex, say v_i, incident with e. By Theorem 8 there is a coloring π' with free components $P_i(\pi_i)$, $i = 1, \ldots, k$. By the maximality of Ψ, the free components P_i are disjoint and hence we have $v_i \neq v_j$ for $P_i \neq P_j$. Since the vertices v_i are endpoints of e, we infer that $k \leq 2$. $\square$

Definition 11. For a free component Q of a coloring (G, π) we define a set of vertices $W(Q) = \{v \in V(G) \; : \; v \in V(Q(\pi')) \text{ for some elementary move } \pi' \text{ of } Q \text{ such that } |E(Q(\pi')) \setminus E(Q)| \leq 1\}$.

Lemma 12. *Let (G, π) be a coloring that maximizes the potential Ψ and let Q be a free component with $|E(Q)| \geq 2$. If $u, v \in W(Q)$ and $u \neq v$ then $\overline{\pi}(u) \cap \overline{\pi}(v) = \emptyset$.*

Proof. Assume that we have two distinct $u, v \in W(Q)$ such that there exists a color $a \in \overline{\pi}(u) \cap \overline{\pi}(v)$. By Lemma 2 the vertices u and v cannot both belong to $V(Q)$. By symmetry we can assume $u \notin V(Q)$.

Since $u \in W(Q)$, one can uncolor an edge ux, $x \in V(Q)$, and color an edge $e \in E(Q)$, obtaining a proper coloring π'. Note that e is incident with x for otherwise we can color e without uncoloring ux, and increase the potential Ψ. Hence $e = xy$ for some $y \in V(Q)$ and $\pi(ux) \in \overline{\pi}(y)$. It follows that $v \notin V(Q)$ for otherwise Q sees the (possibly trivial) component containing u, a contradiction with Lemma 4. Let vp be the edge such that one can uncolor vp and color an edge of Q with color $\pi(vp)$, which exists because $v \in W(Q)$. By Corollary 3, $|\overline{\pi}(Q)| \geq 4$, so there is a color $b \in \overline{\pi}(Q) \setminus \{\pi(ux), \pi(vp)\}$. Let z be the vertex of Q such that $b \in \overline{\pi}(z)$. Note that $a \notin \overline{\pi}(z)$ for otherwise Q sees the (possibly trivial) component containing v, a contradiction with Lemma 4. Consider a maximal path P which starts at z and has edges colored in a and b alternately. Swap the colors a and b on the path. As a result, a becomes free in Q. Also, P touches at most one of the vertices u, v, so a is still free in at least one of them, by symmetry say in u. Since $b \neq \pi(ux)$, the vertex u is still in $W(Q)$. Hence we arrive at the first case again. □

3 Proof of Theorem 1

In this section we consider a colored graph (G, π) which maximizes the potential Ψ.

We put one unit of charge on each colored edge of G. Let P be a free component of (G, π). Each edge in con*(P) sends its charge to P. Morevoer, each edge in I*$(P) \setminus$ con*(P) sends half its charge to P. By lemmas 9 and 10 each edge sends at most one unit of charge. Let ch(P) denote the amount charge sent to P. Then the number of colored edges in G is at least $\sum_P \mathrm{ch}(P) \geq |E| \min_P \frac{\mathrm{ch}(P)}{\mathrm{ch}(P)+|E(P)|}$, where the summation is over all nontrivial free components in (G, π).

Observation 13. *If P is a free component of (G, π) and π' is a move of π then the free component $P(\pi')$ receives in the coloring π' the same amount of charge as does P in π.*

Proof. By Lemma 8 for any move $\pi_1 \in \mathcal{M}(P)$ there is a move $\pi_2 \in \mathcal{M}(P(\pi'))$ such that $P(\pi_1) = P(\pi_2)$ and $\pi_1|_{E(I[P(\pi_1)])} = \pi_2|_{E(I[P(\pi_2)])}$. Thus con*$(P) =$ con*$(P(\pi'))$ and I*$(P) =$ I*$(P(\pi'))$, hence the claim follows. □

Lemma 14. *Let P be a free component of (G, π) with $|E(P)| = 1$. Then P receives at least $\Delta(G)$ charges.*

Proof. We will show that for each color a the component P receives at least one charge from edges colored with a. Suppose that $a \notin \overline{\pi}(P)$ then each of the two vertices of P is incident with an edge colored with a. If those edges are distinct then both belong to I*(P) and send half of a charge. If this is a single edge then it belongs to con*(P) and sends a whole charge. Now suppose that $a \in \overline{\pi}(P)$. Let x and y be the vertices of P, $a \in \overline{\pi}(x)$ and $\pi(yz) = a$. Uncoloring yz and coloring xy is an elementary move so yz is in con*(P) and sends its whole charge to P. □

Lemma 15. *Let P be a free component of (G, π) isomorphic with the 2-path. If both edges of P are single edges in G then P receives at least $2\Delta(G)$ charges.*

Proof. Note that if one of the ends of an edge e belongs to $W(P)$ then $e \in$ I*(P) and if both ends belong to $W(P)$ then $e \in$ con*(P). Thus we may say that each colored edge sends P half of a charge through each end belonging to $W(P)$. If $|W(P)| \geq 5$ then by Lemma 12 for every color there are at least $W(P) - 1 \geq 4$ vertices sending P half a charge from an edge of that color. Thus P receives at least $2\Delta(G)$ charges.

If $W(P) = 4$ then any of the edges of P can be rotated to obtain a component P' containing the vertex in $W(P) \setminus V(P)$. Thus all edges incident with $W(P)$ and colored with a color in $\overline{\pi}(W(P))$ belong to $\mathrm{con}^*(P)$ and send the whole charge to P. So for every color there are either 4 vertices sending P half a charge from an edge of that color or $\lceil \frac{3}{2} \rceil$ edges of that color sending P the whole charge. So P receives at least $2\Delta(G)$ charges. □

Lemma 16. *Let P be a free component of (G, π) consisting of a k-fold xy edge for $k \geq 2$. Assume that $|\overline{\pi}(P)| = \Delta(G)$. Then either $\mathrm{ch}(P) \geq 2\Delta(G)$ or the connected component of G containing P has exactly three vertices.*

Proof. Since $|\overline{\pi}(x)| + |\overline{\pi}(y)| = \Delta(G)$ we see that the sets of colors of edges incident to x and y are disjoint and sum up to the set of all colors. In particular $\deg(x) - k + \deg(y) - k = \Delta(G)$. Consider $N(P)$. If it contains only one vertex z then $N(z) \subset P$ and $I[P]$ is a connected component with three vertices. Assume that z_1 and z_2 are two distinct vertices in $N(P)$. We can move one of the xy edges to an xz_i or yz_i edge obtaining a coloring π_i. Thus $z_1, z_2 \in W(P)$ and $\overline{\pi}(z_i) = \emptyset$. Furthermore, since $|\overline{\pi_i}(P_i)| = \Delta(G)$ we have $I[z_i] \subset \mathrm{con}^*(P)$. So each colored edge in $I[N[P]]$ sends its charge to P. Notice that for each color there are at least two distinct edges in $I[N[P]]$ colored with this color: one incident with P and one incident with the z_i not connected to P by the first edge. Thus P receives at least two charges from edges of a given color and hence the assertion. □

Lemma 17. *Let P be a free component of (G, π) consisting of a k-fold xy edge for $k \geq 2$. Assume that $|\overline{\pi}(P)| = \Delta(G) - 1$. Then either $\mathrm{ch}(P) \geq 2\Delta(G)$ or there is an induced subgraph H of G containing P, containing exactly three vertices and connected with $G \setminus H$ by at most three edges which are all colored with the color not in $\overline{\pi}(P)$.*

Proof. Since $|\overline{\pi}(x)| + |\overline{\pi}(y)| = \Delta(G) - 1$ we see that the sets of colors of edges incident to x and y sum up to the set of all colors and have exactly one common element, denote this color by a. In particular $\deg(x) - (k+1) + \deg(y) - (k+1) = \Delta(G) - 1$. Let $N_{\neq a}(P)$ denote the vertices from $N(P)$ connected with P by an edge of color different from a. Consider $N_{\neq a}(P)$. If it contains only one vertex z then $N_{\neq a}(z) \subset P$ and the induced subgraph on vertices x, y, z is joined with the rest of the graph by at most three edges colored with a. Assume that z_1 and z_2 are two distinct vertices in $N_{\neq a}(P)$. We can move one of the xy edges to an xz_i or yz_i edge obtaining a coloring π_i. As in Lemma 16 we obtain that $z_1, z_2 \in W(P)$ and $\overline{\pi}(z_i) \subset \{a\}$. Furthermore for each color different from a there are at least two distinct edges in $I[N_{\neq a}[P]]$ colored with this color and all of them sent the whole charge to P. It remains to show that P receives at least two charges from edges colored with a.

If $\overline{\pi}(z_1) = \overline{\pi}(z_2) = \emptyset$ then there are four vertices in $N_{\neq a}[P]$ incident with an edge colored with a. Each gives half of a charge, which sums up to two. Since both z_i belong to $W(P)$ it cannot occur that $\overline{\pi}(z_1) = \overline{\pi}(z_2) = \{a\}$. Assume that $\overline{\pi}(z_1) = \{a\}$

and $\overline{\pi}(z_2) = \emptyset$. Both edges colored with a and incident with P are in $\mathrm{con}(P(\pi_1))$ and send the whole charge to P. We are done, unless this is a single xy edge colored with a. Assume this is the case, we will complete the proof by showing that the z_2w edge colored with a belongs to $\mathrm{con}^*(P)$. Notice that we can move $P(\pi_1)$ by uncoloring the xy edge colored with a and coloring the uncolored xz_1 or yz_1 edge and then further by uncoloring an xz_2 or yz_2 edge and coloring an xy edge. This way we obtain a coloring π_3 with the property that $a \in \overline{\pi_3}(P(\pi_3))$, the z_2w edge remains colored with a and z_2 is a vertex of $P(\pi_3)$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Assume that G does not contain a forbidden subgraph.

We start by showing, that if $\Delta(G) = 5$ then we may additionally assume that G does not contain a $2K_3$ as an induced subgraph. Indeed, a $2K_3$ may be connected with the rest of G by at most three edges. Replace every copy of the $2K_3$ in G with a single vertex of degree at most 3 and color the resulting graph G' in such way that at least $\frac{5}{6}$ of the edges are colored. Use the partial coloring of G' to color G. For any copy of $2K_3$ the edges e_i connecting it with the rest of G have all distinct colors, or $\perp$. Color one of the two edges not incident with e_i with the color of e_i or with any admissible color if e_i is uncolored. Then color two of the three remaining edges with the two remaining colors. We have colored 5 of the 6 added edges, so the resulting coloring also has at least $\frac{5}{6}$ of the edges colored.

Now it is enough to show that if a coloring π of a graph without a forbidden subgraph maximizes the potential Ψ then each free component P of π receives at least $|P|\Delta$ charges.

If $|P| = 1$ then we are done by Lemma 14. Assume that $|P| = 2$. We will first consider two special cases and then show that all other cases can be reduced via Observation 13 to those two cases.

If P is a P_2 whose edges are single edges in G then we are done by Lemma 15.

If P is a double edge then either we are done by Lemma 16 or by Lemma 17 or obtain a forbidden subgraph.

Assume that P has edges xy and yz and there are at least two xy edges in G. Let a denote the color of a colored xy edge. If $a \in \overline{\pi}(z)$ then we uncolor the a-colored xy edge and color the yz edge with a. We obtain a component consisting of a double edge and we are done. So assume that $a \notin \overline{\pi}(z)$. This means that $\Delta = 5$ and $a \notin \overline{\pi}(P)$. Let $\overline{\pi}(x) = \{b\}$ and $\overline{\pi}(z) = \{c\}$. Consider the edges yw_b and yw_c colored respectively with b and c. If $w_b = z$ or $w_c = x$ then we can uncolor this edge, color respectively xy or yz and obtain a component consisting of a double edge. If not then $\{x, y\} \cap \{w_b, w_c\} = \emptyset$ and we may move P to a component consisting of yw_b and yw_c. We obtain a double edge or two single edges in G and we are done.

To find the desired coloring in polynomial time one should go through following steps:

1. Collapse all double triangles for $\Delta(G) = 5$.
2. Color the graph using a greedy algorithm.

3. Reduce the size of free components be using repeatedly lemmas 2 and 4 and the recolorings described in the proof of those lemmas.
4. Put charges on the edges and let each component collect the required number of charges. If the sum of charges collected from an edge exceeds 1 then the potential Ψ is not maximal. Move the involved free component as described in the proofs of lemmas 9 and 10 to arrive at a point where the potential can be increased. Then repeat this step. There are only polynomially many possible values of the potential so this step will be repeated only polynomially many times.
5. If $\Delta(G) = 5$ then restore all collapsed triangles and color them. □

4 Approximation Algorithms

Following [1], let $c_k(G)$ be the maximum number of edges of a k-edge-colorable subgraph of G. We use the following result of Kamiński and Kowalik.

Theorem 18 ([8])**.** *Let $\mathcal{G}$ be a family of graphs and let $\mathcal{F}$ be a k-normal family of graphs. Assume there is a polynomial-time algorithm which for every k-matching H of a graph in $\mathcal{G}$, such that $H \notin \mathcal{F}$ finds its k-edge colorable subgraph with at least $\alpha|E(H)|$ edges. Moreover, let*

$$\beta = \min_{\substack{A,B\in\mathcal{F} \\ A \text{ is not } k\text{-regular}}} \frac{c_k(A)+c_k(B)+1}{|E(A)|+|E(B)|+1}, \quad \gamma = \min_{A\in\mathcal{F}} \frac{c_k(A)+1}{|E(A)|+1}$$

$$\text{and} \quad \delta = \min_{A,B\in\mathcal{F}} \frac{c_k(A)+c_k(B)+2}{|E(A)|+|E(B)|+1}.$$

Then, there is an approximation algorithm for the maximum k-ECS problem for graphs in $\mathcal{G}$ with approximation ratio $\min\{\alpha,\beta,\gamma,\delta\}$.

Since the definition of k-normal family is very technical, we refer the reader to [8] for its definition. As a direct consequence of Theorem 1 and Theorem 18 we get the following two results.

Theorem 19. *The maximum 4-ECS problem has a $\frac{5}{7}$-approximation algorithm for multigraphs.*

Proof. Let $\mathcal{F} = \{2K_3\}$. It is easy to check that $\mathcal{F}$ is 4-normal. Now we give the values of parameters α,β,γ and δ from Theorem 18. By Theorem 1, $\alpha = \frac{4}{5}$. We have $c_4(2K_3) = 4$ and $|E(2K_3)| = 6$. Hence, $\beta = \infty$, $\gamma = \frac{5}{7}$ and $\delta = \frac{10}{13}$. □

Theorem 20. *The maximum 5-ECS problem has a $\frac{11}{15}$-approximation algorithm for multigraphs.*

Proof. Let $\mathcal{F} = \{2K_3 + e\}$. It is easy to check that $\mathcal{F}$ is 5-normal. Now we give the values of parameters α,β,γ and δ for Theorem 18. By Theorem 1, $\alpha = \frac{5}{6}$. We have $c_5(2K_3) = 5$ and $|E(2K_3 + e)| = 7$. Hence, $\beta = \frac{11}{15}$, $\gamma = \frac{3}{4}$, $\delta = \frac{4}{5}$. □

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