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Beyond Shannon's bound for at least 6 colors

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Abstract

Let $G = (V, E)$ be a multigraph of maximum degree Δ. We study lower bounds on the size of subgraphs of G that can be colored with Δ colors. Shannon's Theorem gives a bound of $\frac{\Delta}{\lfloor\frac{3}{2}\Delta\rfloor}|E|$. However Kamiński and Kowalik [1] showed that for $\Delta = 3$ there is a 3-edge-colorable subgraph of size at least $\frac{7}{9}|E|$, unless G is isomorphic to $K_3 + e$. Furthermore Farnik, Kowalik and Socała [2] showed that for $\Delta = 4$ there is a 4-edge-colorable subgraph of size at least $\frac{4}{5}|E|$, unless G is isomorphic to $2K_3$ and for $\Delta = 5$ there is a 5-edge-colorable subgraph of size at least $\frac{5}{6}|E|$, unless G contains $2K_3 + e$.

Here we develop the methods introduced in [2] and show that if either Δ is even and G does not contain $\frac{\Delta}{2}K_3$ as a subgraph or Δ is odd and G does not contain $\frac{\Delta-1}{2}K_3 + e$ as a subgraph then G has a Δ-edge colorable subgraph with at least $\frac{\Delta}{\lfloor\frac{3}{2}\Delta\rfloor-1}|E|$ edges.

As a corollary we obtain that

- every connected multigraph G of even maximum degree Δ has a Δ-edge colorable subgraph with at least $\frac{2\Delta}{3\Delta-2}|E|$ edges, unless G is isomorphic to $\frac{\Delta}{2}K_3$,
- every connected multigraph G of odd maximum degree Δ has a Δ-edge colorable subgraph with at least $\frac{2\Delta+1}{3\Delta}|E|$ edges, unless G is isomorphic to $\frac{\Delta-1}{2}K_3 + e$.

Our results have immediate applications in approximation algorithms for the Maximum k-Edge-Colorable Subgraph problem, where given a graph G (without any bound on its maximum degree or other restrictions) one has to find a k-edge-colorable subgraph with maximum number of edges.

In particular, when G is a multigraph then we obtain approximation ratios of $\frac{2k+2}{3k+2}$ for even Δ and $\frac{2k+1}{3k}$ for odd Δ, which improves the best currently known approximations for $6 \leq k \leq 13$.

We also show the existence of a sequence of bottlenecks; we show that for every $t \geq 1.34\Delta$ every multigraph G of maximum degree Δ has a Δ-edge colorable subgraph with at least $\frac{\Delta}{t}|E|$ edges or a subgraph with 3 vertices and more than t edges. We obtain as a corollary a slightly stronger version of the Vizing's theorem for multigraphs, namely that for $\mu(G) \geq 0.45\Delta$ there is a Δ-edge colorable subgraph with at least $\frac{\Delta}{3\mu(G)}|E|$ edges.

1 Introduction

A graph is k-edge-colorable if there exists an assignment of k colors to the edges of the graph, such that every two incident edges receive different colors. By Shannon's theorem [3], $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ colors suffice to color any multigraph, where Δ denotes the maximum degree. Thus is natural to ask how many edges of a graph of maximum degree Δ can be colored with less than $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ colors. The *maximum k-edge-colorable subgraph of G* (maximum k-ECS in short) is a k-edge-colorable subgraph H of G with maximum number of edges. Let $\gamma_k(G)$ denote the ratio $|E(H)|/|E(G)|$; when $|E(G)| = 0$ we define $\gamma_k(G) = 1$. If Δ is the maximum degree of G we write shortly $\gamma(G)$ for $\gamma_\Delta(G)$. We refer to [2] for a detailed discussion on the history and state of art of this subject.

In this paper we study lower bounds on $\gamma(G)$ for multigraphs. Note that we may apply Shannon's theorem to obtain a coloring of G with $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ colors and discard the $\left\lfloor \frac{1}{2}\Delta \right\rfloor$ color classes. This gives us $\gamma(G) \geq \Delta/\lfloor \frac{3}{2}\Delta \rfloor$. This bound is tight by $\frac{\Delta-1}{2}K_3 + e$ if Δ is odd and $\frac{\Delta}{2}K_3$ if Δ is even. However Kamiński and Kowalik [1] showed that $\gamma(G) \geq \frac{7}{9}$ when G is a subcubic multigraph different from $K_3 + e$ and Farnik, Kowalik and Socała [2] showed that $\gamma(G) \geq \frac{4}{5}$ when $\Delta = 4$ and G is different from $2K_3$ and $\gamma(G) \geq \frac{5}{6}$ when $\Delta = 5$ and G does not contain $2K_3 + e$. We extend those results by showing that $\gamma(G) \geq \frac{2\Delta}{3\Delta-2}$ when Δ is even and G is different from $\frac{\Delta}{2}K_3$ and $\gamma(G) \geq \frac{2\Delta+1}{3\Delta}$ when Δ is even and G is different from $\frac{\Delta-1}{2}K_3 + e$.

Using a technique developed by Kamiński and Kowalik [1] (see Theorem 22) we apply our results to obtain approximation algorithms for the Maximum k-ECSubgraph problem, where given a multigraph G one has to find a k-edge-colorable subgraph with maximum number of edges.

Our research exhibits also a sequence of bottlenecks for the maximum Δ-ECS problem: we show that for every $t \geq 1.34\Delta$ every multigraph G of maximum degree Δ has a Δ-edge colorable subgraph with at least $\frac{\Delta}{t}|E|$ edges or a subgraph with 3 vertices and more than t edges. We obtain as a corollary a slightly stronger version of the Vizing's theorem for multigraphs, namely that for $\mu(G) \geq 0.45\Delta$ there is a Δ-edge colorable subgraph with at least $\frac{\Delta}{3\mu(G)}|E|$ edges.

1.1 Main Results

Below we state our main results.

Theorem 1. *Let G be a multigraph of maximum degree Δ and let t be an integer such that $\left\lfloor \frac{3\Delta}{2} \right\rfloor \geq t \geq \left(\frac{1}{2}\sqrt{22} - 1\right)\Delta \approx 1.34\Delta$. Suppose that G does not contain a subgraph with 3 vertices and more than t edges. Then G has a Δ-edge-colorable subgraph with at least $\frac{\Delta}{t}|E|$ edges.*

Moreover, the subgraph and its coloring can be found in polynomial time.

Corollary 2. *Let G be a multigraph of maximum degree Δ and maximal edge multiplicity $\mu \geq \frac{\Delta}{6}\left(\sqrt{22} - 2\right) \approx 0.45\Delta$. Then G has a Δ-edge-colorable subgraph with at least $\frac{\Delta}{3\mu}|E|$ edges.*

Moreover, the subgraph and its coloring can be found in polynomial time.

Theorem 3. *Let G be a multigraph of maximum degree Δ. If Δ is even then suppose G does not contain $\frac{\Delta}{2}K_3$ as a subgraph. If Δ is odd then suppose G does not contain $\frac{\Delta-1}{2}K_3 + e$ as a subgraph. Then G has a Δ-edge colorable subgraph with at least $\frac{\Delta}{\lfloor\frac{3}{2}\Delta\rfloor-1}|E|$ edges.*

Moreover, the subgraph and its coloring can be found in polynomial time.

Theorem 4. *Let G be a connected multigraph of maximum degree Δ. Then G has a Δ-edge-colorable subgraph with at least*

1. *$\frac{2\Delta}{3\Delta-2}|E|$ edges when Δ is even and $G \neq \frac{\Delta}{2}K_3$,*
2. *$\frac{2\Delta+1}{3\Delta}|E|$ edges when Δ is odd and $G \neq \frac{\Delta-1}{2}K_3 + e$.*

Moreover, the subgraph and its coloring can be found in polynomial time.

1.2 Our Approach and Organization of the paper

In order to prove the main results we will base on the techniques developed in [1] and [2]. Thus in Section 1.3 we will recall the basic definitions and results from those papers. As in [2] we will introduce a potential Ψ and consider a coloring that maximimizes the potential Ψ. We will use the notions of free components and charge to estimate the ratio of colored and uncolored edges. In Section 2 we will recall some results from [2] regarding sending charges and introduce a few new ones.

One can construct graphs for which the technique of maximizing Ψ developed in [2] gives results almost as bad as Shannon's Theorem. However those graphs have a very special structure. In Section 3 we develop the method of collapsing subgraphs which reduces the problem of coloring arbitrary graphs to the problem of coloring graphs for which the technique of maximizing Ψ gives a good result.

The paper concludes with Section 4 containing the proofs of the main results and Section 5 with a discussion on the approximation algorithms.

1.3 Preliminaries

We will work with undirected multigraphs (though for simplicity we will call them graphs). Our notation is consitent with the one used in [2], which we recall below.

Let $G = (V, E)$ be a graph. For a vertex $x \in V$ by $N(x)$ we denote the set of neighbors of x and $N[x] = N(x) \cup \{x\}$. For a set of vertices S we denote $N(S) = \bigcup_{x\in S} N(x) \setminus S$ and $N[S] = \bigcup_{x\in S} N[x]$. We also denote the subgraph of G whose set of vertices is $N[S]$ and set of edges is the set of edges of G incident with S by $I[S]$. For a subgraph H of G we denote $N[H] = N[V(H)]$ and $I[H] = I[V(H)]$.

A *partial k-coloring* of a graph $G = (V, E)$ is a function $\pi : E \to \{1, \ldots, k\} \cup \{\bot\}$ such that if two edges $e_1, e_2 \in E$ are incident then $\pi(e_1) \neq \pi(e_2)$, or $\pi(e_1) = \pi(e_2) = \bot$. From now on by a *coloring* of a graph we will mean a partial $\Delta(G)$-coloring. We say that an edge e is *uncolored* if $\pi(e) = \bot$; otherwise, we say that e is *colored.* For a vertex v, $\pi(v)$ is the set of colors of edges incident with v, i.e. $\pi(v) = \{\pi(e) : e \in I[v]\} \setminus \{\bot\}$, while $\overline{\pi}(v) = \{1, \ldots, k\} \setminus \pi(v)$ is the set of free colors at v.

Let $V_\perp = \{v \in V \ : \ \overline{\pi}(v) \neq \emptyset\}$. In what follows, $\perp(G,\pi) = (V_\perp, \pi^{-1}(\perp))$ is called *the graph of free edges.* Every connected component of the graph $\perp(G,\pi)$ is called a *free component.* If a free component has only one vertex, it is called *trivial.*

For a free component Q, by $\overline{\pi}(Q)$ we denote the set of free colors at the vertices of Q, i.e. $\overline{\pi}(Q) = \bigcup_{v \in V(Q)} \overline{\pi}(v)$.

Let Q_1, Q_2 be two distinct free components of (G,π). Assume that for some pair of vertices $x \in V(Q_1)$ and $y \in V(Q_2)$, there is an edge $xy \in E$ such that $\pi(xy) \in \overline{\pi}(Q_1)$. Then we say that Q_1 *sees* Q_2 *with* xy, or shortly Q_1 *sees* Q_2.

Below we state a few lemmas proved in [1] or [2] which will be useful in the present paper. Although the lemmas from [1] were formulated for simple graphs one can easily check that the proofs apply to multigraphs as well.

Lemma 5 ([1], Lemma 7)**.** *Let (G,π) be a colored graph that maximizes the number of colored edges. For any free component Q of (G,π) and for every two distinct vertices $v, w \in V(Q)$*

(a) $\overline{\pi}(v) \cap \overline{\pi}(w) = \emptyset$,

(b) for every $a \in \overline{\pi}(v)$, $b \in \overline{\pi}(w)$ there is an (ab, vw)-path.

Corollary 6 ([1], Lemma 8)**.** *Let (G,π) be a colored graph that maximizes the number of colored edges. For any free component Q of (G,π) we have $|\overline{\pi}(Q)| \geq 2|E(Q)|$. In particular Q has at most $\left\lfloor \frac{\Delta}{2} \right\rfloor$ edges.*

Lemma 7 ([1], Lemma 10)**.** *Let (G,π) be a colored graph that maximizes the number of colored edges. If Q_1, Q_2 are two distinct free components of (G,π) such that Q_1 sees Q_2 then $\overline{\pi}(Q_1) \cap \overline{\pi}(Q_2) = \emptyset$.*

We use the notion of the potential function Ψ introduced in [1]:

$$\Psi(G,\pi) = \left(c, n_{\lfloor \Delta/2 \rfloor}, n_{\lfloor \Delta/2 \rfloor - 1}, \ldots, n_1\right),$$

where c is the number of colored edges, i.e. $c = |\pi^{-1}(\{1,\ldots,\Delta\})|$ and n_i is the number of free components with i edges for every $i = 1, \ldots, \lfloor \Delta/2 \rfloor$.

We also use the notion of elementary moves and moves introduced in [2].

Definition 8. Let (G,π) be a colored graph and let P be a free component of (G,π). An *elementary move* of P in π is a coloring π' such that:

1. π' can be obtained from π by uncoloring k edges of $I(P)$ and coloring k edges of P for some $k \geq 0$,
2. $\pi'|_{I[P]}$ has exactly one nontrivial free component, denote it P'.

If the above holds we say that π' and P' have been obtained respectively from π and P by an elementary move.

A coloring π' is a *move* of π if there is a sequence of colorings $\pi_0 = \pi, \pi_1, \ldots, \pi_k = \pi'$ such that π_i is an elementary move of a free component in π_{i-1}.

We say that a free component P' of π' is obtained from a free component P in π if there are $P_0, \ldots, P_k$ such that for $i = 0, \ldots, k$ P_i is a free component in π_i and either $P_i = P_{i-1}$ or π_i is the elementary move of P_{i-1} to P_i. We denote the free component of a coloring π' obtained from a free component P by $P(\pi')$. Note that if π maximizes the potential Ψ then moving a component P cannot cause a merge of components and hence P' is a free component of π'.

Let P be a free component of (G, π). By $\mathcal{M}(P)$ we denote the set of moves of π that can be obtained by moving only P (and the components obtained from P). We say that vertex v is *controlled* by P when $v \in V(P(\pi'))$ for some $\pi' \in \mathcal{M}(P)$. By $\mathrm{con}(P)$ we denote the set of vertices controlled by P.

After [2] we define a useful subset of $\mathrm{con}(Q)$:

Definition 9. For a free component Q of a coloring (G, π) we define a set of vertices $W(Q) = \{v \in V(G) \,:\, v \in V(Q(\pi'))$ for some elementary move π' of Q such that $|E(Q(\pi')) \setminus E(Q)| \leq 1\}$.

Lemma 10 ([2], Lemma 12)**.** *Let (G, π) be a coloring that maximizes the potential Ψ and let Q be a free component with $|E(Q)| \geq 2$. If $u, v \in W(Q)$ and $u \neq v$ then $\overline{\pi}(u) \cap \overline{\pi}(v) = \emptyset$.*

2 Sending charges

In this section we consider a connected colored graph (G, π) which maximizes the potential Ψ.

We put one unit of charge on each colored edge of G. Every edge divides its charge equally between the nontrivial components that control its endpoints. Let $\mathrm{ch}(P)$ denote the amount of charge sent to P. Then the number of colored edges in G is at least $\sum_P \mathrm{ch}(P) \geq |E| \min_P \frac{\mathrm{ch}(P)}{\mathrm{ch}(P)+|E(P)|}$, where the summation is over all nontrivial free components in (G, π). In what follows we give lower bounds for $\mathrm{ch}(P)$ for various types of free components. Note that this distribution of charges is slightly different from the one described in [2], however the amount of charge received by a component from an edge according to our distribution is not smaller then the amount received according to the distribution from [2]. Hence the results obtained in [2] apply also in our distribution. We start by recalling a few of them:

Observation 11 ([2], Observation 13)**.** *Let P be a free component of (G, π) and let π' be a move of π. Then every edge of G sends the same amount of charge to $P(\pi')$ in the coloring π' as it does to P in π.*

Observation 12 ([2], Lemma 10)**.** *Let P be a free component, let S be a set of colored edges and let $V(S)$ be the set of endpoints of all edges in S. Then edges in S send at least $\frac{1}{2}|V(S) \cap \mathrm{con}(P)|$ of charge to P.*

Let Q be a free component in a colored graph (G, π_1). If a colored edge e is incident with Q, $|\overline{\pi_1}(Q)| \geq \Delta(G) - 1$, and $\pi_1(e) \in \overline{\pi_1}(Q)$, then we say that edge e is *dominated* by Q in the coloring π_1.

Lemma 13 ([2], Lemma 9)**.** *Let P be a free component in (G, π) and let π_1 be a move of π. Let $P' = P(\pi_1)$. Every edge dominated by P' in π_1 sends its whole charge to P.*

Now we will show a few new results:

Lemma 14. *Let P be a free component of (G,π) and let $U \subset \mathrm{con}(P)$ be a set of vertices such that $\overline{\pi}(v) \cap \overline{\pi}(w) = \emptyset$ for every two distinct vertices $v, w \in U$. Then* $\mathrm{ch}(P) \geq (|U|-1)|E(P)|$.

Proof. Let S_a denote the set of edges incident to U and colored with color a. We have $|V(S_a) \cap U| \geq |U| - 1$. Since $U \subseteq \mathrm{con}(P)$, by Observation 12 we infer that S_a sends at least $\frac{|U|-1}{2}$ of charge to P. Taking the sum over all colors we obtain that P receives at least $\frac{\Delta(|U|-1)}{2}$ of charge. Moreover by Lemma 6 we have $|E(P)| \leq \frac{\Delta}{2}$, hence the claim. □

Notice that by Lemma 5 and Lemma 10 both $V(P)$ and $W(P)$ satisfy the assumptions of Lemma 14. We immediately obtain the following:

Corollary 15. *Let Q be a free component of (G,π) such that $|V(Q)| \geq 5$. Then* $\mathrm{ch}(Q) \geq 4|E(Q)|$.

Lemma 16. *Let Q be a free component of (G,π) such that $|V(Q)| = 4$. Then* $\mathrm{ch}(Q) \geq 4|E(Q)|$.

Proof. Notice that Q has at least three edges, consider first the case when Q has exactly three edges. By Corollary 6 we know that $\Delta(G) \geq 6$. If $\Delta(G) \in \{6,7\}$ then $|\overline{\pi}(Q)| \geq \Delta(G) - 1$. Let S_a denote the set of edges incident to $V(Q)$ and colored with color a. By Lemma 5, we have $|V(S_a) \cap V(Q)| \geq |V(Q)| - 1$. If $|V(S_a) \cap V(Q)| = 3$ then every edge in S_a incident with Q is dominated by Q and by Lemma 13 sends 1 to Q. Then S_a sends Q at least $\lceil \frac{3}{2} \rceil = 2$ charge. If $|V(S_a) \cap V(Q)| = 4$ then by Observation 12 S_a sends Q at least $\frac{4}{2} = 2$ charge. Thus $\mathrm{ch}(Q) \geq 2\Delta(G) \geq 12 = 4|E(Q)|$.

If $\Delta \geq 8$ then again by Lemma 5 and Observation 12 we obtain $\mathrm{ch}(Q) \geq \frac{3}{2}\Delta \geq 12 = 4|E(Q)|$.

Now consider the case when Q has at least 4 edges. Note that Q must contain a cycle, we will analyze three cases depending on the length of that cycle.

Suppose Q contains a cycle of length 4. Let u, v, p, q be the consecutive vertices on a cycle of length 4. Let $a \in \overline{\pi}(u)$. By Lemmma 5 we have $a \in \pi v$ and $a \in \pi(q)$. If there is no vq edge colored with a then one of the edges colored with a and incident with v or q, say vx, must be incident with a vertex $x \notin V(Q)$. In that case we may move Q by uncoloring vx and coloring uv with a. The resulting component has 5 vertices so the claim follows by Corollary 15 and Observation 11. On the other hand if there is a vq edge colored with a then there also has to be an px edge colored with a for some $x \notin V(Q)$. In that case we may move Q by uncoloring vq and px and coloring uv and pq with a. Again the resulting component has 5 vertices and the claim follows.

Suppose Q contains a cycle of length 3. Let v, p, q be the vertices on a cycle of length 3 and let u be the fourth vertex in $V(Q)$. Since Q is connected there has to be an edge of Q incident with u, say uv. Let $a \in \overline{\pi}(u)$, by Lemma 5 there is an vx edge colored with a. If $x \notin V(Q)$ then we may move Q by uncoloring vx and coloring uv with a, so $x \in W(Q)$ and the claim follows by Lemma 14. Suppose

$x \in V(Q)$, say $x = p$. Then there is a qy edge colored with a and with $y \notin V(Q)$. Let π' be the elementary move of Q obtained by uncoloring vp and qy and coloring uv and pq with a. We will show that any two distinct vertices in $\{u, v, p, q, y\}$ have disjoint sets of free colors in the coloring π and will be done by Lemma 14. Note that for $z \in \{v, p, q\}$ we have $\overline{\pi}(z) = \overline{\pi}'(z)$, furthermore $\overline{\pi}(y) = \overline{\pi}'(y) \setminus \{a\}$, thus by Lemma 5 applied to P and $P(\pi')$ we only need to show that $\overline{\pi}(u) \cap \overline{\pi}(y) = \emptyset$. Assume for a contradiction that there is a color $b \in \overline{\pi}(u) \cap \overline{\pi}(y)$. Let $c \in \overline{\pi}(p)$, by Lemma 5(b) there is a (cb, pu)-path in (G, π) and a (cb, py)-path in (G, π'). However $\pi|_{\pi^{-1}(\{b,c\})} = \pi'|_{\pi^{-1}(\{b,c\})}$ so the maximal cb path starting at p in (G, π') is also the maximal cb path starting at p in (G, π). Thus $y = u$, a contradiction with $y \notin V(Q)$.

Suppose Q does not contain cycles of length larger than 2. Then Q is a path with multiple edges. Let u, v, p, q be the consecutive vertices on the path. By symmetry we may assume that either uv or vp is a multiple edge in Q. Suppose that uv is a multiple edge. Let $a \in \overline{\pi}(v)$, by Lemma 5 there is an ux edged colored with a. We may move Q by uncoloring ux and coloring uv with a. If $x \notin V(Q)$ then the resulting component has 5 vertices and the claim follows by Corollary 15 and Observation 11. If $x \in V(Q)$ then we obtain a component with 4 vertices and a cycle of length 3 or 4 and we are done by the analysis above and Observation 11. Now suppose that vp is a multiple edge. Let $a \in \overline{\pi}(p)$ and consider the edge vx colored with a. We may move Q by uncoloring ux and coloring uv with a. If $x \notin V(Q)$ then again the resulting component has 5 vertices and the claim follows. If $x \in V(Q)$ then either $x = u$ and we have obtained the case where uv is a multiple edge or $x = q$ and we have obtained the case where Q contains a cycle of length 3. □

3 Collapsing subgraphs

Definition 17. Let G be a multigraph with maximum degree Δ and H an induced subgraph of G such that $|V(H)| = 3$. We say that H is *k-collapsible* if $|E(V(H), V(G) \setminus V(H))| \leq k$ and $|E(V \setminus x)| \geq |E(x, V(G) \setminus V(H))|$ for every vertex $x \in V(H)$

If H is a k-collapsible subgraph of G then we may obtain a graph G' by removing from G all edges in $E(H)$ and identifying the three vertices in $V(H)$ into a single vertex h. Note that G' has the maximum degree not greater than $\max\{k, \Delta(G)\}$. We say that G' is obtained by collapsing H to h in G.

Lemma 18. *Let G' be obtained by collapsing a k-collapsible subgraph H to h in a multigraph G with maximum degree Δ.*

1. *If G does not contain an induced subgraph on three vertices with more than $\Delta + \left\lfloor \frac{k}{2} \right\rfloor$ edges then G' does not contain an induced subgraph on three vertices with more than $\Delta + \left\lfloor \frac{k}{2} \right\rfloor$ edges.*
2. *Given a partial coloring π' of G' which colors at least $p|E(G')|$ edges we may construct a partial coloring π of G which colors at least $\min\{p, \frac{\Delta}{|E(H)|}\}|E(G)|$ edges.*

Proof.

1. Every induced subgraph of G' that does not contain h is isomorphic to an induced subgraph of G hence we only need to consider subgraphs of G' containing h. However a subgraph of G' containing h and two other vertices can have at most $\Delta - \left\lceil \frac{\deg_{G'}(h)}{2} \right\rceil + \deg_{G'}(h) = \Delta + \left\lfloor \frac{\deg_{G'}(h)}{2} \right\rfloor \leq \Delta + \left\lfloor \frac{k}{2} \right\rfloor$ edges.

2. The edges in G' correspond to edges in G that are not in $E(H)$. Thus we may use the partial coloring π' to obtain a partial coloring of $E(G) \setminus E(H)$. The edges in G' incident with h correspond to edges in $E(V(H), V(G) \setminus V(H))$. For a vertex $x \in V(H)$ we use the colors of edges in $E(x, V(G) \setminus V(H))$ to color edges in $E(H \setminus x)$. We use the remaining colors to color arbitrary edges of $E(H)$. We obtained a partial coloring π such that $\min\{\Delta, |E(H)|\}$ edges of $E(H)$ are colored. Thus the claim follows. □

Let us recall the following result:

Lemma 19 ([2], Theorem 1)**.** *Let G be a connected multigraph of maximum degree Δ. Let π be a coloring maximizing Ψ.*

1. If $\Delta = 4$ and $G \neq 2K_3$ then π colors at least $\frac{4}{5}|E|$ edges.

2. If $\Delta = 5$ and G does not contain a 3-collapsible subgraph then π colors at least $\frac{5}{6}|E|$ edges.

In the following Lemma we will use the function ρ. Although the definition may seem artificial it will be validated in the proof.

Let $\rho(\Delta, k, t) = \min(\{\frac{7}{2}\} \cup \{\frac{3\Delta - \alpha}{2e} \mid e, \alpha, \beta \in \mathbb{Z},\ e \geq 2,\ \alpha \geq 2e,\ \beta \geq 0,\ \alpha + \beta \leq \Delta,\ e + \Delta - \beta \leq t,\ 2\beta + \Delta - \alpha \geq k + 1\})$.

Lemma 20. *Let (G, π) be a coloring maximizing Ψ. Let Q be a free component of (G, π) such that $|V(Q)| \in 2, 3$. Let $\Delta \geq 6$ be the maximal degree of G and let $\Delta \leq t \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor$ and $0 \leq k \leq \Delta$ be integers such that G does not contain a k-collapsible subgraph and does not contain an induced subgraph on three vertices with more than t edges. Then $\mathrm{ch}(Q) \geq \rho(\Delta, k, t)|E(Q)|$.*

Proof. We will first consider the case when $|V(Q)| = \{u, v, p\}$ and $|E(Q)| \geq 3$.

Suppose additionally that Q contains a cycle of length 3. Pick any $a \in \overline{\pi}(Q)$, by symmetry assume that $a \in \overline{\pi}(u)$. By Lemma 5 there is an vx edge colored with a. If $x \neq p$ then we may move Q by uncoloring vx and coloring uv with a. We obtain a component with four vertices and the claim follows by Lemma 16 and Observation 11. Thus we may assume that for every color in $\overline{\pi}(Q)$ there is an edge incident with two vertices in $V(Q)$ colored with that color. Let $\alpha = |\overline{\pi}(Q)|$, by Corollary 6 we have $\alpha \geq 2|E(Q)|$. By Observation 12 Q receives at least α charge from edges colored with colors in $\overline{\pi}(Q)$ and at least $\frac{3}{2}(\Delta - \alpha)$ charge from edges colored with colors not in $\overline{\pi}(Q)$. Thus $ch(Q) \geq \frac{1}{2}(3\Delta - \alpha)$.

Note that for any color $b \notin \overline{\pi}(Q)$ there are either two or three distinct edges colored with b and incident with a vertex of Q. Let B and C be the sets of colors such that there are respectively two or three distinct edges colored that color and incident with a vertex of Q. Let $\beta = |B|$, then $|C| = \Delta - \alpha - \beta$. We will show that if $2\beta + \Delta - \alpha \leq k$ and $\beta \leq \|E(Q)|$ then the subgraph of G induced on $V(Q)$ is k-collapsible, contradicting the assumption. Indeed, for $b \in B$ all three edges in $I[Q]$ colored with b belong to $E(V(Q), V(G) \setminus V(Q))$. For $b \in C$ one edge belongs to $E(V(Q), V(G) \setminus V(Q))$ and one has both endpoints in $V(Q)$. In $E(V(Q), V(G) \setminus V(Q))$ there are no edges colored with colors in $\overline{\pi}(Q)$ so $|E(V(Q), V(G) \setminus V(Q))| = 3\beta + (\Delta - \alpha - \beta) \leq k$. For every vertex $x \in V(Q)$ and color $b \in C$ if there is an edge in $E(x, V(G) \setminus V(Q))$ colored with b then there is also an edge in $E(V(Q) \setminus x, V(Q) \setminus x)$ colored with b. So it suffices to show that each pair of vertices in $V(Q)$ is connected by at least β edges in $\pi^{-1}(\overline{\pi}(Q) \cup \{\perp\})$. However for each $x \in V(Q)$ we have $\deg_Q(x) \leq \overline{\pi}(x)$ so we may assign to each edge of Q two colors from $\overline{\pi}(Q)$ using one free color from each of its endpoints. By Lemma 5 we may assign the colors so that every color is assigned at most once. Consider an edge $e \in E(Q)$, say $e = uv$. Let $c_1 \in \overline{\pi}(u)$ and $c_2 \in \overline{\pi}(v)$ be the colors assigned to e, then there is a up edge colored with c_2 and a vp edge colored with c_1. So every edge $e \in E(Q)$ forms a K_3 together with the edges in $E(V(Q), V(Q))$ colored with the assigned colors. Thus each pair of vertices in $V(Q)$ is connected by at least $|E(Q)|$ edges in $\pi^{-1}(\overline{\pi}(Q) \cup \{\perp\})$. Hence if $2\beta + \Delta - \alpha \leq k$ and $\beta \leq \|E(Q)|$ then the subgraph of G induced on $V(Q)$ is k-collapsible, contradicting the assumption. If $\beta > \|E(Q)|$ then $\operatorname{ch}(Q) \geq \frac{1}{2}(3\Delta - \alpha) \geq \frac{1}{2}(2\alpha + 3\beta) \geq \frac{1}{2}(4|E(Q)| + 3|E(Q)|) = \frac{7}{2}|E(Q)|$. If $2\beta + \Delta - \alpha > k$ then by definition $\operatorname{ch}(Q) \geq \rho(\Delta, k, t)$. Either way the claim follows.

Now suppose that Q does not contain a cycle of length 3. Since $|E(Q)| \geq 3$ at least one of the edges of Q is multiple. By symmetry assume that uv is a multiple edge of Q and vp is an edge of Q. Let $a \in \overline{\pi}(v)$, by Lemma 5 there is a ux edge colored with a. We may move Q by uncoloring ux and coloring uv with a. If $x \neq p$ then we obtain a component with four vertices and the claim follows by Lemma 16 and Observation 11. If $x = p$ then we obtain a component with four vertices and a cycle of length 3. Hence the claim follows by Observation 11 and the above analysis.

Now consider the case when $|V(Q)| = 3$ and $|E(Q)| = 2$. If $\Delta \geq 7$ then by Lemma 5 and Observation 12 we have $\operatorname{ch}(Q) \geq \Delta \frac{|V(Q)|-1}{2} \geq 7 = \frac{7}{2}|E(Q)|$. So assume that $\Delta = 6$. Note that if there is a move π' of π such that $W(Q(\pi')) \geq 4$ then by Observation 11, Lemma 10 and Observation 12 we have $\operatorname{ch}(Q) = \operatorname{ch}(Q(\pi')) \geq 6\frac{4-1}{2} = \frac{9}{2}|E(Q)|$. Thus we assume that $W(Q(\pi')) = 3$ for every move π' of π. Let $E(Q) = \{uv, vp\}$ and $c_u \in \overline{\pi}(u)$. By Lemma 5 there is a vx edge colored with c_u. By uncoloring vx and coloring uv with c_u we obtain an elementary move π_1 of π. Hence $x \in W(Q)$, so $x = p$. Thus for every color $c_u \in \overline{\pi}(u)$ there is a vp edge colored with c_u and therefore fore every $c_u \in \overline{\pi_1}(u)$ there is a vp edge colored with c_u in π_1. Let $c_v \in \overline{\pi_1}(v)$ and consider the py edge colored in π_1 with c_v. We may move $Q(\pi_1)$ by uncoloring py and coloring vp with c_v. Hence $y \in W(Q(\pi_1)) = \{u, v, p\}$, so $y = u$. Similarly for every $c_p \in \overline{\pi_1}(p)$ there is a uv edge colored in π_1 with c_p. Thus we have obtained that for every color in $\overline{\pi_1}(Q(\pi_1))$ there is and edge incident with two vertices in $V(Q)$ colored with that color. We obtain the claim by repeating for $Q(\pi_1)$ the analysis done in the case when Q contains a cycle.

Finally consider the case when $|V(Q)| = 2$. If $|E(Q)| = 1$ then $\operatorname{ch}(Q) \geq \Delta \geq 4|E(Q)|$. Suppose $|E(Q)| \geq 2$, let $V(Q) = \{u, v\}$ and $a \in \overline{\pi}(u)$. We may move Q by uncoloring the vx edge colored with a and coloring uv with a. We obtain a free component Q' with three vertices and are done by Observation 11. □

4 Proof of the main results

Now we are ready to describe the algorithm used to find the colorings from Theorem 4, Theorem 1 and Theorem 3 in polynomial time.

Algorithm 21. Let G be a multigraph with maximal degree $\Delta \geq 4$. Let t be an integer such that $\lfloor \frac{3\Delta}{2} \rfloor \geq t \geq 0$ and G does not contain a subgraph with 3 vertices and more than t edges.

1. Let $k = \min \Delta, 2(t - \Delta) + 1$, $i := 0$ and $G_0 := G$.
2. While G_i contains a k-collapsible subgraph H let $i := i+1$ and G_i be the graph obtained by collapsing H in G_{i-1}. By Lemma 18 G_i has maximal degree at most Δ and does not contain a subgraph with 3 vertices and more than t edges. Since $|V(G_i)| = |V(G_{i-1})| - 2$ the loop can be executed only $O(|V(G)|)$ times. The resulting graph G_i does not contain a k-collapsible subgraph.
3. Color G_i using a greedy algorithm.
4. Using repeatedly Lemma 5 and Lemma 7 recolor G_i to obtain a coloring π such that the free components have size at most $\lfloor \frac{\Delta}{2} \rfloor$.
5. For every free component P assign charge to P as described in Section 2. Note that we do not need to find all charge that are sent to P, we only need to find the required amount. For any given component P the Lemmas in Section 2 determine which edges should provide charge to P. Thus assigning charge to a component can be done in constant time.
6. While in step 5 charge is claimed from an uncolored edge or more than 1 unit of charge is claimed from a colored edge do as follows. The Lemmas from Subsection 2 provide a coloring π' with $\Psi(\pi') > \Psi(\pi)$. Replace π with π' and repeat step 5. Since Ψ has only polynomially many values the loop will be executed only polynomially many times.
7. For $j := i$ downto 1 use Lemma 18 to obtain a coloring of G_{i-1} from a coloring of G_i.

Now we will prove that if G satisfies certain conditions then Algorithm 21 constructs a coloring which colors sufficiently many edges.

Proof of Theorem 1. Let G be a multigraph of maximum degree $\Delta \geq 4$. Fix an integer t such that $\lfloor \frac{3\Delta}{2} \rfloor \geq t \geq \left(\frac{1}{2}\sqrt{22} - 1\right)\Delta$ and G does not contain a subgraph with 3 vertices and more than t edges. Let π be the coloring constructed for G and t by Algorithm 21. Let $k = \min\{\Delta, 2(t - \Delta) + 1\}$. By Lemma 18 we may assume that G does not contain k-collapsible subgraphs.

If $\Delta \in \{4, 5\}$ then $\lfloor \frac{3\Delta}{2} \rfloor - 1 < \left(\frac{1}{2}\sqrt{22} - 1\right)\Delta$ so $t = \lfloor \frac{3\Delta}{2} \rfloor$ and $k = \Delta$. Then the claim follows by Lemma 19. Thus assume that $\Delta \geq 6$. We will show that every free component P of (G, π) receives at least $\frac{\Delta}{t-\Delta}|E(P)|$ charge. It follows that

π colors at least $\frac{\Delta}{t-\Delta}|E(G)|/(\frac{\Delta}{t-\Delta}+1) = \frac{\Delta}{t}|E(G)|$ edges, as required. Note that $t \geq \left(\frac{1}{2}\sqrt{22}-1\right)\Delta$ implies $\frac{\Delta}{t-\Delta} \geq \frac{\sqrt{22}+4}{3} \approx 2.9$.

Let P be a free component of (G,π). If $|V(P)| \geq 4$ then by Corollary 15 and Lemma 16 we have $\mathrm{ch}(Q) \geq 4|E(Q)|$. If $|V(P)| \leq 3$ then by Lemma 20 we have $\mathrm{ch}(P) \geq \rho(\Delta,k,t)|E(P)|$. Thus it remains to show that $\rho(\Delta,k,t) \geq \frac{\Delta}{t-\Delta}$.

Let α, β and e be as in definition of $\rho(\Delta,k,t)$. If $k=\Delta$ then $2\beta > \alpha$ so $6\Delta \geq 6\alpha+6\beta > 9\alpha$. Consequently $\frac{3\Delta-\alpha}{2e} = \frac{6\Delta-\alpha}{4e} > \frac{9\alpha-\alpha}{2\alpha} = 4$ and thus $\rho(\Delta,k,t) \geq \frac{\Delta}{t-\Delta}$ as required. So assume $k = 2(t-\Delta)+1$. Then $2\beta+\Delta-\alpha > 2t-2\Delta$ so $2\beta+3\Delta > 2t+\alpha$. It follows that $9\Delta-3\alpha = 7\Delta-3\alpha+2\Delta \geq 7\Delta-3\alpha+2\alpha+2\beta = 4\Delta-\alpha+(2\beta+3\Delta) > 4\Delta-\alpha+2t+\alpha = 4\Delta+2t$. Furthermore $6e \leq 3\alpha < 2(\Delta-\beta)+(2\beta+3\Delta-2t) = 5\Delta-2t$. Thus $\frac{3\Delta-\alpha}{2e} = \frac{9\Delta-3\alpha}{6e} > \frac{4\Delta+2t}{5\Delta-2t}$. It is easy to verify that for $\frac{3\Delta}{2} \geq \frac{t}{\Delta} \geq \frac{1}{2}\sqrt{22}-1$ we have $\frac{4\Delta+2t}{5\Delta-2t} \geq \frac{\Delta}{t-\Delta}$. Thus the claim follows. □

Proof of Theorem 3. Let G be a multigraph of maximum degree Δ. Assume Δ is even and G does not contain $\frac{\Delta}{2}K_3$ as a subgraph. If $\Delta \geq 8$ then $\frac{3\Delta}{2}-1 \geq \left(\frac{1}{2}\sqrt{22}-1\right)\Delta$ and the claim follows by Theorem 1. If $\Delta = 4$ then the claim follows by Lemma 19. So assume $\Delta = 6$. Let $t = \left\lfloor\frac{3\Delta}{2}\right\rfloor - 1 = 8$ and $k=5$ Repeating the argument from the proof of Theorem 1 we obtain that it is sufficient to show that $\rho(6,5,8) \geq \frac{6}{8-6}$. Let α, β and e be as in definition of $\rho(\Delta,k,t)$. We have $2\beta+6-\alpha \geq 6$ and $6 \geq \alpha+\beta$ and thus $12 \geq 2\alpha+2\beta \geq 3\alpha$. Hence $\alpha \leq 4$ and $\frac{3\Delta-\alpha}{2e} \geq \frac{18-4}{4} = \frac{7}{2}$. So $\rho(6,5,8) = \frac{7}{2} > \frac{6}{2}$ and the claim follows.

Now assume Δ is odd and G does not contain $\frac{\Delta-1}{2}K_3+e$ as a subgraph. Similarly as for even Δ, if $\Delta \geq 11$ then $\frac{3}{2}(\Delta-1) \geq \left(\frac{1}{2}\sqrt{22}-1\right)\Delta$ and the claim follows by Theorem 1. If $\Delta = 5$ then the claim follows by Lemma 19. Thus we only need to verify the claim for $\Delta = 7$ and $\Delta = 9$ and this can be achieved by showing respectively $\rho(7,5,9) \geq \frac{7}{9-7}$ and $\rho(9,7,12) \geq \frac{9}{12-9}$. If $\Delta = 7$ then $2\beta+7-\alpha \geq 6$ and $7 \geq \alpha+\beta$ and thus $14 \geq 2\alpha+2\beta \geq 3\alpha-1$. Hence $\alpha \leq 5$ and $e \leq 2$. It follows that $\frac{3\Delta-\alpha}{2e} \geq \frac{21-5}{4} = 4$, so $\rho(7,5,9) = \frac{7}{2}$. If $\Delta = 9$ then $2\beta+9-\alpha \geq 8$ and $9 \geq \alpha+\beta$ and thus $18 \geq 2\alpha+2\beta \geq 3\alpha-1$. Hence $\alpha \leq 6$ and $\frac{3\Delta-\alpha}{2e} \geq \frac{27-6}{6} = \frac{7}{2}$. So $\rho(7,5,9) = \frac{7}{2}$. □

Proof of Theorem 4. Let G be a connected multigraph of maximum degree Δ. Suppose Δ is even. Since $G \neq \frac{\Delta}{2}K_3$ and G is connected it follows that G does not contain $\frac{\Delta}{2}K_3$ as a subgraph. So the claim follows from Theorem 3.

Assume Δ is odd. If G consists of two copies of $\frac{\Delta-1}{2}K_3+e$ joined by an edge then we may color $2\Delta+1$ of the 3Δ edges of G and the claim follows. So assume that G does not consist of two copies of $\frac{\Delta-1}{2}K_3+e$ joined by an edge. Let $\mathcal{H}$ be the set of subgraphs of G isomorphic to $\frac{\Delta-1}{2}K_3+e$. Then each $H \in \mathcal{H}$ is a 2-edge-connected component of G joined with $G \setminus H$ by a single edge e_H. Let $G' = G\left[V(G) \setminus \bigcup_{H\in\mathcal{H}} V(H)\right]$ and $k = |\mathcal{H}|$. By Theorem 3 we may color at least $\frac{2\Delta}{3\Delta-3}|E(G')|$ edges of G'. The partial coloring of G' gives a partial coloring of G. For every $H \in \mathcal{H}$ we may additionally color e_H and Δ of the $\frac{3\Delta-1}{2}$ edges of H. Thus we have colored at least $\frac{2\Delta}{3\Delta-3}|E(G')| + k(\Delta+1) > \frac{2\Delta+1}{3\Delta}|E(G')| + k\frac{2\Delta+1}{3\Delta}\frac{3\Delta-1}{2} = \frac{2\Delta+1}{3\Delta}|E(G)|$ edges of G and the claim follows. □

5 Approximation Algorithms

Let $c_k(G)$ be the maximum number of edges of a k-edge-colorable subgraph of G. We use the following result of Kamiński and Kowalik.

Theorem 22 ([1]). *Let $\mathcal{G}$ be a family of graphs and let $\mathcal{F}$ be a k-normal family of graphs. Assume there is a polynomial-time algorithm which for every k-matching H of a graph in $\mathcal{G}$, such that $H \notin \mathcal{F}$ finds its k-edge colorable subgraph with at least $\alpha|E(H)|$ edges. Moreover, let*

$$\beta = \min_{\substack{A,B\in\mathcal{F} \\ A \text{ is not } k\text{-regular}}} \frac{c_k(A) + c_k(B) + 1}{|E(A)| + |E(B)| + 1} \quad \textit{and} \quad \gamma = \min_{A\in\mathcal{F}} \frac{c_k(A) + 1}{|E(A)| + 1}.$$

Then there is an approximation algorithm for the maximum k-ECS problem for graphs in $\mathcal{G}$ with approximation ratio $\min\{\alpha, \beta, \gamma\}$.

Since the definition of k-normal family is very technical, we refer the reader to [1] for its definition. As a direct consequence of Theorem 4 and Theorem 22 we get the following two results.

Theorem 23. *Let $k \geq 4$ be even. The maximum k-ECS problem has a $\frac{2k+2}{3k+2}$-approximation algorithm for multigraphs.*

Proof. Let $\mathcal{F} = \{\frac{k}{2}K_3\}$. It is easy to check that $\mathcal{F}$ is k-normal. Now we give the values of parameters α, β and γ from Theorem 22. By Theorem 4, $\alpha = \frac{2k}{3k-2}$. We have $c_k(\frac{k}{2}K_3) = k$ and $|E(\frac{k}{2}K_3)| = \frac{3k}{2}$. Hence, $\beta = \infty$ and $\gamma = \frac{2k+2}{3k+2}$. □

Theorem 24. *Let $k \geq 5$ be odd. The maximum k-ECS problem has a $\frac{2k+1}{3k}$-approximation algorithm for multigraphs.*

Proof. Let $\mathcal{F} = \{\frac{k-1}{2}K_3 + e\}$. It is easy to check that $\mathcal{F}$ is k-normal. Now we give the values of parameters α, β, γ and δ for Theorem 22. By Theorem 4, $\alpha = \frac{2k}{3(k-1)}$. We have $c_k(\frac{k-1}{2}K_3 + e) = k$ and $|E(\frac{k-1}{2}K_3 + e)| = \frac{3k-1}{2}$. Hence, $\beta = \frac{2k+1}{3k}$ and $\gamma = \frac{2k+2}{3k+1}$. □

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