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Isomorphic types of spaces of holomorphic functions of one variable

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Abstract

We survey results about isomorphic classification of spaces of holomorphic functions on certain domains in $\mathbb{C}$. In particular we state necessary and sufficient conditions a domain $U \subseteq \mathbb{C}$ must meet so that the space $H(U)$ is isomorphic to $H(\mathbb{D})$, $H(\mathbb{C})$ or $H(\mathbb{C}) \times H(\mathbb{D})$. Even in dimension one the picture is not complete – there are open domains $U \subseteq \mathbb{C}$ such that $H(U)$ does not fit into the classification above.

1 Basic definitions

1.1 Potential theory

The problem of isomorphic classification of the spaces of holomorphic functions turns out to be closely connected with potential theory. This section summarises the fundamental definitions and results needed in the paper. For the proofs, see [Ran95].

Let us remark, that as long as a subset of the Riemann sphere is not equal to the whole sphere, there is a conformal mapping under which the given set is contained in the complex plane. This way we can generalise certain notions defined for subsets of the plane onto the proper subsets of the Riemann sphere.

Let U be a domain in the Riemann sphere $\mathbb{C}_\infty$ and let $f : \partial U \to \mathbb{R}$ be a bounded function. Let us denote by $\Phi(f)$ the set of all real subharmonic functions φ on U such that

$$\forall \xi \in \partial U \quad \limsup_{z \to \xi} \varphi(z) \leq f(\xi).$$

We will call the function $u_f : U \to \mathbb{R}$ given by the formula

$$u_f(z) = \sup_{\varphi \in \Phi(f)} \varphi(z)$$

the **generalised solution of the Dirichlet problem**.

The function u_f is really a solution. In fact, the following theorem holds.

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Theorem 1.1. *[Ran95, 4.1.2] Let $U \subsetneq \mathbb{C}_\infty$ be a domain in the Riemann sphere and let $f : \partial U \to \mathbb{R}$ be a bounded function. Then the generalised solution of the Dirichlet problem $u_f : U \to \mathbb{R}$ is a harmonic function and*

$$\sup_U |u_f| \leq \sup_{\partial U} |f|.$$

The issue of u_f converging to f on the boundary is more delicate. Hence the next definition.

Let $U \subsetneq \mathbb{C}_\infty$ be a domain such that $\mathbb{C}_\infty \setminus U$ contains at least two points. If for every function $f : \partial U \to \mathbb{R}$ continuous at $\zeta_0 \in \partial U$ the condition

$$\lim_{z \to \zeta_0} u_f(z) = f(\zeta_0)$$

holds, then the point ζ_0 is called a **regular point**. If a point is not regular, we call it an **irregular point**.

If every point $\zeta \in \partial U$ is regular, we call a domain U a **regular set**, otherwise we call it **irregular**. The complex plane is not regular as a subset of $\mathbb{C}_\infty$. An open set in the Riemann sphere will be called regular if each of its components is regular. A compact set will be called regular if its complement is regular.

To state the result, we will also need the notion of capacity.

Let $K \subseteq \mathbb{C}$ be compact. If μ is a Borel measure, we define its energy by

$$I(\mu) = -\int_{\mathbb{C} \times \mathbb{C}} \ln |x - y| d\mu(x) d\mu(y).$$

Denote by Θ the set of all probabilistic measures with support contained in K. Then there exists (see [Ran95, 3.3.2]) a measure $\nu_K \in \Theta$ for which

$$\inf_{\mu \in \Theta} I(\mu) = I(\nu_K).$$

By the (logarithmic) **capacity** of K we will mean the number

$$\gamma(K) = e^{-I(\nu_K)}.$$

If $\gamma(K) = 0$, we will call K a **polar set**.

To get a better grasp of the introduced notions let us glance through a couple of theorems and examples.

Theorem 1.2. *[Ran95, 4.2.2] Let $U \subseteq \mathbb{C}_\infty$ be a domain and let $\zeta_0 \in \partial U$. If the component of ∂U containing ζ_0 consists of more than one point, then ζ_0 is a regular point.*

Theorem 1.3 (Kellog's theorem)**.** *[Ran95, 4.2.5] Let $U \subsetneq \mathbb{C}_\infty$ be a domain. The set of irregular boundary points of U is an F_σ polar set.*

Theorem 1.4. *[Ran95, 3.8.5] Every F_σ polar set is totally disconnected.*

Theorem 1.5. *[Ran95, 4.2.3] Let $K \subseteq \mathbb{C}_\infty$ be a compact polar set. Then every point of K is irregular for $\mathbb{C}_\infty \setminus K$.*

Theorem 1.6 (Evans' theorem)**.** *[Tsu59, III.27.] Let $K \subseteq \mathbb{C}$ be a compact polar set. Then there exists a subharmonic function ϕ on $\mathbb{C}$, such that*

$$K = \phi^{-1}(\{-\infty\}).$$

The converse is also true.

Theorem 1.7. *[Ran95, 3.5.1] Let $\phi \not\equiv -\infty$ be a function subharmonic on a domain U in $\mathbb{C}_\infty$. Then then the set $\phi^{-1}(\{-\infty\})$ is polar.*

There is a criterion that gives a full answer whether a given point is regular or not.

Theorem 1.8 (Wiener's criterion)**.** *[Ran95, 5.4.1, 4.2.4] Let K be an F_σ subset of $\mathbb{C}_\infty$ and let $\zeta_0 \in \partial K$. Let θ be a number with $0 < \theta < 1$. Define for $n \geq 1$*

$$F_n = \{z \in F : \theta^n < |z - \zeta_0| \leq \theta^{n-1}\}.$$

Then ζ_0 is a regular boundary point for $\mathbb{C}_\infty \setminus K$ if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\ln \frac{2}{\gamma(F_n)}} = +\infty.$$

Example 1.9. [Ran95, 5.3.7] This last criterion implies, for example, that the standard Cantor set is regular. At the same time if we construct a Cantor type set by throwing out not $\frac{1}{3}$ of each interval in n-th step (like in the standard case), but we throw out $1-2^{2^{-n}}$ in the n-th step – creating a much more sparse structure – we get a polar set.

The criterion also enables us to construct a set which will be the base of a counterexample further on.

Example 1.10. Let $D(a,r)$ denote a disc of radius r centred at a. The set

$$K = \bigcup_{n=2}^{\infty} D\left(e^{-n}, e^{-n^2}\right) \cup \{0\}$$

is neither polar nor regular. Every point on the boundary is regular except for 0. It can be easily derived from the Wiener's criterion as $\gamma\left(D(a,r)\right) = r$.

The set K' of diameters of circles from K, i.e.,

$$K' = \{0\} \cup \bigcup_{n=2}^{\infty} \left[e^{-n} - e^{-n^2}, e^{-n} + e^{-n^2}\right]$$

has the same properties.

1.2 Germs of holomorphic functions

Part of the statement of the theorem is simpler, when we consider compact subsets of $\mathbb{C}$ and spaces of germs of holomorphic functions over these compact set instead of spaces of holomorphic functions over open domains.

Let K be a compact subset of $\mathbb{C}$. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open sets such that $K=\bigcap_{n=0}^{\infty} U_n$ and $U_{n+1}\subset\subset U_n$. We consider the spaces $H^\infty(U_n)$ of bounded holomorphic functions on U_n. We set

$$H(K)=\bigcup_{n=0}^{\infty} H^\infty(U_n),$$

where we identify a function with its restriction to a smaller set. We call $H(K)$ the **space of germs of holomorphic functions**. We equip $H(K)$ with the the topology of the inductive limit, i.e. the strongest locally convex topology under which all the inclusions $i_n : H^\infty(U_n)\to H(K)$ are continuous ([MV04, Chapter 24]).

There is a duality between $H(U)$ and $H_0(\mathbb{C}_\infty\setminus U)$, where the subscript 0 means, that we take only those functions $f\in H(\mathbb{C}_\infty\setminus U)$, for which $f(\infty)=0$. This is the so-called Köthe-Grothendieck-da Silva duality, described in the following theorem (for details consult [LR84, Chapter 9]).

Theorem 1.11. *Let U be an open set in the complex plane. Then*

$$H(U)'\cong H_0(\mathbb{C}_\infty\setminus U).$$

Similarly, if K is a compact subset of $\mathbb{C}$, then

$$H(K)'\cong H_0(\mathbb{C}_\infty\setminus K).$$

The dual pairing for $f\in H(U)$ and $g\in H(\mathbb{C}_\infty\setminus U)$ is given by the formula:

$$\langle f,g\rangle=\int_\gamma f(z)g(z)dz,$$

where γ is a closed curve separating the singularities of f and g and contained in the common area of definition.

2 The survey

We are now ready to state the results.

Theorem 2.1. *Let K be a compact subset of the Riemann sphere $\mathbb{C}_\infty$ and let $U=\mathbb{C}_\infty\setminus K$. The following are equivalent:*

(*a*) *$H(K)\cong H\left(\overline{\mathbb{D}}\right)$ (as locally convex spaces);*

(*b*) *$H(U)\cong H(\mathbb{D})\cong\Lambda_1(n)$;*

(*c*) *$H(U)\cong\Lambda_1(\alpha_n)$ for some sequence (α_n);*

(*d*) *$H(U)$ has property $(\overline{\Omega})$ and U has a finite number of connected components;*

(*e*) *U has a finite number of connected components and is regular;*

(*f*) *U has a finite number of connected components and a positive harmonic function ϕ on U exists, such that for every $\zeta\in\partial U$*

$$\lim_{z\to\zeta}\phi(z)=0.$$

Definition 2.2. A Fréchet space E with the fundamental system of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ has the property (DN) if there exists a $p \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist an $n \in \mathbb{N}$ and a $C > 0$ such that for all $x \in E$

$$\|x\|_k^2 \le C \|x\|_p \|x\|_n .$$

Definition 2.3. A Fréchet space E with the fundamental system of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ has the property $(\overline{\Omega})$ if for every $p \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there is a $C > 0$ such that for all $y \in E'$

$$\|y\|_k^{*2} \le C \|y\|_p^* \|y\|_n^* .$$

Here stars mean we consider dual norms on E':

$$\|y\|_n^* = \sup\{|y(x)| : \|x\|_n \le 1\} \in \mathbb{R} \cup \{\infty\}.$$

Remark 2.4. For a systematic study of properties $(\overline{\Omega})$ and (DN) (and also (Ω) and $(\underline{DN})$) consult [MV04, Chapter 29].

Proof. (of Theorem 2.1)
$(a \Leftrightarrow b)$ This follows from the Köthe-Grothendieck-da Silva duality.
$(b \Leftrightarrow c)$ See [MH71, Lemma 2.1a].
$(b \Leftrightarrow d)$ See [Vog82, Proposition 7.5].
$(b \Leftrightarrow e)$ We will prove this later on.
$(e \Rightarrow f)$ For a regular open set we can always construct a function with these properties – this is the so-called Green's function (see Theorem 3.1).
$(f \Rightarrow e)$ This follows directly from theorem [Ran95, 4.1.5]. □

Theorem 2.5. *Let K be a compact subset of the Riemann sphere $\mathbb{C}_\infty$. The following are equivalent:*

(a) $H(K) \cong H(\{0\})$ *(as locally convex spaces);*

(b) $H(\mathbb{C}_\infty \setminus K) \cong H(\mathbb{C}) \cong \Lambda_\infty(n)$*;*

(c) $H(\mathbb{C}_\infty \setminus K)$ *has property* (DN)*;*

(d) K *is polar;*

(e) *strong Liouville's theorem holds on $\mathbb{C}_\infty \setminus K$, i.e. every function subharmonic on $\mathbb{C}_\infty \setminus K$ and bounded from above is constant.*

Proof.
$(a \Leftrightarrow b)$ This follows from the Köthe-Grothendieck-da Silva duality.
$(b \Leftrightarrow c)$ See [AKT89, Proposition 2.1]
$(b \Leftrightarrow d)$ See [Zah70]
$(d \Rightarrow e)$ See [Ran95, 3.6.7].
$(e \Rightarrow d)$ Follows from the existence of the function $-G_{\mathbb{C}_\infty \setminus K}$ which is negative and harmonic in $\mathbb{C}_\infty \setminus K$, therefore subharmonic (see Theorem 3.1). □

Theorem 2.6. *Let K be a compact subset of the Riemann sphere $\mathbb{C}_\infty$. Then $H(K) \cong H(\overline{\mathbb{D}}) \times H(\{0\})$ if and only if K can be decomposed into two non-empty disjoint compact subsets K_1 and K_2, such that $H(K_1) \cong H(\overline{\mathbb{D}})$ and $H(K_2) \cong H(\{0\})$.*

Proof. For a proof see [Zah70, Theorem 3]. □

Remark 2.7. If K is an irregular compact set of positive capacity, but the decomposition described in Theorem 2.6 is not possible, then $H(K)$ is isomorphic neither to $H(\overline{\mathbb{D}})$ nor to $H(\{0\})$ nor to $H(\overline{\mathbb{D}}) \times H(\{0\})$. An example of such a set is given in Example 1.10.

As for now it is not clear whether $H(K)$ in this case has a basis. It is known that the basis of $H(K)$ **cannot** consist of polynomials such that the n-th basis element has degree n. In particular there is no interpolating sequence (z_k) in K such that $L_n(f) \to f$ in $H(K)$, where $L_n(f)$ is a polynomial interpolating f in $z_0, z_1, \ldots, z_{n-1}$ (see [ZK80]).

Due to size constraints we will prove here only the claim ($b \Leftrightarrow e$) from Theorem 2.1. In fact we will show how to construct a basis of the space $H(K)$ when K is regular. To do this we still need to introduce a lot of necessary machinery.

3 Proof in the case of an open disk

3.1 Green's function

In the case of a simply connected domain $U \subseteq \mathbb{C}_\infty$ one can easily see that $H(U) \cong H(\mathbb{D})$ using the Riemann's Conformal Mapping Theorem. In a multiply connected case we need a similar tool. One can use the Green's function. The following function G_U is only a "slice" of the full-blown Green's function, but it will suffice.

Theorem 3.1. *[Ran95, 4.4.1, 4.4.2, 4.4.3, 4.4.9] Let U be a domain in the Riemann sphere such that $\infty \in U$. If ∂U is non-polar, there exists a function $G_U : U \to \mathbb{C}_\infty$, such that*

- *(i) G_U is harmonic on $U \setminus \infty$;*
- *(ii) $G_U(\infty) = \infty$;*
- *(iii) $G_U(z) > 0$ for all $z \in U$;*
- *(iv) $G_U(z) = \ln|z| + O(1)$ as $|z| \to \infty$;*
- *(v) for all points $\zeta \in \partial U$, we have that $\lim_{z\to\zeta} G_U(z) = 0$ if and only if ζ is a regular point.*

If, as usual, $K = \mathbb{C}_\infty \setminus U$, then we shall also write G_K instead of G_U. We shall also use the notation:

$$\phi_K(z) = e^{G_K(z)}.$$

In the following we shall write

$$K_R = \mathbb{C} \setminus \{z \in \mathbb{C} : \phi_K(z) > R\}$$

for $R > 1$. Observe that if K is regular, then Theorem 3.1 and properties of harmonic functions ensure us that $K = \bigcap_{R>1} K_R$ and $K_r \subset\subset \operatorname{int} K_R$ if $r < R$.

3.2 Approximation by polynomials

Let K be a compact subset of $\mathbb{C}$ and

$$M_n = \sup_{x_1,\ldots,x_n \in K} \left| \prod_{1 \le i < j \le k} (x_i - x_j) \right| .$$

Because of the compactness of K, it follows, that points $\rho_1^{(n)} \ldots, \rho_n^{(n)}$ exist such that

$$M_n = \left| \prod_{1 \le i < j \le k} \left(\rho_i^{(n)} - \rho_j^{(n)} \right) \right| .$$

We will call them **n-Fekete points**. For $\rho_1^{(1)}$ we take any point of K.

If we denote $\delta_n(K) = M_n^{\frac{1}{\binom{n}{2}}}$, then it can be (quite easily) shown that $\delta_1(K) \ge \delta_2(K) \ge \ldots$, so the sequence $(\delta_n(K))_{n=1}^{\infty}$ has a limit. We can say even more:

Theorem 3.2. *[Ran95, 5.5.2] Let $K \subseteq \mathbb{C}$ be a compact set, then*

$$\lim_{n \to \infty} \delta_n(K) = \gamma(K).$$

If K is compact and $\rho_1^{(n)} \ldots, \rho_n^{(n)}$ are n-Fekete points for K, then the polynomial

$$q_n(z) = \left(z - \rho_1^{(n)} \right) \left(z - \rho_2^{(n)} \right) \ldots \left(z - \rho_n^{(n)} \right)$$

will be called an **n-Fekete polynomial**.

We will use the notation

$$\|f\|_K = \sup_{z \in K} |f(z)|.$$

We have that:

Theorem 3.3. *[Ran95, 5.5.4] Let $K \subseteq \mathbb{C}$ be compact and let q_n be the n-th Fekete polynomial, then*

$$\gamma(K) \le \|q_n\|_K^{\frac{1}{n}} \le \delta_n(K).$$

The following lemma gives the property of the Fekete polynomials which is the most important for us:

Lemma 3.4 (Bernstein's Lemma)**.** *[Ran95, 5.5.7] Let K be a non-polar compact subset of $\mathbb{C}$. Let U be the component of $\mathbb{C}_\infty \setminus K$ which contains ∞, then*

$$\|q_n\|_K^{\frac{1}{n}} \phi_K(z) \left| \frac{\gamma(K)}{\delta_n(K)} \right|^{\tau_U(z,\infty)} \le |q_n(z)|^{\frac{1}{n}} \le \|q_n\|_K^{\frac{1}{n}} \phi_K(z)$$

for $z \in U \setminus \{\infty\}$ where τ_U is the so called Harnack's distance (see [Ran95, 1.3.4]).

From the Bernstein's Lemma, using Theorem 3.2 and 3.3, we get that

Corollary 3.5. *With the assumptions of the Bernstein's Lemma*

$$|q_n(z)|^{\frac{1}{n}} \to \gamma(K)\phi_K(z)$$

almost uniformly on U.

Proof. It follows easily as the function τ_U is continuous. □

If K is compact the points

$$\begin{aligned}\rho_1 &= \rho_1^{(1)},\\ \rho_2 &= \rho_1^{(2)},\\ \rho_3 &= \rho_2^{(2)},\\ \rho_4 &= \rho_1^{(3)},\\ &\dots\end{aligned}$$

will be called **Fekete points**. We will also write

$$\begin{aligned}&w_0(z) = 1;\\ &w_n(z) = (z-\rho_1)(z-\rho_2)\dots(z-\rho_n) \quad \text{for } n \geq 1.\end{aligned}$$

and call w_n the **Fekete polynomials**.

Lemma 3.6. *[Wal35, §7.2, Corollary 3] Let K be a compact subset of $\mathbb{C}$ such that $U = \mathbb{C}_\infty \setminus K$ is connected and regular. If $(w_n)_{n=0}^{\infty}$ are the Fekete polynomials for K, then*

$$|w_n(z)|^{\frac{1}{n}} \to \gamma(K)\phi_K(z)$$

almost uniformly on $U \setminus \{\infty\}$.

Proof. First we will show that

$$\left|\left(z-\rho_1^{(1)}\right)\left(z-\rho_1^{(2)}\right)\left(z-\rho_2^{(2)}\right)\left(z-\rho_1^{(3)}\right)\dots\left(z-\rho_n^{(n)}\right)\right|^{\frac{2}{n(n+1)}} \longrightarrow \gamma(K)\phi_K(z) \tag{1}$$

almost uniformly on $U \setminus \{\infty\}$.

We know (Corollary 3.5) that for the n-Fekete polynomials q_n

$$|q_n(z)|^{\frac{1}{n}} = \left|\left(z-\rho_1^{(n)}\right)\dots\left(z-\rho_n^{(n)}\right)\right|^{\frac{1}{n}} \to \gamma(K)\phi_K(z)$$

almost uniformly on $U \setminus \{\infty\}$. We take the sequence of logarithms

$$\ln\left|z-\rho_1^{(1)}\right|, \frac{\ln\left|z-\rho_1^{(2)}\right|\left|z-\rho_2^{(2)}\right|}{2}, \frac{\ln\left|z-\rho_1^{(2)}\right|\left|z-\rho_2^{(2)}\right|}{2},$$
$$\frac{\ln\left|z-\rho_1^{(3)}\right|\left|z-\rho_2^{(3)}\right|\left|z-\rho_3^{(3)}\right|}{3}, \frac{\ln\left|z-\rho_1^{(3)}\right|\left|z-\rho_2^{(3)}\right|\left|z-\rho_3^{(3)}\right|}{3}, \frac{\ln\left|z-\rho_1^{(3)}\right|\left|z-\rho_2^{(3)}\right|\left|z-\rho_3^{(3)}\right|}{3}, \dots$$

Considering the sequence of Cesàro means for this sequence, we get a sequence converging almost uniformly to $\ln(\gamma(K)\phi_K(z))$. Taking a suitable subsequence we get natural logarithms of the sequence (1). By the continuity of the exponential function, (1) follows.

In the general case we get

$$\frac{\ln\left|z-\rho_1^{(1)}\right|\dots\left|z-\rho_{n-1}^{(n-1)}\right|\left|z-\rho_1^{(n)}\right|\dots\left|z-\rho_k^{(n)}\right|}{k+\frac{(n-1)(n-2)}{2}}$$
$$=\underbrace{\frac{\ln\left|z-\rho_1^{(1)}\right|\dots\left|z-\rho_{n-1}^{(n-1)}\right|}{\frac{(n-1)(n-2)}{2}}}_{A(n)}\underbrace{\frac{\frac{(n-1)(n-2)}{2}}{k+\frac{(n-1)(n-2)}{2}}}_{B(n)}+\underbrace{\frac{\ln\left|z-\rho_1^{(n)}\right|\dots\left|z-\rho_k^{(n)}\right|}{k+\frac{(n-1)(n-2)}{2}}}_{C(n)}.$$

We've already shown, that the sequence $A(n)$ converges almost uniformly to the desired limit. $B(n)$ converges to 1, because we can always take $0 \le k < n$. On a compact set disjoint with K all the expressions $\ln\left|z-\rho_j^{(i)}\right|$ are uniformly bounded, so $C(n)$ converges almost uniformly to 0. Lemma follows. □

To prove the result, we will need the following lemma.

Lemma 3.7. *[Wal35, §4.6] Let K be a compact subset of the complex plane such that $U = \mathbb{C}_\infty \setminus K$ is connected and regular. If P is a polynomial of degree at most n and $\|P\|_K \le L$, then*

$$\|P\|_{K_R} \le LR^n.$$

Proof. Let P be a polynomial of degree at most n such that $\|P\|_K \le L$.

Let us consider an open disc $D \subseteq U$. The Green's function G_U is harmonic on D, so it has a harmonic conjugate H on D such that the function

$$F(z) = e^{G_U(z)+iH(z)}$$

is holomorphic on D. As $G_U > 0$ on U we have that also $\frac{P}{F^n}$ is holomorphic on D. Then the function $\left|\frac{P}{F^n}\right| = \frac{|P|}{e^{nG_U}}$ is subharmonic on D. It follows that $\frac{|P|}{e^{nG_U}}$ is subharmonic on $U \setminus \{\infty\}$.

Because U is regular, we have that

$$\lim_{z\to z_0} G_U(z) = 0$$

for all $z_0 \in \partial K$. Then

$$\lim_{z\to z_0} \frac{|P(z)|}{e^{nG_U(z)}} \le L.$$

From the properties of Green's function mentioned in Theorem 3.1 we get, that the function $\frac{|P|}{e^{nG_U}}$ is bounded from above on $U \setminus \{\infty\}$. From the Extended Maximum Principle (see [Ran95, 3.6.9]) we get, that

$$\frac{|P(z)|}{e^{nG_U(z)}} \le L \quad \text{for } z \in U \setminus \{\infty\}.$$

So $|P(z)| \le LR^n$ for $z \in \partial K_R$. The theorem now follows from the Maximum Principle. □

Theorem 3.8. *[Wal35, §4.6, Theorem 6] Let K be a compact subset of the complex plane such that $U = \mathbb{C}_\infty \setminus K$ is connected and regular. Let $f \in H(K)$ and let (p_n) be a sequence of polynomials such that p_n has a degree at most n and*

$$\|f - p_n\|_K \leq \frac{M}{R^n}$$

for some $R > 1$.

Then $p_n \to f$ almost uniformly on $\operatorname{int} K_R$.

Proof. We have that $\|p_n - p_{n+1}\|_K \leq \|p_n - f\|_K + \|f - p_{n+1}\|_K \leq \frac{M}{R^n} + \frac{M}{R^{n+1}} = \frac{1}{R^n}\left(M + \frac{M}{R}\right)$. So for $1 < r < R$ on behalf of Lemma 3.7

$$\|p_n - p_{n+1}\|_{K_r} \leq \frac{r^{n+1}}{R^n}\left(M + \frac{M}{R}\right).$$

By the Weierstrass criterion, the sequence (p_n) converges uniformly to a certain function g on K_r. But $p_n \to f$ on K and K must be infinite (otherwise it would not be regular), so from the Identity Theorem, $f = g$. ☐

Theorem 3.9. *[Wal35, §7.2, Theorem 2] Let K be a compact subset of the complex plane such that $U = \mathbb{C}_\infty \setminus K$ is connected and regular. Let $f \in H(K)$ and let (p_n) be a sequence of polynomials such that p_n is of degree at most n and for the sequence of Fekete points (ρ_j) we have*

$$\begin{aligned} p_n(\rho_1) &= f(\rho_1), \\ &\dots \\ p_n(\rho_{n+1}) &= f(\rho_{n+1}), \end{aligned} \tag{2}$$

where if a certain point ρ_s appears k times, we mean that not only the functions are equal at ρ_s, but also the $k-1$ first derivatives.

Then $p_n \to f$ in $H(K)$.

To prove this theorem (which is crucial to us), we need to know how to represent the polynomials p_n. This is the next lemma.

Lemma 3.10. *Under the assumtions of the Theorem 3.9, if $f \in H(K_{R+\varepsilon})$, then*

$$\begin{aligned} p_n(z) =& \frac{1}{2\pi i}\int_{\partial K_R} \frac{w_{n+1}(t) - w_{n+1}(z)}{w_{n+1}(t)(t-z)} f(t)dt, \\ f(z) - p_n(z) =& \frac{1}{2\pi i}\int_{\partial K_R} \frac{w_{n+1}(z)f(t)}{w_{n+1}(t)(t-z)} dt, \end{aligned}$$

for $z \in \operatorname{int} K_R$, where $(w_n)_{n=0}^\infty$ are the Fekete polynomials for K.

Proof. Let us denote

$$Q(z) = \frac{1}{2\pi i}\int_{\partial K_R} \frac{w_{n+1}(t) - w_{n+1}(z)}{w_{n+1}(t)(t-z)} f(t)dt.$$

Then Q is a polynomial of degree at most n, because

$$(t-z) \mid (w_{n+1}(t) - w_{n+1}(z)).$$

We have that

$$\begin{aligned}\frac{d^j}{dz^j}Q(z) &= \frac{1}{2\pi i}\int_{\partial K_R}\frac{d^j}{dz^j}\left(\frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)(t-z)}\right)f(t)dt\\ &= \frac{1}{2\pi i}\int_{\partial K_R}\left(\frac{j!f(t)}{(t-z)^{j+1}}-\frac{1}{w_{n+1}(t)}\sum_{m=0}^{j}\binom{j}{m}w_{n+1}^{(m)}(z)\frac{(j-m)!f(t)}{(t-z)^{j-m+1}}\right)dt.\end{aligned}$$

Let ρ_s appear k times in the sequence $\rho_1,\dots\rho_{n+1}$, then $w_{n+1}^{(m)}(\rho_s)=0$ for $0\le m<k$, so

$$\frac{d^j}{dz^j}Q(\rho_s)=\frac{1}{2\pi i}\int_{\partial K_R}\left(\frac{j!f(t)}{(t-\rho_s)^{j+1}}\right)dt=\frac{d^j}{dz^j}f(\rho_s)\quad\text{for } 0\le j<k.$$

Because there is only one polynomial p_n satisfying (2), we get that $p_n=Q$.

The formula

$$f(z)-p_n(z)=\frac{1}{2\pi i}\int_{\partial K_R}\frac{w_{n+1}(z)f(t)}{w_{n+1}(t)(t-z)}dt.$$

follows from what we've already proved and the Cauchy theorem.

□

Proof. (of Theorem 3.9)

Let $f\in H(K)$. Then by definition there is a $\rho>1$ such that $f\in H(\operatorname{int}K_\rho)$. We take r and R such that $1<r<R<\rho$.

By Lemma 3.10, we get that

$$f(z)-p_n(z)=\frac{1}{2\pi i}\int_{\partial K_R}\frac{w_{n+1}(z)f(t)}{w_{n+1}(t)(t-z)}dt\quad\text{for } z\in\operatorname{int}K_R.$$

From Corollary 3.6, it follows, that

$$\begin{aligned}|w_n|^{\frac{1}{n}}&\longrightarrow\gamma(K)r\ \text{ uniformly on } \partial K_r;\\ |w_n|^{\frac{1}{n}}&\longrightarrow\gamma(K)R\ \text{ uniformly on } \partial K_R,\end{aligned}$$

where $\gamma(K)>0$.

We choose $0<\delta<\frac{R-r}{3}$. We take n_0 big enough such that for $n>n_0$ we have:

- $|w_{n+1}|^{\frac{1}{n}}\le\gamma(K)(r+\delta)$ on ∂K_r;
- $|w_{n+1}|^{\frac{1}{n}}\ge\gamma(K)(R-\delta)$ on ∂K_R;
- $\left(\frac{\int_{\partial K_R}|f(t)|dt}{2\pi\operatorname{dist}(\partial K_r,\partial K_R)}\right)^{\frac{1}{n}}\frac{r+\delta}{R-\delta}\le\frac{r+2\delta}{R-\delta}$.

Then for $n \geq n_0$:

$$\begin{aligned}
\|f-p_n\|_K^{\frac{1}{n}} &\leq \|f-p_n\|_{K_r}^{\frac{1}{n}} \\
&= \left\| \frac{1}{2\pi i} \int_{\partial K_R} \frac{w_{n+1}(z) f(t)}{w_{n+1}(t)(t-z)} dt \right\|_{K_r}^{\frac{1}{n}} \\
&\leq \sup_{z \in K_r} \left(\frac{1}{2\pi} \int_{\partial K_R} \left| \frac{w_{n+1}(z) f(t)}{w_{n+1}(t)(t-z)} \right| dt \right)^{\frac{1}{n}} \\
&\leq \sup_{z \in K_r} \left(\frac{1}{2\pi} \int_{\partial K_R} \frac{|f(t)|}{|t-z|} dt \right)^{\frac{1}{n}} \frac{r+\delta}{R-\delta} \\
&\leq \left(\frac{\int_{\partial K_R} |f(t)| dt}{2\pi \operatorname{dist}(\partial K_r, \partial K_R)} \right)^{\frac{1}{n}} \frac{r+\delta}{R-\delta} \\
&\leq \frac{r+2\delta}{R-\delta} < 1
\end{aligned}$$

From Theorem 3.8 it now follows, that the sequence (p_n) converges to f in $H(K)$. □

Corollary 3.11. *Let K be a compact subset of $\mathbb{C}$ such that $U = \mathbb{C}_\infty \setminus K$ is connected and regular. The previously defined sequence $(w_n)_{n=0}^\infty$ of Fekete polyniomials is then a Schauder basis of the space $H(K)$.*

Proof. Let p_n be the sequence from Theorem 3.9. We have that $p_n \to f$. From Lemma 3.10 we get that for some $R > 1$

$$\begin{aligned}
p_{n+1}(z) - p_n(z) =& p_{n+1}(z) - f(z) - p_n(z) + f(z) \\
=& \frac{1}{2\pi i} \int_{\partial K_R} f(t) \left(\frac{w_{n+1}(z)}{w_{n+1}(t)(t-z)} - \frac{w_{n+2}(z)}{w_{n+2}(t)(t-z)} \right) dt \\
=& \frac{1}{2\pi i} \int_{\partial K_R} \frac{f(t)}{w_{n+2}(t)} \left(\frac{(t-\rho_{n+2}) - (z-\rho_{n+2})}{(t-z)} \right) dt \; w_{n+1}(z) \\
=& \frac{1}{2\pi i} \int_{\partial K_R} \frac{f(t)}{w_{n+2}(t)} dt \; w_{n+1}(z).
\end{aligned}$$

Then denoting

$$\xi_n(f) = \frac{1}{2\pi i} \int_{\partial K_R} \frac{f(t)}{w_{n+1}(t)} dt$$

and observing the fact, that

$$p_0(z) = \xi_0(f) w_0(z),$$

we get that

$$\begin{aligned}
\sum_{n=0}^{\infty} \xi_n(f) w_n &= \lim_{k\to\infty} \sum_{n=0}^{k} \xi_n(f) w_n \\
&= \lim_{k\to\infty} \left(p_0 + \sum_{n=0}^{k-1} (p_{n+1} - p_n) \right) = \lim_{k\to\infty} p_k = f.
\end{aligned}$$

We see instantly, that the coefficients $\xi_n(f)$ are unique. Indeed, assume that

$$\sum_{n=0}^{\infty} \xi_n w_n(z) = 0 \quad \text{in } H(K).$$

We consider the case when all the Fekete points are different. Observe that for all $n \geq 1$

$$w_n(\rho_n) = w_{n+1}(\rho_n) = \cdots = 0,$$

so we get by induction that

$$\begin{aligned} h(\rho_1) = 0 &\Rightarrow \xi_0 = 0 \\ h(\rho_2) = 0 &\Rightarrow \xi_1 = 0 \\ h(\rho_3) = 0 &\Rightarrow \xi_2 = 0 \\ &\dots \end{aligned}$$

If Fekete points are not different, then we get the same result if we take derivatives into account. □

3.3 Zaharjuta's proof

First we need a simple lemma.

Lemma 3.12. *Let $U \subseteq \mathbb{C}_\infty$ be an open set with an infinite number of components. Then $H(U) \ncong H(G)$, where G is open with a finite number of components.*

Proof. $H(U)$ contains a subspace isomorphic to the space ω of all sequences (by taking functions constant on each component). $H(G)$ does not contain such a subspace, because $H(G)$ (and all its subspaces) has a continuous norm, but ω does not have a continuous norm. □

Let us restate our goal.

Theorem 3.13. *[Zah70, Theorem 1] Let K be a compact (proper) subset of $\mathbb{C}_\infty$. The space $H(K)$ is isomorphic to $H\left(\overline{\mathbb{D}}\right)$ if and only if K is regular and $U = \mathbb{C}_\infty \setminus K$ has a finite number of components.*

Proof.
(⇐)
Let us consider first the case when U has only one component. From Corollary 3.11 and Lemma 3.6 we get that there exists a basis $(w_n)_{n=0}^{\infty}$ of $H(K)$ such that

$$\lim_{n\to\infty} |w_n(z)|^{\frac{1}{n}} = \gamma(K)\phi_K(z) \tag{3}$$

and the convergence is almost uniform on U. As K is regular we have from properties of the Green's function that $K = \bigcap_{\alpha>0} K_{e^\alpha}$ and

$$H(K) = \lim\operatorname{ind}_{\alpha>0} H^\infty\left(\operatorname{int} K_{e^\alpha}\right).$$

From (3) and the Maximum Principle for the polynomials w_n, we get that for each $\varepsilon > 0$ and sufficiently large n:

$$\gamma(K)^n e^{\alpha n} e^{-\varepsilon n} \le \|w_n\|_{K_{e^\alpha}} \le \gamma(K)^n e^{\alpha n} e^{\varepsilon n}.$$

Taking $\varepsilon = \frac{\alpha}{2}$ we get constants $L_1(\alpha), L_2(\alpha)$ such that for all n:

$$L_2(\alpha)\gamma(K)^n e^{\frac{\alpha n}{2}} \le \|w_n\|_{K_{e^\alpha}} \le L_1(\alpha)\gamma(K)^n e^{\frac{3\alpha n}{2}}. \tag{4}$$

Let

$$E_\alpha = \left\{ (\xi_n)_{n=0}^\infty \subseteq \mathbb{C} : |||(\xi_n)|||_\alpha = \sum_{n=0}^\infty |\xi_n| e^{\alpha n} < +\infty \right\}$$

and let $E = \bigcup_{\alpha > 0} E_\alpha$ with the topology of the inductive limit. One can easily see that $E \cong H\left(\overline{\mathbb{D}}\right)$.

We consider the mapping $T : E \to H(K)$ given by the formula

$$(\xi_n)_{n=0}^\infty \longmapsto f = \sum_{n=0}^\infty \frac{\xi_n}{\gamma(K)^n} w_n.$$

We will show that T is the desired isomorphism.

We have

$$\begin{aligned} \|T((\xi_n))\|_{K_{e^\alpha}} &= \left\| \sum_{n=0}^\infty \frac{\xi_n}{\gamma(K)^n} w_n \right\|_{K_{e^\alpha}} \\ &\le \sum_{n=0}^\infty |\xi_n| \gamma(K)^{-n} \|w_n\|_{K_{e^\alpha}} \le L_1(\alpha)\, |||(\xi_n)|||_{\frac{3\alpha}{2}}, \end{aligned}$$

so by [MV04, Proposition 24.7] and the definition of the inductive limit, T is continuous.

We will show now that T is surjective and the inverse mapping T^{-1} is continuous. Let us consider the sequence

$$s_n = \frac{1}{2\pi i} \frac{1}{w_{n+1}}.$$

As it can be easily seen, $(s_n) \subseteq H_0(U) = H(K)'$ is a sequence dual to (w_n) (i.e. $\langle s_n, w_m \rangle = \delta_{nm}$). Let $f \in H(K)$. We define for $f \in H(K_{e^\beta})$

$$\xi_n(f) = \gamma(K)^n \langle s_n, f \rangle = \frac{\gamma(K)^n}{2\pi i} \int_{\partial K_{e^\beta}} \frac{f(z)}{w_{n+1}(z)} dz.$$

By (3) there exists a function $L_3(\beta)$ such that

$$\inf_{z \in \partial K_{e^\beta}} |w_n(z)| \ge L_3(\beta)\gamma(K)^n e^{\frac{\beta n}{2}},$$

so

$$|\xi_n(f)| \le \frac{\gamma(K)^n \|f\|_{K_{e^\beta}}}{2\pi} \int_{\partial K_{e^\beta}} \frac{1}{|w_{n+1}(z)|} dz \le \frac{1}{2\pi\gamma(K) L_3(\beta)} e^{-\frac{\beta(n+1)}{2}} \|f\|_{K_{e^\beta}}.$$

We have

$$\begin{aligned}|||(\xi_n(f))|||_\alpha &= \sum_{n=0}^{\infty} |\xi_n(f)|e^{\alpha n} \\ &\le \frac{1}{2\pi\gamma(K)L_3(\beta)} \|f\|_{K_{e^\beta}} \sum_{n=0}^{\infty} e^{-\frac{\beta(n+1)}{2}+\alpha n} \\ &\le C(\alpha,\beta) \|f\|_{K_{e^\beta}},\end{aligned}$$

provided $\alpha < \frac{\beta}{2}$. Thus $(\xi_n(f)) \in E$, moreover $T((\xi_n(f))) = f$ and T^{-1} is also continuous. Then

$$H(K) \cong E \cong H(\overline{\mathbb{D}}).$$

By the Köthe-Grothendieck-da Silva duality it follows that $H(U) \cong H(\mathbb{D})$.

Let U now have r connected (and regular) components U_j. Then

$$H(U) \cong \prod_{j=1}^{r} H(U_j) \cong \prod_{j=1}^{r} H(\mathbb{D}).$$

So all we have to show is that

$$H(\mathbb{D}) \cong \prod_{j=1}^{r} H(\mathbb{D}).$$

Because we are talking about Fréchet spaces, all we have to do is find a continuous bijection. Let's take $S : \prod_{j=1}^{r} H(\mathbb{D}) \to H(\mathbb{D})$ given by the formula

$$\begin{aligned}(f_1, f_2, \ldots, f_r) &\mapsto f \\ f(z) &= f_1(z^r) + z f_2(z^r) + \ldots + z^{r-1} f_r(z^r).\end{aligned}$$

Directly from the definition, we get that

$$\|Sf\|_{D(0,R)} \le r \max_{1\le i\le r} \max_{z\in D(0,R^r)} |f_i(z)|.$$

So S is continuous and looking at power series of functions from $H(\mathbb{D})$ we instantly get that S a bijection.

$(\Rightarrow)$

Assume now that $H(K) \cong H\left(\overline{\mathbb{D}}\right)$. From Lemma 3.12 and the Köthe-Grothendieck-da Silva duality it follows that U has only a finite number of components. We have to show that U is regular.

There exists an isomorphism $T : H(\mathbb{D}) \to H(U)$.

Let $(U_s)_{s\in\mathbb{N}}$ be a sequence of open sets with a smooth boundry (therefore regular) such that $U = \bigcup_{s\in\mathbb{N}} U_s$ and $U_s \subset\subset U_{s+1}$. We take $h_k(z) = T(z^k)$, because T is an isomorphism, $(h_k)_{k=0}^{\infty}$ is a basis of $H(U)$. By the continuity of T and T^{-1} we get, that there exists a number s_0 such that for any $s \ge s_0$ there exist constants $r(s), R(s) < 1$ and $C_1(s), C_2(s) < \infty$ such that

$$C_2(s)r(s)^k \le \|h_k\|_{U_s} \le C_1(s)R(s)^k, \tag{5}$$

where we can assume that $r(s), R(s) \to 1$ as $s \to \infty$. As U has only a finite number of components, we can assume that s_0 is big enough for U_{s_0} to intersect all of those components.

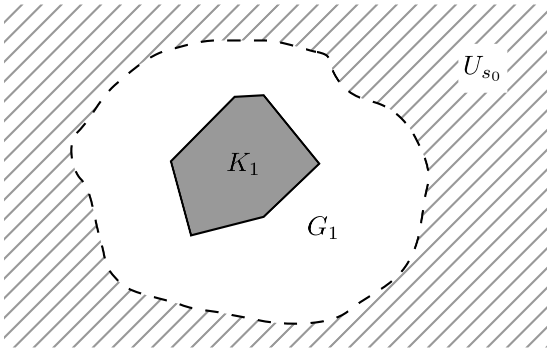

Figure 1: Sets in the proof

Consider the function

$$\psi(z) = \limsup_{\zeta\to z} \limsup_{k\to\infty} \frac{\ln|h_k(\zeta)|}{k}.$$

Function ψ is subharmonic, because $\sup_k \frac{\ln|h_k(\zeta)|}{k}$ is on behalf of (5) locally bounded above (see [Ran95, 3.4.3]).

Because of inequality (5) for any $\zeta \in U_s$ (where $s \geq s_0$) we get, that

$$\limsup_{k\to\infty} \frac{\ln|h_k(\zeta)|}{k} \leq \ln R(s) < 0,$$

so for all $z \in U$

$$\psi(z) \leq 0.$$

Moreover for $z \in U_{s_0}$

$$\psi(z) \leq \ln R(s_0) = \sigma < 0.$$

The set $G = \mathbb{C}_\infty \setminus \overline{U_{s_0}}$ is an open neighbourhood of K and we can assume it has only a finite number of components G_j. Assume that K is not regular. Then one of the sets $K_j = G_j \cap K$ is not regular. We can certainly assume that it is K_1. Figure 1 shows our current setting for visual reference.

Consider the generalised solution u_f for the following Dirichlet problem:

$$f(z) = \begin{cases} \sigma, & z \in \partial G_1 \\ 0, & z \in \partial K_1. \end{cases}$$

From the definition of u_f, Theorem 1.1 and the Maximum Principle, it follows that if $u_f(z) = 0$ for some $z \in G_1 \setminus K_1$ then $u_f \equiv 0$ on some component of $G_1 \setminus K_1$, so some points in $\partial G_1 \subseteq \partial U_{s_0}$ would not be regular (and that is not the case). Then it must be:

$$\psi(z) \leq u_f(z) < 0 \quad \text{for } z \in G_1 \setminus K_1.$$

By the definition of ψ there is a function $C(z)$ such that

$$|h_k(z)| \leq C(z)e^{ku_f(z)}. \tag{6}$$

The sequence (h_k) coresponds to the monomial basis in $H(\mathbb{D})$, so by the Cauchy-Hadamard Theorem it follows that for any $g \in H(U)$ and the expansion $g = \sum_{k=0}^{\infty} \xi_k(g) h_k$ we have

$$\limsup_{k\to\infty} |\xi_k(g)|^{\frac{1}{k}} \leq 1.$$

It means, that there exists a function $L(g, \delta)$ such that for any $\delta > 0$

$$|\xi_k(g)| \leq L(g, \delta) e^{\delta k}. \tag{7}$$

Because K_1 is not regular, we cannot have

$$\forall\, \zeta \in \partial K_1 \qquad \lim_{\substack{z\to\zeta \\ z\in G_1\setminus K_1}} u_f(z) = 0,$$

otherwise u_f would be a barrier (see [Ran95, 4.1.4]) in every point of ∂K_1 and by [Ran95, 4.1.5] K_1 would be regular.

So there is a point $\zeta_0 \in \partial K_1$ and a sequence $(z_n) \subseteq G_1$ such that $z_n \to \zeta_0$ and $u_f(z_n) \to 2\alpha_0 < 0$. We can certainly assume that $u_f(z_n) < \alpha_0$ for all n.

Taking δ so small that $\delta + \alpha_0 < 0$ we get from (6) and (7) that for any $g = \sum_{k=0}^{\infty} \xi_k(g) h_k \in H(U)$

$$|g(z_n)| \leq L(g, \delta) C(z_n) \sum_{k=0}^{\infty} e^{(\delta + u_f(z_n))k} = O(C(z_n)).$$

So we concluded that the values $|g(z_n)|$ are bounded (modulo a constant) by $C(z_n)$ for **each** function $g \in H(U)$. But that is not possible, by [Rud74, Theorem 15.13] it follows in particular that there is a function $g_0 \in H(U)$ such that $g_0(z_n) = nC(z_n)$. This contradiction we arrived at shows that K must be regular. □

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