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Involutions, moments and positivity

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Abstract

We give a proof of an internal condition on extending forms from certain subsets of a semigroup to positive definite forms defined on the semigroup. The proof itself was published in a series of papers and or main goal is to unify it under single notations. The theorem gives rise to some applications, dealing with moment-type problems, in particular the complex moment problem, presented later.

1 Introduction

1.1 Basic definitions and notions

Throughout we will use the symbols $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ in their usual meaning. In the sequel $\mathbb{N} = \{0, 1, 2 \ \ldots\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Definition 1. Let S be a semigroup. If there is an involution $(\cdot)^*$ defined on S with the property that for $s, t \in S$ we have $(st)^* = t^* s^*$, we will call S a **$*$-semigroup**. If S is also a linear space, we require the involution to be antilinear.

A subset T of S will be called symmetric if $T^* = T$.

The set of all hermitian elements in S will be denoted by S_h, i.e.,

$$S_h = \{s \in S : s^* = s\}.$$

Definition 2. Let $\mathcal{D}$ be a complex linear space, by $\mathcal{S}(\mathcal{D})$ we will denote the set of all sesquiliner forms on $\mathcal{D}$.

If S is a $*$-semigroup, then any function $\phi\colon S \to \mathcal{S}(\mathcal{D})$ will be called a **form** on S. The set of all forms with values in $\mathcal{S}(\mathcal{D})$ will be denoted $\mathrm{Form}(S, \mathcal{D})$. A form ϕ will be called **positive definite** if for any $m \geq 1$, $s_1, s_2, \ldots, s_m \in S$ and $h_1, h_2, \ldots, h_m \in \mathcal{D}$

$$\sum_{k,l=1}^{m} \phi(s_l^* s_k)(h_k, h_l) \geq 0.$$

Given a function $c\colon S \to \mathbb{C}$ we consider the form

$$\phi_c(s)(\xi, \eta) = c(s)\xi\overline{\eta}, \quad s \in S,\ \xi, \eta \in \mathbb{C}.$$

We call c positive definite if ϕ_c is positive definite.

This paper was prepared as a semester paper under the guidance of J. Stochel in the framework of joint PhD program ŚSDNM.

Definition 3. Let S be a $*$-semigroup. By $\mathcal{F}_S$ we will denote the complex linear space of all complex functions on S with finite support. If $T \subseteq S$, T not necessarily a $*$-subsemigroup, then $\mathcal{F}_T$ will denote the set of functions in $\mathcal{F}_S$ with support contained in T. Observe that $\mathcal{F}_S$ has a linear basis $\{\delta_s\}_{s\in S}$, where

$$\delta_s(t) = \begin{cases} 1, & t = s; \\ 0, & t \neq s. \end{cases}$$

We turn $\mathcal{F}_S$ into a $*$-algebra with convolution-like multiplication uniquely determined by

$$\delta_s \star \delta_t = \delta_{st}. \quad s, t \in S$$

and involution given by

$$(\delta_s)^* = \delta_{s^*}, \quad s \in S.$$

Remark 4. The subspace $\mathcal{F}_T$ is a $*$-subalgebra of $\mathcal{F}_S$ if and only if T is a $*$-subsemigroup of S.

Definition 5. Let S be a $*$-semigroup with a unit ε. A $*$-homomorphism $\chi\colon S \to \mathbb{C}$ ($\mathbb{C}$ considered a $*$-semigroup with multiplication as semigroup operation and conjugation as involution) will be called a **character** on S if $\chi(\varepsilon) = 1$. The set of all characters on S with pointwise multiplication and complex conjugation as involution (i.e., $\chi^* = \overline{\chi}$) forms a $*$-semigroup with a unit called the **dual $*$-semigroup** of S, which will be denoted by $\widehat{S}$. Each element $s \in S$ induces a character $\hat{s}$ on $\widehat{S}$ by evaluation:

$$\hat{s}(\chi) = \chi(s), \quad \chi \in \widehat{S}.$$

Remark 6. For any linear space X we treat the algebraic tensor product $\mathcal{F}_S \otimes X$ as a space of X-valued functions on S with finite support (i.e., functions that are non-zero on finite sets). If $\{e_\alpha\}_{\alpha\in I}$ is a linear basis of X, then elements of the form

$$\begin{aligned} &\delta_s \otimes e_\alpha\colon S \to X, && s \in S, \alpha \in I \\ &(\delta_s \otimes e_\alpha)(t) = \delta_s(t) e_\alpha, && s, t \in S, \alpha \in I \end{aligned}$$

constitute a linear basis of $\mathcal{F}_S \otimes X$.

If X is a $*$-algebra, we can define multiplication in $\mathcal{F}_S \otimes X$ by

$$(\delta_s \otimes e_\alpha) \star (\delta_t \otimes e_\beta) = \delta_{st} \otimes (e_\alpha \star e_\beta), \quad s, t \in S,\ i, j \in I$$

and involution

$$(\delta_s \otimes e_\alpha)^* = \delta_{s^*} \otimes e_\alpha^* \quad s \in S, \alpha \in I.$$

As before, for $T \subseteq X$, $\mathcal{F}_T \otimes X$ is the space of X-valued functions on S with finite support vanishing off T.

Definition 7. Let S be a $*$-semigroup and $Y \subseteq \widehat{S}$, by $\mathcal{P}(Y)$ we will denote the free linear space spanned by $\hat{s}|_Y$ for $s \in S$. We define multiplication in $\mathcal{P}(Y)$ by the formula:

$$\hat{s}|_Y \star \hat{t}|_Y = \widehat{st}|_Y, \quad s, t \in S,$$

and involution

$$(\hat{s})^* = \widehat{s^*}, \quad s \in S.$$

For $T \subseteq S$ we define $\mathcal{P}_T(Y) = \operatorname{span}\{\hat{s}|_Y : s \in T\}$. $\mathcal{P}_T(Y)$ may fail to be a $*$-subalgebra.

Definition 8. Let S be $*$-semigroup. A set $Y \subseteq \widehat{S}$ will be called **determining** for $\mathcal{P}(S)$ if the unique $*$-algebra homomorphism $\Delta_Y : \mathcal{F}_S \to \mathcal{P}(Y)$ satisfying

$$\Delta_Y(\delta_s) = \hat{s}|_Y, \quad s \in S$$

is a $*$-algebra isomorphism.

Remark 9. It turns out that if $Y \subseteq \widehat{S}$ is determining for $\mathcal{P}(S)$, then Y separates the points of S, i.e., if $\hat{s}|_Y = \hat{t}|_Y$, then $s = t$ (see [3, Prop. 2]), but in general this is not equivalent. If Y is determining and $T \subseteq S$, then it is clear that $\Delta_Y(\mathcal{F}_T) = \mathcal{P}_T(Y)$.

Definition 10. Let $\mathcal{D}$ be a complex linear space with an inner product $\langle \cdot, \cdot \rangle$. Let $f, g \in \mathcal{D}$. By $f \boxtimes g$ we will denote the one dimensional operator on $\mathcal{D}$ given by

$$(f \boxtimes g)h = \langle h, g \rangle f.$$

Let $\{e_\alpha\}_{\alpha \in I}$ be a linear basis of $\mathcal{D}$. By $F^{\#}(\mathcal{D})$ we will denote the linear span of $\{e_\alpha \boxtimes e_\beta\}_{\alpha,\beta \in I}$. We turn $F^{\#}(\mathcal{D})$ into a $*$-algebra with composition as multiplication and involution $(\cdot)^{\#}$ determined uniquely by

$$(e_\alpha \boxtimes e_\beta)^{\#} = e_\beta \boxtimes e_\alpha, \quad \alpha, \beta \in I.$$

One can easily check, that this definition does not depend on the choice of basis $\{e_\alpha\}$. One can also check by straightforward calculation, that $(AB)^{\#} = B^{\#} A^{\#}$, as expected. Observe that for $f, g \in \mathcal{D}$ we have that $f \boxtimes g \in F^{\#}(\mathcal{D})$.

Proposition 11. *Let $\mathcal{D}$ be a complex linear space with an inner product $\langle \cdot, \cdot \rangle$. We have the following:*

- $\lambda(f \boxtimes g) = \lambda f \boxtimes g = f \boxtimes \overline{\lambda} g$ *for* $\lambda \in \mathbb{C}$, $f, g \in \mathcal{D}$.
- $\langle Af, g \rangle = \langle f, A^{\#} g \rangle$ *for* $A \in F^{\#}(\mathcal{D})$, $f, g \in \mathcal{D}$.
- $(f \boxtimes g)^{\#} = g \boxtimes f$ *for* $f, g \in \mathcal{D}$.
- $(f \boxtimes g)(u \boxtimes v) = \langle u, g \rangle f \boxtimes v$ *for* $f, g, u, v \in \mathcal{D}$.
- *If* $A \in F^{\#}(\mathcal{D})$ *and* $\{\eta_j\}_{j=1}^n$ *is some orthonormal basis of* $A(\mathcal{D})$, *then* $A = \sum_{j=1}^n \eta_j \boxtimes A^{\#} \eta_j$.

Proof is mostly a matter of calculation and we omit this, reader may consult [5, Prop. 25].

1.2 Complete positivity

Definition 12. Let S be a $*$-semigroup and Y be a determining subset of $\widehat{S}$ and $T \subseteq S$. Let $\mathcal{D}$ be a complex linear space. $\mathcal{P}_T(Y, l^2)$ will denote the set of all functions from Y to l^2 of the form

$$\chi \mapsto \sum_{s \in T} \hat{s}(\chi) x(s), \quad x \in \mathcal{F}_T \otimes l^2.$$

We will denote

$$
\begin{aligned}
M^m(\mathcal{P}_T(Y)) &= \{[w_{k,l}]_{k,l=1}^m : w_{k,l} \in \mathcal{P}_T(Y)\} \\
M^m_+(\mathcal{P}_T(Y)) &= \{[w_{k,l}]_{k,l=1}^m \in M^m(\mathcal{P}_T(Y)) : \\
&\qquad \forall \chi \in Y \ \ [w_{k,l}(\chi)]_{k,l=1}^m \text{ is positive semidefinite}\} \\
M^m_f(\mathcal{P}_T(Y)) &= M^m(\mathcal{P}_T(Y)) \cap \{[\langle p_k(\cdot), p_l(\cdot)\rangle_{l^2}]_{k,l=1}^m : p_1, \ldots, p_m \in \mathcal{P}_T(Y, l^2)\}
\end{aligned}
$$

One can check that $M^m_f(\mathcal{P}_T(Y)) \subseteq M^m_+(\mathcal{P}_T(Y))$.

$$
\begin{aligned}
M^m(\mathcal{S}(\mathcal{D})) &= \{[\phi_{k,l}]_{k,l=1}^m : \phi_{k,l} \in \mathcal{S}(\mathcal{D})\} \\
M^m_+(\mathcal{S}(\mathcal{D})) &= \{[\phi_{k,l}]_{k,l=1}^m \in M^m(\mathcal{S}(\mathcal{D})) : \\
&\qquad \forall h_1, \ldots, h_m \in \mathcal{D} \ \ \sum_{k,l=1}^m \phi_{k,l}(h_k, h_l) \geq 0\}
\end{aligned}
$$

Sets $M^m_+(\mathcal{P}_T(Y)), M^m_f(\mathcal{P}_T(Y)), M^m_+(\mathcal{S}(\mathcal{D}))$ are convex cones in respective spaces.

Definition 13. Let S be $*$-semigroup and $T \subseteq S$. Let $\mathcal{D}$ be a complex linear space and $\phi \colon T \to \mathcal{S}(\mathcal{D})$. Suppose Y is a determining subset of $\widehat{S}$. By $\Lambda_{\phi,Y} : \mathcal{P}_T(Y) \to \mathcal{S}(\mathcal{D})$ we will denote the linear function satisfying for all $s \in T$

$$\Lambda_{\phi,Y}(\hat{s}|_Y) = \phi(s).$$

Observe that because Y in particular separates the points of S, this mapping is well defined. $\Lambda_{\phi,Y}$ gives rise to a function $\Lambda^{(m)}_{\phi,Y} \colon M^m(\mathcal{P}_T(Y)) \to M^m(\mathcal{S}(\mathcal{D}))$ by

$$\Lambda^{(m)}_{\phi,Y}([w_{k,l}]_{k,l=1}^m) = [\Lambda_{\phi,Y}(w_{k,l})]_{k,l=1}^m$$

Function $\Lambda_{\phi,Y}$ will be called **completely positive** if for every m

$$\Lambda^{(m)}_{\phi,Y}(M^m_+(\mathcal{P}_T(Y))) \subseteq M^m_+(\mathcal{S}(\mathcal{D})),$$

and **completely f-positive** if for every m

$$\Lambda^{(m)}_{\phi,Y}(M^m_f(\mathcal{P}_T(Y))) \subseteq M^m_+(\mathcal{S}(\mathcal{D}))$$

Remark 14. It is clear that complete positivity implies complete f-positivity. It should be noted, that complete f-positivity does not depend on the choice of the determining set Y (see [3, Prop. 10]).

2 Main theorem

2.1 Lemmas

These lemmas may seem a little bit out of place, but they will be needed later in the proof of the main theorem, they are given here in order not to break the flow of the proof.

Definition 15. Let S be a $*$-semigroup and $\mathcal{D}$ a complex linear space with an inner product. Let $e \in \mathcal{D}$. We define

$$\begin{aligned} \mathcal{B}_{S,\mathcal{D}} &= \operatorname{span}\{f^* \star f : f \in \mathcal{F}_S \otimes F^{\#}(\mathcal{D})\}, \\ \mathcal{B}^e_{S,\mathcal{D}} &= \operatorname{span}\{f^* \star f : f = \sum_{s\in S} \delta_s \otimes (e \boxtimes g(s)), g \in \mathcal{F}_S \otimes \mathcal{D}\}. \end{aligned}$$

One can easily check, that these are convex cones in $\mathcal{F}_S \otimes F^{\#}(\mathcal{D})$.

Lemma 16. *[5, Lemma 32] Let S be a $*$-semigroup and $\mathcal{D}$ a complex linear space with an inner product $\langle\cdot,\cdot\rangle$. If $\langle e, e\rangle = 1$, then*

$$\mathcal{B}_{S,\mathcal{D}} = \mathcal{B}^e_{S,\mathcal{D}}.$$

Proof. Only the inclusion $\mathcal{B}_{S,\mathcal{D}} \subseteq \mathcal{B}^e_{S,\mathcal{D}}$ needs to be justified. Take any $f \in \mathcal{F}_S \otimes \mathcal{F}^{\#}(\mathcal{D})$. We can write

$$f = \sum_{k=1}^{n} \delta_{s_k} \otimes A_k, \quad s_k \in S,\ A_k \in F^{\#}(\mathcal{D}).$$

Let $\{\eta_j\}_{j=1}^m$ be an orthonormal basis of $A_1(\mathcal{D}) + A_2(\mathcal{D}) + \ldots + A_n(\mathcal{D})$. Then $A_k = \sum_{j=1}^m \eta_j \boxtimes A_k^{\#}\eta_j$. Thus

$$\begin{aligned} f^* \star f &= \sum_{k,l=1}^{n} \left(\delta_{s_k^*} \otimes A_k^{\#}\right) \star (\delta_{s_l} \otimes A_l) \\ &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{m} \left(\delta_{s_k^*} \otimes (A_k^{\#}\eta_i \boxtimes \eta_i)\right) \star \left(\delta_{s_l} \otimes (\eta_j \boxtimes A_l^{\#}\eta_j)\right) \\ &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{m} \delta_{s_k^* s_l} \otimes \langle \eta_i, \eta_j\rangle (A_k^{\#}\eta_i \boxtimes A_l^{\#}\eta_j) \\ &= \sum_{i=1}^{m} \sum_{k,l=1}^{n} \left(\delta_{s_k^*} \otimes (A_k^{\#}\eta_i \boxtimes e)\right) \star \left(\delta_{s_l} \otimes (e \boxtimes A_l^{\#}\eta_i)\right) \\ &= \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \delta_{s_k} \otimes (e \boxtimes A_k^{\#}\eta_i)\right)^* \star \left(\sum_{l=1}^{n} \delta_{s_l} \otimes (e \boxtimes A_l^{\#}\eta_i)\right) \in \mathcal{B}^e_{S,\mathcal{D}}. \end{aligned}$$

□

Lemma 17. *[5, Lemma 33] Let S be a commutative $*$-semigroup with a unit ε and $\mathcal{D}$ a complex linear space with an inner product. Let T be a symmetric subset of S with $S_h \subseteq T$. Then*

$$(\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h = \mathcal{B}_{S,\mathcal{D}} + (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h.$$

Proof. Certainly $\mathcal{B}_{S,\mathcal{D}} \subseteq (\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h$ and $(\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h \subseteq (\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h$.

Let $f \in (\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h$. We will show that $f \in \mathcal{B}_{S,\mathcal{D}} + (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h$. Proof follows by induction on $|\operatorname{supp} f|$.

- If $|\operatorname{supp} f| = 1$, then $f = \delta_s \otimes A$ for some $s \in S$ and $A \in F^{\#}(\mathcal{D})$. But $f = f^*$, so $s = s^*$ and $A = A^{\#}$, therefore $s \in T$ and $f \in (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h$.

- If $|\operatorname{supp} f| = 2$, then we have two possibilities:
  - $f = \delta_s \otimes A + \delta_t \otimes B$, where $s = s^*, t = t^*, A = A^\#, B = B^\#$, therefore $f \in (\mathcal{F}_T \otimes F^\#(\mathcal{D}))_h$.
  - $f = \delta_s \otimes A + \delta_{s^*} \otimes A^\#$.
    Let $\{e_j\}_{j=1}^m$ be an orthonormal basis of $A(\mathcal{D})$. Take $P = \sum_{j=1}^m e_j \boxtimes e_j$ be the projection onto $A(\mathcal{D})$. We have that $P = P^\# = P^2$, $A = PA$ and $A^\# = A^\# P^\#$, so

    $$f = (\delta_\varepsilon \otimes P + \delta_s \otimes A)^* \star (\delta_\varepsilon \otimes P + \delta_s \otimes A) - \delta_\varepsilon \otimes P - \delta_{s^*s} \otimes A^\# A.$$

    By the assumptions on T this means that $f \in \mathcal{B}_{S,\mathcal{D}} + (\mathcal{F}_T \otimes F^\#(\mathcal{D}))_h$.
- Let $|\operatorname{supp} f| = n \geq 3$ and suppose our claim holds for g such that $|\operatorname{supp} g| < n$. There are two possibilities:
  - There are elements $u \in S$ and $A = A^\# \in \mathcal{F}^\#(\mathcal{D})$ such that $f = \delta_u \otimes A + f'$ and $|\operatorname{supp} f'| = n - 1$.
    Then $(\operatorname{supp} f \setminus \{u\})^* = \operatorname{supp} f \setminus \{u\}$ and hence $f' \in (\mathcal{F}_S \otimes F^\#(\mathcal{D}))_h$, which proves our claim in this case.
  - If for all $u \in \operatorname{supp} f$ we have that $u \neq u^*$, then take any $u \in \operatorname{supp} f$. Because * is an involution – in particular a bijection – it must happen that $(\operatorname{supp} f \setminus \{u, u^*\})^* = \operatorname{supp} f \setminus \{u, u^*\}$. Therefore

    $$f' \in \sum_{\substack{s \in S \\ s \neq u, u^*}} \delta_s \otimes f(s) \in (\mathcal{F}_S \otimes F^\#(\mathcal{D}))_h.$$

    But $f - f' = \delta_u \otimes f(u) + \delta_{u^*} \otimes f(u)^\# \in (\mathcal{F}_S \otimes F^\#(\mathcal{D}))_h$. So $f \in \mathcal{B}_{S,\mathcal{D}} + (\mathcal{F}_T \otimes F^\#(\mathcal{D}))_h$.

□

Lemma 18. *[5, Lemma 34(i)] Let S be a $*$-semigroup with a unit ε and $\mathcal{D}$ be a complex linear space with an inner product. Then the linear mapping $\iota\colon \operatorname{Form}(S, \mathcal{D}) \to (\mathcal{F}_S \otimes F^\#(\mathcal{D}))'$ given by*

$$\iota(\phi)(\delta_s \otimes (f \boxtimes g)) = \phi(s^*)(f, g), \quad \phi \in \operatorname{Form}(S, \mathcal{D}),\ s \in S,\ f, g \in \mathcal{D}$$

is an isomorphism.

Moreover, a form $\phi\colon S \to \mathcal{S}(\mathcal{D})$ is positive definite if and only if $\iota(\phi)(h) \geq 0$ for $h \in \mathcal{B}_{S,\mathcal{D}}$.

Proof. Let $\phi\colon S \to \mathcal{S}(\mathcal{D})$ and let $\{e_\alpha\}_{\alpha \in I}$ be a linear basis of $\mathcal{D}$. Then

$$\delta_s \otimes (e_\alpha \boxtimes e_\beta), \quad s \in S,\ \alpha, \beta \in I$$

is a basis of $\mathcal{F}_S \otimes F_\#(\mathcal{D})$. Therefore there is a unique linear functional $\iota(\phi) \in (\mathcal{F}_S \otimes F^\#(\mathcal{D}))'$ satisfying

$$\iota(\phi)(\delta_s \otimes (e_\alpha \boxtimes e_\beta)) = \phi(s^*)(e_\alpha, e_\beta).$$

The mapping $\phi \mapsto \iota(\phi)$ is linear and is a monomorphism. If $\Lambda \in (\mathcal{F}_S \otimes F^\#(\mathcal{D}))'$, then $\Lambda = \iota(\psi)$, where $\psi(s)(f, g) = \Lambda(\delta_{s^*} \otimes (f \boxtimes g))$, so ι is also an epimorphism.

As for the moreover part, observe that if $f \in \mathcal{F}_S \otimes \mathcal{D}$, then for any $e \in \mathcal{D}$ with $\langle e, e \rangle = 1$ we get that

$$\begin{aligned}
\sum_{s,t \in S} \phi(t^*s)(f(s), f(t)) &= \sum_{s,t \in S} \iota(\phi)[\delta_{s^*t} \otimes (f(s) \boxtimes f(t))] \\
&= \sum_{s,t \in S} \iota(\phi)[\delta_{s^*t} \otimes \langle e, e \rangle (f(s) \boxtimes f(t))] \\
&= \iota(\phi) \left[\sum_{s,t \in S} \delta_{s^*t} \otimes (f(s) \boxtimes e)(e \boxtimes f(t)) \right] \\
&= \iota(\phi) \left[\left(\sum_{s \in S} \delta_{s^*} \otimes (f(s) \boxtimes e) \right) \star \left(\sum_{t \in S} \delta_t \otimes (e \boxtimes f(t)) \right) \right] \\
&= \iota(\phi) \left[\left(\sum_{s \in S} \delta_s \otimes (e \boxtimes f(s)) \right)^* \star \left(\sum_{t \in S} \delta_t \otimes (e \boxtimes f(t)) \right) \right].
\end{aligned}$$

It follows that ϕ is positive definite if and only if $\iota(\phi)(h) \geq 0$ for $h \in \mathcal{B}^e_{S,\mathcal{D}}$. Our claim follows by Lemma 16. □

For convenience of the reader we state here a result attributed to Krein and M. Riesz:

Theorem 19. *[1, 2.2.7] Let M be a linear subspace of a real vector space E, and let P be a convex cone in E such that $M + P = E$. Then every linear functional $f\colon M \to \mathbb{R}$ which is nonnegative on $M \cap P$, can be extended to a linear functional $\tilde{f}\colon E \to \mathbb{R}$ which is nonnegative on P.*

Theorem 20. *[5, Thm. 27, 28, 29],[3, Lemma 13] Let S be a commutative $*$-semigroup with a unit ε. Let T be a symmetric subset of S with $S_h \subseteq T$ and let $Y \subset \widehat{S}$ be determining for S. Let $\mathcal{D}$ be a complex linear space and $\phi\colon T \to \mathcal{S}(\mathcal{D})$. Then the following conditions are equivalent:*

(i) Function ϕ can be extended to a positive definite form $\widetilde{\phi}\colon S \to \mathcal{S}(\mathcal{D})$.

(ii) For finite systems $\{e_i\}_{i=1}^m \subseteq \mathcal{D}$ and $\{\lambda_i\}_{i=1}^m \subseteq \mathcal{F}_S \otimes l_2$ with the property that for $u \in S \setminus T$ and $i, j = 1, \dots, m$

$$\sum_{\substack{s,t \in S \\ t^*s = u}} \langle \lambda_i(s), \lambda_j(t) \rangle_{l_2} = 0 \tag{1}$$

we have

$$\sum_{i,j=1}^{m} \sum_{\substack{s,t \in S \\ t^*s \in T}} \phi(t^*s)(e_i, e_j) \langle \lambda_i(s), \lambda_j(t) \rangle_{l_2} \geq 0. \tag{2}$$

(iii) For any finite systems $\{e_i\}_{i=1}^m \subseteq \mathcal{D}$ of linearly independent vectors and $\{\lambda_i\}_{i=1}^m \subseteq \mathcal{F}_S \otimes \mathbb{C}^n$ (where $n \geq m$) with the property that for $u \in S \setminus T$ and $i, j = 1, \dots, m$

$$\sum_{\substack{s,t \in S \\ t^*s = u}} \langle \lambda_i(s), \lambda_j(t) \rangle_{\mathbb{C}^n} = 0 \tag{3}$$

we have

$$\sum_{i,j=1}^{m} \sum_{\substack{s,t\in S \\ t^*s\in T}} \phi(t^*s)(e_i, e_j)\langle \lambda_i(s), \lambda_j(t)\rangle_{\mathbb{C}^n} \geq 0. \tag{4}$$

(*iv*) *For any inner product on $\mathcal{D}$ and for any finite system $\{f_i\}_{i=1}^n \subseteq \mathcal{F}_S \otimes \mathcal{D}$ with the property that for $u \in S \setminus T$ and $j = 1, \ldots, n$*

$$\sum_{i=1}^{n} \sum_{\substack{s,t\in S \\ t^*s=u}} f_i(s) \boxtimes f_i(t) = 0 \tag{5}$$

we have

$$\sum_{i=1}^{n} \sum_{\substack{s,t\in S \\ t^*s\in T}} \phi(t^*s)(f_i(s), f_i(t)) \geq 0. \tag{6}$$

(*v*) *Function $\Lambda_{\phi,Y}$ is completely f-positive.*

Proof.

(*i*) $\Rightarrow$ (*ii*)

Take $\{e_i\}_{i=1}^m \subseteq \mathcal{D}$ and $\{\lambda_i\}_{i=1}^m \subseteq \mathcal{F}_S \otimes l^2$ satisfying (1). For suitable elements $\{s_k\}_{k=1}^n$ and $\{p_{ik}\}_{i=1}^{m}{}_{k=1}^{n}$ we can write

$$\lambda_i = \sum_{k=1}^{n} \delta_{s_k} \otimes p_{ik}.$$

Let $\{\eta_w\}_{w=1}^d \subseteq l_2$ be any orthonormal basis of the finite dimensional space $\operatorname{span}\{p_{ik}\}$.

By (1) we have

$$\sum_{i,j=1}^{m} \sum_{\substack{s,t\in S \\ t^*s\in T}} \phi(t^*s)(e_i, e_j)\langle \lambda_i(s), \lambda_j(t)\rangle_{l_2}$$
$$= \sum_{i,j=1}^{m} \sum_{s,t\in S} \widetilde{\phi}(t^*s)(e_i, e_j)\langle \lambda_i(s), \lambda_j(t)\rangle_{l_2}.$$

But

$$\sum_{i,j=1}^{m} \sum_{s,t\in S} \widetilde{\phi}(t^*s)(e_i, e_j)\langle \lambda_i(s), \lambda_j(t)\rangle_{l_2}$$
$$= \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \widetilde{\phi}(s_l^* s_k)(e_i, e_j)\langle p_{ik}, p_{jl}\rangle_{l_2}$$
$$= \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \sum_{w=1}^{d} \widetilde{\phi}(s_l^* s_k)(e_i, e_j)\langle p_{ik}, \eta_w\rangle_{l_2} \overline{\langle p_{jl}, \eta_w\rangle_{l_2}}$$
$$= \sum_{w=1}^{d} \sum_{k,l=1}^{n} \widetilde{\phi}(s_l^* s_k) \left(\sum_{i=1}^{m} \langle p_{ik}, \eta_w\rangle_{l_2} e_i, \sum_{j=1}^{m} \langle p_{jl}, \eta_w\rangle_{l_2} e_j \right) \geq 0$$

by positive definiteness of $\widetilde{\phi}$.

$(ii) \Rightarrow (iii)$ This is obvious when one treats $\mathbb{C}^n$ as a subspace of l_2.

$(iii) \Rightarrow (iv)$

Fix any inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{D}$. Take any $\{f_i\}_{i=1}^n \subseteq \mathcal{F}_S \otimes \mathcal{D}$ satisfying (5). Let $\{e_j\}_{j=1}^m$ be any orthonormal basis of span $\bigcup_{i=1}^n f_i(S)$. We can assume that $n \geq m$. Define functions $\lambda_j \colon S \to \mathbb{C}^n$ $(j = 1, \ldots, m)$ by

$$\lambda_j(s) = (\langle f_1(s), e_j \rangle, \langle f_2(s), e_j \rangle, \ldots, \langle f_n(s), e_j \rangle)$$

Then for $u \in S$ we have that

$$\begin{aligned}
\sum_{\substack{s,t \in S \\ t^*s = u}} \sum_{i=1}^n f_i(s) \boxtimes f_i(t) &= \sum_{\substack{s,t \in S \\ t^*s = u}} \sum_{i=1}^n \left(\sum_{k=1}^m \langle f_i(s), e_k \rangle e_k \right) \boxtimes \left(\sum_{l=1}^m \langle f_i(s), e_l \rangle e_l \right) \\
&= \sum_{\substack{s,t \in S \\ t^*s = u}} \sum_{k,l=1}^m \sum_{i=1}^n \langle f_i(s), e_k \rangle \overline{\langle f_i(s), e_l \rangle} e_k \boxtimes e_l \\
&= \sum_{\substack{s,t \in S \\ t^*s = u}} \sum_{k,l=1}^m \langle \lambda_k(s), \lambda_l(t) \rangle_{\mathbb{C}^n} e_k \boxtimes e_l.
\end{aligned}$$

But the system $\{e_k \boxtimes e_l\}_{k,l=1}^m$ is linearly independent, so (5) implies (3). Since

$$\sum_{\substack{s,t \in S \\ t^*s \in T}} \sum_{i=1}^n \phi(t^*s)(f_i(s), f_i(t)) = \sum_{\substack{s,t \in S \\ t^*s \in T}} \sum_{k,l=1}^m \phi(t^*s)(e_k, e_l) \langle \lambda_k(s), \lambda_l(t) \rangle,$$

condition (4) implies (6).

$(iv) \Rightarrow (i)$

We will show that (iv) implies that a certain functional induced by ϕ is positive on a certain cone, then by Hahn-Banach type theorem we will be able to extend the functional, thus getting the desired extension $\widetilde{\phi}$.

From Lemma 16 we have that for any $e \in \mathcal{D}$ with $\langle e, e \rangle = 1$ we have that

$$\mathcal{B}_{S,\mathcal{D}} \cap \mathcal{F}_T \otimes F^{\#}(\mathcal{D}) = \mathcal{B}^e_{S,\mathcal{D}} \cap \mathcal{F}_T \otimes F^{\#}(\mathcal{D}).$$

Therefore if we take $f \in \mathcal{B}_{S,\mathcal{D}} \cap \mathcal{F}_T \otimes F^{\#}(\mathcal{D})$, then we can write $f = \sum_{i=1}^n g_i^* \star g_i$, where

$$g_i = \sum_{s \in S} \delta_s \otimes (e \boxtimes f_i(s)), \quad f_i \in \mathcal{F}_S \otimes \mathcal{D}.$$

So

$$\begin{aligned}
f &= \sum_{i=1}^n \left(\sum_{s \in S} \delta_s \otimes (e \boxtimes f_i(s)) \right)^* \star \left(\sum_{t \in S} \delta_s \otimes (e \boxtimes f_i(t)) \right) \\
&= \sum_{i=1}^n \sum_{s,t \in S} \delta_{s^*t} \otimes (f_i(s) \boxtimes e)(e \boxtimes f_i(t)) \\
&= \sum_{i=1}^n \sum_{s,t \in S} \delta_{s^*t} \otimes (f_i(s) \boxtimes f_i(t)).
\end{aligned}$$

But in particular $f \in \mathcal{F}_T \otimes F^{\#}(\mathcal{D})$, so $f(u) = f(u)^{\#} = 0$ for $u \in S \setminus T$, so the system $\{f_i\}_{i=1}^n$ satisfies condition (5). Let ι be the isomorphism from Lemma 18. Then

$$\begin{aligned}\iota(\phi)(f) &= \iota(\phi)\left(\sum_{i=1}^{n}\sum_{s,t\in S}\delta_{s^*t}\otimes(f_i(s)\boxtimes f_i(t))\right)\\ &= \iota(\phi)\left(\sum_{i=1}^{n}\sum_{\substack{s,t\in S\\ s^*t\in T}}\delta_{s^*t}\otimes(f_i(s)\boxtimes f_i(t))\right)\\ &= \sum_{i=1}^{n}\sum_{\substack{s,t\in S\\ s^*t\in T}}\phi(s^*t)(f_i(s),f_i(t)),\end{aligned}$$

hence $\iota(\phi)(f) \geq 0$ by (6).

We will show that $\iota(\phi)(f^*) = \overline{\iota(\phi)(f)}$ for $f \in \mathcal{F}_T \otimes F^{\#}(\mathcal{D})$. It suffices to show that for $s \in T$, $u, v \in \mathcal{D}$ we have:

$$\iota(\phi)((\delta_s \otimes (u \boxtimes v))^*) = \overline{\iota(\phi)(\delta_s \otimes (u \boxtimes v))}$$

Observe that for $\lambda, \mu \in \mathbb{C}$ a one-element system consisting of $f = \delta_\varepsilon \otimes \lambda u + \delta_s \otimes \mu v$ fulfills condition (5) by the properties of T. Then (6) gives

$$\begin{aligned}&\phi(\varepsilon)(\lambda u,\lambda u)+\phi(s)(\mu v,\lambda u)+\phi(s^*)(\lambda u,\mu v)+\phi(s^*s)(\mu v,\mu v)\\ &\quad=\phi(\varepsilon)(u,u)\lambda\overline{\lambda}+\phi(s)(v,u)\mu\overline{\lambda}+\phi(s^*)(u,v)\lambda\overline{\mu}+\phi(s^*s)(v,v)\mu\overline{\mu}\geq 0\end{aligned}$$

But this is true for any $\lambda, \mu \in \mathbb{C}$, taking (λ, μ) equal to $(0, 1)$, $(1, 0)$, $(1, 1)$ and $(1, i)$ yields that

$$\iota(\phi)((\delta_s \otimes (u \boxtimes v))^*) = \phi(s)(v,u) = \overline{\phi(s^*)(u,v)} = \overline{\iota(\phi)(\delta_s \otimes (u \boxtimes v))}$$

as required.

It follows that the functional $\iota(\phi)_h = \iota(\phi)|_{(\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h} \colon (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h \to \mathbb{R}$ is $\mathbb{R}$-linear. By Lemma 17 we have that

$$(\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h = \mathcal{B}_{S,\mathcal{D}} + (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h$$

We have already shown that $\iota(\phi)_h$ is positive on $\mathcal{B}_{S,\mathcal{D}} \cap \mathcal{F}_T \otimes F^{\#}(\mathcal{D}) = \mathcal{B}_{S,\mathcal{D}} \cap (\mathcal{F}_T \otimes F^{\#}(\mathcal{D}))_h$. Therefore assumptions of Theorem 19 are fulfilled and $\iota(\phi)_h$ can be extended to a functional $\widetilde{\iota(\phi)_h} \colon (\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h \to \mathbb{R}$ positive on $\mathcal{B}_{S,\mathcal{D}}$. As any element $f \in \mathcal{F}_S \otimes F^{\#}(\mathcal{D})$ can be written as

$$f = \frac{f + f^*}{2} + \frac{i(f - f^*)}{2i}$$

and $f + f^*, i(f - f^*) \in (\mathcal{F}_S \otimes F^{\#}(\mathcal{D}))_h$, the functional $\widetilde{\iota(\phi)_h}$ can be extended to a ($\mathbb{C}$-)linear functional $\widetilde{\iota(\phi)} \colon \mathcal{F}_S \otimes F^{\#}(\mathcal{D}) \to \mathbb{C}$ positive on $\mathcal{B}_{S,\mathcal{D}}$. By Lemma 18 $\widetilde{\phi} = \iota^{-1}(\widetilde{\iota(\phi)})$ is the desired positive definite extension of ϕ.

$(ii) \Rightarrow (v)$

Take $w = [\langle p_k(\cdot), p_l(\cdot)\rangle_{l^2}]_{k,l=1}^m \in M_f^m(\mathcal{P}_T(Y))$. Therefore we can find $x_1, \ldots, x_m \in \mathcal{F}_T \otimes l^2$ such that:

$$p_k(\chi) = \sum_{s\in S} \hat{s}(\chi) x_k(s), \quad \chi \in Y, k = 1, \ldots, m.$$

We have that for $k, l = 1, \ldots, m$ and all $\chi \in Y$:

$$\begin{aligned}
\langle p_k(\chi), p_l(\chi)\rangle_{l^2} &= \left\langle \sum_{s\in S} \chi(s) x_k(s), \sum_{t\in S} \chi(t) x_l(t) \right\rangle_{l^2} \\
&= \sum_{s,t\in S} \chi(t^*s) \left\langle x_k(s), x_l(t)\right\rangle_{l^2} \\
&= \sum_{u\in S} \chi(u) \sum_{\substack{s,t\in S \\ t^*s=u}} \left\langle x_k(s), x_l(t)\right\rangle_{l^2}
\end{aligned}$$

But $\langle p_k(\chi), p_l(\chi)\rangle_{l^2} \in \mathcal{P}_T(Y)$, so because Y is determining, we get that for $u \in S \setminus T$

$$\sum_{\substack{s,t\in S \\ t^*s=u}} \left\langle x_k(s), x_l(t)\right\rangle_{l^2} = 0,$$

which shows that the system $\{x_1, \ldots, x_m\} \subseteq \mathcal{F}_T \otimes l^2$ satisfies condition (1). Thus for any finite system $\{e_k\}_{k=1}^m$ we have that

$$\sum_{k,l=1}^m \Lambda_{\phi,Y}(\langle p_k(\cdot), p_l(\cdot)\rangle_{l^2})(e_k, e_l) =$$

$$\begin{aligned}
&= \sum_{k,l=1}^m \Lambda_{\phi,Y} \left(\sum_{u\in T} \sum_{\substack{s,t\in S \\ t^*s=u}} \left\langle x_k(s), x_l(t)\right\rangle_{l^2} \hat{u}|_Y \right) (e_k, e_l) \\
&= \sum_{k,l=1}^m \sum_{u\in T} \sum_{\substack{s,t\in S \\ t^*s=u}} \left\langle x_k(s), x_l(t)\right\rangle_{l^2} \phi(u)(e_k, e_l) \\
&= \sum_{k,l=1}^m \sum_{\substack{s,t\in S \\ t^*s\in T}} \phi(t^*s)(e_k, e_l) \left\langle x_k(s), x_l(t)\right\rangle_{l^2} \geq 0
\end{aligned}$$

by (ii), which shows that $\Lambda_{\phi,Y}^{(m)}(w) \in M_+^m(\mathcal{S}(\mathcal{D}))$, so $\Lambda_{\phi,Y}$ is completely f-positive.

$(v) \Rightarrow (ii)$ One gets a proof by reversing the above reasoning. $\square$

3 Perfectness

Definition 21. Let Σ be a σ-algebra of subsets of some set X. A mapping $\mu\colon \Sigma \to \mathcal{S}(\mathcal{D})$ will be called a **semispectral measure** if for any $f \in \mathcal{D}$ the mapping $\mu(\cdot)(f, f)\colon \Sigma \to \mathbb{R}$ is a finite and positive measure.

Next Lemma is the Naimark Dilation Theorem (e.g. see [4, Thm. 4.6]) (adapted to our setting).

Lemma 22. *Let Σ be a σ-algebra of subsets of some set X, $\mathcal{D}$ a complex linear space and let $\mu\colon \Sigma \to \mathcal{S}(\mathcal{D})$ be a semispectral measure. Then there is a Hilbert space $\mathcal{H}$, a linear mapping $V\colon \mathcal{D} \to \mathcal{H}$ and a spectral measure $\nu\colon \Sigma \to B(\mathcal{H})$ (so-called spectral dilation of μ) such that*

$$\mu(\sigma)(f,g) = \langle \nu(\sigma)Vf, Vg\rangle_{\mathcal{H}}$$

Proof. For any $\sigma \in \Sigma$ the form

$$[f,g] = i(\mu(\sigma)(f,g) - \overline{\mu(\sigma)(g,f)})$$

is hermitian on $\mathcal{D}$, so by Cauchy-Schwarz inequality we have that

$$|[f,g]| \leq \sqrt{[f,f][g,g]} = 0,$$

therefore $\mu(\sigma)(f,g)$ is a semi-inner product. Let $E = \mathcal{D}/\ker(\mu(X)(\cdot,\cdot))$. Monotonicity of measure implies that $\mu'\colon \Sigma \to \mathcal{S}(E)$ given by

$$\mu'(\sigma)([f],[g]) = \mu(\sigma)(f,g)$$

is well defined.

This part by courtesy of prof. Stochel. We can treat Σ as a $*$-semigroup, with intersection as multiplication and identity as involution. We show that μ (eqivalently μ') is positive definite. Take $\sigma_1,\dots,\sigma_n \in \Sigma$, $f_1,\dots,f_n \in \mathcal{D}$. We can find pairwise disjoint sets τ_j, $j = 1,\dots,m$ such that $\sigma_k = \sum_{j\in I_k} \tau_j$. Then

$$\begin{aligned}
\sum_{k,l=1}^{n} \mu'(\sigma_l \cap \sigma_k)([f_k],[f_l]) &= \sum_{k,l=1}^{n} \mu(\sigma_l \cap \sigma_k)(f_k,f_l)\\
&= \sum_{l,k=1}^{n} \sum_{i\in I_l} \sum_{j\in I_k} \mu(\tau_i \cap \tau_j)(f_k,f_l)\\
&= \sum_{l,k=1}^{n} \sum_{i,j=1}^{m} \chi_{I_l}(i)\chi_{I_k}(j)\mu(\tau_i \cap \tau_j)(f_k,f_l)\\
&= \sum_{l,k=1}^{n} \sum_{i=1}^{m} \chi_{I_l}(i)\chi_{I_k}(i)\mu(\tau_i)(f_k,f_l)\\
&= \sum_{l,k=1}^{n} \sum_{i=1}^{m} \mu(\tau_i)(\chi_{I_k}(i)f_k, \chi_{I_l}(i)f_l)\\
&= \sum_{i=1}^{m} \mu(\tau_i)\left(\sum_{k=1}^{n} \chi_{I_k}(i)f_k, \sum_{l=1}^{n} \chi_{I_l}(i)f_l\right) \geq 0.
\end{aligned}$$

Simply by motonicity of measure, we have that $\mu(\sigma \cap \tau)(f,f) \leq \mu(\sigma)(f,f)$, so boundedness condtition [6, (2.12)] is satisfied, and therefore μ' satisfies the assumptions of [6, Wniosek 2.12], which yields our claim. □

Definition 23. Let S be a commutative $*$-semigroup with a unit. By Σ_S we will denote the smallest σ-algebra of subsets of $\widehat{S}$ for which all the evaluations $\hat{s}$, $s \in S$, are measurable. S is called **operator semiperfect** if for any complex linear space $\mathcal{D}$ and any positive definite form $\phi\colon S \to \mathcal{S}(\mathcal{D})$ there exists a semispectral measure $\mu_\phi\colon \Sigma_S \to \mathcal{S}(\mathcal{D})$ such that

$$\phi(s) = \int_{\widehat{S}} \hat{s} d\mu_\phi,$$

i.e., for any $u, v \in \mathcal{D}$

$$\phi(s)(u,v) = \int_{\widehat{S}} \hat{s}(\chi) d\mu_\phi(\chi)(u,v).$$

From Definition 23 we get the following

Corollary 24. *Let S be an operator semiperfect $*$-semigroup. Let T be a symmetric subset of S with $S_h \subseteq T$. Let $\mathcal{D}$ be a complex linear space and $\phi\colon T \to \mathcal{S}(\mathcal{D})$. Then the following are equivalent:*

(i) *The form ϕ extends to a positive definite form $\widetilde{\phi}\colon S \to \mathcal{S}(\mathcal{D})$.*

(ii) *There exists a semispectral measure $\mu_\phi\colon \Sigma_S \to \mathcal{S}(\mathcal{D})$ with*

$$\phi(s) = \int_{\widehat{S}} \hat{s} d\mu_\phi, \quad s \in T.$$

Proof.

$(i) \Rightarrow (ii)$ This follows directly by the definition of semiperfectness.

$(ii) \Rightarrow (i)$ Take $s \in S$. Then $|\hat{s}|^2 = \hat{s}\overline{\hat{s}} = \hat{s}\widehat{s^*} = \widehat{ss^*}$, but $ss^* \in T$, so for each $f, g \in \mathcal{D}$ $\hat{s} \in L^2(\mu_\phi(\cdot)(f,g))$. As the complex measure $\mu_\phi(\cdot)(f,g)$ has finite variation, we have that $\hat{s} \in L^1(\mu_\phi(\cdot)(f,g))$ and we can define the extension by:

$$\widetilde{\phi}(s) = \int_{\widehat{S}} \hat{s} d\mu_\phi, \quad s \in S.$$

We show that $\widetilde{\phi}(s)$ is positive definite. Take any $s_1, \dots, s_m \in S$, $h_1, \dots, h_m \in D$. By Lemma 22 we get a Hilbert space $\mathcal{H}$, a linear mapping $V\colon \mathcal{D} \to \mathcal{H}$ and a spectral measure $\nu_\phi\colon \Sigma \to B(\mathcal{H})$ such that

$$\mu_\phi(\sigma)(f,g) = \langle \nu_\phi(\sigma) V f, V g \rangle_{\mathcal{H}}.$$

Therefore by equation (5), p. 134 and theorems 5.4.4, 5.4.5 from [2] we have that:

$$\begin{aligned}\sum_{k,l=1}^{m} \phi(s_l^* s_k)(h_k, h_l) &= \sum_{k,l=1}^{m} \int_{\widehat{S}} \widehat{s_l^* s_k}(\chi) d\mu_\phi(\chi)(h_k, h_l) \\ &= \sum_{k,l=1}^{m} \int_{\widehat{S}} \overline{\widehat{s_l}(\chi)} \widehat{s_k}(\chi) d\langle \nu_\phi(\chi) V h_k, V h_l \rangle_{\mathcal{H}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{m} \left\langle \left(\int_{\widehat{S}} \overline{\widehat{s_l}(\chi)} \widehat{s_k}(\chi) d\nu_\phi(\chi) \right) V h_k, V h_l \right\rangle_{\mathcal{H}} \\
&= \sum_{k,l=1}^{m} \left\langle \left(\int_{\widehat{S}} \overline{\widehat{s_l}(\chi)} d\nu_\phi(\chi) \right) \left(\int_{\widehat{S}} \widehat{s_k}(\chi) d\nu_\phi(\chi) \right) V h_k, V h_l \right\rangle_{\mathcal{H}} \\
&= \sum_{k,l=1}^{m} \left\langle \left(\int_{\widehat{S}} \widehat{s_k}(\chi) d\nu_\phi(\chi) \right) V h_k, \left(\int_{\widehat{S}} \widehat{s_l}(\chi) d\nu_\phi(\chi) \right) V h_l \right\rangle_{\mathcal{H}} \\
&= \left\| \sum_{k=1}^{m} \left(\int_{\widehat{S}} \widehat{s_k}(\chi) d\nu_\phi(\chi) \right) V h_k \right\|_{\mathcal{H}}^2 \geq 0.
\end{aligned}
$$

□

4 Application to complex moment problem

Assume we are given a double sequence $\{c_{m,n}\}_{m,n\geq 0}$ (or a subsequence thereof). Complex moment problem is the following question: does there exist a positive Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \overline{z}^n d\mu(z).$$

To use Theorem 20 we will be working with $*$-semigroups $\mathfrak{N}_+ = \{(m,n) : m,n \in \mathbb{Z},\ m+n \geq 0\}$ and $\mathfrak{N} = \{(m,n) : m,n \in \mathbb{N}\}$ with coordinatewise defined addition and involution:

$$(m,n)^* = (n,m).$$

Define a function $c\colon T \to \mathbb{C}$ by $c((m,n)) = c_{m,n}$.

We need to identify $\widehat{\mathfrak{N}_+}$. Let $u = (1,0)$ and $v = (1,-1)$. Take any $\chi \in \widehat{\mathfrak{N}_+}$. Let $z = \chi(u)$ and $b = \chi(v)$. Then $1 = \chi((0,0)) = \chi(v+v^*) = |b|^2$, so $|b| = 1$. Additionally $z = \chi(u) = \chi(u^* + v) = \overline{z}b$, so $b = z\overline{z}^{-1}$ or $z = 0$. Any element $(m,n) \in \mathfrak{N}_+$ can be written as

$$(m,n) = pu + qu^* + rv, \quad p,q \in \mathbb{N},\ r \in \mathbb{Z}.$$

Therefore we can identify elements of $\widehat{\mathfrak{N}_+}$ (at least algebraically) with pairs (z,b) satisfying the relations mentioned above. If $z \neq 0$, then $(z,b)((m,n)) = z^p\overline{z}^q b^r = z^m\overline{z}^n$. If $z = 0$, then $(0,b)((m,n)) = b^{m-n/2}$ if $m+n = 0$ and $(0,b)((m,n)) = 0$ if $m+n > 0$.

We have that

Theorem 25. *[5, Prop. 23] Semigroup $\mathfrak{N}_+$ is operator semiperfect, i.e., a form $\phi\colon S \to \mathcal{S}(\mathcal{D})$ is positive definite if and only if there are semispectral $\mathcal{S}(\mathcal{D})$-valued Borel measures μ on $\mathbb{C}^*$ and ν on $\mathbb{T}$ such that*

$$\phi((m,n))(f,g) = \int_{\mathbb{C}^*} z^m\overline{z}^n d\mu(z)(f,g) + \delta_{m+n,0} \int_{\mathbb{T}} b^m\overline{b}^n d\nu(b)(f,g)$$

Therefore, as any character $(z,b) \in \widehat{\mathfrak{N}_+}$ induces a character on $\mathfrak{N}$ by

$$(z,b)((m,n)) = z^m \overline{z}^n,$$

we get from Theorem 20 that:

Corollary 26. *Let $T \subseteq \mathfrak{N}$ and $\mathfrak{N}_h \subseteq T$. Then the following are equivalent:*

(*i*) *$\{c_{m,n}\}_{(m,n)\in T}$ is a (truncated) complex moment sequence, i.e.,*

$$c_{m,n} = \int_{\mathbb{C}} z^m \overline{z}^n d\mu(z), \quad (m,n) \in T$$

for some finite positive Borel measure μ.

(*ii*) *$\{\tilde{c}_{m,n}\}_{(m,n)\in\mathfrak{N}_+}$ extends $\{c_{m,n}\}_{(m,n)\in T}$ and is positive definite, i.e., for each finitely supported sequence $\{\lambda_{m,n}\}_{(m,n)\in\mathfrak{N}_+}$ we have that*

$$\sum_{\substack{(m,n)\in\mathfrak{N}_+\\(p,q)\in\mathfrak{N}_+}} \tilde{c}_{(p,q)^*+(m,n)}\lambda_{m,n}\overline{\lambda_{p,q}} = \sum_{\substack{(m,n)\in\mathfrak{N}_+\\(p,q)\in\mathfrak{N}_+}} \tilde{c}_{m+q,n+p}\lambda_{m,n}\overline{\lambda_{p,q}} \geq 0$$

(*iii*) *Mapping $\Lambda_{c,Y}$ is completely f-positive for any determing subset Y of $\widehat{\mathfrak{N}_+}$.*

We elaborate further on the last condition. Let $\Omega \subseteq \mathbb{C}^*$, denote $\mathcal{Y}_\Omega = \{(z, z\overline{z}^{-1}) : z \in \Omega\} \subseteq \widehat{\mathfrak{N}_+}$. By the above reasoning we get that if $T \subseteq \mathfrak{N}_+$, then $\mathcal{P}_T(\mathcal{Y}_\Omega)$ is the set of all the functions of the form

$$\sum_{(m,n)\in T} a_{m,n} z^m \overline{z}^n, \quad z \in \Omega, \tag{7}$$

where $\{a_{m,n}\}_{(m,n)\in T}$ has finite support.

From [3, Lemma 18] we get in particular, that $\mathcal{Y}_{\mathbb{C}^*}$ is a determining subset of $\widehat{\mathfrak{N}_+}$.

We quote here a result for complex-valued functions on semigroups:

Proposition 27. *Let S be a commutative $*$-semigroup, let $T \subseteq S$ be symmetric and $S_h \subseteq T$. Let $Y \subseteq \widehat{S}$ be determining and let $c\colon T \to \mathbb{C}$. Then $\Lambda_{c,Y}$ is completely f-positive if and only if $\Lambda_{c,Y}(p) \geq 0$ for every $p \in \mathcal{P}_T(Y)$ for which there are finaly many functions $q_1, \dots, q_n \in \mathcal{P}_S(Y)$ such that $p(\chi) = \sum_{i=1}^n |q_i(\chi)|^2$.*

Proof. This can be derived from the proof of [3, Thm. 15(iv)]. □

From this and (7) we get that

Corollary 28. *Let $T \subseteq \mathfrak{N}$ and $\mathfrak{N}_h \subseteq T$. Then $\{c_{m,n}\}_{(m,n)\in T}$ is a complex moment sequence if and only if $\sum_{(m,n)\in T} a_{m,n}c_{m,n} \geq 0$ for any finite system $\{a_{m,n}\}_{(m,n)\in T}$ of complex numbers for which there are finitely many functions $q_1, \dots, q_k \in \mathcal{P}_{\mathfrak{N}_+}(\mathcal{Y}_{\mathbb{C}^*})$ such that $\sum_{(m,n)\in T} a_{m,n}z^m\overline{z}^n = \sum_{i=1}^k |q_i(z)|^2$ for $z \neq 0$.*

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