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Michał Goliński

Uniwersytet A. Mickiewicza w Poznaniu

Invariant subspace problem for the space of rapidly decreasing functions

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Opiekun pracy: Paweł Domański

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Abstract

We present how to construct a continuous linear operator without invariant subspaces on the Schwartz space of rapidly decreasing functions and its dual – the space of tempered distributions.

Construction is carried out by representing the space as a Köthe sequence space and using methods developed by Read for constructing his famous example on the space l_1.

1 Introduction

Existence of non-trivial invariant subspaces for operators on finite dimensional spaces is a simple consequence of the fundamental theorem of algebra, but operators on infinite dimensional spaces pose a much more difficult problem.

The question of existence of non-trivial invariant subspace for continuous operators on the Hilbert space l_2 was posed by von Neumann in the 1930s, who gave an affirmative answer for compact operators, this result was never published. Later many mathematicians have found invariant subspaces for some classes of Hilbert space operators – see [6], [3].

First example of an operator an a Banach space without non-trivial invariant subspaces was given by Enflo in 1976, construction published much later in [4]. Subsequent counterexamples in this direction were given by Read in [8]. Both examples concern artificial, *ad-hoc* constructed spaces. Later Read refined his method and proved, that an operator with much stronger property – not having non-trivial invariant subsets – can be constructed on a large class of (non-reflexive) Banach spaces, including the space l_1, see [10].

In the meantime, Atzmon (see [1]), using other ideas, constructed an operator without invariant subspaces on some (once again artificial) nuclear Fréchet space.

In this paper, using the ideas from the papers of Read, we construct a continuous linear operator without non-trivial invariant subspaces on the space s of rapidly decreasing sequences. This space is important in the structural theory of nuclear spaces (Kōmura-Kōmura theorem, see e.g. [7, 29.8]) and many non-Banach Fréchet spaces of analysis are isomorphic to s as Fréchet spaces.

A similar result along the same lines the author proved for a wider class of so-called Köthe sequence spaces (of which s is an example), see [5], but the proof is a little more complicated and outside the scope of this paper.

2 The space s

We write $\mathbb{N}$ for the the set on non-negative integers and $\mathbb{N}_+$ for the set of positive integers.

Let $\mathbb{K}$ be the scalar field ($\mathbb{R}$ or $\mathbb{C}$). We define

$$s = \left\{ (x_j)_{j=0}^{\infty} \in \mathbb{K}^{\mathbb{N}} : |x|_N = \sum_{j=0}^{\infty} |x_j|(j+1)^N < +\infty \quad \text{for all } N \in \mathbb{N}_+ \right\}$$

and call s the *space of rapidly decreasing sequences.* When endowed with the locally convex topology given by the seminorms $(|\cdot|_N)$, s becomes a nuclear Fréchet space – see [7, 29.4.1].

Many spaces in analysis are in fact isomorphic to s, including

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- the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$, where

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} \left(1+|x|^2\right)^N |f(x)| < +\infty \quad \text{for all } N \in \mathbb{N}_+ \right\},$$

see [7, 31.14]

- the space $C^\infty_{2\pi}(\mathbb{R})$ of periodic smooth functions, see [7, 29.5.(1)];
- the space $C^\infty[0,1]$, for a direct proof see [7, 29.5.(4)];
- the space $C^\infty(K)$ for each compact C^∞-manifold K – see [12];
- the space $\mathcal{D}(K)$ of smooth functions with their support contained in a compact set $K \subset \mathbb{R}^n$, when K has a nonempty interior – see [12];
- the space of all entire Dirichlet series, i.e. space of sequences (a_n) such that the series

$$\sum_{n=1}^{\infty} a_n n^{z_1+\ldots+z_d}$$

is convergent for any $(z_1, \ldots, z_d) \in \mathbb{C}^d$, see [11, 8.4.1].

- the space $A^\infty(\mathbb{D})$ of holomorphic functions on the unit disk with the property that all of their derivatives can be extended to continuous functions on $\overline{\mathbb{D}}$. This follows easily from the fact that $C^\infty_{2\pi}(\mathbb{R}) \cong s$.

Our main result is:

Theorem 1. *There exists a continuous linear operator $T\colon s \to s$ with no non-trivial invariant subspaces.*

The dual space s' is also of importance, as it is isomorphic to the space of tempered distributions. For reflexive spaces we have the following

Fact 2. *If X is a reflexive locally convex space and $T\colon X \to X$ is linear and continuous and has no non-trivial invariant subspaces, then $T'\colon X' \to X'$ has no non-trivial invariant subspaces.*

Proof. With the usual identification in mind, we have that $T'' = T$. It is easy to check that if M is an non-trivial invariant subspace for T, then $M^\perp$ is a non-trivial invariant subspace for T'. This yields our claim. □

As s is reflexive, by Theorem 1 we get

Corollary 3. *There exists a continuous linear operator $T\colon s' \to s'$ without non-trivial invariant subspaces.*

3 Preliminaries

Throughout we will denote by c_{00} the space of all finite sequences – a linear subspace of $\mathbb{K}^\mathbb{N}$. Canonical basis of c_{00} will be denoted by $(e_0, e_1, e_2, \ldots)$ and $E_n = \operatorname{span}\{e_0, e_1, \ldots, e_n\}$.

For a linear basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of E_n and

$$E_n \ni x = \sum_{i=k}^{n} x_i \gamma_i, \quad x_k \neq 0$$

we write $\operatorname{val}_{\widetilde{\gamma}}(x) = k$. For a set $K \subseteq E_n$ we write $\operatorname{val}_{\widetilde{\gamma}}(K) = \sup_{y\in K} \operatorname{val}_{\widetilde{\gamma}}(y)$.

Remark 4. Observe that if $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ and $\widetilde{\mu} = (\gamma_0, \gamma_1, \ldots, \gamma_n, \mu_{n+1}, \ldots, \mu_m)$ are bases of E_n and E_m respectively, then for $y \in E_n$

$$\operatorname{val}_{\widetilde{\gamma}}(y) = \operatorname{val}_{\widetilde{\mu}}(y).$$

The following simple observation will be important for us:

Proposition 5. *Let E be linear topological space and let $T\colon E \to E$ be a continuous linear operator. Let y by cyclic vector with respect to T. Then for every vector $x \in E$, if $y \in \overline{\operatorname{span}\{x, Tx, T^2x, \ldots\}}$, then x is also cyclic with respect to T.*

Proof. Take any element $z \in E$ and any neighbourhood U of z. Because y is cyclic, for some polynomial we have that $P(T)y \in U$. But $P(T)$ is also continuous, so for some neighbourhood V of y we have that $P(T)(V) \subseteq U$. By hypothesis, we can find a polynomial Q such that $Q(T)x \in V$. Then $(P \circ Q)(T)x \in U$. □

Further on, for a polynomial $P \in \mathbb{K}[t]$ with $P(t) = \sum_{l \in M} p_l t^l$ we will write shortly that $\operatorname{sc} P \subseteq M$ (sc stands for the 'support of the sequence of coefficients') and $|P| = \sum_{l \in Z} |p_l|$.

Observe that for each $N \in \mathbb{N}_+$ the sequence $\left(\left(\frac{j+1}{j}\right)^N\right)_j$ is decreasing and converges to 1, therefore there exists an increasing sequence $(k_N)_{N \in \mathbb{N}_+}$, $k_1 = 3$, with the property that

$$\forall N \in \mathbb{N}_+ \quad \sup_{j \in \mathbb{N}} \frac{(j + k_N + 1)^N}{(j + k_N)^N} = \frac{(k_N + 1)^N}{k_N^N} \leq \frac{3}{2}. \tag{1}$$

Clearly,

$$\frac{(j + k_N)^N}{(j + k_{N+1})^{N+1}} \leq \frac{1}{j + k_{N+1}} < \frac{1}{2} \tag{2}$$

for all $j \in \mathbb{N}, N \in \mathbb{N}_+$

Further on, we write

$$\|(x_j)\|_N = \sum_{j=0}^{\infty} A_{N,j} x_j.$$

For further reference let us restate condition (1):

$$\forall N \in \mathbb{N}_+ \;\; \forall j \in \mathbb{N} \quad \frac{A_{N,j+1}}{A_{N,j}} \leq \frac{3}{2}. \tag{3}$$

One can easily check that each seminorm $\|\cdot\|_N$ is equivalent to $|\cdot|_N$, thereofre $(s, |\cdot|_N)$ is isomorphic to $(s, \|\cdot\|_N)$. Moreover, unit balls of seminorms $\|\cdot\|_N$ form a fundamental system of neighbourhoods of zero by (2).

4 Perturbed weighted forward shifts

A sequence $\widetilde{\mu} = (\mu_0, \mu_1, \mu_2, \ldots) \subset c_{00}$ (finite or infinite) will be called a *perturbed canonical basis* if $\mu_j = \sum_{i=0}^{j} \gamma_{i,j} e_i$, where $\gamma_{j,j} \neq 0$. Similarly, we will call an operator $T\colon c_{00} \to c_{00}$ a *perturbed weighted forward shift* if for any $j \in \mathbb{N}_+$

$$T^j e_0 = \sum_{i=0}^{j} \gamma_{i,j} e_i, \quad \text{where } \gamma_{j,j} \neq 0.$$

We will work with perturbed weighted forward shifts of a very specific form.

Proposition 6. *Assume that a linear map $T\colon c_{00} \to c_{00}$ satisfies for all $j \in \mathbb{N}$*

$$T^j e_0 = \delta_j e_j + \sigma_j T^{j - p_j} e_0, \quad \textit{for } j = 1, 2, 3, \ldots, \tag{4}$$

where $(\delta_j) \subset \mathbb{R} \setminus \{0\}$, $(\sigma_j) \subset \mathbb{R}$, $(p_j) \subset \mathbb{N}$ and $p_j \leq j$. Then

(i) the operator T is a perturbed weighted forward shift;

(ii) vectors $T^j e_0$ are linearly independent;

(iii) $\operatorname{span}\{e_0, Te_0, \ldots, T^n e_0\} = \operatorname{span}\{e_0, e_1, \ldots, e_n\}$*;*

(iv) $\operatorname{span}\{e_0, Te_0, T^2 e_0, \ldots\} = c_{00}$*;*

(v) $e_j = \frac{1}{\delta_j}\left(T^j e_0 - \sigma_j T^{j-p_j} e_0\right)$;

(vi) $Te_j = \frac{\delta_{j+1}}{\delta_j} e_{j+1} + \frac{\sigma_{j+1}}{\delta_j} T^{j+1-p_{j+1}} e_0 - \frac{\sigma_j}{\delta_j} T^{j+1-p_j} e_0$.

Proof. Claims (i), (ii), (iii) and (iv) are obvious because the numbers δ_j in (4) are all non-zero. Claim (v) follows easily from (4). Applying T to (v) yields (vi). □

5 The lemma

For pertubed weighted forward shifts we have the following lemma, which will be crucial for us. This lemma can be traced to [9, Lemma 5.1], but our formulation, exposition and the proof is an almost verbatim repetition of that in [2, Chapter 12]. We give a complete proof for the convenience of the reader.

Lemma 7. *Assume that for some integers $a, \Delta > 0$ there is given a perturbed canonical basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{a+\Delta-1})$ of $E_{a+\Delta-1}$, with $\gamma_a = \varepsilon e_a + e_0$ and $\gamma_0 = e_0$. Let $\|\cdot\|$ be any weighted l_1-norm on c_{00} and K be any compact set with respect to the topology induced by $\|\cdot\|$ satisfying*

$$K \subseteq \{y \in E_{a+\Delta-1} : \mathrm{val}_{\widetilde{\gamma}}(y) \leq a\}.$$

Let $v := a - \mathrm{val}_{\widetilde{\gamma}}(K)$. Then there is a constant $C = C(\widetilde{\gamma}, K, \|\cdot\|)$ such that for any $y \in K$ there is a polynomial P with $\mathrm{sc}\, P \subseteq [v, a+\Delta)$ and $|P| \leq C$ such that for each perturbed weighted forward shift $T \colon c_{00} \to c_{00}$ with

$$T^j e_0 = \gamma_j, \qquad \text{if } j = 1, 2, \ldots, a+\Delta-1 \tag{5}$$

and

$$T^{b+v+j} e_0 = e_{b+v+j} + DT^{v+j} e_0, \qquad \text{if } j = 0, 1, \ldots, a+\Delta-v-1 \tag{6}$$

for some integer $b > a+\Delta$ and number $D > 0$, we have that

$$\left\| \frac{T^b}{D} P(T) y - e_0 \right\| \leq 2\varepsilon \|e_a\| + \frac{C}{D}\left(\max_{b+v \leq i < b+a+\Delta} \|e_i\| + \max_{b+a+\Delta \leq j \leq b+2(a+\Delta-1)} \left\|T^j e_0\right\| \right).$$

Proof. Define a linear operator $T' \colon E_{a+\Delta-1} \to E_{a+\Delta-1}$ by the relation

$$T'(\gamma_j) = \begin{cases} \gamma_{j+1}, & \text{if } j < a+\Delta-1, \\ 0, & \text{if } j = a+\Delta-1. \end{cases}$$

Take any $z \in E_{a+\Delta-1}$, $z \neq 0$. Then the vectors

$$z, T'z, T'^2 z, \ldots, T'^{a+\Delta-1-\mathrm{val}_{\widetilde{\gamma}}(z)} z$$

form a linear basis of $\mathrm{span}\{\gamma_{\mathrm{val}_{\widetilde{\gamma}}(z)}, \gamma_{\mathrm{val}_{\widetilde{\gamma}}(z)+1}, \ldots, \gamma_{a+\Delta-1}\}$. In particular, if $z \in K$, then $\mathrm{val}_{\widetilde{\gamma}}(z) \leq a$ and, consequently, there is a polynomial P_z with $\mathrm{sc}\, P_z \subseteq [a - \mathrm{val}_{\widetilde{\gamma}}(z), a+\Delta)$ such that

$$P_z(T')z = \gamma_a = \varepsilon e_a + e_0.$$

For some neighbourhood B_z of z in $E_{a+\Delta-1}$ we have therefore

$$\|P_z(T')y - e_0\| < 2\varepsilon\|e_a\|, \quad \text{for every } y \in B_z.$$

By the compactness of K we get that for any $y \in K$ there is a polynomial P for which

$$\|P(T')y - e_0\| < 2\varepsilon\|e_a\| \tag{7}$$

and $\mathrm{sc}\, P \subseteq [v, a+\Delta)$, $|P| \leq C$ for some $C > 0$.

Now let $T \colon c_{00} \to c_{00}$ be any linear operator satisfying (5) and (6).

Fix $y = \sum_{k=0}^{a+\Delta-1} y_k \gamma_k \in K$ and fix $P(t) = \sum_{l=v}^{a+\Delta-1} p_l t^l$ chosen so as (7) holds. Then

$$P(T)y = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} y_k p_l T^l \gamma_k = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} y_k p_l T^{k+l} e_0 = \sum_{j=v}^{2(a+\Delta-1)} \lambda_j T^j e_0, \tag{8}$$

where

$$\lambda_j := \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{j,k+l} y_k p_l, \tag{9}$$

where $\delta_{m,n}$ is the Kronecker symbol. By the boundedness of K, we can assume (taking a bigger C if necessary) that $\sum_{j=v}^{2(a+\Delta-1)} |\lambda_j| \le C$.

But we get that

$$\begin{aligned} P(T')y = P(T') \sum_{k=0}^{a+\Delta-1} y_k \gamma_k &= \sum_{j=v}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{j,k+l} y_k p_l \gamma_j \overset{(9)}{=} \sum_{j=v}^{a+\Delta-1} \lambda_j \gamma_j \\ &\overset{(5)}{=} \sum_{j=v}^{a+\Delta-1} \lambda_j T^j e_0 = \sum_{j=0}^{a+\Delta-v-1} \lambda_{j+v} T^{j+v} e_0. \end{aligned} \tag{10}$$

It follows once again by (5) and (6) that

$$\begin{aligned} \frac{T^b}{D} P(T')y &= \sum_{j=0}^{a+\Delta-v-1} \lambda_{j+v} \frac{T^{b+v+j} e_0}{D} \\ &= \sum_{j=0}^{a+\Delta-v-1} \frac{\lambda_{j+v}}{D} e_{b+v+j} + \sum_{j=0}^{a+\Delta-v-1} \lambda_{j+v} T^{j+v} e_0 \\ &= \sum_{j=0}^{a+\Delta-v-1} \frac{\lambda_{j+v}}{D} e_{b+v+j} + P(T')y. \end{aligned}$$

Therefore

$$\left\| \frac{T^b}{D} P(T')y - P(T')y \right\| \le \frac{C}{D} \cdot \sup_{b+v \le j < b+a+\Delta} \|e_j\|. \tag{11}$$

But, by (8) and (10), we get that

$$T^b P(T)y - T^b P(T')y = T^b \sum_{j=a+\Delta}^{2(a+\Delta-1)} \lambda_j T^j e_0,$$

so

$$\left\| \frac{T^b}{D} P(T)y - \frac{T^b}{D} P(T')y \right\| \le \frac{C}{D} \cdot \sup_{b+a+\Delta \le j \le b+2(a+\Delta-1)} \|T^j e_0\|. \tag{12}$$

Taking (7), (11) and (12) into account we get the final conclusion. □

6 The operator

Motivated by Lemma 7, assume that there are given sequences (Δ_n), (a_n), (b_n) of natural numbers such that

$$\begin{aligned} 1 = \Delta_1 < a_1 < a_1 + \Delta_1 < b_1 < b_1 + a_1 + \Delta_1 = \\ = \Delta_2 < a_2 < a_2 + \Delta_2 < b_2 < b_2 + a_2 + \Delta_2 = \\ = \Delta_3 < \dots, \end{aligned}$$

a sequence $(v_n) \subseteq \mathbb{N}$ satisfying $v_n < a_n$, numbers $\alpha_j > 0$ for $j \in [\Delta_n, a_n) \cup [a_n + \Delta_n, b_n + v_n)$ and a sequence (η_n). We define a perturbed weighted forward shift $T\colon c_{00} \to c_{00}$ by the relation

$$T^j e_0 = \begin{cases} \frac{1}{A_{N_n,a_n}} e_j + T^{j-a_n} e_0, & \text{if } j \in [a_n, a_n + \Delta_n), \\ e_j + \eta_n^{b_n/3} T^{j-b_n} e_0, & \text{if } j \in [b_n + v_n, b_n + a_n + \Delta_n), \\ \alpha_j e_j, & \text{otherwise}, \end{cases} \tag{13}$$

where

$$(N_n) = (1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots)$$

is a sequence containing any positive integer infinitely many times. By a suitable choice of parameters, we will be able to extend T to a continuous linear operator on s without non-trivial invariant subspaces.

By Proposition 6.(vi), we get that (13) implies the following corollary.

Corollary 8.

$$Te_j = \begin{cases} \alpha_1 e_1, & \text{if } j = 0, \\ \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}, & \text{if } j \in [\Delta_n, a_n - 1), \\ \frac{1}{\alpha_{a_n-1}} \left(\frac{1}{A_{N_n,a_n}} e_{a_n} + e_0 \right), & \text{if } j = a_n - 1, \\ e_{j+1}, & \text{if } j \in [a_n, a_n + \Delta_n - 1), \\ \alpha_{a_n+\Delta_n} A_{N_n,a_n} e_{a_n+\Delta_n} - A_{N_n,a_n} T^{\Delta_n} e_0, & \text{if } j = a_n + \Delta_n - 1, \\ \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}, & \text{if } j \in [a_n + \Delta_n, b_n + v_n - 1), \\ \frac{1}{\alpha_{b_n+v_n-1}} \left(e_{b_n+v_n} + \eta_n^{b_n/3} T^{v_n} e_0 \right), & \text{if } j = b_n + v_n - 1, \\ e_{j+1}, & \text{if } j \in [b_n + v_n, b_n + a_n + \Delta_n - 1), \\ \alpha_{b_n+a_n+\Delta_n} e_{b_n+a_n+\Delta_n} - \eta_n^{b_n/3} \alpha_{a_n+\Delta_n} e_{a_n+\Delta_n}, & \text{if } j = b_n + a_n + \Delta_n - 1. \end{cases}$$

7 Choosing the parameters

Assume that we have already defined Δ_k, v_k, a_k, b_k for $k < n$ and numbers α_j for $j \in [\Delta_k, a_k) \cup [a_k + \Delta_k, b_k + v_k)$ for $k < n$. We put $\Delta_n = b_{n-1} + a_{n-1} + \Delta_{n-1}$ (we put $\Delta_1 = 1$). Let $\eta_n = 2^{\frac{1}{b_{n-1}+v_{n-1}}}$ (we put $\eta_1 = \frac{4}{3}$). Let (for $n \geq 2$)

$$s_{n-1} = \max_{j \in [a_{n-1}+\Delta_{n-1}, b_{n-1}+v_{n-1})} \frac{A_{N_{n-1}, b_{n-1}+v_{n-1}+j}}{\alpha_j A_{N_{n-1}+1,j}} + 1. \tag{14}$$

We take a_n to be any number satisfying the following conditions:

$$a_n \geq 4\Delta_n \quad \text{and} \quad 4 \mid a_n; \tag{15}$$

$$a_n \geq 2^{\Delta_n} \|e_0\|_1; \tag{16}$$

$$\frac{\eta_n^{a_n - \Delta_n - 1}}{2A_{N_n,a_n} \left[s_{n-1} A_{N_{n-1}, 3b_{n-1}} \eta_{n-1}^{b_{n-1}/3} \right]} \geq 2^{[b_{n-1}+v_{n-1}]+1} \|e_0\|_n, \tag{17}$$

where the parts in brackets are assumed to be 1 and 0 respectively for $n = 1$. Observe that satisfying the last inequality is possible as $A_{N_n,a_n} = (a_n + k_N)^{k_N}$ increases polynomially fast while $\eta_n^{a_n}$ increases exponentially fast (with respect to a_n).

We put

$$\alpha_j = \frac{\eta_n^{j-\Delta_n}}{2A_{N_n,a_n} \left[s_{n-1} A_{N_{n-1}, 3b_{n-1}} \eta_{n-1}^{b_{n-1}/3} \right]} \tag{18}$$

for $j \in [\Delta_n, a_n)$.

Even though the rest of parameters is not yet known, we can define a system

$$\widetilde{\gamma}_n = (e_0, Te_0, \ldots, T^{a_n+\Delta_n-1} e_0)$$

using the already defined parameters and the definition (13). Then $\widetilde{\gamma}_n$ is a perturbed canonical basis of $E_{a_n+\Delta_n-1}$ and $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ given by

$$\sum_{i=0}^{a_n+\Delta_n-1} x_i T^i e_0 \overset{\tau_n}{\longmapsto} \sum_{i=0}^{a_n/4} x_i T^i e_0 \tag{19}$$

is a well-defined linear projection on $E_{a_n+\Delta_n-1}$. We put

$$K_n = \left\{ x \in E_{a_n+\Delta_n-1} : \|x\|_1 \leq 1 \quad \text{and} \quad \|\tau_n x\|_1 \geq \frac{1}{2} \right\}. \tag{20}$$

By the definition (19), we have for $y \in K_n$ that $\mathrm{val}_{\widetilde{\gamma}_n}(y) \leq \frac{a_n}{4}$, in fact it is easy to see that $\mathrm{val}_{\widetilde{\gamma}_n}(K_n) = \frac{a_n}{4}$. Let us define (in accordance with symbols used in Lemma 7) $v_n = a_n - \mathrm{val}_{\widetilde{\gamma}_n}(K_n) = \frac{3a_n}{4}$. Now $a = a_n, \Delta = \Delta_n, K = K_n, v = v_n$ and $\|\cdot\| = \|\cdot\|_{N_n}$ fulfill the assumptions of Lemma 7. Let $C_n = C(\widetilde{\gamma}_n, K_n, \|\cdot\|_{N_n})$ be the constant from Lemma 7.

We take b_n to be any number greater that $a_n + \Delta_n$ such that:

$$b_n \geq a_n + \Delta_n + [b_{n-1} + v_{n-1}] \tag{21}$$

$$\frac{\eta_n^{b_n+v_n-1-a_n-\Delta_n}}{2\eta_n^{2b_n/3}} \geq 2^{v_n+[b_{n-1}+v_{n-1}]+1} \|e_0\|_n \max(\alpha_{v_n}, 1); \tag{22}$$

$$\eta_n^{b_n/3} \geq 2^{\Delta_n+a_n} A_{N_n, b_n+a_n+\Delta_n-1} \max(C_n, 1); \tag{23}$$

$$\frac{A_{N_n,3b_n}}{A_{N_n+1,b_n}} \leq 1; \tag{24}$$

$$\text{the sequence } \left(\frac{A_{N_n, j+b_n+v_n}}{A_{N_n+1,j}} \right)_{j=0}^{\infty} \text{ is decreasing.} \tag{25}$$

The first three conditions can be satisfied as the right hand sides are constant or increase polynomially fast while the left hand sides tend to infinity exponentially fast (with respect to b_n). The left hand in the fourth condition tends to 0 as b_n tends to $+\infty$. As for the fifth condition, it can be easily seen that it is satisfied for b_n big enough, when one calculates the corresponding derivative.

We put

$$\alpha_j = \frac{\eta_n^{j-a_n-\Delta_n}}{2\eta_n^{b_n/3}} \tag{26}$$

for $j \in [a_n + \Delta_n, b_n + v_n)$.

Corollary 9. *For j such that both α_j and α_{j+1} are defined we have that*

$$\frac{\alpha_{j+1}}{\alpha_j} \leq \frac{4}{3}. \tag{27}$$

For $j \geq \Delta_{n+1}$ such that both α_j and $\alpha_{j+b_n+v_n}$ are defined we have that

$$\frac{\alpha_{j+b_n+v_n}}{\alpha_j} \leq 2. \tag{28}$$

Proof. The first part is trivial. For the second part observe that if both α_j and $\alpha_{j+b_n+v_n}$ are defined and $j \geq \Delta_{n+1}$, then j and $j + b_n + v_n$ must lie in the same interval $[\Delta_p, a_p)$ or $[a_p + \Delta_p, b_p + v_p)$ for some $p > n$. By (18) and (26), it follows that

$$\frac{\alpha_{j+b_n+v_n}}{\alpha_j} = \eta_p^{b_n+v_n} \leq 2.$$

□

Corollary 10. *If all the parameters are chosen as described in this section, then*

$$\alpha_{\Delta_n} \overset{(18)}{=} \frac{1}{2A_{N_n,a_n}\left[s_{n-1}A_{N_{n-1},3b_{n-1}}\eta_{n-1}^{b_{n-1}/3}\right]} < \frac{1}{2} \tag{29}$$

$$\alpha_{a_n-1} \overset{(18),(17)}{\geq} 2^{[b_{n-1}+v_{n-1}]+1}\|e_0\|_n \tag{30}$$

$$\alpha_{a_n+\Delta_n} \overset{(26)}{=} \frac{1}{2\eta_n^{b_n/3}} \tag{31}$$

$$\alpha_{b_n+v_n-1} \overset{(26),(22)}{\geq} \eta_n^{b_n/3}2^{v_n+[b_{n-1}+v_{n-1}]+1}\|e_0\|_n \max(\alpha_{v_n},1) \tag{32}$$

Corollary 11. *If all the parameters are chosen as described in this section, then for every $y \in K_n$ there is a polynomial P with $\operatorname{sc} P \subseteq [v_n, a_n+\Delta_n)$ and $|P| \leq C_n$ such that*

$$\left\|\frac{T^{b_n}}{\eta_n^{b_n/3}}P(T)y - e_0\right\|_{N_n} \leq 4,$$

where $n \in \mathbb{N}_+$ and $T\colon c_{00} \to c_{00}$ is defined in (13).

Proof. By Lemma 7, with $a = a_n$, $\Delta = \Delta_n$, $\widetilde{\gamma} = \widetilde{\gamma}_n$, $\varepsilon = \frac{1}{A_{N_n,a_n}}$, $K = K_n$, $\|\cdot\| = \|\cdot\|_{N_n}$, $v = v_n$, $b = b_n$, $D = \eta_n^{b_n/3}$ and T as in (13), we get that for every $y \in K_n$ there is a polynomial P with $\operatorname{sc} P \subseteq [v_n, a_n+\Delta_n)$ and $|P| \leq C_n$ such that

$$\begin{aligned}
&\left\|\frac{T^{b_n}}{\eta_n^{b_n/3}}P(T)y - e_0\right\|_{N_n}\\
&\leq 2\frac{\|e_{a_n}\|_{N_n}}{A_{N_n,a_n}} + \frac{C_n}{\eta_n^{b_n/3}}\left(\max_{b_n+v_n\leq i<b_n+a_n+\Delta_n}\|e_i\|_{N_n} + \max_{b_n+a_n+\Delta_n\leq j\leq b_n+2(a_n+\Delta_n-1)}\left\|T^je_0\right\|_{N_n}\right)\\
&= 2 + \frac{C_n}{\eta_n^{b_n/3}}\left(A_{N_n,b_n+a_n+\Delta_n-1} + \max_{b_n+a_n+\Delta_n\leq j\leq b_n+2(a_n+\Delta_n-1)}\left\|T^je_0\right\|_{N_n},\right)\\
&\overset{(23)}{\leq} 3 + \frac{C_n}{\eta_n^{b_n/3}}\left(\max_{b_n+a_n+\Delta_n\leq j\leq b_n+2(a_n+\Delta_n-1)}\left\|\alpha_je_j\right\|_{N_n}\right) &(*)\\
&\leq 3 + \frac{C_nA_{N_n,b_n+2(a_n+\Delta_n)}}{\eta_n^{b_n/3}}\left(\max_{b_n+a_n+\Delta_n\leq j\leq b_n+2(a_n+\Delta_n-1)}\alpha_j\right)\\
&\overset{(3)}{\leq} 3 + \frac{C_n(\frac{3}{2})^{a_n+\Delta_n+1}A_{N_n,b_n+a_n+\Delta_n-1}}{\eta_n^{b_n/3}}\left(\max_{b_n+a_n+\Delta_n\leq j\leq b_n+2(a_n+\Delta_n-1)}\alpha_j\right)\\
&\overset{(23)}{\leq} 3 + \max_{\underbrace{b_n+a_n+\Delta_n}_{\Delta_{n+1}}\leq j\leq b_n+2(a_n+\Delta_n-1)}\alpha_j = 3 + \alpha_{b_n+2(a_n+\Delta_n-1)}\\
&\overset{(28)}{\leq} 3 + 2\alpha_{\Delta_{n+1}} &(**)\\
&\overset{(29)}{\leq} 4.
\end{aligned}$$

Inequalities (*) nad (**) are true as $b_n+2(a_n+\Delta_n-1) = \Delta_{n+1}+a_n+\Delta_n-2 < \Delta_{n+1}+b_n+v_n$. ☐

8 Continuity

Lemma 12. *When all the parameters are chosen in the way specified in Section 7, the operator T given by (13) is continuous with respect to the topology induced on c_{00} by s, in fact for each $N \in \mathbb{N}_+$*

$$\|Tx\|_N \leq 2\|x\|_N.$$

Proof. Because of the special form of the norms we are considering it suffices to show that for any $j \in \mathbb{N}, N \in \mathbb{N}_+$

$$\frac{\|Te_j\|_N}{\|e_j\|_N} \leq 2.$$

To do that we calculate Te_j for various j using Corollary 8. We get that:

- If $j = 0$, then
$$\frac{\|Te_0\|_N}{\|e_0\|_N} = \frac{\|\alpha_1 e_1\|_N}{\|e_0\|_N} = \alpha_{\Delta_1} \frac{A_{N,1}}{A_{N,0}} \overset{(29),(3)}{\leq} \frac{3}{2} < 2.$$

- If $j \in [\Delta_n, a_n - 1)$ or $j \in [a_n + \Delta_n, b_n + v_n - 1)$ for some $n \in \mathbb{N}_+$, we have
$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\left\|\frac{\alpha_{j+1}}{\alpha_j} e_{j+1}\right\|_N}{\|e_j\|_N} = \frac{\alpha_{j+1}}{\alpha_j} \frac{A_{N,j+1}}{A_{N,j}} \overset{(27),(3)}{\leq} \frac{4}{3} \cdot \frac{3}{2} = 2.$$

- If $j \in [a_n, a_n + \Delta_n - 1)$ or $j \in [b_n + v_n, b_n + a_n + \Delta_n - 1)$ for some $n \in \mathbb{N}_+$, we have
$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\|e_{j+1}\|_N}{\|e_j\|_N} = \frac{A_{N,j+1}}{A_{N,j}} \overset{(3)}{\leq} \frac{3}{2} < 2.$$

- If $j = a_n - 1$ for some $n \in \mathbb{N}_+$, we have
$$\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} = \frac{\left\|\frac{1}{\alpha_{a_n-1}}\left(\frac{1}{A_{N_n,a_n}} e_{a_n} + e_0\right)\right\|_N}{\|e_{a_n-1}\|_N}$$
$$\leq \frac{A_{N,a_n}}{\alpha_{a_n-1} A_{N_n,a_n} A_{N,a_n-1}} + \frac{A_{N,0}}{\alpha_{a_n-1} A_{N,a_n-1}}.$$

  By our choice of the parameters, it follows that
$$\frac{1}{\alpha_{a_n-1} A_{N_n,a_n}} \frac{A_{N,a_n}}{A_{N,a_n-1}} \overset{(30),(3)}{\leq} \frac{1}{A_{N_n,a_n}} \cdot \frac{3}{2} \leq \frac{1}{A_{1,a_1}} \cdot \frac{3}{2} = \frac{1}{a_1 + 3} \cdot \frac{3}{2} < 1,$$
$$\frac{1}{\alpha_{a_n-1}} \frac{A_{N,0}}{A_{N,a_n-1}} \overset{(30)}{<} 1,$$

  so
$$\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} \leq 2.$$

- If $j = a_n + \Delta_n - 1$ for some $n \in \mathbb{N}_+$, we have that
$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} = \frac{\|\alpha_{a_n+\Delta_n} A_{N_n,a_n} e_{a_n+\Delta_n} - A_{N_n,a_n} T^{\Delta_n} e_0\|_N}{\|e_{a_n+\Delta_n-1}\|_N}$$
$$\overset{(13)}{\leq} \frac{\alpha_{a_n+\Delta_n} A_{N_n,a_n} A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} + \frac{A_{N_n,a_n} \alpha_{\Delta_n} A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}}.$$

  By our choice of the parameters, it follows that
$$\alpha_{a_n+\Delta_n} A_{N_n,a_n} \frac{A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} \overset{(31),(3)}{\leq} \frac{A_{N_n,a_n}}{2\eta_n^{b_n/3}} \cdot \frac{3}{2} \overset{(23)}{\leq} 1,$$
$$\alpha_{\Delta_n} A_{N_n,a_n} \frac{A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}} \overset{(29)}{\leq} 1,$$

  so
$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \leq 2.$$

- If $j = b_n + v_n - 1$ for some $n \in \mathbb{N}_+$, then we have that

$$\begin{aligned}\frac{\|Te_{b_n+v_n-1}\|_N}{\|e_{b_n+v_n-1}\|_N} &= \frac{\left\|\frac{1}{\alpha_{b_n+v_n-1}}\left(e_{b_n+v_n} + \eta_n^{b_n/3} T^{v_n} e_0\right)\right\|_N}{\|e_{b_n+v_n-1}\|_N} \\ &\le \frac{A_{N,b_n+v_n}}{\alpha_{b_n+v_n-1} A_{N,b_n+v_n-1}} + \frac{\eta_n^{b_n/3}\alpha_{v_n} A_{N,v_n}}{\alpha_{a_{b_n+v_n-1}} A_{N,b_n+v_n-1}},\end{aligned}$$

where $T^{v_n} e_0 = \alpha_{v_n} e_{v_n}$ because $v_n = \frac{3a_n}{4} \in [\Delta_n, a_n)$. By our choice of the parameters, it follows that

$$\begin{aligned}\frac{1}{\alpha_{b_n+v_n-1}} \frac{A_{N,b_n+v_n}}{A_{N,b_n+v_n-1}} &\overset{(32),(3)}{\le} 1, \\ \frac{\eta_n^{b_n/3}\alpha_{v_n}}{\alpha_{a_{b_n+v_n-1}}} \frac{A_{N,v_n}}{A_{N,b_n+v_n-1}} &\overset{(32)}{\le} 1,\end{aligned}$$

so

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \le 2.$$

- $j = b_n + a_n + \Delta_n - 1$ for some $n \in \mathbb{N}_+$.

$$\begin{aligned}\frac{\|Te_{b_n+a_n+\Delta_n-1}\|_N}{\|e_{b_n+a_n+\Delta_n-1}\|_N} &= \frac{\left\|\alpha_{b_n+a_n+\Delta_n} e_{b_n+a_n+\Delta_n} - \eta_n^{b_n/3}\alpha_{a_n+\Delta_n} e_{a_n+\Delta_n}\right\|_N}{\|e_{b_n+a_n+\Delta_n-1}\|_N} \\ &\le \frac{\alpha_{b_n+a_n+\Delta_n} A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} + \frac{\eta_n^{b_n/3}\alpha_{a_n+\Delta_n} A_{N,a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}}.\end{aligned}$$

By our choice of the parameters, it follows that

$$\begin{aligned}\alpha_{b_n+a_n+\Delta_n} \frac{A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} = \alpha_{\Delta_{n+1}} \frac{A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} &\overset{(29),(3)}{\le} 1, \\ \eta_n^{b_n/3}\alpha_{a_n+\Delta_n} \frac{A_{N,a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} &\overset{(31)}{\le} 1,\end{aligned}$$

so

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \le 2.$$

□

Corollary 13. *The operator T given by (13) can be uniquely extended to a continuous linear operator $T\colon s \to s$.*

9 Cyclic vectors

In Section 7 we have defined projections τ_n by the formula

$$\sum_{i=0}^{a_n+\Delta_n-1} x_i T^i e_0 \overset{\tau_n}{\longmapsto} \sum_{i=0}^{a_n/4} x_i T^i e_0.$$

Because $(e_0, Te_0, \ldots, T^{a_n+\Delta_n-1}e_0)$ is a perturbed weighted canonical basis, and by (15), $\Delta_n \le \frac{a_n}{4}$, a straightforward computation using Proposition 6.(v) yields that in the canonical basis τ_n is given by the following formula:

$$\tau_n e_j = \begin{cases} e_j, & \text{if } j \le \frac{a_n}{4} \\ 0, & \text{if } j \in \left(\frac{a_n}{4}, a_n\right) \\ -A_{N_n,a_n} T^{j-a_n} e_0, & \text{if } j \in [a_n, a_n+\Delta_n). \end{cases} \tag{33}$$

We have also defined (see (20)) compact sets

$$K_n = \left\{x \in E_{a_n+\Delta_n-1} : \|x\|_1 \le 1 \quad \text{and} \quad \|\tau_n x\|_1 \ge \frac{1}{2}\right\}. \tag{34}$$

Proposition 14. *For any $n \in \mathbb{N}_+$ the projection $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ satisfies*

$$\|\tau_n x\|_1 \le \|x\|_{N_n+1}.$$

Proof. Once again, by the special form of weighted l_1-norms we need to show the claim for e_j, where $j = 0, 1, \dots a_n + \Delta_n - 1$. For $j \in [0, a_n)$ we get by (33) that

$$\|\tau_n e_j\|_1 \le \|e_j\|_1 \le \|e_j\|_{N_n+1}.$$

For $j \in [a_n, a_n + \Delta_n)$ we get by Lemma 12 that

$$\frac{\|\tau_n e_j\|_1}{\|e_j\|_{N_n+1}} \overset{(33)}{=} \frac{A_{N_n,a_n}\|T^{j-a_n}e_0\|_1}{\|e_j\|_{N_n+1}} \le \frac{A_{N_n,a_n}}{A_{N_n+1,a_n}} 2^{\Delta_n}\|e_0\|_1 < \frac{1}{a_n}2^{\Delta_n}\|e_0\|_1 \overset{(16)}{\le} 1$$

□

For a set $M \subseteq \mathbb{N}$ we denote by π_M the canonical projection from s onto $\operatorname{span}\{e_j : j \in M\}$. We write $\pi_m = \pi_{[0,m]}$

Lemma 15. *Let $N \in \mathbb{N}_+$ and let (n_k) be a sequence of natural numbers such that $N_{n_k} = N$. Let $y \in s$ be a vector such that $\|y\|_1 = 1$. If parameters in the definition (13) are chosen as in Section 7, then for all but finitely many k*

$$\pi_{a_{n_k}+\Delta_{n_k}-1}(y) \in K_{n_k},$$

where K_{n_k} is defined in (34).

Proof. We have that $\|\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 \le \|y\|_1 \le 1$, so we only need to show that

$$\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 \ge \frac{1}{2}$$

for almost all k. But by (33) and Proposition 14 we get that

$$\begin{aligned}\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 &= \|\pi_{a_{n_k}/4}(y) + \tau_{n_k}\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\|_1\\ &\ge \|\pi_{a_{n_k}/4}(y)\|_1 - \|\tau_{n_k}\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\|_1\\ &\ge \|\pi_{a_{n_k}/4}(y)\|_1 - \|\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\|_{N+1} \xrightarrow[k\to\infty]{} 1,\end{aligned}$$

so indeed, for k big enough, $\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 \ge \frac{1}{2}$. □

Proposition 16. *For each $n \in \mathbb{N}_+$ we have for $j \ge a_n + \Delta_n$ that*

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \le 2.$$

Proof. The proof consists of checking all the possible cases of j.

- If $j \in [a_n + \Delta_n, b_n + v_n)$, then by Proposition 6.(v) and (13) we have that

$$T^{b_n+v_n}e_j = T^{b_n+v_n}\left(\frac{1}{\alpha_j}T^j e_0\right) = \frac{1}{\alpha_j}\alpha_{b_n+v_n+j}e_{b_n+v_n+j},$$

  where last equality is true as $b_n + a_n + \Delta_n < b_n + v_n + j < \Delta_{n+1} + b_n + v_n$. Therefore

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|Te_j\|_{N_n+1}} = \alpha_{b_n+v_n+j}\frac{A_{N_n,b_n+v_n+j}}{\alpha_j A_{N_n+1,j}} \overset{(14)}{\le} \alpha_{b_n+v_n+j}s_n \overset{(28)}{\le} 2\alpha_{\Delta_{n+1}}s_n \overset{(29)}{\le} 1.$$

- If $j \in [b_n + v_n, b_n + a_n + \Delta_n) = [b_n + v_n, \Delta_{n+1})$, then by Proposition 6.(v) and (13)

$$T^{b_n+v_n}e_j = T^{b_n+v_n}\left(T^j e_0 - \eta_n^{b_n/3}T^{j-b_n}e_0\right) = T^{j+b_n+v_n}e_0 - \eta_n^{b_n/3}T^{j+v_n}e_0.$$

  Observe that $2v_n = \frac{3}{2}a_n \overset{(15)}{\ge} a_n + \Delta_n$, it follows that

$$\Delta_{n+1} = b_n + a_n + \Delta_n \le b_n + 2v_n \le j + v_n < j + b_n + v_n < \Delta_{n+1} + b_n + v_n \qquad (35)$$

and by (13) we get

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \overset{(35),(13)}{\leq} \frac{\alpha_{j+b_n+v_n}A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + \frac{\alpha_{j+v_n}\eta_n^{b_n/3}A_{N_n,j+v_n}}{A_{N_n+1,j}}$$
$$\overset{(35),(28)}{\leq} \frac{2\alpha_{\Delta_{n+1}}A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + \frac{2\alpha_{\Delta_{n+1}}\eta_n^{b_n/3}A_{N_n,j+v_n}}{A_{N_n+1,j}}.$$

But

$$\alpha_{\Delta_{n+1}}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} \overset{(29),(25)}{\leq} \frac{1}{2}\cdot\frac{A_{N_n,2b_n+v_n}}{A_{N_n+1,b_n}} \leq \frac{1}{2}\cdot\frac{A_{N_n,3b_n}}{A_{N_n+1,b_n}} \overset{(24)}{\leq} \frac{1}{2},$$
$$\alpha_{\Delta_{n+1}}\eta_n^{b_n/3}\frac{A_{N_n,j+v_n}}{A_{N_n+1,j}} \overset{(29)}{\leq} \frac{1}{2}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} \overset{(25),(24)}{\leq} \frac{1}{2},$$

so we get that

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leq 2.$$

- If $j \geq \Delta_{n+1}$ and for some $p > n$ both j and $j+b_n+\Delta_n$ lie in the same interval $[\Delta_p, a_p)$, $[a_p, a_p+\Delta_p)$, $[a_p+\Delta_p, b_p+v_p)$ or $[b_p+v_p, b_p+a_p+\Delta_p)$, then, by Proposition 6.(v) and (13), $T^{b_n+v_n}e_j = \frac{\alpha_{j+b_n+v_n}}{\alpha_j}e_{j+b_n+v_n}$ or $T^{b_n+v_n}e_j = e_{j+b_n+v_n}$, in either case

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \overset{(27)}{\leq} 2\cdot\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} \overset{(25),(24)}{\leq} 2.$$

- If for some $p > n$ we have that $j < a_p \leq j+b_n+v_n$ then by Proposition 6.(v) and (13), we get that

$$T^{b_n+v_n}e_j = \frac{1}{\alpha_j}T^{j+b_n+v_n}e_0 = \frac{1}{\alpha_j}\left(\frac{1}{A_{N_p,a_p}}e_{j+b_n+v_n} + T^{j+b_n+v_n-a_p}e_0\right),$$

so, by Lemma 12,

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leq \frac{A_{N_n,j+b_n+v_n}}{\alpha_jA_{N_p,a_p}A_{N_n+1,j}} + \frac{2^{b_n+v_n}\|e_0\|_{N_n}}{\alpha_jA_{N_n+1,j}}.$$

But

$$\frac{1}{\alpha_jA_{N_p,a_p}}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} \overset{(28),(25),(24)}{\leq} \frac{2}{\alpha_{a_p-1}A_{N_p,a_p}} \overset{(30)}{\leq} 1$$
$$\frac{2^{b_n+v_n}\|e_0\|_{N_n}}{\alpha_{a_j}}\frac{1}{A_{N_n+1,j}} \overset{(28)}{\leq} \frac{2^{b_n+v_n+1}\|e_0\|_n}{\alpha_{a_p-1}} \overset{(30)}{\leq} 1,$$

hence

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leq 2.$$

- If for some $p > n$ we have that $j < a_p+\Delta_p \leq j+b_n+v_n$, then, by (21), $j+b_n+v_n < b_p < b_p+v_p$. Therefore, by Proposition 6.(v) and (13), we get that

$$T^{b_n+v_n}e_j = A_{N_p,a_p}T^{b_n+v_n}\left(T^je_0 - T^{j-a_p}e_0\right)$$
$$= \alpha_{j+b_n+v_n}A_{N_p,a_p}e_{j+b_n+v_n} - \alpha_{j+b_n+v_n-a_p}A_{N_p,a_p}e_{j+b_n+v_n-a_p}.$$

But $\Delta_p \leq j+b_n+v_n-a_p < \Delta_p+b_n+v_n \leq \Delta_p+b_{p-1}+v_{p-1}$, so

$$\frac{\|T^{b_n+v_n}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \overset{(28)}{\leq} 2\alpha_{a_p+\Delta_p}A_{N_p,a_p}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + 2\alpha_{\Delta_p}A_{N_p,a_p}\frac{A_{N_n,j+b_n+v_n-a_p}}{A_{N_n+1,j}}$$
$$\overset{(31),(23),(29),(25),(24)}{\leq} 2$$

- If for some $p > n$ we have that $j < b_p + v_p \leq j + b_n + v_n$, then, by Proposition 6.(v) and (13), we get

$$T^{b_n+v_n} e_j = \frac{1}{\alpha_j} T^{j+b_n+v_n} e_0 = \frac{1}{\alpha_j}\left(e_{j+b_n+v_n} + \eta_p^{b_p/3} T^{j+b_n+v_n-b_p} e_0\right),$$

hence, by Lemma 12,

$$\begin{aligned}\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} &\leq \frac{1}{\alpha_j}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + \frac{\eta_p^{b_p/3} 2^{v_p+b_n+v_n}\|e_0\|_{N_n}}{\alpha_j}\frac{1}{A_{N_n+1,j}} \\ &\overset{(28),(25),(24)}{\leq} \frac{2}{\alpha_{b_p+v_p-1}} + \frac{2\eta_p^{b_p/3} 2^{v_p+b_n+v_n}\|e_0\|_n}{\alpha_{b_p+v_p-1}} \overset{(32)}{\leq} 2.\end{aligned}$$

- If for some $p > n$ we have that $j < b_p + a_p + \Delta_p = \Delta_{p+1} \leq j + b_n + v_n$, then Proposition 6.(v) and (13) yield

$$\begin{aligned}T^{b_n+v_n} e_j &= T^{b_n+v_n}\left(T^j e_0 - \eta_p^{b_p/3} T^{j-b_p} e_0\right) \\ &= \alpha_{j+b_n+v_n} e_{j+b_n+v_n} - \eta_p^{b_p/3}\alpha_{j+b_n+v_n-b_p} e_{j+b_n+v_n-b_p},\end{aligned}$$

so by (28)

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leq 2\alpha_{\Delta_{p+1}}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + 2\alpha_{a_p+\Delta_p}\eta_p^{b_p/3}\frac{A_{N_n,j+b_n+v_n-b_p}}{A_{N_n+1,j}} \overset{(29),(31),(25),(24)}{\leq} 2.$$

□

Theorem 17. *Every non-zero vector $x \in s$ is a cyclic vector of T.*

Proof. It suffices to show that any vector x with $\|x\|_1 = 1$ is cyclic.

As e_0 is cyclic for T by Proposition 6.(iv), by Proposition 5 it suffices to show that e_0 is an element of the closed linear span of the orbit of x.

Take any $N \in \mathbb{N}_+$. We will show that there is a polynomial P with $\|P(T)x - e_0\|_N \leq 6$ and it completes the proof since the unit balls of seminorms $\|\cdot\|_N$ form the basis of zero-neighbourhoods in s.

Lemma 15 implies that there is a number n such that $N_n = N$, $y = \pi_{a_n+\Delta_n-1}x \in K_n$ and $\|\pi_{[a_n+\Delta_n,\infty)}x\|_{N+1} \leq 1$. From Corollary 11 we get a polynomial P with $|P| \leq C_n$ and $\operatorname{sc} P \subseteq [v_n, a_n + \Delta_n)$ such that

$$\left\|\frac{T^{b_n}}{\eta_n^{b_n/3}} P(T) y - e_0\right\|_N \leq 4.$$

We have for the same polynomial P

$$\begin{aligned}\left\|\frac{T^{b_n} P(T)}{\eta_n^{b_n/3}} x - e_0\right\|_N &\leq \left\|\frac{T^{b_n} P(T)}{\eta_n^{b_n/3}} y - e_0\right\|_N + \left\|\frac{T^{b_n} P(T)}{\eta_n^{b_n/3}} \pi_{[a_n+\Delta_n,\infty)} x\right\|_N \\ &\leq 4 + \left\|\frac{T^{b_n} P(T)}{\eta_n^{b_n/3}} \pi_{[a_n+\Delta_n,\infty)} x\right\|_N.\end{aligned}$$

Let $P(t) = \sum_{l=v_n}^{a_n+\Delta_n-1} p_l t^l$. By Lemma 12, for any $z \in s$ we have that

$$\left\|T^{b_n} P(T) z\right\|_N = \left\|\sum_{l=0}^{a_n+\Delta_n-v_n-1} p_{l+v_n} T^l T^{b_n+v_n} z\right\|_N \leq C_n 2^{a_n+\Delta_n}\left\|T^{b_n+v_n} z\right\|_N.$$

By Proposition 16, it now follows that

$$\begin{aligned}\left\|\frac{T^{b_n} P(T)}{\eta_n^{b_n/3}} \pi_{[a_n+\Delta_n,\infty)} x\right\|_N &\leq \frac{C_n 2^{a_n+\Delta_n}}{\eta_n^{b_n/3}}\|T^{b_n+v_n}\pi_{[a_n+\Delta_n,\infty)} x\|_N \\ &\overset{(23)}{\leq} 2\|\pi_{[a_n+\Delta_n,\infty)} x\|_{N+1} \leq 2.\end{aligned}$$

□

This completes the proof of Theorem 1.

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Adam Mickiewicz University,
Faculty of Mathematics and Comp. Sci.,
Umultowska 87,
61-614 Poznań, Poland
E-mail address: golinski@amu.edu.pl