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Operators on Fréchet spaces without nontrivial invariant subspaces

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Abstract

The main part of this thesis presents a construction of continuous linear operators without nontrivial invariant subspaces on many Fréchet spaces appearing naturally in analysis, in particular, spaces of holomorphic functions and the Schwartz space of rapidly decreasing smooth functions. The construction is based on the ideas that C. J. Read used in his famous proof of the existence of such operators on the space ℓ_1. Because of the different structure of a non-normable Fréchet space, the proof can be considerably simplified.

In the last chapter, using similar methods, we construct on the Schwartz space of rapidly decreasing smooth functions an operator for which all non-zero vectors are hypercyclic.

Streszczenie

Główna część rozprawy opisuje konstrukcję ciągłych operatorów liniowych bez nietrywialnych podprzestrzeni niezmienniczych na wielu pojawiających się naturalnie w analizie przestrzeniach Frécheta, m.in. przestrzeniach funkcji holomorficznych czy przestrzeni Schwartza gładkich funkcji szybko malejących. Konstrukcja oparta jest o idee użyte przez C. J. Reada w jego słynnym dowodzie istnienia takich operatorów na przestrzeni ℓ_1. Ze względu na inną strukturę nienormowalnej przestrzeni Frécheta, dowód może być znacznie uproszczony.

W ostatnim rozdziale, używając podobnych metod, konstruujemy na przestrzeni Schwartza gładkich funkcji szybko malejących operator, dla którego wszystkie niezerowe wektory są hipercykliczne.

Contents

Chapter 1

Introduction

Let X be a locally convex space and let $T\colon X \to X$ be a continuous linear operator. A *closed* subspace (subset) M of X is called invariant for T if $T(M) \subseteq M$. Of course, the zero space and the whole space are invariant subspaces (subsets) for any operator – they are called trivial. The important question is the existence of other invariant subspaces. The most pertinent case is the famous invariant subspace problem – whether any continuous linear operator on the separable Hilbert space has a nontrivial invariant subspace.

The invariant subspace problem goes back to J. von Neumann's work in the 1930s regarding operators on Hilbert spaces. Von Neumann himself was able to prove that any compact operator on the separable Hilbert space has a nontrivial invariant subspace, as noted in [3], but this result was never published. Still, further research into invariant subspaces of operators was conducted, and in 1954 N. Aronszajn and K. T. Smith proved in [3] that any compact operator on a Banach space has a nontrivial invariant subspace.

In 1973 V. I. Lomonosov proved in [21], *inter alia*, that merely commuting with one compact operator is already enough for an operator to have a nontrivial invariant subspace. In fact he proved, that if S is a compact operator on a Banach space X, then there is a closed subspace M of X, $\{0\} \neq M \neq X$, such that for every operator T on X, if $ST = TS$, then $T(M) \subseteq M$. The proof uses the Schauder fixed point theorem and, at the time it was published, Lomonosov's result was stronger and had simpler proof than anything known until then. The proof can also be found in [32, 10.35].

Not long after Lomonosov's paper counterexamples began to appear. The first was done by P. Enflo and first presented at a seminar in 1975/76 [13]. Unfortunately, the full preprint was not submitted until 1981 and on top of that, due to a lengthy publishing process, it was accepted for publication not sooner than in 1985 and published only in 1987 in [14]. In this proof an artificial Banach space is built together with an operator that has no nontrivial invariant subspace. The space is built by constructing a norm on the space of all polynomials of one variable x and the operator is just the multiplication by x. When one represents polynomials as finite sequences, then we are just dealing with a forward shift. A construction using Enflo's ideas, albeit simplified and giving an operator with stronger properties, can also be found in [8].

An analogous construction to Enflo's was carried out later by C. J. Read in his PhD thesis, the result was published in 1984 in [26]. In this construction a linear basis of c_{00} is built and the operator is just the forward shift in this basis. The difficult part is constructing a norm which makes the shift continuous and does not allow it to have nontrivial invariant subspaces. Read was later able to change his construction so that the resulting space was isomorphic to ℓ_1, see [27], and then carry out the construction on the space ℓ_1 itself, see [28]. A simplification of Read's proof on the space ℓ_1 can be found in [9, Ch. XIV]. Another exposition of Read's construction, presented in a very didactical, step-by-step way, can be found in the last chapter of [7].

A. Atzmon researched the invariant subspace problem for non-normable Fréchet spaces. First, he constructed in [5] a nuclear Fréchet space and an invertible operator with no common nontrivial invariant subspace with its inverse. Later, he constructed and published in [4] an example of a nuclear Fréchet space together with an operator without nontrivial invariant subspaces. This beautiful construction is very different from those of Enflo and Read.

Atzmon's construction from [4] starts with building a union (inductive limit) of weighted Bergman spaces on a decreasing sequence of planar domains with empty intersection, like in Figure 1.1.

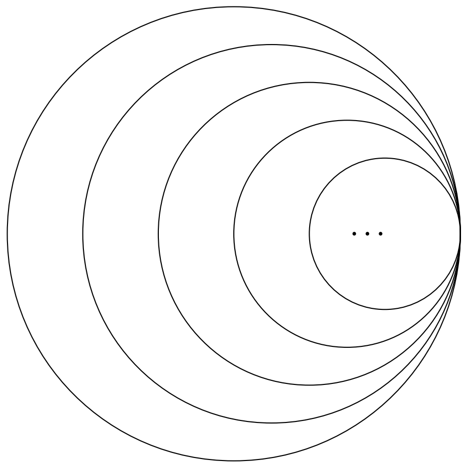


Figure 1.1: Sets in Atzmon's construction

Appropriate choice of the weights makes the resulting space a nuclear topological algebra with respect to pointwise multiplication of functions. Then one shows that this algebra has no closed nontrivial ideals, which in turn gives an operator without nontrivial invariant subspaces on the dual Fréchet space. It should be noted, that the resulting operator is the backward shift, while Enflo's and Read's constructions are more akin to the forward shift.

Atzmon later showed in [6] that one can build some special Fréchet space consisting of entire functions, closed with respect to differentiation, such that the differentiation operator has no nontrivial invariant subspace in that space.

The study of invariant subspaces led to contemporary interest in linear dynamics, i.e., investigating the orbits $\mathrm{Orb}(x,T) = \{T^i x : i \in \mathbb{N}\}$ of linear operators. A vector $x \in X$ is called *hypercyclic* if $\mathrm{Orb}(x,T)$ is dense in X. An operator is called *hypercyclic* if it has at least one hypercyclic vector.

The theory of hypercyclicity and related phenomena for linear operators has been developing rapidly, which has fruited in the recent publication of two monographs on the subject, one by F. Bayart and É. Mathéron [7] and the other by K.-G. Grosse-Erdmann and A. Peris [19].

Hypercyclicity is not so rare a phenomenon as one might initially think. It was known for a long time that some simple operators are hypercyclic, in particular:

- the translation operators: $f(z) \mapsto f(z+a)$ on the space of entire functions with the usual Fréchet topology of almost uniform convergence, where $a \neq 0$, see the Birkhoff Theorem, [19, 2.20];
- the differentiation operator: $f \mapsto f'$ on the space of entire functions, see the MacLane Theorem, [19, 2.21];
- the weighted backward shifts: λB, where $|\lambda| > 1$ and B is the backward shift on l_p or on c_0, see the Rolewicz Theorem, [19, 2.22].

It was shown independently by S. I. Ansari and L. Bernal-González (see [1, 10]) that any Banach space supports a hypercyclic operator. Ansari also formulated the result for Fréchet spaces, but unfortunately in this case the proof was flawed. This was later corrected by J. Bonet and Peris in [11].

By a Baire category argument, any hypercyclic operator on a Fréchet space has a dense G_δ set of hypercyclic vectors, see [19, 2.19]. There is an interest in finding operators with even larger sets of hypercyclic vectors. This is closely connected with invariant subsets. Indeed, observe, that if every non-zero vector is hypercyclic for T then T has no nontrivial invariant subset, since the closure of an orbit is always an invariant subset. Recall that, by definition, invariant subsets are assumed closed.

After further refinement of his methods Read was able to prove with a quite complicated construction [29], that there are continuous linear operators without nontrivial invariant subsets on any space $\ell_1 \oplus W$, where W is a separable Banach space.

In 2003 S. Grivaux and M. Roginskaya applied Read's techniques to Hilbert spaces. They were able to construct operators on ℓ_2 with "large" sets of hypercyclic vectors, with different meanings of "large". In addition they were able to construct an operator such that the closure of each of its orbits is a subspace and another one, for which the closures of its orbits are totally ordered by inclusion. For details see [18].

W. Śliwa applied Read's techniques and showed in [33], that one can construct operators without nontrivial invariant subspaces on any non-archimedean Banach space of countable type. This research also allowed Śliwa to construct an explicit example [34] on the space ℓ_1, i.e., an operator for which all the parameters can be effectively calculated.

One should also mention the recent paper [2], where S. A. Argyros and R. G. Haydon constructed a hereditarily indecomposable Banach space, such that each continuous operator on it is of the form $\lambda \mathrm{id} + K$ with K compact. By Lomonosov's result, each continuous linear operator on this space has a nontrivial invariant subspace. This is the first infinite dimensional Banach space with this property.

In Chapter 4 of this thesis we present our first main result (Theorem 4.1): a construction of operators without nontrivial invariant subspaces on Köthe sequence spaces, which are isomorphic to many spaces occurring naturally in analysis. In particular, this encompasses the space of holomorphic functions on the complex plane, the space of holomorphic functions on the disc and the Schwartz space of test functions for tempered distributions, as well as the space of smooth functions on the closed interval (see conditions in Section 4.1 and a long list of natural spaces of analysis isomorphic to Köthe sequence sapces in Section 2.4).

The construction is based on the ideas of Read, but it is adapted to the setting of Fréchet spaces. Our notation is compatible with [7, Ch. 12]. It turns out that the setting allows for some simplifications that let the underlying idea to be more easily understood. The constructed operator is just a slightly perturbed forward shift on the Köthe space. To the author's best knowledge this is the first example of such an operator on *natural* non-normable (in particular, nuclear) Fréchet spaces. By duality, our example gives also operators without nontrivial invariant subspaces on spaces of germs of holomorphic functions and on the space of tempered distributions with their natural topologies (see Corollary 4.3).

The author's initial construction of this kind was done on the space s of rapidly decreasing sequences, and the proof for this single case is contained in the semester paper [15]. A more general version of the construction was later published in [16]. The proof presented there was, as it turned out, more complicated than necessary and did not include the space $H(\mathbb{C})$, which is remedied in this thesis.

In Chapter 5 we present our second main result (Theorem 5.1): a construction of an operator without nontrivial invariant subsets on the space s of rapidly decreasing sequences. Note that s is isomorphic to the space of smooth functions on the unit interval or on an arbitrary smooth manifold (see Section 2.4.1.3 for a long list of natural spaces of analysis isomorphic to s). To the author's best knowledge, this is the first example of such an operator on nuclear Fréchet spaces.

The operator constructed in Chapter 5 is also a perturbed forward shift, but its structure is more complicated. Again, our notation is kept largely compatible with [7, Ch. 12].

In the next Chapter 2 we will recall the definitions and notions necessary in this thesis. Chapter 3 introduces the main tool we will be using in the construction – a lemma heavily used in the inductive step of the construction.

Chapter 2

Preliminaries

We will use freely notions and result of functional analysis as contained in [23]. In this chapter we restrict ourselves to defining those notions and notations which are crucial for our thesis.

2.1 Linear dynamics

Let X be a Fréchet space (i.e., metrizable, complete locally convex space) or a strong dual of a Fréchet space. Let $T: X \to X$ be a continuous linear operator.

Let us recall the crucial notion from the introduction. A closed subspace (subset) $M \subseteq X$ is called an *invariant subspace* (*subset*) of T if $T(M) \subseteq M$. An invariant subspace (subset) M is called *nontrivial* if $\{0\} \neq M \neq X$.

If for a vector $x \in X$ the set

$$\mathrm{Lin}(x, T) = \{P(T)x : P - \text{polynomial}\},$$

called the *linear orbit* of x, is a dense subset of X, then x is called a *cyclic* vector. Observe that the closure of $\mathrm{Lin}(x, T)$ is the smallest invariant subspace containing x. Therefore showing that an operator has no nontrivial invariant subspace is equivalent to showing that every non-zero vector is cyclic.

If the set $\mathrm{Orb}(x, T) = \{T^i x : i \in \mathbb{N}\}$, called the *orbit* of x, is dense, then x is called a *hypercyclic* vector. If an operator has at least one hypercyclic vector, the operator is called *hypercyclic*.

The closure of $\mathrm{Orb}(x, T)$ is the smallest invariant subset containing x, so an operator has no nontrivial invariant subset if and only if each non-zero vector is hypercyclic.

We will not need any more notions from linear dynamics, interested reader is advised to consult [7] or [19] for more details on this vast theory.

2.2 Köthe spaces

We will concentrate on the so-called Köthe sequence spaces.

Definition 2.1 (Köthe space). Let $A = \left[A_{N,j}\right]_{N,j}$ be an infinite matrix of positive numbers such that $A_{N,j} \leq A_{N+1,j}$. For the sake of convenience we will assume that $N = 1, 2, \ldots$ and

$j = 0, 1, 2, \ldots$. Such a matrix is called a *Köthe matrix*. We call the sequence space

$$\lambda^1(A) = \left\{ x = (x_j)_{j=0}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \|x\|_N = \sum_{j=0}^{\infty} |x_j| A_{N,j} < \infty \text{ for every } N = 1, 2, \ldots \right\}$$

the *Köthe λ^1 echelon space* associated with the matrix A.

Remark 2.2. One can also define λ^p echelon spaces and a dual notion, the so-called co-echelon spaces. As we will not be interested in these, from now on, we will speak of Köthe spaces having Köthe λ^1 echelon spaces in mind. For a more systematic treatment of the basic theory of these spaces, see [23, Ch. 27]. Additional information can be found in [31], [25], [12] and [20].

Fact 2.3 ([23, 27.1]). *Köthe spaces are Fréchet spaces, i.e., locally convex and metrizable.*

2.3 Power series spaces

In this section we give some examples of Köthe spaces arising naturally in analysis.

Definition 2.4. Let $\alpha = (\alpha_j)_{j=0}^{\infty}$ be a nonnegative real sequence tending monotonically to $+\infty$ and let $R \in \mathbb{R} \cup \{+\infty\}$. We take any sequence $t_N \nearrow R$, and define

$$\Lambda_R(\alpha) = \lambda^1\left(\left[\exp(t_N \alpha_j)\right]\right).$$

The resulting space does not depend on the choice of the sequence (t_N) but only on the number R. We call $\Lambda_R(\alpha)$ the *power series space* corresponding to the sequence α with radius R. This is the most important class of Köthe spaces (and one that is reasonably well understood).

Remark 2.5. In the definition above we used in fact weighted ℓ_1 norms by considering λ^1 echelon spaces. Unfortunately, there seems to be no universal choice for the norm exponent in the literature. The book [23], which is the main reference for us, uses ℓ_2 norms (by considering λ^2 echelon spaces), so one has to be careful. In most cases, namely when the resulting space is nuclear, all norm exponents give exactly the same set of sequences with the same topology, see [23, 28.16].

Fact 2.6 ([23, p. 368]). *There are just two distinct cases in Definition 2.4. For a given sequence α all the spaces $\Lambda_R(\alpha)$ with $R \in \mathbb{R}$ are isomorphic as Fréchet spaces and not isomorphic to any of the spaces $\Lambda_\infty(\beta)$.*

Remark 2.7. For reasons that will become clear in the next chapter, we restrict ourselves to $R \in \{1, \infty\}$ (the standard choice would be $R \in \{0, \infty\}$). The spaces with $R = 1$ are called finite type power series spaces, the spaces with $R = \infty$ are called infinite type power series spaces.

Remark 2.8. It is well known that $\Lambda_R(\alpha)$ is always a Schwartz space (see [23, 27.10]).

Fact 2.9 ([23, 29.6]). *The space $\Lambda_\infty(\alpha)$ is nuclear if and only if* $\sup_j \frac{\log j}{\alpha_j} < \infty$. *The space $\Lambda_1(\alpha)$ is nuclear if and only if* $\lim_{n\to\infty} \frac{\log j}{\alpha_j} = 0$.

2.4 SPECIFIC EXAMPLES OF POWER SERIES SPACES

Many classical spaces appearing in analysis are isomorphic to power series spaces. In this section we collect the most important examples.

2.4.1 Infinite type power series spaces

2.4.1.1 Entire functions

Let $\mathbb{K} = \mathbb{C}$. We have that $\Lambda_\infty(j) \cong H(\mathbb{C})$ with isomorphism given by the usual Taylor expansion of a holomorphic function ([23, 29.4(2)], [31, 8.2.1]). Here the space $H(\mathbb{C})$ is equipped with the natural topology of uniform convergence on compact subsets of $\mathbb{C}$.

Indeed, if we consider the Banach spaces

$$\Lambda_\infty(j)_N = \left\{ (x_j) \in \mathbb{C}^{\mathbb{N}} : \sum_{j=0}^{\infty} |x_j| N^j < \infty \right\}$$

$$H^\infty(N\mathbb{D}) = \left\{ f \in H(N\mathbb{D}) : \sup_{|z|<N} |f(z)| < \infty \right\}$$

and let $\varphi\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{C}}$ be given by

$$(x_j) \overset{\varphi}{\longmapsto} \left(z \mapsto \sum_{j=0}^{\infty} x_j z^j \right),$$

then $\varphi\left(\Lambda_\infty(j)_N\right) \subset H^\infty((N-1)\mathbb{D})$ by the properties of power series. On the other hand $\varphi^{-1}(H^\infty((N+1)\mathbb{D})) \subset \Lambda_\infty(j)_N$ by the Cauchy inequality. Therefore the projective systems $(\Lambda_\infty(j)_N)_N$ and $(H^\infty(N\mathbb{D}))_N$ (with natural inclusions as linking maps) give rise to isomorphic Fréchet spaces and so $\Lambda_\infty(j) \cong H(\mathbb{C})$ as claimed.

In fact by a result of V. P. Zahariuta [37, Thm. 1], for an open domain U in the Riemann sphere $\widehat{\mathbb{C}}$, the space $H(U)$ is isomorphic to $H(\mathbb{C})$ if and only if the logarithmic capacity of the set $\widehat{\mathbb{C}} \setminus U$ equals 0.

2.4.1.2 Entire functions in higher dimensions

When one takes $\alpha_j = \sqrt[d]{j}$ for some $d \geq 2$, then $\Lambda_\infty(\alpha) \cong H(\mathbb{C}^d)$, where $H(\mathbb{C}^d)$ is again equipped with the compact-open topology ([31, 8.3.2]). Still, the proof is a little more complicated in this case. The Taylor coefficients of a function from $H(\mathbb{C}^d)$ form not a sequence but a d-dimensional matrix, which has to be rearranged to give a sequence from the corresponding power series space.

A result of Zahariuta gives necessary and sufficient conditions for a holomorphically convex domain U in $\mathbb{C}^d$, so that $H(U) \cong H(\mathbb{C}^d)$, for details see the survey [38, 3.3.5].

2.4.1.3 Rapidly decreasing sequences

A very important space is the result of taking $\alpha_j = \log(j+1)$. The space $\Lambda_\infty(\log(j+1))$ is called the space of rapidly decreasing sequences and is usually denoted by s. If we plug the sequence α_j into the definition of a power series space, we can see that s consists of sequences (x_j) such that $(P(j)x_j)$ is bounded for any polynomial P. Hence the name of the space.

This space is important in the structural theory of nuclear locally convex spaces, as by the Kōmura–Kōmura theorem, every nuclear locally convex space is isomorphic to a subspace of s^I for a suitable set I, see [23, 29.8].

Many spaces naturally occurring in analysis are isomorphic to s as Fréchet spaces. In particular this is the case for the following spaces:

- $C^\infty_{2\pi}(\mathbb{R})$ – the space of all 2π-periodic smooth functions on $\mathbb{R}$, with the topology of uniform convergence in all derivatives see [23, 29.5] for a direct proof;
- $\mathbb{C}^\infty_{2\pi}(\mathbb{R}^d)$ – the multidimensional analogue of the above, smooth functions on $\mathbb{R}^d$ that are periodic with respect to a d-dimensional lattice, see [23, 31.8];
- $\mathcal{S}(\mathbb{R}^d)$ – the Schwartz space of rapidly decreasing functions, space of test functions for tempered distributions, see [23, 29.5(2)] and [23, 31.14];
- $\mathcal{D}(K)$ – the space of smooth functions on $\mathbb{R}^d$ with the support contained in a given compact set $K \subset \mathbb{R}^d$ with non-empty interior, equipped with the topology of uniform convergence in all derivatives, see [23, 31.12];
- $C^\infty(K)$ – the space of smooth functions on a compact smooth manifold with the topology of uniform convergence in all derivatives, see [36, 2.3]. Observe that when the manifold in question is a torus, we have just smooth periodic functions;
- some spaces of entire Dirichlet series, in particular the space

  $$S_{(\log n)} = \left\{ f \in H(\mathbb{C}) : \exists (a_n) \subset \mathbb{C} \quad f(z) = \sum_{n=1}^{\infty} a_n \exp(z \log n) \right\}$$

  with its natural topology, see [31, 8.4.1].
- the space A^∞ of holomorphic functions on the open unit disc which can be extended continuously to the closed unit disc together with all their derivatives, equipped with the topology of uniform convergence in all derivatives. The isomorphism with s can be proved along the same lines as [24, 1.13].

2.4.2 Finite type power series spaces

2.4.2.1 Holomorphic functions on the disc

If one takes $\alpha_j = j$, but considers the finite type power series space instead, the resulting space $\Lambda_1(\alpha)$ is isomorphic to the space of holomorphic functions on the disc with a proof very similar to what we did for entire functions, see [23, 29.4(2)].

By a result of Zahariuta, for an open set $U \subset \mathbb{C}$, $H(U) \cong H(\mathbb{D})$ if and only if U has finitely many components and the Dirichlet problem is solvable in U, see [37, Thm. 2].

2.4.2.2 Holomorphic functions on the polydisc

It should be no surprise that when one takes $\alpha_j = \sqrt[d]{j}$, then the $\Lambda_1(\alpha)$ is isomorphic to the space of holomorphic functions on the d-dimensional polydisc, see [31, 8.3.2].

By another result of Zahariuta [38, 3.3.1], if a holomorphically convex domain in $\mathbb{C}^d$ is sufficiently regular, then $H(U) \cong H(\mathbb{D}^d)$ (see the paper [38] for the precise formulation).

2.5 Duals of Köthe spaces

As we will be interested in operators without nontrivial invariant subspaces, the following fact will be useful. Let us recall that invariant subspaces and subsets are, by definition, assumed to be closed. The dual space X' of locally convex space X is always assumed to be equipped with the strong topology, i.e., the topology of uniform convergence on bounded sets.

Fact 2.10. *Let $T\colon X \to X$ be a continuous linear operator without nontrivial invariant subspaces and let X be a reflexive locally convex space. Then the adjoint operator $T'\colon X' \to X'$ also has no nontrivial invariant subspace.*

Proof. It is easy to check that if $M \subset X'$ is a invariant subspace for T', then

$$M^\circ = \{x^{**} \in X'' : x^{**}(M) = \{0\}\}$$

is an invariant subspace for T''. But, because of reflexivity, $T'' = T$, so M° is either the zero space or the whole of X. By the Hahn–Banach Theorem, any invariant subspace of T' must be trivial. □

Therefore we are interested in the reflexivity of the Köthe space $\lambda^1(A)$. We have the following theorem:

Theorem 2.11 (Dieudonné–Gomes, [23, 27.9])**.** *Let A be a Köthe matrix. Then $\lambda^1(A)$ is reflexive if and only if for each infinite subset I of $\mathbb{N}$ and each N there exists a K such that*

$$\inf_{j \in I} \frac{A_{N,j}}{A_{K,j}} = 0.$$

Corollary 2.12. *The power series space $\Lambda_R(\alpha)$ is always reflexive.*

Duals of the spaces that we have considered are also of importance in analysis. Let us just shortly identify them:

Fact 2.13. *The dual space to the Fréchet space $H(\mathbb{C}^d)$ is isomorphic to the space of germs of holomorphic functions of d variables in a point, denoted by $H(\{0\}^d)$.*

Fact 2.14. *The dual space to the Fréchet space $H(\mathbb{D}^d)$ is isomorphic to the space of germs of holomorphic functions of d variables on the closed polydisc $H(\overline{\mathbb{D}}^d)$.*

For one variable the preceding two facts follow from the so-called Grothendieck–Köthe–da Silva duality, which represents duals of spaces of holomorphic functions on a planar set as spaces of holomorphic functions on the set's complement with respect to the Riemann sphere, for details see [22, 9.12]. For the many variables case this can be done only in certain cases, see [35, Satz 1], fortunately this covers the cases $\mathbb{C}^d$ and $\mathbb{D}^d$.

Note that the natural topology on the space $H(K)$ of germs of holomorphic functions over a compact set $K \subseteq \mathbb{C}^d$ is defined as the inductive limit in the category of locally convex spaces

$$H(K) = \lim\operatorname{ind}_{K \Subset U} H(U),$$

where U runs through all open neighbourhoods of K in $\mathbb{C}^d$. The space $H(K)$ is *not* metrizable.

By definition the dual space to the space $s \cong \mathcal{S}(\mathbb{R}^d)$ (with the strong topology) is isomorphic to the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions.

Chapter 3

Notation and the basic lemma

In this chapter, first we introduce some rather standard notation that will be very useful throughout the thesis. Then we present a basic lemma – crucial for the proofs.

3.1 Basic notation

Let $\mathbb{K}$ be the field of real or complex numbers. For the construction it does not matter which particular field we choose. We fix $\mathbb{N} = \{0, 1, \ldots\}$. By c_{00} we will denote the linear space of all finite sequences with elements from $\mathbb{K}$. The elements of the canonical basis of c_{00} will be denoted e_j and for convenience we assume that $j = 0, 1, 2, \ldots$.

By E_n we will denote the finite dimensional subspace of c_{00} consisting of the vectors of the form $\sum_{j=0}^{n} \mu_j e_j$, $\mu_j \in \mathbb{K}$. We call a linear basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of E_n a *perturbed canonical basis* if $\gamma_j = \sum_{i=0}^{j} \mu_{j,i} e_i$ for suitable coefficients. Observe that, in particular, this implies that $\mu_{j,j} \neq 0$.

Given a basis $\widetilde{\gamma}$ of E_n and a vector $x \in E_n$ such that $x = \sum_{j=\nu}^{n} \mu_j e_j$ with $\mu_\nu \neq 0$, we write that $\operatorname{val}_{\widetilde{\gamma}}(x) = \nu$. Given a set $K \subset E_n$ we define

$$\operatorname{val}_{\widetilde{\gamma}}(K) = \sup_{x \in K} \operatorname{val}_{\widetilde{\gamma}}(x).$$

For a subset M of $\mathbb{N}$ we will write π_M to denote the standard projection of a sequence onto $\operatorname{span}\{e_j : j \in M\}$. For brevity we will write π_n to denote $\pi_{[0,n]}$. This should be clear from the context, as will be the domain of the projection.

Throughout we will be working with polynomials a lot. From now on we fix a norm $|\cdot|$ on the set of all polynomials, simply taking $|P|$ to be the sum of the moduli of the coefficients of P. If $P(t) = \sum_l p_l t^l$ is a polynomial, then we write

$$\operatorname{supp} \operatorname{coeff} P = \{l \in \mathbb{N} : p_l \neq 0\}.$$

Definition 3.1. An operator $T\colon c_{00} \to c_{00}$ will be called a *perturbed forward shift* if

$$Te_j = \lambda_j e_{j+1} + \lambda_{j,j} e_j + \lambda_{j,j-1} e_{j-1} + \ldots + \lambda_{j,0} e_0,$$

where $\lambda_j \neq 0$.

Remark 3.2. We will encounter linear operators $T\colon c_{00} \to c_{00}$ satisfying the recursive relation

$$T^j e_0 = \mu_j e_j + P_j(T)e_0, \quad j = 1, 2, \dots \tag{3.1}$$

with $\mu_j \neq 0$ and $\deg P < j$. We always tacitly mean that $T^0 = \mathrm{id}$. It can be easily proved inductively that

$$T^j e_0 \in \operatorname{span}\left\{e_0, e_1, \dots, e_j\right\}.$$

Then T is necessarily a perturbed forward shift. Indeed, from (3.1) one can calculate that

$$Te_j = T\left(\frac{T^j e_0 - P_j(T)e_0}{\mu_j}\right) = \frac{T^{j+1}e_0 - P_j(T)Te_0}{\mu_j},$$

so by (3.1), T is a perturbed forward shift. In fact, in the proofs further on, calculating the values of Te_j more explicitly will be a substantial part of our work.

3.2 The lemma

The following lemma is the very cornerstone of the constructions presented in this thesis. It surely deserves its own chapter, even a very short one. Its root can be traced back to [26, 5.6], but it shows up in some form in subsequent Read's constructions, see [27, 6.1], [28, 5.1], [30, 5.2], [29, 7.2]. The exposition and the proof below are taken from the last chapter of [7].

Lemma 3.3. *Assume that for some integers a and Δ there is given a perturbed canonical basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_{a+\Delta-1})$ of $E_{a+\Delta-1}$ with $\gamma_0 = e_0$ and*

$$\gamma_a = \varepsilon e_a + e_0.$$

Let $\|\cdot\|$ be any norm on c_{00} and $K \subset E_{a+\Delta-1}$ be a compact set in the induced topology, such that $\mathrm{val}_{\widetilde{\gamma}}(K) \leq a$. We denote $v := a - \mathrm{val}_{\widetilde{\gamma}}(K)$.

Then there is a number $D \geq 1$ and a finite family of polynomials $\mathcal{P} = \{P_l\}_{l=1}^{L}$ all of which satisfy

$$\operatorname{supp}\operatorname{coeff} P_l \subseteq [v, a+\Delta) \quad \text{and} \quad |P_l| \leq D,$$

such that for any $y \in K$ there is a polynomial $P \in \mathcal{P}$ such that for each perturbed forward shift $T\colon c_{00} \to c_{00}$ with

$$T^j e_0 = \gamma_j, \qquad \text{for } j = 1, 2, \dots, a+\Delta-1 \tag{3.2}$$

we have that

$$\|P(T)y - e_0\| \leq 2\varepsilon\|e_a\| + D \times \max_{a+\Delta \leq j \leq 2(a+\Delta-1)} \left\|T^j e_0\right\|. \tag{3.3}$$

Moreover, if $y \in K$ has an expansion in the basis $\widetilde{\gamma}$ of the form

$$y = \sum_{j=0}^{a+\Delta-1} \lambda_j \gamma_j, \quad \lambda_j \in \mathbb{K},$$

then $\sum_{j=0}^{a+\Delta-1} |\lambda_j| \leq D$.

The most important aspect in this lemma is the fact that both the family of polynomials $\mathcal{P}$ and the constant D depend only on $\widetilde{\gamma}$, the set K and the norm $\|\cdot\|$, while there is no dependence on the vectors $T^{a+\Delta}e_0, T^{a+\Delta+1}e_0, \ldots$.

Proof. We define a linear operator $S\colon E_{a+\Delta-1} \to E_{a+\Delta-1}$ by the formula

$$S(\gamma_j) = \begin{cases} \gamma_{j+1}, & \text{if } j < a+\Delta-1, \\ 0, & \text{if } j = a+\Delta-1. \end{cases}$$

Take any vector $z \in E_{a+\Delta-1}$, $z \neq 0$. Then z can be expanded in the basis $\widetilde{\gamma}$ as follows:

$$z = \sum_{j=\operatorname{val}_{\widetilde{\gamma}}(z)}^{a+\Delta-1} \mu_j \gamma_j.$$

Then for any $k \le a+\Delta-1-\operatorname{val}_{\widetilde{\gamma}}(z)$ the vectors

$$S^k z = \sum_{j=\operatorname{val}_{\widetilde{\gamma}}(z)+k}^{a+\Delta-1} \mu_{j-k}\gamma_j,$$

$$\ldots$$

$$S^{a+\Delta-1-\operatorname{val}_{\widetilde{\gamma}}(z)-1} z = \mu_{\operatorname{val}_{\widetilde{\gamma}}(z)}\gamma_{a+\Delta-2} + \mu_{\operatorname{val}_{\widetilde{\gamma}}(z)+1}\gamma_{a+\Delta-1},$$

$$S^{a+\Delta-1-\operatorname{val}_{\widetilde{\gamma}}(z)} z = \mu_{\operatorname{val}_{\widetilde{\gamma}}(z)}\gamma_{a+\Delta-1}$$

form a linear basis of the space

$$\operatorname{span}\left\{\gamma_{\operatorname{val}_{\widetilde{\gamma}}(z)+k}, \gamma_{\operatorname{val}_{\widetilde{\gamma}}(z)+k+1}, \ldots, \gamma_{a+\Delta-1}\right\}.$$

In particular, if $z \in K$, then $\operatorname{val}_{\widetilde{\gamma}}(z) \le a$ and, consequently, there is a polynomial P_z with $\operatorname{supp\,coeff} P_z \subseteq [a-\operatorname{val}_{\widetilde{\gamma}}(z), a+\Delta)$ such that

$$P_z(S)z = \gamma_a = \varepsilon e_a + e_0.$$

Therefore, for some neighbourhood B_z of z in $E_{a+\Delta-1}$ we have

$$\|P_z(S)y - e_0\| < 2\varepsilon\|e_a\|, \quad \text{for every } y \in B_z.$$

By the compactness of K, we can cover K by a finite family of neighbourhoods B_z and get a finite number of polynomials P_z – these polynomials form the family $\mathcal{P}$. So for any $y \in K$ there is a polynomial $P \in \mathcal{P}$ for which

$$\|P(S)y - e_0\| < 2\varepsilon\|e_a\| \tag{3.4}$$

and $\operatorname{supp\,coeff} P \subseteq [v, a+\Delta)$, $|P| \le D$ for some $D > 0$. Once again by the compactness of K, we can assume that D is so large, that it satisfies the "moreover" claim of the lemma. Of course we can assume that $D \ge 1$.

Now let $T\colon c_{00} \to c_{00}$ be any perturbed forward shift satisfying (3.2).

Fix $y = \sum_{k=0}^{a+\Delta-1} \alpha_k \gamma_k \in K$ and let $P(t) = \sum_{l=v}^{a+\Delta-1} p_l t^l$, $P \in \mathcal{P}$, be chosen so that (3.4) holds. Then

$$\begin{aligned} P(T)y &= \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \alpha_k p_l T^l \gamma_k = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \alpha_k p_l T^{k+l} e_0 \\ &= \sum_{j=v}^{2(a+\Delta-1)} \lambda_j T^j e_0, \end{aligned} \tag{3.5}$$

where

$$\lambda_j = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{j,k+l} \alpha_k p_l, \tag{3.6}$$

where $\delta_{i,j}$ is the Kronecker delta. By the boundedness of K and finiteness of $\mathcal{P}$, we can assume (taking larger D if necessary) that $\sum_{j=v}^{2(a+\Delta-1)} |\lambda_j| \le D$. Note that this can be done independently of T.

We get that

$$P(S)y = P(S) \sum_{k=0}^{a+\Delta-1} \alpha_k \gamma_k = \sum_{j=v}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{k+l,j} \alpha_k p_l \gamma_j.$$

We now use (3.6) and the assumption (3.2) and we get that

$$P(S)y = \sum_{j=v}^{a+\Delta-1} \lambda_j \gamma_j = \sum_{j=v}^{a+\Delta-1} \lambda_j T^j e_0. \tag{3.7}$$

Hence, by (3.6) and (3.7), we get that

$$P(T)y - P(S)y = \sum_{j=a+\Delta}^{2(a+\Delta-1)} \lambda_j T^j e_0 \tag{3.8}$$

and consequently

$$\|P(T)y - P(S)y\| \le D \times \sup_{a+\Delta \le j \le 2(a+\Delta-1)} \left\| T^j e_0 \right\|. \tag{3.9}$$

Inequalities (3.4) and (3.9) yield the final conclusion. $\square$

Chapter 4

Invariant subspaces

In this chapter we will describe the construction of an operator without nontrivial invariant subspaces on Köthe spaces $\lambda^1(A)$ satisfying some easily checkable assumptions on the Köthe matrix A. We will prove the following main result.

Theorem 4.1. *Let $A = [A_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ be a positive Köthe matrix satisfying the following assumptions:*

$$\forall N \quad \sup_{j\in\mathbb{N}} \frac{A_{N,j+1}}{A_{N,j}} < +\infty; \tag{4.1}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j\to\infty} \frac{A_{N,j}}{A_{N+1,j}} = 0; \tag{4.2}$$

$$\forall N \in \mathbb{N}_+ \quad \text{the sequence } (A_{N,j})_j \text{ tends monotonically to } +\infty. \tag{4.3}$$

Then there exists a continuous linear operator $T\colon \lambda^1(A) \to \lambda^1(A)$ without nontrivial invariant subspaces.

The condition (4.1) means that the forward shift operator S on $\lambda^1(A)$, $Se_j = e_{j+1}$, is continuous without loss of norm, i.e., $\forall N \exists C_N \; \|Sx\|_N \le C_N \|x\|_N$.

The condition (4.2), by [16, 27.10], means that $\lambda^1(A)$ is a Schwartz space.

This theorem was initially proved in [16] with the additional assumption that the quantity in (4.1) must be bounded by a single number, regardless of the index N. This unfortunately excludes the space of entire functions. Here we give a modified proof that is moreover simplified a bit, for which the assumptions above suffice.

4.1 Corollaries

We collect the examples of natural Fréchet spaces of analysis which support an operator without nontrivial invariant subspaces.

Corollary 4.2. *Let (α_j) be a positive sequence tending monotonically to $+\infty$. If*

$$\sup(\alpha_{j+1} - \alpha_j) < +\infty$$

and $R \in \mathbb{R} \cup \{\infty\}$, then there exists a continuous linear operator $T\colon \Lambda_R(\alpha) \to \Lambda_R(\alpha)$ without nontrivial invariant subspaces.

Proof. Let $t_N \nearrow R$. It is easy to check that the Köthe matrix $\left[\exp(t_N\alpha_j)\right]$ satisfies condition (4.2). If $t_N > 0$ (which can be assumed if we restrict ourselves to $R \in \{1, \infty\}$), then also condition (4.3) is satisfied. We are left with the condition (4.1).

Fix $t_N > 0$. Then

$$\frac{\exp(t_N\alpha_{j+1})}{\exp(t_N\alpha_j)} = \exp(t_N(\alpha_{j+1} - \alpha_j)),$$

so (4.1) is equivalent to $\sup(\alpha_{j+1} - \alpha_j) < +\infty$. □

Taking Fact 2.10 into account together with examples from Section 2.4 (see also Facts 2.13, 2.14), we get the next corollary.

Corollary 4.3. *The following power series spaces support a continuous linear operator without nontrivial invariant subspaces:*

- $H(\mathbb{D}^d)$, $d \geq 1$;
- $H(\mathbb{C}^d)$, $d \geq 1$;
- *s (or, equivalently,* $\mathcal{S}(\mathbb{R}^d)$; $C^\infty(K)$, K *– compact* C^∞*-manifold).*

Moreover, the same is true for their dual nuclear (DF)-spaces:

- $H(\overline{\mathbb{D}}^d)$;
- $H(\{0\}^d)$;
- s' *(or, equivalently, the space* $\mathcal{S}'(\mathbb{R}^d)$ *of tempered distributions).*

To the author's best knowledge, so far there were known no natural non-Banach spaces supporting operators without nontrivial invariant subspaces.

4.2 The strategy

We start with a very simple observation. Let $T\colon X \to X$ be a continuous linear operator on a locally convex space X. It is easy to see that the smallest invariant subspace containing the given vector $x \in X$ is in fact the closure in X of the subspace $\mathrm{Lin}(x, T)$. If x_0 is a cyclic vector for T, then in order to show that some other vector x is cyclic for T it suffices to prove that $x_0 \in \overline{\mathrm{Lin}(x, T)}$.

More specifically, assume that $c_{00} \subset X$ and $T\colon c_{00} \to c_{00}$ is a perturbed forward shift, i.e.,

$$Te_j = \mu_j e_{j+1} + \sum_{i=0}^{j} \mu_{j,i} e_i, \quad \mu_j \neq 0.$$

Then it is easy to see that $\mathrm{Lin}(e_0, T) = c_{00}$, so if c_{00} is dense in X and we can extend T to a continuous operator $\widetilde{T}\colon X \to X$, then e_0 is a cyclic vector for $\widetilde{T}$. The operator $\widetilde{T}$ would have no nontrivial invariant subspace if any other non-zero vector x was also cyclic for $\widetilde{T}$. So we have to ensure that the smallest invariant subspace containing x is the whole space X. One way to that would be to show that e_0 is in this smallest invariant subspace. In other words, to show that

$$\forall X \ni x \neq 0 \quad \forall U \in \mathfrak{U}(X) \quad \exists P \qquad e_0 \in P\left(\widetilde{T}\right) x + U,$$

where $\mathfrak{U}(X)$ denotes the family of all the neighbourhoods of zero in X and P is a one-variable polynomial. If X is a Fréchet space, then $\mathfrak{U}(X)$ is given by a countable fundamental system of seminorms $\|\cdot\|_N$, so we may rewrite the above condition as

$$\forall X \ni x \neq 0 \quad \forall N \quad \forall \varepsilon > 0 \quad \exists P \qquad \left\| P\left(\widetilde{T}\right) x - e_0 \right\|_N \leq \varepsilon.$$

If the norms $\|\cdot\|_N$ are chosen carefully, so that their unit balls constitute a basis of neighbourhoods of zero of X, we only need to show that

$$\forall X \ni x \neq 0 \quad \forall N \quad \exists P \qquad \left\| P\left(\widetilde{T}\right) x - e_0 \right\|_N \leq 1.$$

Observe that if x is a cyclic vector, then any non-zero multiple of x is also cyclic. Therefore if, say $\|\cdot\|_1$ is a norm, then it would suffice to show that

$$\forall x \in X, \|x\|_1 = 1 \quad \forall N \quad \exists P \qquad \left\| P\left(\widetilde{T}\right) x - e_0 \right\|_N \leq 1. \tag{4.4}$$

This is in essence what we will do. Of course the crucial point is to construct a specific perturbed forward shift such that (4.4) holds.

We have a tool to implement the above scheme, namely the seemingly quite complicated Lemma 3.3. It allows us to find suitable P in (4.4), at least for some vectors, if we additionally assume that $x \in c_{00}$. If we were lucky and were able to truncate an infinitely supported vector $x \in X$ and get a vector $y \in c_{00}$ that works for our lemma, then by triangle inequality we would have that

$$\left\| P\left(\widetilde{T}\right) x - e_0 \right\|_N \leq \left\| P\left(\widetilde{T}\right) y - e_0 \right\|_N + \left\| P\left(\widetilde{T}\right) (x - y) \right\|_N.$$

As the lemma takes care of the first term we have to estimate the second one. On ℓ_1, as in Read's paper [28], we have only a single norm, so $\left\| P\left(\widetilde{T}\right)(x-y) \right\|$ has to be estimated somehow with $\|x - y\|$. In a Fréchet space setting, we have a sequence of nonequivalent norms at our disposal, so we will try to estimate $\left\| P\left(\widetilde{T}\right)(x-y) \right\|_N$ with $\|x - y\|_{N+1}$. This turns out to leave room for some simplifications. Further on we will not write $\widetilde{T}$ for the extension, but T. It should be clear from the context which operator we have in mind.

4.3 The matrix

We will show that we can get much better properties of the matrix A than the ones assumed in Theorem 4.1 without changing the resulting space $\lambda^1(A)$.

Lemma 4.4. *If a Köthe matrix $B = [B_{N,j}]_{N \in \mathbb{N}_+, j \in \mathbb{N}}$ of positive numbers satisfies*

$$\forall N \in \mathbb{N}_+ \quad \sup_{j \in \mathbb{N}} \frac{B_{N,j+1}}{B_{N,j}} < +\infty, \tag{4.5}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j \to \infty} \frac{B_{N,j}}{B_{N+1,j}} = 0, \tag{4.6}$$

$$\forall N \in \mathbb{N}_+ \quad \text{the sequence } (B_{N,j})_j \text{ tends monotonically to } +\infty, \tag{4.7}$$

then there exists a Köthe matrix $A = [A_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ *such that* $\lambda^1(B) = \lambda^1(A)$ *(equality of topological spaces) together with a non-increasing null-sequence* (ω_j) *and a sequence* (M_N) *with terms greater than 1 such that:*

$$\forall N \in \mathbb{N}_+ \ \ \forall j \in \mathbb{N} \quad A_{N,j} \geq 1, \tag{4.8}$$

$$\forall N \in \mathbb{N}_+ \ \ \forall j \in \mathbb{N} \quad \frac{A_{N,j+1}}{A_{N,j}} \leq M_N, \tag{4.9}$$

$$\forall N \in \mathbb{N}_+ \ \ \exists C_N \ \ \forall j \in \mathbb{N} \quad \frac{A_{N,j}}{A_{N+1,j}} \leq C_N \omega_j, \tag{4.10}$$

$$\forall N \in \mathbb{N}_+ \ \ \forall h \in \mathbb{N} \quad \lim_{j\to\infty} \frac{A_{N,j+h}}{A_{N+1,j}} = 0, \tag{4.11}$$

$$\forall N \in \mathbb{N}_+ \quad \textit{the sequence } (A_{N,j})_j \textit{ tends monotonically to } +\infty, \tag{4.12}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j\to\infty} \frac{M_N^j}{A_{N,j}} = +\infty \tag{4.13}$$

and the sequence of the unit balls of the norms $(U_N)_{N\in\mathbb{N}_+}$,

$$U_N = \left\{ x = (x_j) \in \mathbb{K}^{\mathbb{N}} : \|x\|_N = \sum_{j=0}^{\infty} |x_j| A_{N,j} \leq 1 \right\},$$

forms a basis of neighbourhoods of zero in $\lambda^1(A)$.

Proof. Take $M_N > 1$ to be a number greater than $\limsup_{j\to\infty} \frac{B_{N,j+1}}{B_{N,j}}$, existing by (4.5).

Let $(k_N)_{N\in\mathbb{N}_+}$ be an increasing sequence of positive integers chosen so that for $j \geq k_N$

$$B_{N,j} \geq 1, \qquad \frac{B_{N,j}}{B_{N+1,j}} < \frac{1}{2} \quad \text{and} \quad \frac{B_{N,j+1}}{B_{N,j}} \leq M_N.$$

This can be done by (4.7), (4.6) and the definition of M_N respectively.

We define $A_{N,j} = B_{N,j+k_N}$. Then $A = [A_{N,j}]$ fulfils automatically (4.8), (4.9), (4.12).

The definition of the numbers M_N implies that for any N there is a number $\varepsilon > 0$ and an index j_0 such that for $j \geq j_0$

$$\frac{A_{N,j+1}}{A_{N,j}} < M_N - \varepsilon.$$

Hence for $j \geq j_0$

$$A_{N,j} = A_{N,j_0} \frac{A_{N,j_0+1}}{A_{N,j_0}} \frac{A_{N,j_0+2}}{A_{N,j_0+1}} \cdots \frac{A_{N,j}}{A_{N,j-1}} \leq A_{N,j_0} (M_N - \varepsilon)^{j-j_0}$$

and consequently

$$\frac{(M_N - \varepsilon)^{j_0}}{A_{N,j_0}} \cdot \left(\frac{M_N}{M_N - \varepsilon} \right)^j \leq \frac{M_N^j}{A_{N,j}},$$

which implies (4.13).

For any $h \geq 0$ we have that

$$\frac{A_{N,j+h}}{A_{N+1,j}} < M_N^h \frac{A_{N,j}}{A_{N+1,j}} = M_N^h \frac{B_{N,j+k_N}}{B_{N+1,j+k_{N+1}}} \leq M_N^h \frac{B_{N,j+k_N}}{B_{N+1,j+k_N}} \xrightarrow[j\to\infty]{} 0,$$

which shows (4.11). By the choice of k_N, we have that

$$\frac{A_{N,j}}{A_{N+1,j}} = \frac{B_{N,j+k_N}}{B_{N+1,j+k_{N+1}}} \leq \frac{B_{N,j+k_N}}{B_{N+1,j+k_N}} < \frac{1}{2},$$

so $A_{N,j} < \frac{1}{2} A_{N+1,j}$. It follows that $U_{N+1} \subseteq \frac{1}{2} U_N$ and by the definition of the topology on $\lambda^1(A)$, $(U_N)_{N\in\mathbb{N}_+}$ is a basis of neighbourhoods of zero.

We will now show that the formal identity

$$\lambda^1(B) \ni (x_j) \longmapsto (x_j) \in \lambda^1(A)$$

is an isomorphism. Indeed, we have, by the monotonicity of the sequence $(B_{N,j})_j$, that

$$\begin{aligned}\left\|(x_j)\right\|_N^{\lambda^1(A)} &= \sum_{j=0}^{\infty} |x_j| A_{N,j} = \sum_{j=0}^{\infty} |x_j| B_{N,j+k_N} \\ &\geq \sum_{j=0}^{\infty} |x_j| B_{N,j} = \left\|(x_j)\right\|_N^{\lambda^1(B)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|(x_j)\right\|_N^{\lambda^1(A)} &= \sum_{j=0}^{\infty} |x_j| A_{N,j} = \sum_{j=0}^{\infty} |x_j| B_{N,j+k_N} \\ &\leq \sup_j \frac{B_{N,j+k_N}}{B_{N,j}} \sum_{j=0}^{\infty} |x_j| B_{N,j} \\ &= \sup_j \frac{B_{N,j+k_N}}{B_{N,j}} \left\|(x_j)\right\|_N^{\lambda^1(B)},\end{aligned}$$

where $\sup_j \frac{B_{N,j+k_N}}{B_{N,j}}$ is finite in view of (4.5). We still need to show (4.10), but for clarity we will turn this part of the proof into a separate lemma. □

Lemma 4.5. *Let $(\delta_i)_i$, where $\delta_i = (\delta_{ij})_j$, be a sequence of null-sequences. Then there exists a non-increasing null-sequence η which tends to zero slower than each of the sequences δ_i, more precisely, for each i there is a constant C_i such that $\delta_{ij} \leq C_i \eta_j$.*

Proof. Let $\vartheta_{ij} = \max\{\delta_{1j}, \delta_{2j}, \ldots, \delta_{ij}\}$. We define $\vartheta_i = (\vartheta_{ij})_j$. Then ϑ_i is a null-sequence.

Let j_1 be an integer so large that $\vartheta_{1j} \leq 1$ for $j \geq j_1$. Given j_{n-1}, let j_n be so large that $\vartheta_{nj} \leq \frac{1}{n}$ for $j \geq j_n$ and $j_n \geq j_{n-1}$.

We define the sequence η:

$$\eta_j = \begin{cases} 1, & j \leq j_1; \\ \frac{1}{n}, & j_n \leq j < j_{n+1}. \end{cases}$$

Then $\eta_j \to 0$. Moreover, for $i \leq n$ and $j_n \leq j < j_{n+1}$ we have that $\delta_{ij} \leq \vartheta_{nj} \leq \frac{1}{n} = \eta_j$. Consequently, for $j_i \leq j$ we have that $\delta_{ij} \leq \eta_j$. Therefore, it suffices to take

$$C_i = 1 + \max_{j \leq j_i} \frac{\delta_{ij}}{\eta_j}.$$

□

4.4 The operator

From now on we fix a sequence:

$$(N_n) = (1, 2, 1, 2, 3, 1, 2, 3, 4, \dots). \tag{4.14}$$

The only two features of this sequence which are important is that it contains any integer infinitely many times and starts with 1.

We will also need the following sequence:

$$\widetilde{M}_n = \max\{M_{N_1}, M_{N_2}, \dots, M_{N_n}\}, \tag{4.15}$$

where M_N is the sequence from Lemma 4.4.

Now assume that there is an increasing sequence of integers

$$(\Delta_1, a_1, \Delta_2, s_2, a_2, \Delta_3, s_3, a_3, \dots), \tag{4.16}$$

such that

$$\begin{aligned} 1 = \Delta_1 &< a_1 < a_1 + \Delta_1 \\ = \Delta_2 &< s_2 < a_2 < a_2 + \Delta_2 \\ = \Delta_3 &< s_3 < a_3 < a_3 + \Delta_3 \\ = \Delta_4 &< \dots. \end{aligned} \tag{4.17}$$

Recall that by $\{e_j\}_{j=0}^{\infty}$ we have denoted the canonical basis of c_{00}. We define a linear operator $T\colon c_{00} \to c_{00}$ through the following formula:

$$T^j e_0 = \begin{cases} \frac{1}{A_{N_n,a_n}} e_j + T^{j-a_n} e_0, & j \in [a_n, a_n + \Delta_n); \\ \alpha_j e_j, & j \in [\Delta_n, a_n). \end{cases} \tag{4.18}$$

The definition of the non-zero numbers α_j will be given below and it is somewhat involved, as it requires successive applications of Lemma 3.3. Observe that under the assumption that $\alpha_j \neq 0$, Remark 3.2 implies that (4.18) uniquely defines a linear operator $T\colon c_{00} \to c_{00}$.

The construction of T is carried out inductively over the intervals $[\Delta_n, a_n + \Delta_n)$. With each such interval we will associate a number $D_n \geq 1$ which will be important in the subsequent interval. There is only a very slight difference in the construction between the first and the next intervals, so we only present the general step of the inductive procedure, marking the adjustments necessary for the first interval.

Assume the procedure has been carried out up to the interval $[\Delta_{n-1}, a_{n-1} + \Delta_{n-1})$ and the number D_{n-1} has been fixed. Consistently with (4.17) we put $\Delta_n = a_{n-1} + \Delta_{n-1}$ (with $\Delta_1 = 1$).

We choose s_n to be a number much greater than Δ_n and then we choose a_n to be much greater than s_n (on the first interval we put $s_1 = \Delta_1$). We then put

$$\alpha_j = \begin{cases} \frac{1}{A_{N_n,a_n}} \left(\frac{1}{D_{n-1}\widetilde{M}_n} \right)^{j-\Delta_n}, & j \in [\Delta_n, s_n]; \\ \alpha_{s_n} M_{N_n}^{j-s_n}, & j \in (s_n, a_n). \end{cases} \tag{4.19}$$

Note that α_j's are decreasing on the interval $[\Delta_n, s_n]$ and they increase on the interval $[s_n, a_n)$.

Now all the vectors Te_0, T^2e_0, …, $T^{a_n+\Delta_n-1}e_0$ are defined through (4.18). These vectors then, together with $T^0e_0 = e_0$, form a perturbed canonical basis

$$\widetilde{\gamma}_n = (e_0, Te_0, \dots, T^{a_n+\Delta_n-1}e_0)$$

of $E_{a_n+\Delta_n-1}$. Therefore, the mapping

$$\sum_{j=0}^{a_n+\Delta_n-1} x_j T^j e_0 \overset{\tau_n}{\longmapsto} \sum_{j=0}^{a_n-1} x_j T^j e_0 \tag{4.20}$$

is a well-defined linear projection in $E_{a_n+\Delta_n-1}$. We define

$$K_n = \left\{ y \in E_{a_n+\Delta_n-1} : \|y\|_1 \le 1 \wedge \|\tau_n y\|_1 \ge \frac{1}{2} \right\}. \tag{4.21}$$

Observe that because of the definition (4.20) and the second condition in the definition of K_n, for every element $y \in K_n$ we have that $\mathrm{val}_{\widetilde{\gamma}_n}(y) \le a_n - 1$.

Therefore $\widetilde{\gamma} = \widetilde{\gamma}_n$, $\|\cdot\| = \|\cdot\|_{N_n}$, $K = K_n$ satisfy the assumptions of Lemma 3.3 and so we get an appropriate number $D_n \ge 1$ from this lemma. This allows us to carry on with the construction on the next interval.

We will show in the subsequent sections that if the sequence $(a_1, s_2, a_2, s_3, \dots)$ is chosen to increase rapidly enough, then the operator $T\colon c_{00} \to c_{00}$ defined by (4.18) can be extended to $\lambda^1(A)$ and this extension has no nontrivial invariant subspace.

For clarity, let us illustrate the dependence of all the parameters. First, we have "global" objects – derived simply from the Köthe matrix. These are:

- the "enhanced" Köthe matrix $[A_{N,j}]_{N,j}$ from Lemma 4.4;
- the sequences (M_N) and (ω_j) therefrom;
- the sequence (N_n) from (4.14);
- the sequence $(\widetilde{M}_n)$ from (4.15).

Then the sequence (4.16) will be inductively constructed with dependence between various parameters as in the following diagram – dashed arrows show where a choice is possible, solid arrows depict (sometimes complicated) functions.

$$\begin{array}{l}
1 = \Delta_1 = s_1 \dashrightarrow a_1 \xrightarrow{(4.19)} \{\alpha_j\}_{j=\Delta_1}^{a_1-1} \xrightarrow{(4.18)} \{T^j e_0\}_{j=\Delta_1}^{a_1+\Delta_1-1} \xrightarrow{(4.20),\,(4.21)} \tau_1, K_1 \xrightarrow{\text{Lemma 3.3}} D_1 \\
\hookrightarrow \Delta_2 \dashrightarrow s_2 \dashrightarrow a_2 \xrightarrow{(4.19)} \{\alpha_j\}_{j=\Delta_2}^{a_2-1} \xrightarrow{(4.18)} \{T^j e_0\}_{j=\Delta_2}^{a_2+\Delta_2-1} \xrightarrow{(4.20),\,(4.21)} \tau_2, K_2 \xrightarrow{\text{Lemma 3.3}} D_2 \\
\hookrightarrow \Delta_3 \dashrightarrow s_3 \dashrightarrow a_3 \cdots
\end{array} \tag{4.22}$$

Corollary 4.6. *If s_{n+1} is sufficiently large compared to a_n and a_{n+1} is sufficiently large compared to s_{n+1}, then there exists a constant $D_n \ge 1$ such that for any $y \in K_n$ there exists a polynomial P such that $|P| \le D_n$ and* supp coeff $P \subseteq [1, a_n + \Delta_n)$ *such that*

$$\|P(T)y - e_0\|_{N_n} \le 3.$$

Proof. We have already applied Lemma 3.3 in the procedure above with $a = a_n$, $\Delta = \Delta_n$, $K = K_n$, $\|\cdot\| = \|\cdot\|_{N_n}$ and $\widetilde{y} = (e_0, Te_0, \dots, T^{a_n+\Delta_n-1}e_0)$. Then (4.20) and (4.21) imply that $\operatorname{val}_{\widetilde{y}}(K_n) = a_n - 1$, so by the claim of Lemma 3.3 for any $y \in K_n$ there is a polynomial P satisfying the claims of our corollary (observe that D_n is chosen so that $|P| \leq D_n$) such that

$$\begin{aligned}\|P(T)y - e_0\|_{N_n} &\leq \frac{2}{A_{N_n,a_n}} \|e_{a_n}\|_{N_n} + D_n \times \max_{a_n+\Delta_n \leq j \leq 2(a_n+\Delta_n-1)} \|T^j e_0\|_{N_n} \\ &= 2 + D_n \times \max_{a_n+\Delta_n \leq j \leq 2(a_n+\Delta_n-1)} \|\alpha_j e_j\|_{N_n}.\end{aligned}$$

We may assume that $2(a_n + \Delta_n - 1) < s_{n+1}$, then, remembering that $a_n + \Delta_n = \Delta_{n+1}$, by (4.19) we get that

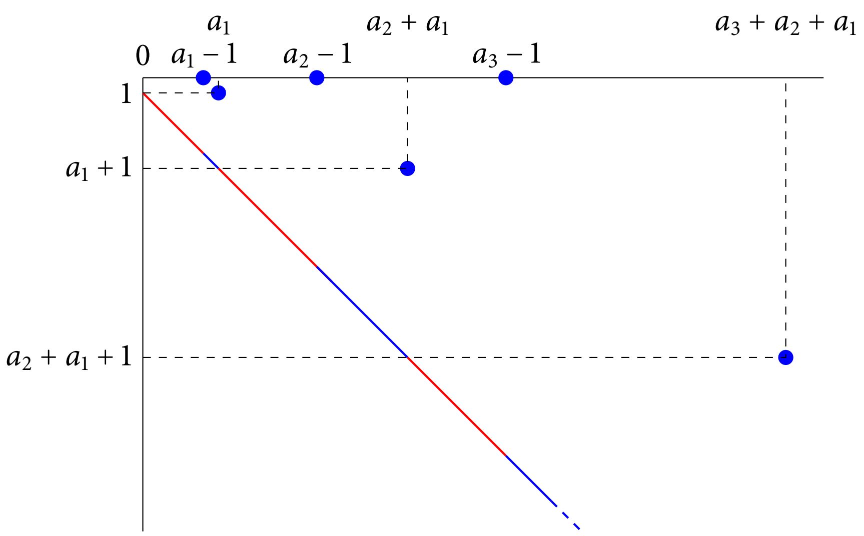
$$\max_{a_n+\Delta_n \leq j \leq 2(a_n+\Delta_n-1)} \alpha_j = \alpha_{\Delta_{n+1}} = \frac{1}{A_{N_{n+1},a_{n+1}}}.$$

Hence

$$\|P(T)y - e_0\|_{N_n} \leq 2 + D_n \alpha_{a_n+\Delta_n} A_{N_n,2(a_n+\Delta_n-1)} = 2 + \frac{D_n A_{N_n,2(a_n+\Delta_n-1)}}{A_{N_{n+1},a_{n+1}}}.$$

The fraction $\frac{D_n A_{N_n,2(a_n+\Delta_n-1)}}{A_{N_{n+1},a_{n+1}}}$ can be made smaller than 1 by choosing a_{n+1} so large that $A_{N_{n+1},a_{n+1}}$ is larger than $D_n A_{N_n,2(a_n+\Delta_n-1)}$. Observe that this is possible by (4.12), as a_{n+1} is chosen when Δ_n, a_n and D_n are already fixed. □

Remark 4.7. The linear operator T given by (4.18) has quite a simple matrix and it might be helpful to see it. While the calculations leading to it will be hidden inside the subsequent proofs, the picture below depicts a part of the (infinite) matrix of T. The coloured diagonal just below the main diagonal corresponds to the “weighted forward shift” part of T, while the isolated blue dots above it constitute the “perturbation”. All the other elements are zero.



4.5 CONTINUITY

We will show that if enough caution is taken with choices in (4.22), then the resulting linear operator is continuous in the topology of $\lambda^1(A)$.

Theorem 4.8. *If the sequence $(a_1, s_2, a_2, s_3, \dots)$ increases rapidly enough, then the linear operator $T\colon c_{00} \to c_{00}$ defined by* (4.18) *is continuous in the topology of $\lambda^1(A)$.*

Proof. As the norms considered on $\lambda^1(A)$ are weighted ℓ_1 norms, it suffices to show that for any N

$$\sup_j \frac{\|Te_j\|_N}{\|e_j\|_{N+1}} < +\infty. \tag{4.23}$$

To prove that, we will consider all the possible cases for j. As we will not be concerned with the exact bound in (4.23), we may omit a finite number of indices. Therefore we restrict ourselves only to $j \geq \Delta_2$.

- If $j \in [\Delta_n, a_n - 1)$, then (4.18) implies that

$$Te_j = T\left(\frac{1}{\alpha_j} T^j e_0\right) = \frac{1}{\alpha_j} T^{j+1} e_0 = \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}.$$

  Hence, by (4.9), (4.10) and (4.19),

$$\frac{\|Te_j\|_N}{\|e_j\|_{N+1}} = \frac{\alpha_{j+1}}{\alpha_j}\frac{A_{N,j+1}}{A_{N+1,j}} \leq M_{N_n} M_N C_N \omega_j \leq M_{N_n} M_N C_N \omega_{\Delta_n},$$

  which is bounded whenever $\omega_{\Delta_n} < \frac{1}{M_{N_n}}$. This can be done by choosing a_{n-1} (hence Δ_n) large enough.

- If $j \in [a_n, a_n + \Delta_n - 1)$, then (4.18) implies that

$$\begin{aligned} Te_j &= T\left(A_{N_n,a_n}\left(T^j e_0 - T^{j-a_n} e_0\right)\right) \\ &= A_{N_n,a_n}\left(T^{j+1} e_0 - T^{j-a_n+1} e_0\right) \\ &= A_{N_n,a_n}\left(\frac{1}{A_{N_n,a_n}} e_{j+1} + T^{j+1-a_n} e_0 - T^{j-a_n+1} e_0\right) = e_{j+1}. \end{aligned}$$

  Hence, by (4.9), (4.10),

$$\frac{\|Te_j\|_N}{\|e_j\|_{N+1}} = \frac{A_{N,j+1}}{A_{N+1,j}} \leq M_N C_N \omega_j,$$

  which is bounded since $\omega_j \to 0$.

- If $j = a_n - 1$, then from (4.18) we get

$$Te_{a_n-1} = \frac{1}{\alpha_{a_n-1}} T^{a_n} e_0 = \frac{1}{\alpha_{a_n-1}}\left(\frac{1}{A_{N_n,a_n}} e_{a_n} + e_0\right).$$

Therefore, by (4.9), (4.10) and (4.19),

$$\begin{aligned}\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_{N+1}} &= \frac{1}{\alpha_{a_n-1}}\left(\frac{1}{A_{N_n,a_n}}\frac{A_{N,a_n}}{A_{N+1,a_n-1}} + \frac{A_{N,0}}{A_{N+1,a_n-1}}\right)\\ &\leq \frac{A_{N_n,a_n}\left(D_{n-1}\widetilde{M}_n\right)^{s_n-\Delta_n}}{M_{N_n}^{a_n-s_n-1}}\left(\frac{1}{A_{N_n,a_n}}M_N C_N \omega_{a_n-1}+1\right)\\ &\leq \frac{A_{N_n,a_n}}{M_{N_n}^{a_n}}\left(D_{n-1}\widetilde{M}_n\right)^{s_n-\Delta_n} M_{N_n}^{s_n+1}\left(M_N C_N\omega_{a_n-1}+1\right).\end{aligned}$$

As $\omega_j \to 0$, $M_N C_N \omega_{a_n-1}$ is bounded. Because a_n is fixed after s_n, by (4.13), we can make $\frac{A_{N_n,a_n}}{M_{N_n}^{a_n}}$ small enough to keep $\frac{A_{N_n,a_n}}{M_{N_n}^{a_n}}\left(D_{n-1}\widetilde{M}_n\right)^{s_n-\Delta_n} M_{N_n}^{s_n+1}$ bounded by 1.

- If $j = a_n + \Delta_n - 1$, then, by (4.18),

$$\begin{aligned}Te_{a_n+\Delta_n-1} &= T\left(A_{N_n,a_n}\left(T^{a_n+\Delta_n-1}e_0 - T^{\Delta_n-1}e_0\right)\right)\\ &= A_{N_n,a_n}\left(T^{a_n+\Delta_n}e_0 - T^{\Delta_n}e_0\right)\\ &= A_{N_n,a_n}\alpha_{a_n+\Delta_n}e_{a_n+\Delta_n} - A_{N_n,a_n}\alpha_{\Delta_n}e_{\Delta_n}.\end{aligned}$$

  Consequently, because $a_n + \Delta_n = \Delta_{n+1}$, we get by (4.9), (4.10) and (4.19),

$$\begin{aligned}\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_{N+1}} &= \frac{A_{N_n,a_n}}{A_{N_{n+1},a_{n+1}}}\frac{A_{N,a_n+\Delta_n}}{A_{N+1,a_n+\Delta_n-1}} + \frac{A_{N_n,a_n}}{A_{N_n,a_n}}\frac{A_{N,\Delta_n}}{A_{N+1,a_n+\Delta_n-1}}\\ &< \frac{A_{N_n,a_n}}{A_{N_{n+1},a_{n+1}}}M_N C_N\omega_{a_n+\Delta_n-1}+1.\end{aligned}$$

  When we choose a_{n+1}, then a_n is already fixed, so by (4.12) we can assume that $A_{N_{n+1},a_{n+1}}$ is larger then A_{N_n,a_n} (regardless of actual values of N_n and N_{n+1}). So the resulting quantity is bounded, since $\omega_j \to 0$. □

Corollary 4.9. *If the sequence* $(a_1, s_2, a_2, s_3, \ldots)$ *increases rapidly enough, then the linear operator* $T\colon c_{00} \to c_{00}$ *defined by* (4.18) *can be uniquely extended to a continuous linear operator on* $\lambda^1(A)$*. We will still denote this extension* T*.*

4.6 CONTINUITY REVISITED

We will need another inequality of continuity type, this time with the same norm in the numerator and the denominator.

Lemma 4.10. *If the sequence* $(a_1, s_2, a_2, s_3, \ldots)$ *increases rapidly enough, then for* $j < s_n$

$$\frac{\|Te_j\|_N}{\|e_j\|_N} \leq M_N\widetilde{M}_n + 1.$$

Proof. We will use the same strategy as in the proof of Lemma 4.8, using also the values for Te_j we have calculated.

- If $j = 0$, then straight from (4.18), by (4.8) and (4.9) we get that

$$\frac{\|Te_0\|_N}{\|e_0\|_N} = \frac{1}{A_{N_1,a_1}} \frac{A_{N,1}}{A_{N,0}} \leq M_N.$$

- If $j \in [\Delta_p, a_p - 1)$ for some $p \leq n$, then, by (4.19) and (4.9),

$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\alpha_{j+1}}{\alpha_j} \frac{A_{N,j+1}}{A_{N,j}} \leq M_{N_p} M_N \leq \widetilde{M}_n M_N.$$

- If $j \in [a_p, a_p + \Delta_p - 1)$ for some $p < n$, then, by (4.9),

$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{A_{N,j+1}}{A_{N,j}} \leq M_N.$$

- If $j = a_p - 1$ for some $p < n$, then

$$\begin{aligned}\frac{\|Te_{a_p-1}\|_N}{\|e_{a_p-1}\|_N} &= \frac{1}{\alpha_{a_p-1}} \left(\frac{1}{A_{N_p,a_p}} \frac{A_{N,a_p}}{A_{N,a_p-1}} + \frac{A_{N,0}}{A_{N,a_p-1}} \right) \\ &= \frac{A_{N_p,a_p} \left(D_{p-1}\widetilde{M}_p\right)^{s_p-\Delta_p}}{M_{N_p}^{a_p-s_p-1}} \left(M_N + 1\right).\end{aligned}$$

We encountered a very similar quantity in the case "$j = a_n - 1$" in the proof of Lemma 4.8. As before, a_p is the last parameter chosen, and by (4.13) we can make $\frac{A_{N_p,a_p}}{M_{N_p}^{a_p}}$ small enough to meet our bound.

- If $j = a_p + \Delta_p - 1$ for some $p < n$, then

$$\begin{aligned}\frac{\|Te_{a_p+\Delta_p-1}\|_N}{\|e_{a_p+\Delta_p-1}\|_N} &= \frac{A_{N_p,a_p}}{A_{N_{p+1},a_{p+1}}} \frac{A_{N,a_p+\Delta_p}}{A_{N,a_p+\Delta_p-1}} + \frac{A_{N_p,a_p}}{A_{N_p,a_p}} \frac{A_{N,\Delta_p}}{A_{N,a_p+\Delta_p-1}} \\ &< \frac{A_{N_p,a_p}}{A_{N_{p+1},a_{p+1}}} M_N + 1.\end{aligned}$$

We reason as in the case "$j = a_n + \Delta_n - 1$" in the proof of Lemma 4.8. As a_{p+1} is chosen after a_p, we can make sure that $A_{N_{p+1},a_{p+1}}$ is larger than A_{N_p,a_p}. □

Because norms on the space $\lambda^1(A)$ are weighted ℓ_1 norms we get the following

Corollary 4.11. *If $x \in E_{s_n}$ and $M_N \leq \widetilde{M}_n$, (e.g., when $N \in \{N_1, N_2, \ldots, N_n\}$), we have that*

$$\|Tx\|_N \leq \left(\widetilde{M}_n^2 + 1\right) \|x\|_N.$$

Remark 4.12. Observe that if the sequence (M_N) is bounded, then also the sequence $(\widetilde{M}_n)$ is bounded, and Corollary 4.11 gives us continuity of T without loss of norms. Unfortunately, it is not always possible to have (M_N) bounded. In particular, this is the case for the space $H(\mathbb{C})$, and that is why we also need Lemma 4.8.

4.7 Tails

The following lemma will allow us to extend cyclicity from finitely supported vectors to infinitely supported ones.

Lemma 4.13. *If the sequence* $(a_1, s_2, a_2, s_3, \dots)$ *increases rapidly enough, then*

$$\frac{\|T^i e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \le \frac{1}{D_n} \quad \textit{for } j \ge a_n + \Delta_n \textit{ and } i \in [1, a_n + \Delta_n).$$

Proof. We check all the possible cases for j.

- If $j \in [a_n + \Delta_n, s_{n+1} - a_n - \Delta_n)$, then by (4.18),

$$T^i e_j = \frac{1}{\alpha_j} T^{j+i} e_0 = \frac{\alpha_{j+i}}{\alpha_j} e_{j+i}.$$

  Hence, by the definition (4.19) of the numbers α_j and (4.9):

$$\frac{\|T^i e_j\|_{N_n}}{\|e_j\|_{N_n+1}} = \left(\frac{1}{D_n \widetilde{M}_{n+1}}\right)^i \frac{A_{N_n, j+i}}{A_{N_n+1, j}} \le \frac{1}{D_n^i} \frac{M_{N_n}^i}{\widetilde{M}_{n+1}^i} \frac{A_{N_n, j}}{A_{N_n+1, j}} \le \frac{1}{D_n},$$

  where the last inequality is true, because $M_{N_n} \le \widetilde{M}_{n+1}$ (by the definition (4.15) of the numbers $\widetilde{M}_n$) and $A_{N_n, j} \le A_{N_n+1, j}$.

- If $j \in [s_p - a_n - \Delta_n, s_{p+1} - a_n - \Delta_n)$ for some $p > n$, then, by Corollary 4.11:

$$\frac{\|T^i e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \le \frac{\left(\widetilde{M}_{p+1}^2 + 1\right)^{a_n + \Delta_n} A_{N_n, j}}{A_{N_n+1, j}}.$$

  Observe that, by (4.11), $\frac{A_{N_n, j}}{A_{N_n+1, j}} \to 0$. At the same time, $(\widetilde{M}_{p+1}^2 + 1)^{a_n + \Delta_n}$ does not depend on s_p. Therefore, we can make sure that s_p is large enough so that

$$\frac{A_{N_n, j}}{A_{N_n+1, j}} \le \frac{1}{D_n (\widetilde{M}_{p+1}^2 + 1)^{a_n + \Delta_n}} \quad \text{for } j \ge s_p - a_n - \Delta_n. \qquad \square$$

From the lemma we just proved, we immediately have:

Corollary 4.14. *If the sequence* $(a_1, s_2, a_2, s_3, \dots)$ *increases rapidly enough, then for vectors* $x \in \lambda^1(A)$ *such that* $\operatorname{supp} x \subseteq [a_n + \Delta_n, +\infty)$ *and for* $1 \le i < a_n + \Delta_n$ *we have that*

$$\|T^i x\|_{N_n} \le \frac{1}{D_n} \|x\|_{N_n+1}.$$

4.8 THE SETS K_n

In this section we will prove that the compact sets K_n defined in (4.21) cover the part of the unit sphere of the norm $\|\cdot\|_1$ that is contained in c_{00}. Moreover, they form, in some sense, an increasing sequence. This is made precise in Lemma 4.17. First, we need to establish some facts about the projections τ_n defined in Section 4.4.

Lemma 4.15. *Projection $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ defined in* (4.20) *acts on the unit vectors as follows:*

$$\tau_n e_j = \begin{cases} e_j, & j < a_n; \\ -A_{N_n,a_n} T^{j-a_n} e_0, & j \in [a_n, a_n + \Delta_n). \end{cases} \tag{4.24}$$

Proof. Because $(T^j e_0)_{j=0}^{a_n+\Delta_n-1}$ is a perturbed canonical basis of $E_{a_n+\Delta_n-1}$, we always have that $e_j = \sum_{i=0}^{j} \lambda_i T^i e_0$ for some coefficients λ_i. Therefore (4.20) implies that $\tau_n(e_j) = e_j$ for $j < a_n$. On the other hand, for $j \in [a_n, a_n + \Delta_n)$ we have by (4.18) and (4.20) that

$$\tau_n\left(e_j\right) = \tau_n\left(A_{N_n,a_n}\left(T^j e_0 - T^{j-a_n} e_0\right)\right) = -A_{N_n,a_n} T^{j-a_n} e_0. \qquad \square$$

Lemma 4.16. *If the sequence $(a_1, s_2, a_2, s_3, \dots)$ increases rapidly enough, then*

$$\|\tau_n y\|_1 \le \|y\|_{N_n+1}, \tag{4.25}$$

where $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ is defined in (4.20).

Proof. Because the norms are weighted ℓ_1 norms, it is sufficient to show (4.25) only for the basic vectors e_j, $j < a_n + \Delta_n$.

- If $j < a_n$, then by (4.24), $\left\|\tau_n e_j\right\|_1 = \left\|e_j\right\|_1 \le \left\|e_j\right\|_{N_n+1}$.
- If $j \in [a_n, a_n + \Delta_n)$, then (4.24) implies

$$\frac{\left\|\tau_n e_j\right\|_1}{\left\|e_j\right\|_{N_n+1}} \le \frac{A_{N_n,a_n}\left\|T^{j-a_n} e_0\right\|_1}{A_{N_n+1,a_n}}$$

Observe that, by (4.18), $T^i e_0 \in E_{\Delta_n} \subset E_{s_n}$ for $i \le \Delta_n$. Given that $N_1 = 1 \le \widetilde{M}_n$, by Corollary 4.11, we can further estimate:

$$\frac{A_{N_n,a_n}\left\|T^{j-a_n} e_0\right\|_1}{A_{N_n+1,a_n}} \le \frac{A_{N_n,a_n}\left(\widetilde{M}_n^2 + 1\right)^{\Delta_n} A_{1,0}}{A_{N_n+1,a_n}}.$$

The resulting quantity will be smaller than 1, as required, if a_n is chosen large enough, as $\frac{A_{N_n,a_n}}{A_{N_n+1,a_n}}$ can be made arbitrarily small by (4.11). $\square$

Recall that π_m denotes the truncation operator onto E_m.

Lemma 4.17. *Let (n_k) be a sequence such that $N_{n_k} = N$. Take $x \in \lambda^1(A)$ such that $\|x\|_1 = 1$. Then for all but finitely many k*

$$\pi_{a_{n_k}+\Delta_{n_k}-1} x \in K_{n_k}.$$

Proof. In view of (4.21) we need only to show that $\left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\right\|_1 \geq \frac{1}{2}$ holds for all but finitely many k. But with the help of (4.24) and (4.25) we get that:

$$\begin{aligned}\left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\right\|_1 &= \left\|\tau_{n_k}\pi_{a_{n_k}-1}x + \tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\right\|_1 \\ &= \left\|\pi_{a_{n_k}-1}x + \tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\right\|_1 \\ &\geq \left\|\pi_{a_{n_k}-1}x\right\|_1 - \left\|\tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\right\|_1 \\ &\geq \left\|\pi_{a_{n_k-1}}x\right\|_1 - \left\|\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\right\|_{N+1} \xrightarrow[k\to\infty]{} 1.\end{aligned}$$ □

4.9 The proof

After the laborious work done in the previous sections, we are in position to prove Theorem 4.1.

Take $x \in \lambda^1(A)$ with $\|x\|_1 = 1$. We will show that x is a cyclic vector for T. Keeping in mind the discussion in Section 4.2, we need to show that $e_0 \in \overline{\mathrm{Lin}(x,T)}$. As the unit balls of the norms form a basis of neighbourhoods of zero, it suffices to show that for any N there is a polynomial P such that $\|P(T)x - e_0\|_N \leq 4$.

In order to do that, first we find by Lemma 4.17 a number n such that simultaneously

- $N_n = N$,
- $\pi_{a_n+\Delta_n-1}x \in K_n$,
- $\left\|\pi_{[a_n+\Delta_n,+\infty)}x\right\|_{N+1} \leq 1$.

Then, by Corollary 4.6, there exists a polynomial P with $|P| \leq D_n$ and $\operatorname{supp}\operatorname{coeff} P \subseteq [1, a_n + \Delta_n)$ such that

$$\begin{aligned}\|P(T)x - e_0\|_N &\leq \|P(T)\pi_{a_n+\Delta_n-1}x - e_0\|_N + \left\|P(T)\pi_{[a_n+\Delta_n,+\infty)}x\right\|_N \\ &\leq 3 + |P| \sup_{1\leq i<a_n+\Delta_n} \left\|T^i\pi_{[a_n+\Delta_n,+\infty)}x\right\|_N \\ &\leq 3 + D_n\frac{\left\|\pi_{[a_n+\Delta_n,+\infty)}x\right\|_{N+1}}{D_n} \leq 4,\end{aligned}$$

where in the penultimate inequality we have used Corollary 4.14. This proves Theorem 4.1.

Remark 4.18. The reader might be wondering, how fast does the sequence (4.16) have to increase. An article [34] by Śliwa gives some insight into that. Using Read's methods, but with a more explicit version of Lemma 3.3, Śliwa was able to show, that if one takes the sequence

$$\begin{cases} d_0 = 2, \\ d_n = 8^{2d_{n-1}}, n \geq 1 \end{cases}$$

and puts $a_n = d_{2n-1}$, $b_n = d_{2n}$ (note, that in our construction we have ultimately not needed the analogue of the numbers b_n, but they are still necessary on ℓ_1), then the operator constructed in [34], similar to (4.18), has no nontrivial invariant subspace. While one cannot of course apply this result directly, it gives at least some idea about the necessary growth of (a_n).

Chapter 5

Invariant subsets

In the previous chapter we constructed, in particular, a continuous linear operator on the space s of rapidly decreasing sequences for which every non-zero vector is *cyclic*. In this chapter we want to push it even further, namely ensure that every non-zero vector is *hypercyclic*. Observe that such an operator has not only no nontrivial invariant subspace but also no nontrivial invariant (closed) subset.

Theorem 5.1. *There exists a continuous linear operator $T\colon s \to s$ for which all non-zero vectors are hypercyclic.*

Construction is built on ideas similar to the ones used in the previous chapter – in particular heavily relies on Lemma 3.3 – but the operator is even more complicated. This construction is based on another Read's article [29] but, once again, can be made much simpler due to the structure of a non-normable Fréchet space. The construction is described also in the paper [17] of the author.

5.1 The strategy

In the previous chapter we were able to construct T such that for any x with $\|x\|_1 = 1$ and any norm $\|\cdot\|_N$ we could find a polynomial P such that

$$\|P(T)x - e_0\|_N \leq 3.$$

As a consequence x was cyclic for T.

In this chapter we want to show that x can be made hypercyclic so, remembering that e_0 was a cyclic vector for T, it would suffice to show that for any polynomial S, any non-zero x and any norm $\|\cdot\|_N$,

$$\left\|T^i x - S(T)e_0\right\|_N \leq 3,$$

for some integer i.

We want to follow along the lines of what we have done in the previous chapter. For finitely supported vectors x we will try to find polynomials P such that $\|P(T)x - e_0\|_N$ is much smaller than 1. Then, as long as a polynomial S is not "too big", by continuity of T,

$$\|S(T)P(T)x - S(T)e_0\|_N \leq 1.$$

The vector $S(T)P(T)x$ has still finite support, and we will try to make T behave in such a way, that some very high iterate T^i "resembles" $S(T)P(T)$, at least for finitely supported arguments. Then

$$\|T^i x - S(T)P(T)x\|_N \le 1.$$

Doing this, we should also keep in mind the infinitely supported vectors. Given such a vector $y \in s$ we want to define a finitely supported vector x and estimate by the triangle inequality

$$\|T^i y - S(T)e_0\|_N \le \|T^i(y-x)\|_N + \|T^i x - S(T)e_0\|_N.$$

Of course the problem lies in defining T in such a way, that the above procedure can be carried out for every $y \in s$. As in the previous chapter, the Fréchet space topology will allow us to estimate $\|T^i(y-x)\|_N$ with $\|y-x\|_{N+1}$, which is much easier than estimating with $\|y-x\|_N$.

5.2 The matrix

In this chapter we will be working with a single Köthe space only.

Recall that $s = \Lambda_\infty(\log(j+1))$, so $s = \lambda^1([(j+1)^N])$. By looking at the proof of Lemma 4.4 we get the following corollary.

Corollary 5.2. *There is a Köthe matrix A such that $\lambda^1(A) = s$ and*

$$\forall N \in \mathbb{N}_+ \ \forall j \in \mathbb{N} \quad A_{N,j} \ge 1, \tag{5.1}$$

$$\forall N \in \mathbb{N}_+ \ \forall j \in \mathbb{N} \quad \frac{A_{N,j+1}}{A_{N,j}} \le \frac{3}{2}, \tag{5.2}$$

$$\forall N \in \mathbb{N}_+ \ \forall h \in \mathbb{N} \quad \lim_{j\to\infty} \frac{A_{N,j+h}}{A_{N+1,j}} = 0, \tag{5.3}$$

$$\forall N \in \mathbb{N}_+ \quad \text{the sequence } (A_{N,j})_j \text{ tends monotonically to } +\infty, \tag{5.4}$$

$$\forall N \in \mathbb{N}_+ \quad \text{the sequence } (A_{N,j})_j \text{ is a polynomial of degree } N \text{ in } j. \tag{5.5}$$

Moreover, the unit balls of the norms with respect to the matrix A form a basis of neighbourhoods of zero in s.

From now on we assume that the matrix $[A_{N,j}]_{N,j}$ satisfies the assumptions above.

Remark 5.3. The reader might wonder at this point why, in contrast to the previous chapter, the theorem is not stated for a wider class of Köthe spaces. While the author thinks this indeed is possible, the only natural space in this class would be s.

The reason is buried deep inside the proof: at some point of the construction one has to compare two functions of j, $\left(A_{N,j^k}\right)^k$ and 2^j, where N and k are fixed integers. We want to have that 2^j increases faster than $\left(A_{N,j^k}\right)^k$. This is certainly true, if $A_{N,j}$ is a polynomial in j, but fails dramatically if, e.g., $A_{N,j} = (1+\varepsilon)^j$. While this might seem like a mere technicality, the author was unable to overcome this difficulty.

5.3 The operator

As in Chapter 4, let

$$N_n = (1, 2, 1, 2, 3, 1, 2, 3, 4, \dots).$$

Assume we are given an integer sequence (μ_n) and an increasing sequence

$$(\Delta_1, a_1, c_1, \Delta_2, a_2, c_2, \dots) \tag{5.6}$$

such that

$$\begin{aligned}
1 = \Delta_1 &< a_1 < a_1 + \Delta_1 \\
&< c_1 < c_1 + a_1 + \Delta_1 < c_1^2 < c_1^2 + a_1 + \Delta_1 < c_1^3 < \dots < c_1^{\mu_1} < c_1^{\mu_1} + a_1 + \Delta_1 \\
= \Delta_2 &< a_2 < a_2 + \Delta_2 \\
&< c_2 < c_2 + a_2 + \Delta_2 < c_2^2 < c_2^2 + a_2 + \Delta_2 < c_2^3 < \dots < c_2^{\mu_1} < c_2^{\mu_1} + a_2 + \Delta_2 \\
= \Delta_3 &< a_3 < a_3 + \Delta_3 \\
&< c_3 < c_3 + a_3 + \Delta_3 < c_3^2 < \dots
\end{aligned}$$

together with polynomials $Q_{n,k}$ for $1 \le k \le \mu_n$ such that

$$\deg Q_{n,k} < c_n^k.$$

By Remark 3.2, assuming that all the numbers α_j appearing in the formula below are non-zero, the following relation is satisfied by a precisely one linear operator $T\colon c_{00} \to c_{00}$:

$$T^j e_0 = \begin{cases} \frac{1}{n2^n A_{N_n,a_n}} e_j + T^{j-a_n} e_0, & j \in [a_n, a_n + \Delta_n); \\ \frac{1}{D_n A_{N_n,c_n^k}} e_j + Q_{n,k}(T) T^{j-c_n^k} e_0, & j \in [c_n^k, c_n^k + a_n + \Delta_n), 1 \le k \le \mu_n; \\ \alpha_j e_j, & \text{otherwise.} \end{cases} \tag{5.7}$$

Remark 5.4. The sequence (μ_n), the sequence from (5.6), the polynomials $Q_{n,k}$, numbers D_n and α_j will be fixed in the inductive procedure in the next section.

Further on we will show that with appropriate choices for all the parameters T can be made continuous (see Lemma 5.7).

The role of the intervals $[a_n, a_n + \Delta_n)$ will be the same as in the previous chapter. For vectors $x \in c_{00}$, they will allow us to find a polynomial P, such that $P(T)x$ is very close to e_0. If now S is any polynomial, then by continuity, $S(T)P(T)x$ is still close to $S(T)e_0$.

The new intervals $[c_n^k, c_n^k + a_n + \Delta_n)$ will allow us to approximate $S(T)P(T)x$ with $T^{c_n^k}x$, making x hypercyclic.

5.4 The parameters

As we have already said, the procedure resembles a lot the procedure from the previous chapter, and will be presented in a similar fashion. This time the basic interval for the induction is $[\Delta_n, c_n^{\mu_n} + a_n + \Delta_n)$.

Assume that all the numbers a_k and c_k and μ_k for $k < n$ have been fixed. We put $\Delta_n = c_{n-1}^{\mu_{n-1}} + a_{n-1} + \Delta_{n-1}$ ($\Delta_1 = 1$). We choose a_n to be some number much larger than Δ_n and we put

$$\alpha_j = \frac{\left(\frac{4}{3}\right)^{\frac{j-\Delta_n}{\Delta_n}}}{n2^n A_{N_n,a_n}}, \quad \text{for } j \in [\Delta_n, a_n). \tag{5.8}$$

Together with (5.7) this defines the vectors $T^j e_0$ for $j < a_n + \Delta_n$. Once again, we can define a projection $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ by

$$\sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0 \overset{\tau_n}{\longmapsto} \sum_{j=0}^{a_n-1} \lambda_j T^j e_0 \tag{5.9}$$

and with its help a compact set

$$K_n = \left\{ y \in E_{a_n+\Delta_n-1} : \|y\|_1 \le 1 \text{ and } \|\tau_n y\|_1 \ge \frac{1}{2} \right\}. \tag{5.10}$$

Now, similarly to what we did in the previous chapter, we can use Lemma 3.3 with $\widetilde{\gamma} = (e_0, Te_0, T^2e_0, \dots, T^{a_n+\Delta_n-1}e_0)$, $\|\cdot\| = \|\cdot\|_{N_n}$ and $K = K_n$. From the lemma we get a number $D_n \ge 1$ and a finite family $\mathcal{P}_n = \{P_{n,l}\}_{l=1}^{L_n}$ of polynomials of degree smaller than $a_n + \Delta_n$ and satisfying $|P_{n,l}| \le D_n$.

Let $\mathcal{S}_n = \{S_{n,w}\}_{w=1}^{W_n}$ be a finite $\frac{1}{2^n A_{N_n,0}}$-net (with respect to the norm $|\cdot|$) in the set of all polynomials of degree at most n and sum of coefficients at most n. We put $\mu_n = W_n \cdot L_n$ and fix any bijection $\rho_n\colon \{1, 2, \dots, L_n\} \times \{1, 2, \dots, W_n\} \to \{1, 2, \dots, \mu_n\}$. We put in (5.7)

$$Q_{n,k} = P_{n,l} S_{n,w}, \text{ where } k = \rho_n(l, w). \tag{5.11}$$

We now choose c_n to be a number much greater than $a_n + \Delta_n$ and put

$$\alpha_j = \begin{cases} \dfrac{2^{\frac{j-a_n-\Delta_n}{a_n}}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}, & j \in [a_n + \Delta_n, c_n); \\[2ex] \dfrac{2^{\frac{j-c_n^k-a_n-\Delta_n}{c_n^k}}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^{k+2}}, & j \in [c_n^k + a_n + \Delta_n, c_n^{k+1}), \quad 1 \le k \le \mu_n. \end{cases} \tag{5.12}$$

Once again we give an illustration of the order of choices made in the procedure above. As before a dashed arrow indicates a choice, while a solid arrow shows where the parameters are defined through functions:

$$\begin{array}{l}
1 = \Delta_1 \dashrightarrow a_1 \xrightarrow{(5.8)} \{\alpha_j\}_{j=\Delta_1}^{a_1-1} \xrightarrow{(5.7)} \{T^j e_0\}_{j=\Delta_1}^{a_1+\Delta_1-1} \xrightarrow{(5.9),\,(5.10)} \tau_1, K_1 \xrightarrow{\text{Lemma 3.3}} D_1, \mathcal{P}_1 \\
\mathcal{S}_1 \to \mu_1 \longrightarrow \{Q_{1,k}\}_{k=1}^{\mu_1} \dashrightarrow c_1 \xrightarrow{(5.12)} \{\alpha_j\}_{j=a_1+\Delta_1}^{c_1^{\mu_1}-1} \xrightarrow{(5.7)} \{T^j e_0\}_{j=a_1+\Delta_1}^{c_1^{\mu_1}-1} \\
\to \Delta_2 \dashrightarrow a_2 \xrightarrow{(5.8)} \{\alpha_j\}_{j=\Delta_2}^{a_2-1} \xrightarrow{(5.7)} \{T^j e_0\}_{j=\Delta_2}^{a_2+\Delta_2-1} \xrightarrow{(5.9),\,(5.10)} \tau_2, K_2 \xrightarrow{\text{Lemma 3.3}} D_2, \mathcal{P}_2 \\
\mathcal{S}_2 \to \mu_2 \longrightarrow \{Q_{2,k}\}_{k=1}^{\mu_2} \dashrightarrow c_2 \xrightarrow{(5.12)} \{\alpha_j\}_{j=a_2+\Delta_2}^{c_2^{\mu_2}-1} \xrightarrow{(5.7)} \{T^j e_0\}_{j=a_1+\Delta_1}^{c_2^{\mu_2}-1} \\
\to \Delta_3 \dashrightarrow a_3 \cdots
\end{array} \tag{5.13}$$

Remark 5.5. In the diagram above the items $\{\alpha_j\}_{j=a_1+\Delta_1}^{c_1^{\mu_1}-1}$ and $\{\alpha_j\}_{j=a_2+\Delta_2}^{c_2^{\mu_2}-1}$ might be a bit misleading, as in fact not all the indicated numbers exist. The items should be viewed as a convenient shorthand for what (5.12) actually does.

Corollary 5.6. *If c_n is chosen large enough compared to a_n, then for each $y \in K_n$ there exists a polynomial $P_{n,k} \in \mathcal{P}_n$ with $|P_{n,k}| \leq D_n$ and $\deg P_{n,k} < a_n + \Delta_n$ such that*

$$\|P_{n,k}(T)y - e_0\|_{N_n} \leq \frac{3}{n2^n}.$$

Moreover, if $y = \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0$, then $\sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \leq D_n$.

Proof. In the inductive procedure carried out in this section we have already used Lemma 3.3 with $a = a_n$, $\Delta = \Delta_n$, $K = K_n$, $\widetilde{y} = (e_0, Te_0, T^2e_0, \ldots, T^{a_n+\Delta_n-1})$ and $\|\cdot\| = \|\cdot\|_{N_n}$. The lemma implies the existence of $\mathcal{P}_n$ and the "moreover" part of the claim. We also get from Lemma 3.3, by (5.7) and (5.12), that (assuming that $c_n > 2(a_n + \Delta_n - 1)$):

$$\begin{aligned}
\|P_{n,k}(T)y - e_0\|_{N_n} &\leq \frac{2}{n2^n A_{N_n,a_n}} \|e_{a_n}\|_{N_n} + D_n \times \max_{a_n+\Delta_n \leq j \leq 2(a_n+\Delta_n-1)} \left\|T^j e_0\right\|_{N_n} \\
&= \frac{2}{n2^n} + D_n \times \max_{a_n+\Delta_n \leq j \leq 2(a_n+\Delta_n-1)} \left\|\alpha_j e_j\right\|_{N_n} \\
&\leq \frac{2}{n2^n} + \frac{D_n 2^{\frac{a_n+\Delta_n-2}{a_n}} A_{N_n,2(a_n+\Delta_n-1)}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}.
\end{aligned}$$

Observe now, that the numerator in the last fraction does not depend on c_n, so by taking c_n very large, we can make the whole fraction arbitrarily small by (5.4). □

5.5 Continuity

We will now prove the following lemma.

Lemma 5.7. *If the sequence $(a_1, c_1, a_2, c_2, \ldots)$ increases rapidly enough, then the operator T given by* (5.7) *satisfies*

$$\frac{\left\|Te_j\right\|_N}{\left\|e_j\right\|_N} \leq 2 \quad \textit{for all } j \in \mathbb{N}.$$

Proof. We will prove the inequality in all the possible cases for j. The proof is very similar to what we have already seen in the previous chapter, but most calculations have to be done once again, as the operator is different.

- If $j = 0$, then by (5.2) and (5.8)

$$\frac{\|Te_0\|_N}{\|e_0\|_N} = \frac{\|\alpha_1 e_1\|_N}{\|e_0\|_N} = \frac{\alpha_1 A_{N,1}}{A_{N,0}} \leq \frac{3}{4A_{N_1,a_1}} \leq \frac{3}{4}.$$

- If $j \in [\Delta_n, a_n - 1) \cup [a_n + \Delta_n, c_n - 1) \cup \bigcup_{k=1}^{\mu_n - 1} [c_n^k + a_n + \Delta_n, c_n^{k+1} - 1)$ for some n, then it is easy to check that $Te_j = \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}$. Hence by (5.2) and the definitions (5.8) and (5.12),
$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\alpha_{j+1}}{\alpha_j} \frac{A_{N,j+1}}{A_{N,j}} \leq \frac{4}{3} \cdot \frac{3}{2} = 2,$$
since $2^{1/a_n}$ and $2^{1/c_n^k}$ are smaller than $\frac{4}{3}$ for a_n and c_n large enough.

- If $j \in [a_n, a_n + \Delta_n - 1) \cup \bigcup_{k=1}^{\mu_n} [c_n^k, c_n^k + a_n + \Delta_n - 1)$, then by (5.7), $Te_j = e_{j+1}$. Indeed, if $j \in [a_n, a_n + \Delta_n - 1)$, then
$$\begin{aligned} Te_j &= T\left(n2^n A_{N_n,a_n}\left(T^j e_0 - T^{j-a_n} e_0\right)\right) \\ &= n2^n A_{N_n,a_n}\left(T^{j+1} e_0 - T^{j-a_n+1} e_0\right) \\ &= n2^n A_{N_n,a_n}\left(\frac{1}{n2^n A_{N_n,a_n}} e_{j+1} + T^{j-a_n+1} e_0 - T^{j-a_n+1} e_0\right) = e_{j+1}. \end{aligned}$$
On the other hand, if $j \in \bigcup_{k=1}^{\mu_n} [c_n^k, c_n^k + a_n + \Delta_n - 1)$, then
$$\begin{aligned} Te_j &= T\left(D_n A_{N_n,c_n^k}\left(T^j e_0 - Q_{n,k}(T) T^{j-c_n^k} e_0\right)\right) \\ &= D_n A_{N_n,c_n^k}\left(T^{j+1} e_0 - T^{j-c_n^k+1} e_0\right) \\ &= D_n A_{N_n,c_n^k}\left(\frac{1}{D_n A_{N_n,c_n^k}} e_{j+1} + T^{j-c_n^k+1} e_0 - T^{j-c_n^k+1} e_0\right) = e_{j+1}. \end{aligned}$$
Consequently, by (5.2),
$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{A_{N,j+1}}{A_{N,j}} \leq \frac{3}{2}.$$

- If $j = a_n - 1$ for some n, then by (5.7),
$$Te_{a_n-1} = T\left(\frac{1}{\alpha_{a_n-1}} T^{a_n-1} e_0\right) = \frac{1}{\alpha_{a_n-1}}\left(\frac{1}{n2^n A_{N_n,a_n}} e_{a_n} + e_0\right).$$
Consequently, by (5.8), (5.1), (5.2) and (5.4),
$$\begin{aligned} \frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} &= \frac{n2^n A_{N_n,a_n}}{\left(\frac{4}{3}\right)^{\frac{a_n-\Delta_n-1}{\Delta_n}}}\left(\frac{1}{n2^n A_{N_n,a_n}} \frac{A_{N,a_n}}{A_{N,a_n-1}} + \frac{A_{N,0}}{A_{N,a_n-1}}\right) \\ &\leq \frac{n2^n A_{N_n,a_n}}{\left(\frac{4}{3}\right)^{\frac{a_n-\Delta_n-1}{\Delta_n}}}\left(\frac{3}{2} + 1\right). \end{aligned}$$
Observe that the fraction can be made arbitrarily small by taking a_n large enough, since, by (5.5), the numerator increases polynomially with a_n, while the denominator increases exponentially.

- If $j = a_n + \Delta_n - 1$, then

$$\begin{aligned} Te_{a_n+\Delta_n-1} &= T\left(n2^n A_{N_n,a_n}\left(T^{a_n+\Delta_n-1}e_0 - T^{\Delta_n-1}e_0\right)\right) \\ &= n2^n A_{N_n,a_n}\left(\alpha_{a_n+\Delta_n} e_{a_n+\Delta_n} - \alpha_{\Delta_n} e_{\Delta_n}\right). \end{aligned}$$

  Together with (5.2), (5.8) and (5.12) this implies that

$$\begin{aligned} \frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} &= n2^n A_{N_n,a_n}\left(\frac{1}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}\frac{A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} + \frac{1}{n2^n A_{N_n,a_n}}\frac{A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}}\right) \\ &\leq \frac{n2^n A_{N_n,a_n}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}\cdot\frac{3}{2} + 1. \end{aligned}$$

  The fraction can be made arbitrarily small when c_n is taken sufficiently large.

The remaining two cases need induction over j in their proofs. Observe that, because the norms are weighted ℓ_1 norms, proving our claim for $j \leq s$ implies that $\|Tx\|_N \leq 2\|x\|_N$ for $x \in E_s$. This is our inductive hypothesis.

- If $j = c_n^k - 1$ for some n, k and $k = \rho_n(l, w)$, then by (5.7)

$$Te_{c_n^k-1} = T\left(\frac{1}{\alpha_{c_n^k-1}}T^{c_n^k-1}e_0\right) = \frac{1}{\alpha_{c_n^k-1}}\left(\frac{1}{D_n A_{N_n,c_n^k}}e_{c_n^k} + Q_{n,k}(T)e_0\right).$$

  We have, by the definition (5.11), that

$$\deg Q_{n,k} = \deg P_{n,l}S_{n,w} < a_n + \Delta_n + n.$$

  Hence we can assume that $\deg Q_{n,k}$ is much smaller than c_n, so by the inductive hypothesis, we can estimate

$$\begin{aligned} \|Q_{n,k}(T)e_0\|_N &\leq \|P_{n,l}(T)S_{n,w}(T)e_0\|_N \\ &\leq |P_{n,l}||S_{n,w}|2^{a_n+\Delta_n+n}\|e_0\|_N \\ &\leq nD_n 2^{a_n+\Delta_n+n}A_{N,0}. \end{aligned}$$

  Therefore, by (5.2),

$$\begin{aligned} \frac{\|Te_{c_n^k-1}\|_N}{\|e_{c_n^k-1}\|_N} &\leq \frac{1}{\alpha_{c_n^k-1}}\left(\frac{1}{D_n A_{N_n,c_n^k}}\frac{A_{N,c_n^k}}{A_{N,c_n^k-1}} + nD_n 2^{a_n+\Delta_n+n}\frac{A_{N,0}}{A_{N,c_n^k-1}}\right) \\ &\leq \frac{1}{\alpha_{c_n^k-1}}\left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n}\right) \end{aligned}$$

  We now consider two subcases:

  - ◇ If $k = 1$, then the appropriate definition from (5.12) implies

$$\frac{\|Te_{c_n-1}\|_N}{\|e_{c_n-1}\|_N} \leq \frac{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}{2^{\frac{c_n-a_n-\Delta_n-1}{a_n}}}\left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n}\right).$$

◇ If $k > 1$, then by (5.12) we get

$$\frac{\left\| Te_{c_n^k-1} \right\|_N}{\left\| e_{c_n^k-1} \right\|_N} \le \frac{\left(A_{N_n,2c_n^{\mu_n}}\right)^{k+1}}{2^{\frac{c_n^k - c_n^{k-1} - a_n - \Delta_n - 1}{c_n^{k-1}}}} \left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n} \right)$$
$$\le \frac{\left(A_{N_n,2c_n^{\mu_n}}\right)^{\mu_n+1}}{2^{c_n-2}} \left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n} \right).$$

In both cases the resulting quantity can be made arbitrarily small by taking c_n sufficiently large, as denominators increase exponentially with c_n, while numerators increase only at a polynomial rate (see (5.5)).

- If $j = c_n^k + a_n + \Delta_n - 1$ for some n, k and $k = \rho_n(l, w)$, then (5.7) implies that

$$Te_{c_n^k+a_n+\Delta_n-1} = T\left(D_n A_{N_n,c_n^k} \left(T^{c_n^k+a_n+\Delta_n-1} e_0 - Q_{n,k}(T) T^{a_n+\Delta_n-1} e_0 \right) \right)$$
$$= D_n A_{N_n,c_n^k} \left(\alpha_{c_n^k+a_n+\Delta_n} e_{c_n^k+a_n+\Delta_n} - Q_{n,k}(T) \alpha_{a_n+\Delta_n} e_{a_n+\Delta_n} \right).$$

Similarly as in the previous case, we may assume by induction that

$$\| Q_{n,k}(T) e_{a_n+\Delta_n} \|_N \le nD_n 2^{a_n+\Delta_n+n} \| e_{a_n+\Delta_n} \|_N = nD_n 2^{a_n+\Delta_n+n} A_{N,a_n+\Delta_n}.$$

By (5.12), we have for $1 \le k < \mu_n$ that

$$\alpha_{c_n^k+a_n+\Delta_n} \le \alpha_{c_n+a_n+\Delta_n} < \alpha_{a_n+\Delta_n} = \frac{1}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}.$$

Similarly, for $k = \mu_n$, by (5.8),

$$\alpha_{c_n^{\mu_n}+a_n+\Delta_n} = \alpha_{\Delta_{n+1}} = \frac{1}{(n+1)2^{n+1} A_{N_{n+1},a_{n+1}}} < \frac{1}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2}$$

if a_{n+1} is large enough.

Therefore we have, by (5.2) and (5.4), that

$$\frac{\left\| Te_{c_n^k+a_n+\Delta_n-1} \right\|_N}{\left\| e_{c_n^k+a_n+\Delta_n-1} \right\|_N} \le \frac{D_n A_{N_n,c_n^k}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2} \left(\frac{A_{N,c_n^k+a_n+\Delta_n}}{A_{N,c_n^k+a_n+\Delta_n-1}} + nD_n 2^{a_n+\Delta_n+n} \frac{A_{N,a_n+\Delta_n}}{A_{N,c_n^k+a_n+\Delta_n-1}} \right)$$
$$\le \frac{D_n A_{N_n,c_n^{\mu_n}}}{\left(A_{N_n,2c_n^{\mu_n}}\right)^2} \left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n} \right)$$
$$\le \frac{D_n}{A_{N_n,2c_n^{\mu_n}}} \left(\frac{3}{2} + nD_n 2^{a_n+\Delta_n+n} \right).$$

By (5.4), in this last quantity the denominator can be made arbitrarily large compared to the rest, because c_n is chosen after all the other parameters are fixed. □

From Lemma 5.7 we get immediately:

Corollary 5.8. *If the sequence* $(a_1, c_1, a_2, c_2, \dots)$ *increases rapidly enough, then the linear operator* $T\colon c_{00} \to c_{00}$ *defined by* (5.7) *can be uniquely extended to a continuous linear operator* $T\colon s \to s$ *satisfying for each* $x \in s$ *and each* N

$$\|Tx\|_N \le 2\|x\|_N.$$

5.6 Tails

In this section we will prove an analogue of Lemma 4.13. In some sense this is a much stronger version – we are dealing with much higher powers of T. That is why the proof will be much longer and more complicated than the proof of Lemma 4.13.

Lemma 5.9. *If the sequence $(a_1, c_1, a_2, c_2, \ldots)$ increases sufficiently rapidly, then for each $j \in [a_n + \Delta_n, +\infty)$ and $1 \le k \le \mu_n$ we have that*

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \le 3.$$

Proof. Let us fix k and assume that $k = \rho_n(l, w)$. We check all the possible cases for j. First, we will consider what happens if $j < \Delta_{n+1}$.

- If $j \in [c_n^p, c_n^p + a_n + \Delta_n)$ for some $1 \le p \le \mu_n$, then (5.7) implies that

$$\begin{aligned} T^{c_n^k} e_j &= D_n A_{N_n, c_n^p} \left(T^{j + c_n^k} e_0 - Q_{n,p}(T) T^{j - c_n^p + c_n^k} e_0 \right) \\ &= D_n A_{N_n, c_n^p} T^{j + c_n^k} e_0 - \frac{D_n A_{N_n, c_n^p}}{D_n A_{N_n, c_n^k}} Q_{n,p}(T) e_{j - c_n^p + c_n^k} \\ &\quad - D_n A_{N_n, c_n^p} Q_{n,p}(T) Q_{n,k}(T) T^{j - c_n^p} e_0. \end{aligned}$$

  Corollary 5.8 and the definition (5.11) of $Q_{n,k}$ and $Q_{n,p}$ imply

$$\left\| Q_{n,p}(T) e_{j - c_n^p + c_n^k} \right\|_{N_n} \le n D_n 2^{a_n + \Delta_n + n} \left\| e_{j - c_n^p + c_n^k} \right\|_{N_n}$$

  and

$$\left\| Q_{n,p}(T) Q_{n,k}(T) T^{j - c_n^p} e_0 \right\|_{N_n} \le \left(n D_n 2^{a_n + \Delta_n + n} \right)^2 2^{a_n + \Delta_n} \left\| e_0 \right\|_{N_n}.$$

  Observe that $j + c_n^k \in [c_n^p + c_n^k, c_n^p + c_n^k + a_n + \Delta_n)$. Therefore, if $\max(p, k) < \mu_n$,

$$\begin{aligned} \left[c_n^p + c_n^k, c_n^p + c_n^k + a_n + \Delta_n \right) &\subseteq \left[c_n^{\max(p,k)} + a_n + \Delta_n, 2 c_n^{\max(p,k)} + a_n + \Delta_n \right) \\ &\subseteq \left[c_n^{\max(p,k)} + a_n + \Delta_n, c_n^{\max(p,k)+1} \right). \end{aligned}$$

  If $\max(p, k) = \mu_n$ and a_{n+1} is large enough, then

$$\begin{aligned} \left[c_n^p + c_n^k, c_n^p + c_n^k + a_n + \Delta_n \right) &\subseteq \left[c_n^{\mu_n} + a_n + \Delta_n, 2 c_n^{\mu_n} + a_n + \Delta_n \right) \\ &\subseteq \left[\Delta_{n+1}, a_{n+1} \right). \end{aligned}$$

Consequently, regardless of p and k, we have that $T^{j+c_n^k} e_0 = \alpha_{j+c_n^k} e_{j+c_n^k}$. Thus

$$\begin{aligned}\frac{\left\|T^{c_n^k} e_j\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} &\le D_n A_{N_n,c_n^p} \alpha_{j+c_n^k} \frac{A_{N_n,j+c_n^k}}{A_{N_n+1,j}} + \frac{A_{N_n,c_n^p}}{A_{N_n,c_n^k}} n D_n 2^{a_n+\Delta_n+n} \frac{A_{N_n,j-c_n^p+c_n^k}}{A_{N_n+1,j}} \\ &\qquad + n^2 D_n^3 2^{3(a_n+\Delta_n)+2n} \frac{A_{N_n,c_n^p} A_{N_n,0}}{A_{N_n+1,j}} \\ &\le D_n A_{N_n,c_n^p} \alpha_{c_n^p+c_n^k+a_n+\Delta_n} \frac{A_{N_n,c_n^p+c_n^k+a_n+\Delta_n}}{A_{N_n+1,c_n^p}} \\ &\qquad + \frac{A_{N_n,c_n^p}}{A_{N_n,c_n^k}} n D_n 2^{a_n+\Delta_n+n} \frac{A_{N_n,c_n^k+a_n+\Delta_n}}{A_{N_n+1,c_n^p}} \\ &\qquad + n^2 D_n^3 2^{3(a_n+\Delta_n)+2n} \frac{A_{N_n,c_n^p} A_{N_n,0}}{A_{N_n+1,c_n^p}}.\end{aligned} \tag{5.14}$$

We will deal with the three terms in (5.14) individually. For further reference, observe that, by (5.5), we have that $\frac{A_{N_n,j+a_n+\Delta_n}}{A_{N_n,j}} \xrightarrow[j\to\infty]{} 1$ and $\frac{A_{N_n,2j}}{A_{N_n+1,j}} \xrightarrow[j\to\infty]{} 0$. Therefore, by a suitable choice of c_n we can ascertain that

$$\frac{A_{N_n,j+a_n+\Delta_n}}{A_{N_n,j}} \le 2 \qquad \text{for } j \ge c_n \tag{5.15}$$

and

$$\frac{A_{N_n,2j}}{A_{N_n,j}} \le \frac{1}{2} \qquad \text{for } j \ge c_n \tag{5.16}$$

and, by (5.3),

$$\frac{A_{N_n,j}}{A_{N_n+1,j}} \text{ can be made arbitrarily small for } j \ge c_n, \tag{5.17}$$

where “arbitrarily small” means that we can assume that it is smaller than some quantity depending on the parameters in (5.13) up to, but not including, c_n (appropriate quantity will become clear later).

- ◇ For the first term in (5.14) we have by (5.12) and (5.8) that

$$\alpha_{c_n^p+c_n^k+a_n+\Delta_n} \le 2\alpha_{\max(c_n^p,c_n^k)+a_n+\Delta_n} \le \frac{2}{A_{N_n,2c_n^{\mu_n}}},$$

where, if $\max(p,k) = \mu_n$, we need to assume that a_{n+1} is large enough. So, using (5.15),

$$D_n A_{N_n,c_n^p} \alpha_{c_n^p+c_n^k+a_n+\Delta_n} \frac{A_{N_n,c_n^p+c_n^k+a_n+\Delta_n}}{A_{N_n+1,c_n^p}} \le \frac{4 D_n A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} \frac{A_{N_n,2c_n^{\mu_n}}}{A_{N_n,2c_n^{\mu_n}}}.$$

The fraction we are left with can be made as small as we wish because of (5.17).

◇ For the second term in (5.14) we have, by (5.15), that:

$$\frac{A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} nD_n 2^{a_n+\Delta_n+n} \frac{A_{N_n,c_n^k+a_n+\Delta_n}}{A_{N_n,c_n^k}} \le \frac{A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} nD_n 2^{a_n+\Delta_n+n+1}.$$

Once again the resulting quantity can be assumed small by (5.17).

◇ The last term can also be made small because of (5.17).

Now we will consider all the intervals from (5.12). It will be convenient to discuss separately the cases $j < c_n^k$ and $j > c_n^k$.

- If $j \in [a_n + \Delta_n, c_n) \cup \bigcup_{p=2}^{k} [c_n^{p-1} + a_n + \Delta_n, c_n^p)$, then (5.7) implies that

$$T^{c_n^k} e_j = \frac{1}{\alpha_j} T^{c_n^k + j} e_0.$$

We have that $c_n^k + j \in [c_n^k + a_n + \Delta_n, c_n^{k+1})$ or $c_n^k + j \in [\Delta_{n+1}, a_{n+1})$ (if $k = \mu_n$ and a_{n+1} is large enough). In either case

$$T^{c_n^k} e_j = \frac{\alpha_{c_n^k+j}}{\alpha_j} e_{c_n^k+j}.$$

Definitions (5.12) and (5.8) imply that on each contiguous interval where α_j are defined, they are in fact increasing. Hence, if $j \in [a_n + \Delta_n, c_n)$, then $\alpha_j \ge \alpha_{a_n+\Delta_n}$. Similarly, if $j \in [c_n^{p-1} + a_n + \Delta_n, c_n^p)$, then $\alpha_j \ge \alpha_{c_n^{p-1}+a_n+\Delta_n}$. Therefore in the considered case we have that

$$\alpha_j \ge \begin{cases} \alpha_{a_n+\Delta_n}, & \text{for } k = 1; \\ \alpha_{c_n^{k-1}+a_n+\Delta_n}, & \text{for } k > 1. \end{cases}$$

At the same time

$$\alpha_{c_n^k+j} \le \alpha_{c_n^k+c_n^k} < \alpha_{2c_n^k+a_n+\Delta_n} = \begin{cases} 2\alpha_{c_n^k+a_n+\Delta_n}, & \text{for } k < \mu_n; \\ \frac{4}{3}\alpha_{\Delta_{n+1}}, & \text{for } k = \mu_n. \end{cases}$$

Therefore, if $k = 1$, then by (5.12),

$$\begin{aligned} \frac{\|T^{c_n^k} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} &= \frac{\alpha_{c_n+j}}{\alpha_j} \frac{A_{N_n,c_n+j}}{A_{N_n+1,j}} \\ &\le \frac{2\alpha_{c_n+a_n+\Delta_n}}{\alpha_{a_n+\Delta_n}} A_{N_n,2c_n} \\ &= \frac{2A_{N_n,2c_n}}{A_{N_n,2c_n^{\mu_n}}} \le 2. \end{aligned}$$

If $1 < k < \mu_n$, then by (5.12),

$$\frac{\|T^{c_n^k}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} = \frac{\alpha_{c_n^k+j}}{\alpha_j}\frac{A_{N_n,c_n^k+j}}{A_{N_n+1,j}} \leq \frac{2\alpha_{c_n^k+a_n+\Delta_n}}{\alpha_{c_n^{k-1}+a_n+\Delta_n}}A_{N_n,2c_n^k} = \frac{2A_{N_n,2c_n^k}}{A_{N_n,2c_n^{\mu_n}}} \leq 2.$$

Finally, if $k = \mu_n$, then by (5.12) and (5.8),

$$\frac{\|T^{c_n^k}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} = \frac{\alpha_{c_n^{\mu_n}+j}}{\alpha_j}\frac{A_{N_n,c_n^{\mu_n}+j}}{A_{N_n+1,j}} \leq \frac{4}{3}\frac{\alpha_{\Delta_{n+1}}}{\alpha_{c_n^{\mu_n-1}+a_n+\Delta_n}}A_{N_n,2c_n^{\mu_n}} = \frac{4}{3}\frac{\left(A_{N_n,2c_n^{\mu_n}}\right)^{\mu_n+1}}{(n+1)2^{n+1}A_{N_{n+1},a_{n+1}}}A_{N_n,2c_n^{\mu_n}}.$$

This quantity can be made smaller than 2 by taking a_{n+1} very large compared to c_n.

The next three cases cover the intervals from (5.12) that still have to be checked, i.e., when $j \in \bigcup_{p=k+1}^{\mu_n}[c_n^{p-1} + a_n + \Delta_n, c_n^p)$. The length of all these intervals is much larger than c_n^k. It will be convenient to discern when j and $j + c_n^k$ lie in the same contiguous interval; when $j + c_n^k$ falls into $[c_n^p, c_n^p + a_n + \Delta_n)$ and when $c_n^p + a_n + \Delta_n \leq j + c_n^k$.

- If $j \in [c_n^{p-1} + a_n + \Delta_n, c_n^p - c_n^k)$ for some $k < p \leq \mu_n$, then $j + c_n^k \in [c_n^{p-1} + a_n + \Delta_n, c_n^p)$, so (5.7) implies that

  $$T^{c_n^k}e_j = \frac{1}{\alpha_j}T^{j+c_n^k}e_0 = \frac{\alpha_{j+c_n^k}}{\alpha_j}e_{j+c_n^k}.$$

  Observe that j and $j + c_n^k$ lie in one contiguous interval appearing in (5.12). Therefore, remembering that $k < p$, it is easy to calculate that

  $$\frac{\alpha_{j+c_n^k}}{\alpha_j} = 2^{c_n^k/c_n^{p-1}} \leq 2.$$

  Hence, by (5.16),

  $$\frac{\|T^{c_n^k}e_j\|_{N_n}}{\|e_j\|_{N_n+1}} = \frac{\alpha_{j+c_n^k}}{\alpha_j}\frac{A_{N_n,j+c_n^k}}{A_{N_n+1,j}} \leq 2\frac{A_{N_n,2j}}{A_{N_n+1,j}} \leq 1.$$

- If $j \in [c_n^p - c_n^k, c_n^p - c_n^k + a_n + \Delta_n)$ for some $k < p \leq \mu_n$ then $j + c_n^k \in [c_n^p, c_n^p + a_n + \Delta_n)$. Therefore by (5.7)

  $$T^{c_n^k}e_j = \frac{1}{\alpha_j}T^{j+c_n^k}e_0 = \frac{1}{\alpha_j}\left(\frac{1}{D_nA_{N_n,c_n^p}}e_{j+c_n^k} + Q_{n,p}(T)T^{j+c_n^k-c_n^p}e_0\right).$$

By Corollary 5.8 and the definition (5.11), we have that

$$\left\| Q_{n,p}(T) T^{j+c_n^k-c_n^p} e_0 \right\|_{N_n} \le n D_n 2^{a_n+\Delta_n+n} 2^{a_n+\Delta_n} \| e_0 \|_{N_n}.$$

Therefore we can estimate, using (5.4),

$$\begin{aligned} \frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} &\le \frac{1}{\alpha_j} \left(\frac{1}{D_n A_{N_n,c_n^p}} \frac{A_{N_n,j+c_n^k}}{A_{N_n+1,j}} + \frac{\left\| Q_{n,p}(T) T^{j+c_n^k-c_n^p} e_0 \right\|_{N_n}}{A_{N_n+1,j}} \right) \\ &\le \frac{1}{\alpha_j} \left(\frac{A_{N_n,2j}}{A_{N_n+1,j}} + \frac{n D_n 2^{a_n+\Delta_n+n} 2^{a_n+\Delta_n} A_{N_n,0}}{A_{N_n+1,c_n^2-c_n}} \right). \end{aligned}$$

The sum in the parentheses poses no problems, as in the first fraction we can use (5.16) and the second fraction can be made arbitrarily small by taking c_n large enough. So we just have to estimate α_j from below. By (5.12), α_j's are monotonically increasing for $j \in [c_n^{p-1} + a_n + \Delta_n, c_n^p)$. But $j \ge c_n^p - c_n^{p-1}$ in the case we are now considering and $p \ge 2$, hence

$$\alpha_j \ge \alpha_{c_n^p - c_n^{p-1}} = \frac{2^{\frac{c_n^p - 2c_n^{p-1} - a_n - \Delta_n}{c_n^{p-1}}}}{\left(A_{N_n,2c_n^{\mu_n}} \right)^{p+1}} > \frac{2^{c_n-3}}{\left(A_{N_n,2c_n^{\mu_n}} \right)^{\mu_n+1}} \ge 1. \tag{5.18}$$

The last fraction can indeed be made larger than 1, if c_n is chosen large enough, because, by (5.5), the denominator increases only polynomially with c_n, while the numerator increases exponentially.

- If $j \in [c_n^p - c_n^k + a_n + \Delta_n, c_n^p)$ for some $k < p \le \mu_n$, then $j + c_n^k \ge c_n^p + a_n + \Delta_n$, i.e., j is near the end of one of the intervals in (5.12) while $j + c_n^k$ "jumps over" to the beginning of the next interval. Note that the next interval may in fact be $[\Delta_{n+1}, a_{n+1})$ if $p = \mu_n$. In particular, by (5.7),

$$T^{c_n^k} e_j = \frac{1}{\alpha_j} T^{j+c_n^k} e_0 = \frac{\alpha_{j+c_n^k}}{\alpha_j} e_{j+c_n^k}.$$

  Both (5.12) and (5.8) imply that

$$\alpha_{j+c_n^k} \le \alpha_{c_n^p+c_n^k} \le 2\alpha_{c_n^p+a_n+\Delta_n} \le 2.$$

  Because $j \ge c_n^p - c_n^{p-1} + a_n + \Delta_n > c_n^p - c_n^{p-1}$, we can use the estimate from (5.18) to obtain

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} = \frac{\alpha_{j+c_n^k}}{\alpha_j} \frac{A_{N_n,j+c_n^k}}{A_{N_n+1,j}} \le 2 \frac{A_{N_n,2j}}{A_{N_n+1,j}},$$

  therefore we get the desired estimate by (5.16).

Now we are left only with $j \ge \Delta_{n+1}$ and for these j it is quite easy to get the result. We can (and will) assume that $a_{n+1} - c_n^{\mu_n} \ge \Delta_{n+1}$.

- If $j \in [\Delta_{n+1}, a_{n+1} - c_n^{\mu_n})$, then by (5.7), $T^{c_n^k} e_j = \frac{\alpha_{j+c_n^k}}{\alpha_j} e_{j+c_n^k}$. From (5.8) we get that

$$\frac{\alpha_{j+c_n^k}}{\alpha_j} \le \frac{4}{3},$$

which leads, by (5.16), to

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} = \frac{\alpha_{j+c_n^k}}{\alpha_j} \frac{A_{N_n, j+c_n^k}}{A_{N_n+1,j}} \le \frac{4}{3} \frac{A_{N_n,2j}}{A_{N_n+1,j}} \le 1.$$

- If $j \ge a_{n+1} - c_n^{\mu_n}$, then by Corollary 5.8,

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \le 2^{c_n^k} \frac{A_{N_n,j}}{A_{N_n+1,j}} \le 2^{c_n^{\mu_n}} \frac{A_{N_n,j}}{A_{N_n+1,j}}.$$

Because of (5.5), $\frac{A_{N_n,j}}{A_{N_n+1,j}} \xrightarrow[j\to\infty]{} 0$, so if a_{n+1} is much larger than $c_n^{\mu_n}$, then

$$\frac{A_{N_n,j}}{A_{N_n+1,j}} \le 2^{-c_n^{\mu_n}} \quad \text{for } j \ge a_{n+1} - c_n^{\mu_n}.$$ □

5.7 The sets K_n

This section is analogous to Section 4.8. We will prove three lemmata with almost exactly the same formulation (although one has to keep in mind that the operator T we are dealing with is a bit different now). Proofs are almost verbatim repetitions of those in Section 4.8, but we repeat them for the convenience of the reader.

Lemma 5.10. *Projection $\tau_n \colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ defined in* (5.9) *acts on the unit vectors as follows:*

$$\tau_n e_j = \begin{cases} e_j, & j < a_n; \\ -n2^n A_{N_n,a_n} T^{j-a_n} e_0, & j \in [a_n, a_n + \Delta_n). \end{cases} \tag{5.19}$$

Proof. Because $(T^j e_0)_{j=0}^{a_n+\Delta_n-1}$ is a perturbed canonical basis of $E_{a_n+\Delta_n-1}$, we always have that $e_j = \sum_{i=0}^{j} \lambda_i T^i e_0$ for some coefficients λ_i. Therefore (5.9) implies that $\tau_n(e_j) = e_j$ for $j < a_n$. On the other hand, for $j \in [a_n, a_n + \Delta_n)$ we have by (5.7) and (5.9) that

$$\tau_n \left(e_j \right) = \tau_n \left(n2^n A_{N_n,a_n} \left(T^j e_0 - T^{j-a_n} e_0 \right) \right) = -n2^n A_{N_n,a_n} T^{j-a_n} e_0.$$

□

Lemma 5.11. *If the sequence $(a_1, c_1, a_2, c_2, a_3, \dots)$ increases rapidly enough, then*

$$\left\| \tau_n y \right\|_1 \le \left\| y \right\|_{N_n+1}, \tag{5.20}$$

where $\tau_n \colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ is defined as in (5.9).

Proof. Because we deal with weighted ℓ_1 norms, it suffices to show (5.20) only for the basic vectors e_j, $j < a_n + \Delta_n$.

- If $j < a_n$, then (5.19) implies that $\|\tau_n e_j\|_1 = \|e_j\|_1 \le \|e_j\|_{N_n+1}$.
- If $j \in [a_n, a_n + \Delta_n)$, then by (5.19) and Corollary 5.8, we have the estimate

$$\frac{\|\tau_n e_j\|_1}{\|e_j\|_{N_n+1}} \le \frac{n2^n A_{N_n,a_n} 2^{\Delta_n} A_{1,0}}{A_{N_n+1,a_n}}.$$

This quantity is smaller than 1 if a_n is chosen large enough, as $\frac{A_{N_n,a_n}}{A_{N_n+1,a_n}}$ can be made arbitrarily small by (5.5). □

Lemma 5.12. *Take a sequence (n_k) such that $N_{n_k} = N$. Take $x \in \lambda^1(A)$ such that $\|x\|_1 = 1$. Then for all but finitely many k*

$$\pi_{a_{n_k}+\Delta_{n_k}-1} x \in K_{n_k}.$$

Proof. With the preceding lemma in mind, the proof is identical to the proof of Lemma 4.17. For the convenience of the reader we repeat the proof here.

In view of (5.10) we need only to show that $\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\|_1 \ge \frac{1}{2}$ holds for all but finitely many k, but with the help of (5.19) we have that:

$$\begin{aligned}
\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\|_1 &= \|\tau_{n_k}\pi_{a_{n_k}-1}x + \tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\|_1 \\
&= \|\pi_{a_{n_k}-1}x + \tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\|_1 \\
&\ge \|\pi_{a_{n_k}-1}x\|_1 - \|\tau_{n_k}\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\|_1 \\
&\ge \|\pi_{a_{n_k-1}}x\|_1 - \|\pi_{[a_{n_k},a_{n_k}+\Delta_{n_k}-1)}x\|_{N+1} \xrightarrow[k\to\infty]{} 1.
\end{aligned}$$

□

5.8 The proof

Now we are able to prove Theorem 5.1. Let us fix all the parameters appearing in the diagram (5.13) in a way that all the results of sections 5.5, 5.6 and 5.7 hold true. Then we have a continuous linear operator T on s given by (5.7). For this operator we have the following result.

Theorem 5.13. *Every non-zero vector from s is hypercyclic for T.*

Proof. Observe that if a vector is hypercyclic, then all of its scalar multiples are also hypercyclic. Therefore it is enough to show, that given $x \in s$ such that $\|x\|_1 = 1$, any $z \in s$ and N we will find a number i such that

$$\|T^i x - z\|_N \le 10.$$

This is indeed enough, because the unit balls of the norms form a basis of neighbourhoods of zero in s.

As T is a perturbed forward shift, it is clear that e_0 is a cyclic vector for T, so we can find a polynomial S such that

$$\|S(T)e_0 - z\|_N \leq 1. \tag{5.21}$$

Now by Lemma 5.12 we can find n so that the following are simultaneously satisfied:

- $\deg S \leq n$,
- $|S| \leq n$,
- $N_n = N$,
- $y := \pi_{a_n+\Delta_n-1}x \in K_n$,
- $\|x-y\|_{N+1} = \left\|\pi_{[a_n+\Delta_n,+\infty)}x\right\|_{N+1} \leq 1$.

By the definition of the net $\mathcal{S}_n$, there is a polynomial $S_{n,w} \in \mathcal{S}_n$ with $\deg S_{n,w} \leq n$ and $|S_{n,w}| \leq n$ such that

$$|S - S_{n,w}| \leq \frac{1}{2^n A_{N,0}}.$$

In particular, by Corollary 5.8,

$$\|S(T)e_0 - S_{n,w}(T)e_0\|_N \leq 2^n|S - S_{n,w}|\|e_0\|_N \leq 1. \tag{5.22}$$

By Corollary 5.6, there is a polynomial $P_{n,l} \in \mathcal{P}_n$ such that:

$$\|P_{n,l}(T)y - e_0\|_N \leq \frac{3}{n2^n},$$

so, by Corollary 5.8,

$$\|S_{n,w}(T)(P_{n,l}(T)y - e_0)\|_N \leq 2^n|S_{n,w}|\,\|P_{n,l}(T)y - e_0\|_N \leq 3. \tag{5.23}$$

For suitable coefficients λ_j we have that

$$y = \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0. \tag{5.24}$$

As $y \in K_n$, by Corollary 5.6, we also have that

$$\sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \leq D_n. \tag{5.25}$$

Now, let $k = \rho_n(l,w)$. Then $k \leq \mu_n$, hence by Lemma 5.9,

$$\left\|T^{c_n^k}(x-y)\right\|_N \leq 3\,\|x-y\|_{N+1} \leq 3. \tag{5.26}$$

We can estimate now by the triangle inequality:

$$\begin{aligned}\left\|T^{c_n^k}x - z\right\|_N \leq{}& \left\|T^{c_n^k}(x-y)\right\|_N \\ &+ \left\|T^{c_n^k}y - P_{n,l}(T)S_{n,w}(T)y\right\|_N \\ &+ \|S_{n,w}(T)\,(P_{n,l}(T)y - e_0)\|_N \\ &+ \|S_{n,w}(T)e_0 - S(T)e_0\|_N \\ &+ \|S(T)e_0 - z\|_N\,.\end{aligned}$$

Using the inequalities (5.26), (5.23), (5.22) and (5.21), we get that

$$\left\| T^{c_n^k} x - z \right\|_N \leq \left\| T^{c_n^k} y - P_{n,l}(T) S_{n,w}(T) y \right\|_N + 8.$$

By expanding y from (5.24) and noting that, by (5.11), $Q_{n,k}(T) = P_{n,l}(T)S_{n,w}(T)$, we get

$$T^{c_n^k} y - P_{n,l}(T) S_{n,w}(T) y = \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^{c_n^k+j} e_0 - \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j Q_{n,k}(T) T^j e_0.$$

Now, by (5.7), for $j \leq a_n + \Delta_n - 1$ we have that $T^{c_n^k+j} e_0 = \frac{1}{D_n A_{N_n,c_n^k}} e_{c_n^k+j} + Q_{n,k}(T) T^j e_0$, hence

$$\begin{aligned}
\left\| \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^{c_n^k+j} e_0 - \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j Q_{n,k}(T) T^j e_0 \right\|_N &= \left\| \sum_{j=0}^{a_n+\Delta_n-1} \frac{\lambda_j}{D_n A_{N_n,c_n^k}} e_{c_n^k+j} \right\|_N \\
&\leq \sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \left\| \frac{1}{D_n A_{N_n,c_n^k}} e_{c_n^k+j} \right\|_N \\
&\leq \frac{A_{N,c_n^k+a_n+\Delta_n}}{D_n A_{N_n,c_n^k}} \sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \\
&\leq 2
\end{aligned}$$

by (5.15) and (5.25). So finally, $\left\| T^{c_n^k} x - z \right\|_N \leq 10$ as required. □

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