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Mean-value principle under anticipated utility theory

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Abstract

In this paper we study properties of modified (adjusted to rank-dependent utility theory) mean-value principle without the requirements of concavity or differentiability of utility function u and probability distortion function g. Functions u and g of such type appear in the latest papers by economists analysing the problem of optimal choices. Previous results (see e.g. Gerber [6]) were accomplished by solving differential equations, whereas in this paper we find solutions of functional equations.

Introduction

Expected utility theory by von Neumann-Morgenstern is a simple model describing human behaviour under risk and uncertainty. Mean-value principle and zero utility principle, both based on this theory, are widely described in insurance literature. As far as we know, these premiums were firstly suggested by Pratt [17], though Hardy, Littlewood and Polya [9] examined the quasilinear mean-value functional $u^{-1}(Eu(X))$ earlier. Calculation of both premiums is based on the utility function u. Gerber [6] in the book constituting the foundations of determining different kinds of premiums, as well as Goovaerts, De Vylder, Haezendonck [8] or Rolski, Schmidli, Schmidt, Teugels [20] study the properties of mean-value and/or zero utility principle assuming concavity (convexity) and twice or even triple differentiability of the utility function u. Famous paradoxes (e.g. Allais [3], Rabin [19], Yaari [25]) demonstrate the weakness of classical expected utility theory and lead to other classes of utility functions which are not differentiable.

Friedman and Savage [5] propose utility functions which are differentiable, but have two inflection points. As the critics of this paper, Markowitz [15] finds it plausible to use utility functions which have only one inflection point located nearby the investor's current wealth. Influenced by these two ideas, Gillen and Markowitz [7] suggest a class of utility functions which are differentiable but piecewise convex and concave. The analysis of certain subclasses of these functions allows us to determine the importance of wealth for the investor, as well as characterize aversion or willingness to risk. Schmidt and Zank [22] apply piecewise linear, that is not differentiable, utility functions to portfolio selection and insurance demand.

In a breakthrough paper Kahneman and Tversky [11] suggest utility functions which are convex for negative arguments and concave for positive arguments such that $u'_+(0) < u'_-(0)$, where u'_+ i u'_- denote the right- and left-sided derivatives of u, respectively. This class of utility functions was set apart based on numerous experiments whose aim was to study how people behave under uncertainty. Prospect theory introduced by Kahneman and Tversky has already been modified by some economists.

In completion to the prospect theory, Kőszegi and Rabin [12] notice that making decisions under uncertainty increases risk aversion if the risk is expected. They introduce the notion of reference points which influence a decision maker to take a certain action under uncertainty. Reference points are allocated basing on the beliefs of a decision maker concerning a possible outcome and they can be determined in a stochastic way. Taking action relies on maximizing the functional

$$E_F \int u(w|r)\, dG(r),$$

where u is the utility function proposed by Kahneman and Tversky, w is the wealth with distribution F and G is a probability distribution function of a discrete random variable R with finite support.

Kahneman and Tversky also posite that probabilities are distorted while making decisions under risk. This discovery was a motivation for economists (e.g. Yaari [25], Segal [23], Puppe [18], Abdellaoui [1]) who examined and justified these phenomena. Schmidt, Starmer and Sugden [21] develop the results by Kahneman and Tversky describing reference dependent utility models, focusing simultaneously on comparing consequences which arise from making a certain decision. In this case utility functions depend on utility reference, they are piecewise concave and convex and not differentiable in many points. The results are obtained under the assumption that probabilities while making decisions are distorted by a certain function.

Based on these observations, the rank-dependent utility model originated (see e.g. Segal [23]), in which we assume that probabilities characterizing random variable X are distorted by a function $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$ and g is non-decreasing, called probability distortion function. We denote briefly by $g \in \mathcal{G}$ if g is a probability distortion function satisfying aforementioned three requirements. For fixed $g \in \mathcal{G}$ and random variable X let us define

$$E_g X := \int_{-\infty}^{0} \left(g\left(P\left(X > t\right)\right) - 1\right) dt + \int_{0}^{\infty} g\left(P\left(X > t\right)\right) dt, \tag{1}$$

provided both integrals are finite. The functional $E_g X$ is called the Choquet integral. If random variable X takes values $x_1 < x_2 < ... < x_n$ with probabilities $P(X = x_i) = p_i > 0$, then $E_g X = x_1 + \sum_{i=1}^{n-1} g(q_i)(x_{i+1} - x_i)$, where $q_i = \sum_{k=i+1}^{n} p_k$; in particular for $n = 2$ we have $E_g X = x_1(1 - g(p_2)) + g(p_2)x_2$. The Choquet integral is a functional which is additive, positively homogenous, monotonous (i.e. $E_g X \geq E_g Y$ if $X \geq Y$ a.e.) and $E_g(c) = c$ for all $c \in \mathbb{R}$. Moreover

$$E_g(-X) = -E_{\overline{g}} X, \tag{2}$$

where $\overline{g}(x) = 1 - g(1 - x)$ (see Denneberg [4]).

The aim of this paper is to analyse the properties of mean-value principle with possibly weak requirements concerning functions u and g. In the second section we introduce the mean-value principle based on rank-dependent utility theory. In Section 3 we study the properties of mean-value principle. Section 4 establishes the necessary and sufficient conditions for mean-value principle to satisfy risk loading property. Results presented in this paper are original and achieved along with Marek Kałuszka.

Mean-value principle

Let us assume that $u : \mathbb{R} \to \mathbb{R}$ is a utility function and X is an arbitrary, non-negative random variable. Consider an individual that owns an initial wealth $w \geq 0$ and is exposed to a random loss X. To protect oneself from the loss, an individual buys an insurance policy which pays the monetary equivalent of loss, provided it happens. If w and $g \in \mathcal{G}$ are fixed, then mean-value principle $H(X)$ for insuring X is the solution of

$$u(w - H(X)) = E_g[u(w - X)]. \tag{3}$$

We assume that all random variables are defined on a probability space $(\Omega, \mathcal{A}, P)$. We also denote $X \in \mathcal{X}_2$ if $P(X = 0) = 1 - q$, $P(X = s) = q$, where $s > 0$ and $q \in [0, 1]$ are arbitrary.

Now, let us establish the weakest assumptions which function u should satisfy so that the premium $H(X)$ always exists and is determined uniquely. It is commonly accepted for u to be non-decreasing. However, if u were constant on any interval, then the premium would not be determined uniquely. Therefore we assume that u is increasing. It is also required from u to be continuous. Otherwise, equation (3) may have no solutions. Without loss of generality we can assume that $u(0) = 0$. Reassuming, we will consider utility functions u which are increasing, continuous and $u(0) = 0$. To simplify notation we write $u \in \mathcal{U}$ if u satisfies these three conditions. We also write $u \in \mathcal{U}_0$ if $u(x) = cx$, $u(x) = (e^{cx} - 1)/a$ or $u(x) = (1 - e^{-cx})/a$ for all $x \in \mathbb{R}$ and some $a, c > 0$.

If utility function is linear, i.e. $u(x) = cx$, $c > 0$, then the right-hand side of equation (3) takes the form

$$E_g[u(w - X)] = cE_g[w - X] = c(w + E_g(-X)) = c(w - E_{\overline{g}}(X)). \tag{4}$$

Thus

$$H(X) = E_{\overline{g}}(X) \text{ for } u(x) = cx. \tag{5}$$

Properties of premium determined from (3) were examined by Luan [14] under additional assumptions of convexity and twice differentiability of function g. These strong assumptions are not satisfied by some important families of probability distortion functions. For instance, $g(x) = \mathbf{1}_{\{x>p\}}$ corresponding to VaR principle is not continuous and both functions

$$g(x) = \begin{cases} (1+r)\,x & \text{for } 0 \le x < 0,5 \\ r + (1-r)\,x & \text{for } 0,5 \le x \le 1 \end{cases}$$

and

$$g(x) = \begin{cases} \frac{x}{1-\alpha} & \text{for } x < 1-\alpha \\ 1 & \text{for } x \ge \alpha \end{cases}$$

describing Denneberg absolute deviation principle and CVaR principle, respectively, are not differentiable (e.g. Wang [24]).

Further, we will discuss properties of premium $H(X)$ which is the solution of (3). Notation and the names of properties are the same as in [26].

Properties of mean-value principle

1. No rip-off: $H(X) \le \sup X$.

This property is satisfied for all $u \in \mathcal{U}$ and $g \in \mathcal{G}$. Since $w - X \ge w - \sup X$, it follows from the monotonicity of u and Choquet's integral that

$$u(w - H(X)) = E_g[u(w - X)] \ge E_g[u(w - \sup X)] = u(w - \sup X).$$

Thus $H(X) \le \sup X$.

2. No unjustified risk loading: $H(a) = a$ *for all* $a \ge 0$.

This property is also satisfied for all $u \in \mathcal{U}$ and $g \in \mathcal{G}$. We have

$$u(w - H(a)) = E_g[u(w - a)] = u(w - a).$$

Hence $H(a) = a$.

3. Translation invariance: $H(X + b) = H(X) + b$ *for all* $b \ge 0$.

From (3) we have

$$u(w - b - H(X)) = E_g[u(w - b - X)] = E_g[u(w - (X + b))] \tag{6}$$

$$= u(w - H(X + b)) = u(w - b - (H(X + b) - b)). \tag{7}$$

The monotonicity of u implies that $H(X + b) = H(X) + b$. However, let us notice that in the left-hand side of (6) premium is calculated under the initial wealth $w - b$, whereas in the right-hand side of (7) premium is calculated under the initial capital w. According to this observation, the premium is translation invariant if it does not depend on w. Further results concerning translation invariance of the mean-value principle are stated in Theorem 1.

Theorem 1 *Let $u \in \mathcal{U}$ and $g \in \mathcal{G}$.*
(i) If $u \in \mathcal{U}_0$, then $H(X)$ is translation invariant.
(ii) If g is continuous and $H(X)$ is translation invariant, then $u \in \mathcal{U}_0$.

Proof. (i) From (5) for $u(x) = ax$ we have

$$H(X+b) = E_{\overline{g}}(X+b) = E_{\overline{g}}X + b = H(X) + b,$$

which means that $H(X)$ is translation invariant. If $u(x) = (e^{cx} - 1)/a$, then $H(X) = -c^{-1}\ln\left(E_g e^{-cX}\right)$. From this and positive homogeneity of the Choquet integral it follows that

$$H(X+b) = -\frac{1}{c}\ln\left(E_g e^{-c(X+b)}\right) = -\frac{1}{c}\left[-cb + \ln\left(E_g e^{-cX}\right)\right] = b + H(X),$$

which proves the translation invariance of $H(X)$. The similar conclusion may be drawn for $u(x) = (1 - e^{-cx})/a$.
(ii) Assume that $H(X+b) = H(X) + b$ for all $b \geq 0$. Consider $X \in \mathcal{X}_2$. From (3) for $w = 0$ we obtain

$$g(q) = \frac{u(-H(X))}{u(-s)}. \tag{8}$$

Since $H(X) = 0$ for $q = 0$ and $H(X) = s$ for $q = 1$, the monotonicity and continuity of u imply that $H(X)$ is a constant and non-decreasing function of probability q and takes all values from $[0, s]$. By the translation invariance of $H(X)$ it follows that equation (3) for random variable $X + b$ can be rewritten as

$$u(-H(X) - b) = (1 - g(q))\, u(-b) + g(q)\, u(-s-b). \tag{9}$$

Substituting (8) into (9), denoting $x = -H(X)$, $y = -b$ and dividing both sides by $u(x)\, u(-s)$ yields

$$f(x, y) = f(-s, y) \tag{10}$$

for all $y \leq 0$, $s > 0$ and $-s \leq x \leq 0$, where $f(x, y) = (u(x+y) - u(y))/u(x)$. Putting $s = 1$ in (10) gives

$$f(x, y) = f(-1, y) \tag{11}$$

for all $y \leq 0$, $-1 \leq x \leq 0$. Setting $x = -1$ in (10) implies

$$f(-s, y) = f(-1, y) \tag{12}$$

for all $y \leq 0$ and $-s \leq -1$. From (11) and (12) we have $f(x, y) = f(-1, y)$ for all $x, y \leq 0$. Hence for a fixed y we obtain

$$u(x+y) = c(y)\, u(x) + u(y) \tag{13}$$

for all $x \leq 0$, where $c(y)$ is a certain function. Consider two cases.

1° $u(-\infty) > -\infty$. Without loss of generality we may assume that $u(-\infty) = -1$. Letting $x \to -\infty$ in (13) we have $c(y) = u(y) + 1$, which after substituting into (13) gives

$$u(x+y) = u(x)\,u(y) + u(x) + u(y)$$

for all $x, y \leq 0$. Writing $h(x) = u(x) + 1$ we have

$$h(x+y) = h(x)\,h(y)$$

for all $x, y \leq 0$. Since h is increasing and continuous, the only solution is $h(x) = e^{ax}$ for all $x \leq 0$ and some $a \geq 0$ (see [13, p. 349]). Thus the unique solution of (13) is $u(x) = e^{ax} - 1$ for all $x \leq 0$ and some $a > 0$.

2° $u(-\infty) = -\infty$. Dividing both sides of (13) by $u(x)$ and letting $x \to -\infty$ gives

$$c(y) = \lim_{x \to -\infty} \frac{u(x+y)}{u(x)}.$$

According to Lemma 12 (see Appendix) and equation (13) we have

$$u(x+y) = u(x)\,e^{-by} + u(y) \tag{14}$$

for all $x, y \leq 0$ and some $b \leq 0$. If $b = 0$, then we obtain the Cauchy equation, whose unique continuous and increasing solution is $u(x) = cx$, for all $x \leq 0$ and some $c > 0$ (see [13, p. 129]). Applying Lemma 13 (see Appendix) for $b > 0$ we conclude that the solution of (14) is

$$u(x) = \varphi_y(x)\,e^{-bx} + \frac{u(y)}{1 - e^{-by}}$$

for all $x \leq 0$, where $\varphi_y(\cdot)$ is an arbitrary periodic function with period y. It suffices to show that $x \to \varphi_y(x)$ is constant for all $y < 0$. For arbitrary $y_1 \neq y_2$ we have

$$\varphi_{y_1}(x) - \varphi_{y_2}(x) = e^{bx}\left(\frac{u(y_1)}{1 - e^{-by_1}} - \frac{u(y_2)}{1 - e^{-by_2}}\right) \tag{15}$$

for all $x \leq 0$. Notice that the function in the right-hand side of (15) is unbounded with respect to x if $\frac{u(y_1)}{1-e^{-by_1}} \neq \frac{u(y_2)}{1-e^{-by_2}}$, while the difference of continuous and periodic functions can not be unbounded. Thus $\frac{u(y)}{1-e^{-by}}$ is constant, so $u(x) = \left(1 - e^{-bx}\right)/a$ for all $x \leq 0$ and some $a, b > 0$. Define

$$\widehat{u}(x) = u(w+x) - u(w) \tag{16}$$

for all $x \leq 0$. By the translation invariance of $H(X)$ we deduce that u is linear or exponential for negative arguments. Set $x = -w$ in (16). Since w is arbitrary, it follows that u is either linear or exponential for $x \geq 0$. Thus $u \in \mathcal{U}_0$. ■

4. Scale invariance: $H(aX) = aH(X)$ for all $a > 0$.

Theorem 2 *Let $u \in \mathcal{U}$ and $g \in \mathcal{G}$.*
(i) If $u(x) = cx$ for some $c > 0$, then $H(aX) = aH(X)$ for some $c > 0$.
(ii) If g is continuous and $H(aX) = aH(X)$ for all $a > 0$, then $u(x) = cx$ for all $x \in \mathbb{R}$ and some $c > 0$.

Proof. If $u(x) = cx$, then it is easy to verify that $H(X)$ is scale invariant. Assume now that $H(aX) = aH(X)$. For $X \in \mathcal{X}_2$ we have

$$u(-ah) = g(q)\, u(-as) \tag{17}$$

for all $a > 0$, where $h = H(X)$. Putting $f(x) = -u(-x)$ and calculating $g(q)$ from (17) for $a = 1$, we can rewrite (17) as

$$f(ah) = f(as)\frac{f(h)}{f(s)} \tag{18}$$

for $s > 0$ and $0 \leq h \leq s$. Setting $s = 1$ in (18) we get

$$z(ah) = z(a)\, z(h)$$

for $0 \leq h \leq 1$ and $a > 0$, where $z(x) = f(x)/u(-1)$. From (18) for $h = 1$ we have

$$z(a)\, z(s) = z(as)$$

for $s \geq 1$ and $a > 0$. From the last two equations it follows that

$$z(ax) = z(a)\, z(x)$$

for $x > 0$ and $a > 0$. Since z is continuous, the only solution is $z(x) = x^d$ for all $x \geq 0$ and some $d > 0$ (see [13, p. 349]). Thus $u(x) = -c(-x)^d$ for all $x \leq 0$, some $c = -u(-1) > 0$ and $d > 0$. Now, let $\widehat{u}(x) = u(x+w) - u(w)$ for $x \leq 0$ and $w \geq 0$. Scale invariance of $H(X)$ implies

$$u(w+x) - u(w) = -c(-x)^d. \tag{19}$$

Setting $x = -w$ in (19) gives $u(w) = cw^d$ for all $w \geq 0$. Hence $u(x) = cx\,|x|^{d-1}$ for $x \in \mathbb{R}$. For such determined function u we put $x = -1$ and $w = \frac{1}{2}$ in (19) to obtain $2/2^d = 1$. Thus $d = 1$. ■

5. Additivity for comonotonic risks.

Theorem 3 *Let $u \in \mathcal{U}$ and $g \in \mathcal{G}$ be continuous. Then $H(X)$ is additive for comonotonic risks if and only if $u(x) = cx$ for some $c > 0$.*

Proof. If $u(x) = cx$, then $H(X) = E_g(X)$ and the additivity of the Choquet integral for comonotonic risks proves the first part of the theorem. If premium is additive for comonotonic risks, then it is scale invariant. From Theorem 2 it follows that $u(x) = cx$. ■

6. Additivity for independent risks.

Theorem 4 *(i) If $g(p) = p$ and $u \in \mathcal{U}_0$, then $H(X)$ is additive for independent risks.*
(ii) If $g(p) = p$ and $H(X)$ is additive for independent risks, then $u \in \mathcal{U}_0$.
(iii) Let $u \in \mathcal{U}_0$. If $g \in \mathcal{G}$ is right continuous at 0, left continuous at 1 and has the left-sided derivative at 1, then $H(X)$ is additive for independent risks if and only if $g(p) = p$.

Proof. (i) One can easily verify that the relevant premiums are additive for independent risks.

(ii) Proof is based on the idea of Gerber [6]. Assume that $H(X+Y) = H(X) + H(Y)$ for arbitrary, independent risks X, Y. In particular, if X is an arbitrary risk and Y is constant, i.e. $P(Y = d) = 1$ for some $d > 0$, then additivity for independent risks and no unjustified risk loading imply that the premium is translation invariant. From Theorem 1 we conclude that $u \in \mathcal{U}_0$.

(iii) Let $u(x) = cx$. We assume that $H(X)$ is additive for independent risks. Let $X, Y \in \mathcal{X}_2$ be independent random variables and $P(X = 1) = p$, $P(Y = 1) = q$. Then from (5) we have

$$H(X) = \overline{g}(p), H(Y) = \overline{g}(q) \tag{20}$$

and

$$H(X+Y) = \overline{g}(p+q-pq) + \overline{g}(pq). \tag{21}$$

Additivity for independent risks, (20) and (21) imply that

$$\overline{g}(p+q-pq) + \overline{g}(pq) = \overline{g}(p) + \overline{g}(q) \tag{22}$$

for all $0 \leq p, q \leq 1$. Set $q = c - p$, where $0 \leq c \leq 1$, and define a sequence $(p_n)_{n \in \mathbb{N}}$ such that $p_0 = c/2$ and $p_{n+1} = p_n(c - p_n)$. Then $(p_n)_{n \in \mathbb{N}}$ is generated by logistic difference equation (see Polyanin, Manzhirow [16, p. 875]). From (22) it follows that

$$\overline{g}(c - p_{n+1}) + \overline{g}(p_{n+1}) = \overline{g}(c - p_n) + \overline{g}(p_n) = \ldots = 2\overline{g}\left(\frac{c}{2}\right). \tag{23}$$

We have $p_{n+1}/c = c \cdot p_n/c \cdot (1 - p_n/c)$, where $c \leq 1$, hence $\lim_{n \to \infty} p_n = 0$. Function $\overline{g}$ is continuous at 0 and at 1, thus letting $n \to \infty$ in (23) we obtain

$$\overline{g}(c) = 2\overline{g}\left(\frac{c}{2}\right) \tag{24}$$

for all $0 \leq c \leq 1$. As $\overline{g}$ has the right-sided derivative at 0 (we allow $\overline{g}'(0) = \infty$), by Lemma 14 (see Appendix) it follows that $\overline{g}(p) = p$. Hence $g(p) = p$.
Assume now $u(x) = (1 - e^{-cx})/a$, $a, c > 0$. Then $H(X) = c^{-1} \ln\left(E_{\overline{g}} e^{cX}\right)$. For independent random variables $X, Y \in \mathcal{X}_2$ such that $P(X = s) = p$ and $P(Y = s) = q$ we have

$$H(X) = \frac{1}{c} \ln\left(1 + \overline{g}(p)\left(e^{cs} - 1\right)\right), H(Y) = \frac{1}{c} \ln\left(1 + \overline{g}(q)\left(e^{cs} - 1\right)\right) \tag{25}$$

$$H(X+Y)=\frac{1}{c}\ln\left(1+\overline{g}(p+q-pq)\left(e^{cs}-1\right)+\overline{g}(pq)\left(e^{2cs}-e^{cs}\right)\right). \tag{26}$$

From additivity for independent risks, (25) and (26) it follows that

$$\overline{g}(p)+\overline{g}(q)+\overline{g}(p)\,\overline{g}(q)\left(e^{cs}-1\right)=\overline{g}(p+q-pq)+\overline{g}(pq)\left(e^{cs}-1\right) \tag{27}$$

for all $0 \le p, q \le 1$. Equation (27) holds for all s which imply that $\overline{g}$ satisfies (22).
By a similar argument we prove the theorem for $u(x)=\left(e^{cx}-1\right)/a$, where $a, c>0$. ■

Remark 5 *As far as we know all functions g analysed in the actuarial literature satisfy the assumptions of Theorem 4 (e.g. Wang [24]).*

7. Subadditivity.

Theorem 6 *Let $u(x)=cx$ for some $c>0$ and $g \in \mathcal{G}$. Then $H(X)$ is subadditive if and only if g is convex.*

Proof. Let $u(x)=cx$. From (5) it follows that $H(X)=E_{\overline{g}}X$. Assume g is convex. We know that $E_g(X+Y) \le E_gX+E_gY$ if and only if g is concave. For $\overline{g}$ which is concave we have

$$H(X+Y)=E_{\overline{g}}(X+Y) \le E_{\overline{g}}X+E_{\overline{g}}Y=H(X)+H(Y). \tag{28}$$

Assume now $H(X)$ is subadditive. Then inequality (28) holds, so $\overline{g}$ is concave. ■

8. Stop-loss order preserving: $X \le_{sl} Y \Longrightarrow H(X) \le H(Y)$.

It depends on the shape of function u whether the property holds or not. If u is convex, then the premium preserves stop-loss order. Indeed, let $X \le_{sl} Y$. Then $E_{\overline{g}}[-u(w-X)] \le E_{\overline{g}}[-u(w-Y)]$. Hence $E_g[u(w-X)] \ge E_g[u(w-Y)]$, because $E_{\overline{g}}[-u(X)]=-E_g[u(X)]$. By the definition of $H(X)$ we have

$$u(w-H(Y))=E_g[u(w-Y)] \le E_g[u(w-X)]=u(w-H(X)).$$

From the monotonicity of u we get $H(Y) \ge H(X)$.

Remark 7 *In our considerations we overrode the factor of time. In more general model we can assume that the decision maker's wealth after buying (or not buying) the policy is invested for one period under risk-free interest rate r. Under this assumption we can rewrite (3) as*

$$u((w-H(X))(1+r))=E_gu(w(1+r)-X), \tag{29}$$

Denoting $\widehat{w}=w(1+r)$, $\widehat{H}(X)=H(X)(1+r)$ we can write equivalents of all properties of principle $H(X)$ determined from (29). For instance, let us consider unjustified risk loading. We have

$$u\left(\widehat{w}-\widehat{H}(X)\right)=u(\widehat{w}-a).$$

Hence $\widehat{H}(X) = a$, *so* $H(X) = a/(1+r)$ *which corresponds to the premium calculated from (3) discounted by* $(1+r)$*.*

We can easily notice that for $u \in \mathcal{U}_0$ *premium* $\widehat{H}(X)$ *has the following property:* $\widehat{H}(X) = H(X)/(1+r)$*, i.e.* $\widehat{H}(X)$ *equals to the premium* $H(X)$ *discounted by* $(1+r)$*.*

Risk loading

One of the properties characterizing the premium is risk loading, i.e. $H(X) \geq E(X)$. If this condition is not satisfied, then obviously no insurance company would decide to sell the policy. The following propositions describe in terms of rank-dependent utility theory two groups of people: those who either can afford to buy an insurance or refuse to be insured.

Proposition 8 *If* $u \in \mathcal{U}$ *is concave,* $g \in \mathcal{G}$ *is convex, then* $H(X) \geq E(X)$*.*

Proof. Equation (3) and Lemma 15 (see Appendix) imply that

$$u(w - H(X)) = E_g(u(w - X)) \leq u(E_g(w - X)),$$

thus from the monotonicity of u we have

$$E_g(X) \leq H(X). \tag{30}$$

Since g is convex, $\overline{g}$ is concave from which follows that $E_{\overline{g}}(X) \geq E(X)$. This fact along with (30) complete the proof. ■

Clearly, the condition $H(X) \geq E(X)$ is satisfied if and only if

$$E(X) \leq w - u^{-1}(E_g u(w - X)). \tag{31}$$

As the right-hand side of (31) is usually complicated to calculate, we present some sufficient conditions for risk loading.

Proposition 9 *Let* $u \in \mathcal{U}$, $g \in \mathcal{G}$, X *be non-negative and bounded random variable and* $w < s = \sup X$*. Then* $H(X)$ *satisfies* $H(X) \geq E(X)$ *if*

$$E(X) \leq w - u^{-1}\left[g(P(X < w))\,u(w) + \overline{g}(P(X = s))\,u(w - s)\right]. \tag{32}$$

If X *takes only values from the set* $\{0, w, s\}$*, then (32) is equivalent to* $H(X) \geq E(X)$*.*

Proof. Define $Y = 0$ if $X < w$, $Y = w$ if $w \leq X < s$ and $Y = s$ if $X = s$. Since $Y \leq X$, from the monotonicity of the Choquet integral we have

$$\begin{aligned} u(w - H(X)) &= E_g u(w - X) \leq E_g u(w - Y) \\ &= u(w - s) - g(P(X < s))\,u(w - s) + g(P(X < w))\,u(w) \\ &= g(P(X < w))\,u(w) + \overline{g}(P(X = s))\,u(w - s). \end{aligned}$$

This inequality and (32) imply $H(X) \geq E(X)$. ■

Proposition 10 *Let $u \in \mathcal{U}$ and $g \in \mathcal{G}$. Then $H(X) \geq E(X)$ if*

$$E(X) \leq w - u^{-1}(u(w)\, g(P(X < w))). \tag{33}$$

If $P(X = 0) + P(X = w) = 1$, then (33) is equivalent to $H(X) \geq E(X)$.

Proof. Define $Y = 0$ for $X < w$ and $Y = w$ for $X \geq w$. Then $X = Y$ if $P(X = 0) + P(X = w) = 1$. Clearly $Y \leq X$, so

$$\begin{aligned} u(w - H(X)) &= E_g u(w - X) \leq E_g u(w - Y) \\ &= g(P(X < w))\, u(w). \end{aligned}$$

This inequality along with (33) complete the proof. ■

Notice that in Proposition 10 we do not assume that X is bounded. We will now present a sufficient condition for customer to refuse to buy an insurance, if the decision is taken under the Kahneman-Tversky utility function.

Proposition 11 *Let $g \in \mathcal{G}$ and $u \in \mathcal{U}$ be convex for negative arguments, concave for positive arguments and $u'_+(0) < u'_-(0)$. Let $0 \leq X \leq s$, where $s > w$. If*

$$E(X) \geq w - u^{-1}\left(\frac{u(w)}{w} g(P(X \leq w)) - u'_-(0)\, E_{\overline{g}}(X - w)_+\right), \tag{34}$$

then $H(X) < E(X)$. If $P(X = 0) + P(X = w) = 1$, then (34) is equivalent to $H(X) < E(X)$.

Proof. Functions $h_1(x) = \frac{u(w)}{w}\mathbf{1}_{\{x \leq w\}}$ and $h_2(x) = u'_-(0)(w - x)\mathbf{1}_{\{x > w\}}$ are non-decreasing, so random variables $\frac{u(w)}{w}\mathbf{1}_{\{X \leq w\}}$, $u'_-(0)(w - X)\mathbf{1}_{\{X > w\}}$ are comonotonic. Moreover, $u(x) \geq u'_-(0)$ for $x < 0$ and $u(x) > xu(w)/w$ for $0 < X \leq w$. Thus (34), (2) and additivity of the Choquet integral for comonotonic risks gives

$$\begin{aligned} u(w - H(X)) &= E_g u(w - X) \geq E_g\left[\frac{u(w)}{w}\mathbf{1}_{\{X \leq w\}} + u'_-(0)(w - X)\mathbf{1}_{\{X > w\}}\right] \\ &= -E_{\overline{g}}\left[-\frac{u(w)}{w}\mathbf{1}_{\{X \leq w\}} + u'_-(0)(X - w)_+\right] \\ &= \frac{u(w)}{w} g(P(X \leq w)) - u'_-(0)\, E_{\overline{g}}(X - w)_+ > u(w - E(X)). \end{aligned}$$

■

Appendix

Lemma 12 *If u is increasing and continuous, then for some $b > 0$ we have*

$$\lim_{x \to -\infty} \frac{u(x + y)}{u(x)} = e^{-by}, \tag{35}$$

as far as the limit exists and is finite.

Proof. Let $x, y \leq 0$. Then

$$g(x+y) := \lim_{z \to -\infty} \frac{u(z+x+y)}{u(z)} = \lim_{z+y \to -\infty} \frac{u(z+y+x)}{u(z+y)} \cdot \lim_{z \to -\infty} \frac{u(z+y)}{u(z)} = g(x)\, g(y)\, .$$

Function u is increasing, so $u(x+y) < u(x) < 0$ for $x, y < 0$ from which it follows that $g(x) > 1$ for $x < 0$. Notice that $h(x) = \ln g(x)$ satisfies the Cauchy equation, whose unique non-negative solution is a linear function (see Aczél [2, p. 34]). This fact and the monotonicity of g imply that $g(x) = e^{-bx}$ for some $b > 0$. ■

Lemma 13 *Let $a, b > 0$ and $c \in \mathbb{R}$. The unique solution of the functional equation $u(x+a) = bu(x) + c$, $x \geq 0$ (or $x \leq 0$, or $x \in \mathbb{R}$) is:*
1° $u(x) = \varphi(x) + \frac{c}{a}x$ if $b = 1$;
2° $u(x) = \varphi(x)\, b^{x/a} + \frac{c}{1-b}$ if $b \neq 1$,
where φ is an arbitrary periodic function with period a.

Proof. Lemma may be found in [16, p. 890]. Below we present the sketch of the proof.
1° Consider $\varphi(x) = u(x) - \frac{c}{a}x$. One can easily verify that φ is a periodic function with period a, thus u takes the form as in the thesis of 1°.
2° It is easy to check that $\varphi(x) = \left[u(x) - \frac{c}{1-b}\right] b^{-x/a}$ is periodic with period a, which follows that $u(x) = \varphi(x)\, b^{x/a} + \frac{c}{1-b}$. ■

Lemma 14 *If the domain of u is $[0, 1/2]$, then the general solution of*

$$u(2x) = 2u(x) \tag{36}$$

is $u(x) = xh(\ln x)$, where h is a periodic function with period $\ln 2$ and $0 \cdot h(-\infty) = 0$. If we additionally assume that u has the right-sided derivative at $x = 0$ (we allow $u'_{+}(0) = \infty$), then the only solution is $u(x) = cx$ for some $c > 0$.

Proof. Putting $x = 0$ in (36) implies $u(0) = 0$. Let u be a solution of (36). It is easy to check that $h(t) = e^{-t}u(e^{t})$ is periodic with period 2. Setting $x = e^{t}$ we have $u(x) = xh(\ln x)$ for $x > 0$, where h is an arbitrary periodic function with period $\ln 2$. Since u has the right-sided derivative at $x = 0$ and

$$u'(0) = \lim_{x \to 0^{+}} \frac{u(x)}{x} = \lim_{x \to 0^{+}} h(\ln x)\, ,$$

h is constant, as a periodic function which has the limit in $-\infty$. ■

Lemma 15 *If u is non-decreasing and convex, then for arbitrary $g \in \mathcal{G}$ and all random variables X such that $E_g |X| < \infty$ we have $E_g(u(X)) \geq u(E_g(X))$. If u is concave, then $E_g(u(X)) \leq u(E_g(X))$.*

Proof. Proof of lemma can be found in [10]. ■

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